

# Kernel of the monodromy operator for semistable curves

By

Valentina DI PROIETTO\*

## Abstract

This is an announcement of results, obtained in collaboration with B. Chiarellotto, R. Coleman and A. Iovita, whose proofs will be published elsewhere. We study the relation between the kernel of the monodromy operator acting on the first log crystalline cohomology group of the special fiber of a semistable curve and the first rigid cohomology group with non trivial coefficients.

## § 1. Notations and preliminaries

Let  $k$  be a perfect field of characteristic  $p > 0$ . We denote by  $W$  the ring of Witt vectors of  $k$  and by  $K$  its fraction field.

Let  $X$  be a projective curve over  $\text{Spec}(W)$  with semistable reduction. We denote by  $X_k$  the special fiber of  $X$ , which we suppose is a union of smooth irreducible components and by  $X_K$  its generic fiber. We suppose that  $X_k$  is connected and that it has at least two irreducible components. For simplicity, we suppose also that all the intersection points of irreducible components of  $X_k$  are  $k$ -rational.

In this situation, we consider the notion of convergent  $F$ -isocrystals on  $X_k/\text{Spf}(W)$ , as defined by Berthelot in [1] and Ogus in [9]. Since  $X$  is projective over  $\text{Spec}(W)$ , we can embed  $X_k$  into a  $p$ -adic formal scheme  $P$  which is formally smooth over  $\text{Spf}(W)$ . Hence, as described in [1], we can construct the tube of  $X_k$  in  $P$ , denoted by  $]X_k[_P$ . Every convergent  $F$ -isocrystal  $E$  on  $X_k/\text{Spf}(W)$  naturally induces a module with integrable connection  $(\mathcal{E}, \nabla)$  on  $]X_k[_P$  and the first rigid cohomology group is defined as  $H_{\text{rig}}^1(X_k/K, E) := H_{\text{dR}}^1(]X_k[_P, (\mathcal{E}, \nabla))$ .

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\*Graduate School of Mathematical Sciences, University of Tokyo, Komaba 153-8914 Tokyo, Japan.  
e-mail: [proietto@ms.u-tokyo.ac.jp](mailto:proietto@ms.u-tokyo.ac.jp)

One can also consider the rigid analytic space  $X_K^{\text{rig}}$  associated to the  $p$ -adic completion of  $X$ . Since we have an immersion  $X_K^{\text{rig}} \hookrightarrow X_k[P]$ , every convergent  $F$ -isocrystal induces a module with integrable connection on  $X_K^{\text{rig}}$  by pullback, which we also denote by  $(\mathcal{E}, \nabla)$ . Then we can consider its first de Rham cohomology  $H_{\text{dR}}^1(X_K^{\text{rig}}, (\mathcal{E}, \nabla))$ .

Note that we can equip the curve  $X$  with a log structure associated to the special fiber  $X_k$  which is a divisor with simple normal crossing and we can equip  $\text{Spec}(W)$  with a log structure associated to the closed point. By pulling them back, we can also equip  $X_k$  and  $\text{Spec}(k)$  with log structures, and we denote the resulting log schemes by  $X_k^\times$  and  $\text{Spec}(k)^\times$ , respectively. The log structure on  $\text{Spec}(W)$  induces a log structure on  $\text{Spf}(W)$ ; again, we denote the resulting formal log scheme by  $\text{Spf}(W)^\times$ . We can consider the first log crystalline cohomology with coefficients in a convergent  $F$ -isocrystal  $E$ , which we denote by  $H_{\text{log-crys}}^1(X_k^\times/\text{Spf}(W)^\times, E)$ ; thanks to [7] Section 2 (see also [6] Section 2.1), we have an isomorphism  $H_{\text{log-crys}}^1(X_k^\times/\text{Spf}(W)^\times, E) \cong H_{\text{dR}}^1(X_K^{\text{rig}}, (\mathcal{E}, \nabla))$  and we have the monodromy operator acting on this cohomology group, which we denote by  $N_{\mathcal{E}}$  in this paper.

