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On v-adic periods of t-motives: a resume

By

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Abstract

This is a resume of our results ([7]) on v-adic periods of t-motives, where v is a "finite" place of the rational function field over a finite field. For a t-motive M, we define v-adic periods of M and the fundamental group of the Tannakian category generated by M. Our main result is the transcendental degree of the extension generated by the v-adic periods is equal to the dimension of the fundamental group.

§ 1. Introduction

The special values $\zeta(n)$ of the Riemann zeta function are important objects in number theory. However we do not know how many algebraic relations are there among them over $\mathbb{Q}$. Euler proved that if $n \geq 2$ is an even integer, then we have $\zeta(n)/\pi^n \in \mathbb{Q}$. The odd integer points are more mysterious and we have the following conjecture:

Conjecture 1.1. *For each integer $n \geq 2$, we have the equality*

$$\mathrm{tr.deg}_{\mathbb{Q}}\, \mathbb{Q}\,(\pi, \zeta(2), \ldots, \zeta(n)) = n - \lfloor n/2 \rfloor.$$

Here, for a real number x, we denote by $\lfloor x \rfloor$ the largest integer not greater than x.

To prove Conjecture 1.1 seems to be very difficult, but the function field analogue of this conjecture was proved by Chang and Yu ([4]). We explain this briefly.

Let $\mathbb{F}_q$ be the finite field with q elements, p the characteristic of $\mathbb{F}_q$, $K := \mathbb{F}_q(\theta)$ the rational function field over $\mathbb{F}_q$, and $K_\infty := \mathbb{F}_q(\!(\theta^{-1})\!)$ the ∞-adic completion of K. For each integer $n \geq 1$, the Carlitz zeta value is defined by

$$\zeta_C(n) := \sum_{a \in \mathbb{F}_q[\theta],\mathrm{monic}} a^{-n} \in K_\infty.$$

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This is the function field analogue of the Riemann zeta values. Fix a $(q-1)$st root $(-\theta)^{\frac{1}{q-1}}$ of $-\theta$ and set

$$\tilde{\pi} := \theta(-\theta)^{\frac{1}{q-1}} \prod_{i=1}^{\infty}(1-\theta^{1-q^i})^{-1} \in K_\infty((-\theta)^{\frac{1}{q-1}}).$$

This is an analogue of $2\pi\sqrt{-1}$. By the definition of ζ_C, we have the equalities $\zeta_C(np^m) = \zeta_C(n)^{p^m}$ for all integers $m, n \geq 1$. Carlitz proved that if n is divisible by $q-1$, then we have $\zeta_C(n)/\tilde{\pi}^n \in K$. This is an analogue of the corresponding fact about the Riemann zeta values at positive even integers (note that $q-1$ and 2 are respectively the cardinalities of $\mathbb{F}_q[\theta]^\times$ and $\mathbb{Z}^\times$). Chang and Yu proved that these are essentially the only relations among the special values.

Theorem 1.2 ([4, Corollary 4.6]). *For each integer $n \geq 1$, we have the equality*

$$\operatorname{tr.deg}_K K(\tilde{\pi}, \zeta_C(1), \ldots, \zeta_C(n)) = n + 1 - \lfloor n/p \rfloor - \lfloor n/(q-1) \rfloor + \lfloor n/p(q-1) \rfloor.$$

The proof of Theorem 1.2 uses Papanikolas' result on periods of t-motives. We explain this result. Let t be a variable independent of θ. A t-motive over K is a free $K[t]$-module M of finite rank equipped with a "Frobenius action" satisfying certain conditions. Let $\mathbb{C}_\infty$ be the ∞-adic completion of an algebraic closure of K_∞ and $|\cdot|_\infty$ its valuation. Set $\mathbb{T} := \{f \in \mathbb{C}_\infty[\![t]\!] | f \text{ converges on } |t|_\infty \leq 1\}$ and $\mathbb{L} := \operatorname{Frac}\mathbb{T}$ the fraction field of $\mathbb{T}$. For a t-motive M, a Betti realization $H_B(M) \subset \mathbb{L} \otimes_{K[t]} M$ is defined. This is an $\mathbb{F}_q(t)$-vector space and we have $\dim_{\mathbb{F}_q(t)} H_B(M) \leq \operatorname{rank}_{K[t]} M$. Assume that the equality holds. Such t-motives are called *rigid analytically trivial.* Fix bases $\mathbf{x}$ of $H_B(M)$ and $\mathbf{m}$ of M. We obtain the matrix $\Psi = (\Psi_{ij})_{i,j} \in \mathrm{GL}_r(\mathbb{L})$ such that $\mathbf{m} = \Psi\mathbf{x}$ in $\mathbb{L} \otimes M$, where r is the rank of M. We can construct a "good" category of rigid analytically trivial t-motives and this category forms a neutral Tannakian category with fiber functor H_B. Thus we obtain an algebraic group $\Gamma \subset \mathrm{GL}_r$ over $\mathbb{F}_q(t)$ which corresponds to the Tannakian subcategory generated by M via the Tannakian duality. In this situation, Papanikolas proved the following theorem:

Theorem 1.3 ([8, Theorem 4.3.1, 4.5.10]). *Let M, Ψ and Γ be as above. Then we have*

$$\operatorname{tr.deg}_{\bar{K}(t)} \bar{K}(t)(\Psi_{11}, \Psi_{12}, \ldots, \Psi_{rr}) = \dim \Gamma.$$

Note that, each component of Ψ converges at $t = \theta$, and moreover Papanikolas proved the equality

$$\operatorname{tr.deg}_{\bar{K}} \bar{K}(\Psi_{11}|_{t=\theta}, \Psi_{12}|_{t=\theta}, \ldots, \Psi_{rr}|_{t=\theta}) = \dim \Gamma$$

by using the "ABP-criterion" ([2]). Anderson and Thakur showed ([3]) that the Carlitz zeta values are described by linear combinations of entries of $\Psi|_{t=\theta}$ over K for certain t-motives. Hence Theorem 1.2 is proved by calculations of algebraic groups.

For a finite place v, there exist v-adic zeta values and v-adic realizations of t-motives. In [7] we proved a v-adic analogue of Theorem 1.3. However we do not know whether we can apply this result to the v-adic zeta values.

§ 2. t-motives

Before we state our results, we review t-motives. The notion of t-motive was introduced by Anderson in [1]. Let K and t be as in Section 1. We define an endomorphism σ on $K[t]$ by

$$\sigma\colon K[t] \to K[t]; \quad \sum_i a_i t^i \mapsto \sum_i a_i^q t^i.$$

Definition 2.1. A *t-motive* over K is a free $K[t]$-module M of finite rank equipped with a σ-semilinear map $\varphi\colon M \to M$ such that

- $\det \varphi = c(t-\theta)^n$ $(c \in K^\times, n \geq 0)$,
- M is finitely generated over $K[\varphi]$.

Note that $\det \varphi$ in the first condition is the determinant of the matrix $A \in \mathrm{GL}_r(K[t])$ which satisfies $\varphi\mathbf{m} = A\mathbf{m}$ for a fixed basis $\mathbf{m}$ of M, where r is the $K[t]$-rank of M. Since $K[t]^\times = K^\times$, the validity of the first condition is independent of the choice of $\mathbf{m}$.

Remark. There exists an anti-equivalence of categories between the category of t-motives over K and the category of "abelian t-modules" over K. Roughly speaking, an abelian t-module is an algebraic group $\mathbb{G}_a^d$ over K for some $d \geq 0$ equipped with an $\mathbb{F}_q[t]$-action which satisfies certain conditions.

Example 2.2 (Carlitz t-motive)**.** As a $K[t]$-module, set $M := K[t]$. We define a φ-action on M by

$$\varphi(a) := (t-\theta)\sigma(a)$$

for each $a \in M$. This forms a t-motive. We call this t-motive the *Carlitz t-motive.* The Carlitz t-motive corresponds to an abelian t-module $\mathbb{G}_a$ equipped with an $\mathbb{F}_q[t]$-action defined by $\mathbb{F}_q[t] \to \mathrm{End}(\mathbb{G}_a); t \mapsto (x \mapsto \theta x + x^q)$.

§ 3. v-adic case

Let $v \in \mathbb{F}_q[t]$ be an irreducible monic polynomial of degree d. Set $K^{\mathrm{sep}}(t)_v := K^{\mathrm{sep}}(t) \otimes_{K^{\mathrm{sep}}[t]} \varprojlim_n K^{\mathrm{sep}}[t]/v^n$, the v-adic completion of $K^{\mathrm{sep}}(t)$, where K^{sep} is a separable closure of K. We define $K(t)_v$ and $\mathbb{F}_q(t)_v$ similarly. The endomorphism σ on $K[t]$

naturally extends to an endomorphism on $K^{\mathrm{sep}}(t)_v$, and we have $(K^{\mathrm{sep}}(t)_v)^\sigma = \mathbb{F}_q(t)_v$, where $(-)^\sigma$ is the σ-fixed part. For a t-motive M over K, we set

$$V(M) := \left(K^{\mathrm{sep}}(t)_v \otimes_{K[t]} M\right)^{\sigma\otimes\varphi}.$$