In [6] the authors describe the action of the monodromy operator  $N_{\mathcal{E}}$  on  $H_{\text{dR}}^1(X_K^{\text{rig}}, (\mathcal{E}, \nabla))$  explicitly by using certain admissible covering of  $X_K^{\text{rig}}$ . In this paper, we recall their description and we announce some results on the relation between the kernel of the monodromy operator  $N_{\mathcal{E}}$  and the rigid cohomology  $H_{\text{rig}}^1(X_k/K, E)$  which we prove by using their description: in the case of trivial coefficients, we give another proof of a theorem of B. Chiarellotto ([2]) saying that the kernel of the monodromy operator is isomorphic to the rigid cohomology  $H_{\text{rig}}^1(X_k/K)$ . Moreover, we announce a theorem that, if  $E$  is a unipotent convergent  $F$ -isocrystal, the kernel of the monodromy operator  $N_{\mathcal{E}}$  on  $H_{\text{dR}}^1(X_K^{\text{rig}}, (\mathcal{E}, \nabla))$  does not necessarily coincide with the rigid cohomology group  $H_{\text{rig}}^1(X_k/K, E)$ .

## § 2. A Mayer-Vietoris exact sequence

First we recall the construction of the graph  $Gr(X_k)$  attached to  $X_k$  (e.g. [5]): we associate a vertex  $v$  to every irreducible component  $C_v$  of  $X_k$ , and we associate an edge  $e$  connecting  $v$  and  $w$  to every intersection point  $C_e$  of the irreducible components  $C_v, C_w$  of  $X_k$ . We can regard  $Gr(X_k)$  as an oriented graph: for any edge of the graph  $e$  we choose an orientation of it and given two vertices of the graph  $v$  and  $w$ , we denote the set of edges connecting  $v$  and  $w$ , with origin  $v$  and end  $w$ , by  $[v, w]$ . We denote by  $\mathcal{V}$  the set of vertices and by  $\mathcal{E}$  the set of oriented edges.

As described in [1] there is a specialization map  $\text{sp} : X_K^{\text{rig}} \rightarrow X_k$ . For every  $v \in \mathcal{V}$  we define  $X_v := \text{sp}^{-1}(C_v)$  and for every  $e \in \mathcal{E}$  we define  $X_e := \text{sp}^{-1}(C_e)$ . Then the set  $X_e$  is an open annulus in  $X_K^{\text{rig}}$  and  $X_v$  is what Coleman calls a wide open subspace ([4] proposition 3.3), namely an open of  $X_K^{\text{rig}}$  isomorphic to  $X_K^{\text{rig}}$  minus a finite number of

closed disks contained in residue classes.

The covering  $\{X_v\}_{v \in \mathcal{V}}$  is an admissible covering of  $X_K^{\text{rig}}$  and wide opens are Stein spaces, since every wide open  $X_v$  can be written as a union of an affinoid (inverse image via the specialization map of an affine subscheme of the special fiber) and a finite number of open annuli. So we can use the covering  $\{X_v\}_{v \in \mathcal{V}}$  to calculate the de Rham cohomology of  $X_K^{\text{rig}}$  via a Čech complex. Note also that every intersection of three distinct elements in  $\{X_v\}_{v \in \mathcal{V}}$  is empty. Hence, for a convergent  $F$ -isocrystal  $E$ , we have the Mayer-Vietoris exact sequence:

$$(2.1) \quad \begin{array}{c} \oplus_{v \in \mathcal{V}} H_{\text{dR}}^0(X_v, (\mathcal{E}, \nabla)) \xrightarrow{\alpha} \oplus_{e \in \mathcal{E}} H_{\text{dR}}^0(X_e, (\mathcal{E}, \nabla)) \longrightarrow H_{\text{dR}}^1(X_K^{\text{rig}}, (\mathcal{E}, \nabla)) \\ \longleftarrow \oplus_{v \in \mathcal{V}} H_{\text{dR}}^1(X_v, (\mathcal{E}, \nabla)) \xrightarrow{\beta} \oplus_{e \in \mathcal{E}} H_{\text{dR}}^1(X_e, (\mathcal{E}, \nabla)) \end{array}$$

We remark that we can calculate the cohomologies in the above exact sequence except  $H_{\text{dR}}^1(X_K^{\text{rig}}, (\mathcal{E}, \nabla))$  as the cohomology of the complex of global sections, since wide opens are Stein spaces. We remark also that, since there is a comparison theorem between the de Rham cohomology of wide open subspaces and the de Rham cohomology of certain algebraic curves ([4] theorem 4.2), the cohomologies in the above exact sequences are finite dimensional. From the exact sequence (2.1), we can deduce the short exact sequence

$$(2.2) \quad 0 \longrightarrow \text{Coker}(\alpha) \xrightarrow{\gamma} H_{\text{dR}}^1(X_K^{\text{rig}}, (\mathcal{E}, \nabla)) \longrightarrow \text{Ker}(\beta) \longrightarrow 0.$$

When  $E$  is the trivial isocrystal, then  $\text{Coker}(\alpha)$  coincides with the first Betti cohomology of the graph  $Gr(X_k)$ .

### § 3. Monodromy operator

Let the notations be as in the previous section. We recall here the explicit definition of the monodromy operator  $N_{\mathcal{E}}$  on the first de Rham cohomology  $H_{\text{dR}}^1(X_K^{\text{rig}}, (\mathcal{E}, \nabla))$  given by Coleman and Iovita in [6] section 2.2.

Since wide opens are Stein spaces, every cohomology class  $[\omega]$  in  $H_{\text{dR}}^1(X_K^{\text{rig}}, (\mathcal{E}, \nabla))$  is represented by a hypercocycle  $((\omega_v)_{v \in \mathcal{V}}, (f_e)_{e \in \mathcal{E}})$ , where  $\omega_v \in \Gamma(X_v, \Omega_{X_K^{\text{rig}}}^1 \otimes \mathcal{E})$  and  $f_e \in \Gamma(X_e, \mathcal{E})$ , which satisfies  $\omega_v|_{X_e} - \omega_w|_{X_e} = \nabla(f_e)$  for any  $v, w$  with  $e \in [v, w]$ .

The orientation on  $e$  induces an orientation on  $X_e$ . Using this, we define a residue map

$$\text{Res} : H_{\text{dR}}^1(X_e, (\mathcal{E}, \nabla)) \longrightarrow H_{\text{dR}}^0(X_e, (\mathcal{E}, \nabla))$$

as follows. The module with connection  $(\mathcal{E}, \nabla)$  has a basis of horizontal sections  $e_1, \dots, e_n$  on  $X_e$ , because  $X_e$  is a residue class (lemma 2.2 of [6]). Hence if  $z$  is a uniformizer of the oriented annulus  $X_e$  every differential form  $\mu_e \in H_{\text{dR}}^1(X_e, (\mathcal{E}, \nabla))$  can be written as  $\mu_e = \sum_{i=1}^n (e_i \otimes \sum_j a_{i,j} z^j dz)$  with  $a_{i,j} \in K$ . Then we define the morphism Res by  $\text{Res}(\mu_e) := \sum_{i=1}^n a_{i,-1} e_i$ .

One can prove that the morphism

$$\tilde{N}_{\mathcal{E}} : H_{\text{dR}}^1(X_K^{\text{rig}}, (\mathcal{E}, \nabla)) \longrightarrow \bigoplus_{e \in \mathcal{E}} H_{\text{dR}}^0(X_e, (\mathcal{E}, \nabla))$$

given by  $[\omega] = ((\omega_v)_{v \in \mathcal{V}}, (f_e)_{e \in \mathcal{E}}) \mapsto (\text{Res}(\omega_v|_{X_e})_{e \in \mathcal{E}})$  is well-defined. Also, we have the morphism  $i : \bigoplus_{e \in \mathcal{E}} H_{\text{dR}}^0(X_e, (\mathcal{E}, \nabla)) \longrightarrow \text{Coker}(\alpha) \longrightarrow H_{\text{dR}}^1(X_K^{\text{rig}}, (\mathcal{E}, \nabla))$  induced by the Mayer-Vietoris sequence, which can be explicitly written as

$$i((f_e)_{e \in \mathcal{E}}) = \text{the class of } (0, (f_e)_{e \in \mathcal{E}}).$$