We call $V(M)$ the *v-adic realization* of M. This is an $\mathbb{F}_q(t)_v$-vector space and the absolute Galois group $G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ of K acts on $V(M)$ naturally. For any t-motive M, we can prove that $\dim_{\mathbb{F}_q(t)_v} V(M) = \mathrm{rank}_{K[t]} M$. Thus if we fix bases $\mathbf{x}$ of $V(M)$ and $\mathbf{m}$ of M, we obtain the matrix $\Psi = (\Psi_{ij})_{i,j} \in \mathrm{GL}_r(K^{\mathrm{sep}}(t)_v)$ such that $\mathbf{m} = \Psi\mathbf{x}$ in $K^{\mathrm{sep}}(t)_v \otimes M$. Each component of the matrix Ψ is called a *v-adic period* of M. If we factorize $v = \prod_{l\in\mathbb{Z}/d}(t-\lambda_l)$ in $\overline{\mathbb{F}_q}$, we have the decomposition $K^{\mathrm{sep}}(t)_v = \prod_l K^{\mathrm{sep}}(\!(t-\lambda_l)\!)$. Thus we can write $\Psi_{ij} = (\Psi_{ijl})_{l\in\mathbb{Z}/d}$ where $\Psi_{ijl} \in K^{\mathrm{sep}}(\!(t-\lambda_l)\!)$ for each i, j and l.

To construct the v-adic analogue of the algebraic group Γ, we consider a certain subcategory of the category of φ-modules over $K(t)_v$. A φ-module over $K(t)_v$ is a finite-dimensional $K(t)_v$-vector space N equipped with a σ-semilinear map $\varphi\colon N \to N$. A morphism between φ-modules is a $K(t)_v$-linear map which commutes with φ's. For a φ-module N, we set

$$V(N) := \left(K^{\mathrm{sep}}(t)_v \otimes_{K(t)_v} N\right)^{\sigma\otimes\varphi}.$$

We have a natural injection

$$K^{\mathrm{sep}}(t)_v \otimes_{\mathbb{F}_q(t)_v} V(N) \to K^{\mathrm{sep}}(t)_v \otimes_{K(t)_v} N.$$

Let $\mathcal{C}$ be the full subcategory of the category of φ-modules over $K(t)_v$ whose objects are the φ-modules such that the above map is an isomorphism. We can prove that $K(t)_v \otimes_{K[t]} M$ is an object of $\mathcal{C}$ for each t-motive M. Recall that a neutral Tannakian category over a field k is a rigid abelian k-linear tensor category $\mathcal{A}$ for which $k \xrightarrow{\sim} \mathrm{End}(\mathbf{1})$ and there exists an exact faithful k-linear tensor functor $\omega : \mathcal{A} \to \mathbf{Vec}(k)$, where $\mathbf{1}$ is the unit object of $\mathcal{A}$ and $\mathbf{Vec}(k)$ is the category of finite-dimensional k-vector spaces (cf. [5, Definition 2.19]). Any such functor ω is said to be a fiber functor for $\mathcal{A}$. We can prove that the category $\mathcal{C}$ forms a neutral Tannakian category over $\mathbb{F}_q(t)_v$, and the functor V is a fiber functor for $\mathcal{C}$. For a t-motive M over K, we set $\mathcal{C}_M$ to be the Tannakian subcategory of $\mathcal{C}$ generated by $K(t)_v \otimes M$. Let Γ_v be the algebraic group over $\mathbb{F}_q(t)_v$ corresponding to $\mathcal{C}_M$ via the Tannakian duality. Thus we obtain the matrix Ψ and the algebraic group Γ_v from a t-motive M. We have the following theorem, which is a v-adic analogue of Theorem 1.3:

Theorem 3.1 ([7, Theorem 4.14, 5.15])**.** *Let M, Ψ and Γ_v be as above. Then we have*

$$\mathrm{tr.deg}_{K(t)_v} K(t)_v(\Psi_{11l}, \Psi_{12l}, \ldots, \Psi_{rrl}) = \dim \Gamma_v$$

for all $l \in \mathbb{Z}/d$.