Then the monodromy operator  $N_{\mathcal{E}}$  on  $H_{\text{dR}}^1(X_K^{\text{rig}}, (\mathcal{E}, \nabla))$  is defined as  $N_{\mathcal{E}} = i \circ \tilde{N}_{\mathcal{E}}$ .

#### § 4. Monodromy and rigid cohomology

Keep the same hypothesis as before. Since  $X$  is projective,  $X_k$  can be embedded in a  $p$ -adic formal scheme  $P$  smooth over  $\text{Spf}(W)$ . Let us denote by  $P_k$  its special fiber and by  $P_K^{\text{rig}}$  the rigid analytic space associated to  $P$ . We have the following diagram:

$$\begin{array}{ccccc} & & & P_K^{\text{rig}} & \\ & & & \downarrow & \\ & & \text{sp}_P & \swarrow & \\ X_k & \longrightarrow & P_k & \longrightarrow & P, \end{array}$$

where the map between  $P_K^{\text{rig}}$  and  $P_k$  is the specialization. We put  $]X_k[_P := \text{sp}_P^{-1}(X_k)$  and define the first rigid cohomology group with coefficients in a convergent  $F$ -isocrystal  $E$  as  $H_{\text{rig}}^1(X_k/K, E) := H_{\text{dR}}^1(]X_k[_P, (\mathcal{E}, \nabla))$ . On the other hand, we can complete the above diagram as follows:

$$\begin{array}{ccccc} X_K^{\text{rig}} & & & P_K^{\text{rig}} & \\ \downarrow \text{sp}_X & & & \downarrow & \\ X_k & \longrightarrow & P_k & \longrightarrow & P, \end{array}$$

where the arrow  $\text{sp}_X : X_K^{\text{rig}} \rightarrow X_k$  is again the specialization map, as in section 2. We have a morphism

$$\varphi : X_K^{\text{rig}} \longrightarrow ]X_k[_P$$

induced by the immersion of  $X$  into  $P$ . It induces a map

$$(4.1) \quad \varphi^* : H_{\text{rig}}^1(X_k/K, E) = H_{\text{dR}}^1(\lrcorner X_k \lrcorner P, (\mathcal{E}, \nabla)) \longrightarrow H_{\text{dR}}^1(X_K^{\text{rig}}, (\mathcal{E}, \nabla)).$$

Using  $\varphi^*$  and  $N_{\mathcal{E}}$  we can form the following sequence

$$(4.2) \quad H_{\text{rig}}^1(X_k/K, E) \xrightarrow{\varphi^*} H_{\text{dR}}^1(X_K^{\text{rig}}, (\mathcal{E}, \nabla)) \xrightarrow{N_{\mathcal{E}}} H_{\text{dR}}^1(X_K^{\text{rig}}, (\mathcal{E}, \nabla)).$$

In [2] (proposition 4.10), Chiarellotto proved the following theorem, with the additional hypothesis that  $k$  is a finite field. (We remark also that higher dimensional versions of theorem 4.1 are proven by Nakajima in [8] theorem 8.3 and theorem 9.1.)

**Theorem 4.1.** *If  $(\mathcal{E}, \nabla)$  is the module with connection  $(\mathcal{O}_{X_K^{\text{rig}}}, d)$  induced by the trivial  $F$ -isocrystal  $\mathcal{O}$ , then, in the sequence (4.2), the map  $\varphi^*$  is injective and  $\text{Im}(\varphi^*) = \text{Ker}(N_{\mathcal{O}})$ .*

We study the sequence (4.2) when  $E$  is not necessarily trivial. We can prove the following proposition for any convergent  $F$ -isocrystal  $E$ .

**Proposition 4.2.** *If  $E$  is a convergent  $F$ -isocrystal and  $(\mathcal{E}, \nabla)$  is the module with integrable connection induced by it, then the map  $\varphi^*$  in the sequence (4.2) is injective and*

$$N_{\mathcal{E}} \circ \varphi^* = 0.$$

To study the exactness of the sequence (4.2) we find a criterion to detect when an element in  $H_{\text{dR}}^1(X_K, (\mathcal{E}, \nabla))$  is in the image of  $\varphi^*$ .

**Proposition 4.3.** *Let us take  $[\omega] \in H_{\text{dR}}^1(X_K, (\mathcal{E}, \nabla))$  and, as before, a representative  $((\omega_v)_{v \in \mathcal{V}}, (f_e)_{e \in \mathcal{E}})$ . Then  $\text{Res}_{X_e}(\omega_v|_{X_e}) = 0$  for every  $e \in \mathcal{E}$  if and only if  $[\omega] \in \text{Im}(\varphi^*)$ .*

When  $E = \mathcal{O}$  is the trivial  $F$ -isocrystal, we prove that if  $[\omega] \in H_{\text{dR}}^1(X_K^{\text{rig}}, (\mathcal{O}_{X_K^{\text{rig}}}, d))$  belongs to  $\text{Ker}(N_{\mathcal{O}})$ , then  $\text{Res}_{X_e}(\omega_v|_{X_e}) = 0$  for every  $e \in \mathcal{E}$ , i.e. the condition in proposition 4.3 is fulfilled. We prove this fact using graph theory. This implies that the sequence (4.2) is exact, giving a new proof of theorem 4.1.

## § 5. Main result: the case of unipotent coefficients

In this section, we announce our result that (4.2) fails to remain exact in general when  $E$  is a unipotent convergent  $F$ -isocrystal.

Consider the following non-split exact sequence of convergent  $F$ -isocrystals

$$(5.1) \quad 0 \longrightarrow E \longrightarrow G \longrightarrow \mathcal{O} \longrightarrow 0,$$

and assume  $E$  is a unipotent convergent  $F$ -isocrystal for which the sequence (4.2) is exact. Let

$$(5.2) \quad 0 \longrightarrow (\mathcal{E}, \nabla_{\mathcal{E}}) \xrightarrow{\iota} (\mathcal{G}, \nabla_{\mathcal{G}}) \longrightarrow (\mathcal{O}_{X_K^{\text{rig}}}, d) \longrightarrow 0,$$

be the exact sequence of modules with integrable connections on  $X_K^{\text{rig}}$  induced by (5.1) and let  $0 \neq x \in H_{\text{rig}}^1(X_k/K, E)$  be the element corresponding to the exact sequence (5.1). The next result shows that, although the sequences (4.2) attached to  $E$  and  $\mathcal{O}$  are exact, this may fail for that attached to  $G$ .

**Theorem 5.1.** *Let us suppose that there exists  $y \in H_{\text{dR}}^1(X_K^{\text{rig}}, (\mathcal{E}, \nabla_{\mathcal{E}}))$  such that  $\varphi^*(x) = N_{\mathcal{E}}(y)$ . Let*

$$\iota_* : H_{\text{dR}}^1(X_K^{\text{rig}}, (\mathcal{E}, \nabla_{\mathcal{E}})) \longrightarrow H_{\text{dR}}^1(X_K^{\text{rig}}, (\mathcal{G}, \nabla_{\mathcal{G}}))$$

be the map in cohomology induced by the map  $\iota$  in the exact sequence (5.2). Then

$$\text{Ker}(N_{\mathcal{G}}) = \text{Im}(\varphi^*) \oplus K \cdot \iota_*(y).$$

In fact, we have an explicit example of a semistable curve  $X$  and a unipotent convergent  $F$ -isocrystal  $E$  on  $X_k/\text{Spf}(W)$  for which the sequence (4.2) is not exact.

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