Example 3.2. Let M be the Carlitz t-motive defined in Example 2.2. Then we have $\Gamma_v = \mathbb{G}_m$, the multiplicative group over $\mathbb{F}_q(t)_v$, and the transcendental degree is one.

By using Theorem 3.1, we can prove the following proposition:

Proposition 3.3 ([7, Proposition 6.4, Corollary 7.4]). *Fix an integer* $n \geq 1$.

(1) *For each* $\alpha \in K$, *there exists an element* $L_{\alpha,n} = (L_{\alpha,n,l})_l \in K^{\mathrm{sep}}[\![t]\!]_v = \prod_l K^{\mathrm{sep}}[\![t-\lambda_l]\!]$ *such that* $\sigma(L_{\alpha,n}) = \sigma(\alpha) + L_{\alpha,n}/(t-\theta)^n$.

(2) *Fix elements* $\alpha_1, \ldots, \alpha_r \in K$. *If* $L_{\alpha_1,n,l}, \ldots, L_{\alpha_r,n,l}, 1$ *are linearly independent over* $K(t)_v$ *for some* $l \in \mathbb{Z}/d$, *then* $L_{\alpha_1,n,l'}, \ldots, L_{\alpha_r,n,l'}$ *are algebraically independent over* $K(t)_v$ *for each* $l' \in \mathbb{Z}/d$.

Remark. In the ∞-adic case, an analogous element of $L_{\alpha,n}$ is constructed explicitly, and its value at $t = \theta$ is the n-th Carlitz polylogarithm of α. Thus we consider $L_{\alpha,n}$ as a v-adic formal polylogarithm.

§ 4. Outline of the proof of Theorem 3.1

In this section, we will sketch the proof of Theorem 3.1. We will construct an algebraic group Γ' defined over $\mathbb{F}_q(t)_v$ such that the dimension of Γ' is equal to the transcendental degree in Theorem 3.1, and there exists an isomorphism $\Gamma' \xrightarrow{\sim} \Gamma$. We continue to use the notations of the previous sections, and factorize $v = \prod_{l\in\mathbb{Z}/d}(t-\lambda_l)$ in $\overline{\mathbb{F}_q}$ so that $\lambda_l^q = \lambda_{l+1}$ for each l. Set $F := \mathbb{F}_q(t)_v$, $E := K(t)_v$, $L := K^{\mathrm{sep}}(t)_v$ and $L_l := K^{\mathrm{sep}}(\!(t-\lambda_l)\!)$ for each l.

Let $X := (X_{ij})$ be an $r \times r$ matrix of independent variables X_{ij}, and set $\Delta := \det(X)$. We set $E[X, \Delta^{-1}] := E[X_{11}, X_{12}, \ldots, X_{rr}, \Delta^{-1}]$. We define E-algebra homomorphisms $\nu : E[X, \Delta^{-1}] \to L$; $X_{ij} \mapsto \Psi_{ij}$ and $\nu_l : E[X, \Delta^{-1}] \to L_l$; $X_{ij} \mapsto \Psi_{ijl}$, and set

$$Z := \operatorname{Spec} E[X, \Delta^{-1}]/\operatorname{Ker}\nu$$

and

$$Z_l := \operatorname{Spec} E[X, \Delta^{-1}]/\operatorname{Ker}\nu_l$$

for each l. It is clear that the dimension of Z_l is equal to the transcendental degree which we want to calculate. To construct Γ', we define matrices $\widetilde{\Psi} = (\widetilde{\Psi}_{ij})_{i,j} := (\Psi_{ij} \otimes 1)_{i,j}^{-1}(1 \otimes \Psi_{ij})_{i,j} \in \mathrm{GL}_r(L \otimes_E L)$ and $\widetilde{\Psi}_{lm} = (\widetilde{\Psi}_{ijlm})_{i,j} := (\Psi_{ijl} \otimes 1)_{i,j}^{-1}(1 \otimes \Psi_{ijm})_{i,j} \in \mathrm{GL}_r(L_l \otimes_E L_m)$ for each $l, m \in \mathbb{Z}/d$. We define F-algebra homomorphisms

$\mu : F[X, \Delta^{-1}] \to L \otimes_E L;\ X_{ij} \mapsto \widetilde{\Psi}_{ij}$ and $\mu_{lm} : F[X, \Delta^{-1}] \to L_l \otimes_E L_m;\ X_{ij} \mapsto \widetilde{\Psi}_{ijlm}$ for each l and m. Set

$$\Gamma' := \operatorname{Spec} F[X, \Delta^{-1}] / \operatorname{Ker} \mu$$

and

$$\Gamma'_{lm} := \operatorname{Spec} F[X, \Delta^{-1}] / \operatorname{Ker} \mu_{lm}.$$

By a simple calculation, we have $\operatorname{Ker} \mu_{0,m} = \operatorname{Ker} \mu_{1,m+1} = \cdots$. Thus we can set $\Gamma'_m := \Gamma'_{0,m} = \Gamma'_{1,m+1} = \cdots$. For each scheme Y over F, we set $Y_E := Y \times_{\operatorname{Spec} F} \operatorname{Spec} E$.

Proposition 4.1 ([7, Proposition 4.11]). (1) *Let $\psi : Z \times_E Z \to Z \times_E \mathrm{GL}_{r/E}$ be the morphism of affine E-schemes defined by $(u, v) \mapsto (u, u^{-1}v)$. Then ψ factors through an isomorphism $\psi' : Z \times_E Z \xrightarrow{\sim} Z \times_E \Gamma'_E$ of affine E-schemes.*

(2) *For any l and m, let $\psi_{lm} : Z_l \times_E Z_{l+m} \to Z_l \times_E \mathrm{GL}_{r/E}$ be the morphism of affine E-schemes defined by $(u, v) \mapsto (u, u^{-1}v)$. Then ψ factors through an isomorphism $\psi'_{lm} : Z_l \times_E Z_{l+m} \xrightarrow{\sim} Z_l \times_E \Gamma'_{m,E}$ of affine E-schemes.*

$$\begin{tikzcd}
Z \times Z \arrow[rr, "\psi:(u,v)\mapsto(u,u^{-1}v)"] \arrow[dr, dotted, "\psi'"'] & & Z \times \mathrm{GL}_{r/E} \\
& Z \times \Gamma'_E \arrow[ur, hook, "natural"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
Z_l \times Z_{l+m} \arrow[rr, "\psi_{lm}:(u,v)\mapsto(u,u^{-1}v)"] \arrow[dr, dotted, "\psi'_{lm}"'] & & Z_l \times \mathrm{GL}_{r/E} \\
& Z_l \times \Gamma'_{m,E} \arrow[ur, hook, "natural"'] &
\end{tikzcd}$$

By Proposition 4.1, we conclude that

- Γ' is a closed subgroup scheme of $\mathrm{GL}_{r/F}$ and Z is a Γ'_E-torsor under right multiplication,
- Γ'_0 is a closed subgroup scheme of $\mathrm{GL}_{r/F}$ and Z_l is a $\Gamma'_{0,E}$-torsor under right multiplication for each l,
- Γ'_m is a Γ'_0-torsor under right and left multiplications for each m.

In particular, we have $\dim \Gamma' = \dim \Gamma'_m = \dim Z_l = \operatorname{tr.deg}_E E(\Psi_{11l}, \Psi_{12l}, \ldots, \Psi_{rrl})$ for each l and m.

For any object N in $\mathcal{C}_M$, we can define a Γ'-action on $V(N)$. This is functorial in N and we have a functor

$$\xi \colon \mathcal{C}_M \to \mathbf{Rep}_F(\Gamma')$$

which is compatible with tensor products, where $\mathbf{Rep}_F(\Gamma')$ is the category of finite-dimensional F-representations of Γ'. Thus we obtain the morphism $f \colon \Gamma' \to \Gamma_v$ of algebraic groups over F which corresponds to the functor ξ via the Tannakian duality. We can check easily that the morphism f is a closed immersion. To prove that f is an isomorphism, we consider the Galois representation $G_K \to \mathrm{GL}(V(M))$ which is obtained naturally by the definition of V. This representation factors as follows:

$$G_K \to \Gamma'(F) \hookrightarrow \Gamma_v(F) \hookrightarrow \mathrm{GL}(V(M)).$$

Since the functor V induces an equivalence of categories $V\colon \mathcal{C} \xrightarrow{\sim} \mathbf{Rep}_F(G_K)$, where $\mathbf{Rep}_F(G_K)$ is the category of finite-dimensional continuous F-representations of G_K (cf. [6, Appendix]), the set of F-valued points $\Gamma_v(F)$ is dense in Γ_v. Therefore we conclude that the immersion $f\colon \Gamma' \hookrightarrow \Gamma_v$ is an isomorphism. This is an essentially different point from Papanikolas' proof for the ∞-adic case, in which the Zariski density is not proved and other facts are used to show this isomorphism.

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