RIMS Kôkyûroku Bessatsu
B44 (2013), 67–83

Torsion representations arising from $(\varphi, \hat{G})$-modules: A resume

By

Yoshiyasu Ozeki*

Abstract

In this paper, we announce some results on torsion representations arising from torsion $(\varphi, \hat{G})$-modules and properties of the category of torsion $(\varphi, \hat{G})$-modules. This study is related with torsion semi-stable representations.

§ 1. Introduction

1.1. Let K be a complete discrete valuation field of mixed characteristics $(0, p)$ with perfect residue field, and let G denote the absolute Galois group of K. There are established several theories describing $\mathbb{Z}_p$-representations of G by linear algebra data, by Fontaine-Laffaille, Breuil, Wach-Berger, Kisin and Liu. The main purpose of this article is to give a survey of several results proved in [O] on torsion $(\varphi, \hat{G})$-modules and torsion p-adic representations of G associated with them. (The notion of $(\varphi, \hat{G})$-modules is defined by Tong Liu [Li2], and we recall the definition in Section 2.)

We explain a motivation of our study more precisely. Let $\mathrm{Rep}_{\mathbb{Z}_p}(G)$ (resp. $\mathrm{Rep}_{\mathrm{tor}}(G)$) denote the category of $\mathbb{Z}_p$-representations of G, free of finite rank over $\mathbb{Z}_p$ (resp. torsion of finite type over $\mathbb{Z}_p$). Let $\mathcal{C}$ be a full subcategory of $\mathrm{Rep}_{\mathbb{Z}_p}(G)$. We define a full subcategory of $\mathrm{Rep}_{\mathrm{tor}}(G)$ by

$$\mathrm{Rep}^{\mathcal{C}}_{\mathrm{tor}}(G) := \left\{ T \in \mathrm{Rep}_{\mathrm{tor}}(G); \begin{array}{c} \text{there exists an inclusion } L_1 \subset L_2 \text{ in } \mathcal{C} \\ \text{such that } L_2/L_1 \text{ is isomorphic to } T \end{array} \right\}.$$

It is natural to raise the following:

Received March 27, 2012. Revised November 22, 2012 and December 11, 2012.
2000 Mathematics Subject Classification(s): 11E95, 11S99.
Key Words: semi-stable representations, Kisin modules.
Partly supported by the Grant-in-Aid for Research Activity Start-up, The Ministry of Education, Culture, Sports, Science and Technology, Japan.
*Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606-8502, Japan.
e-mail: yozeki@kurims.kyoto-u.ac.jp

Question 1.2. For which full subcategory $\mathcal{C}$ of $\mathrm{Rep}_{\mathbb{Z}_p}(G)$ does the equality $\mathrm{Rep}^{\mathcal{C}}_{\mathrm{tor}}(G) = \mathrm{Rep}_{\mathrm{tor}}(G)$ hold?

Let r be a non-negative integer. Then Question 1.2 has a negative answer when $\mathcal{C}$ is the full subcategory consisted of semi-stable representations with Hodge-Tate weights in $[0, r]$. This follows from boundedness of ramifications for torsion semi-stable representations as is shown by Caruso and Liu ([CL2], Theorem 5.4). However Question 1.2 is still open when $\mathcal{C}$ is the full subcategory consisted of semi-stable representations of G. In this case Question 1.2 is related with non-emptiness of crystalline points on a rigid analytic space associated with a deformation space of a mod p representation (for example, see [Na], Section 4). Therefore, we may raise the following:

Question 1.3. Let $\mathcal{C}$ be the full subcategory consisted of semi-stable representations of G and put $\mathrm{Rep}^{\mathrm{ss}}_{\mathrm{tor}}(G) = \mathrm{Rep}^{\mathcal{C}}_{\mathrm{tor}}(G)$. Does the equality $\mathrm{Rep}^{\mathrm{ss}}_{\mathrm{tor}}(G) = \mathrm{Rep}_{\mathrm{tor}}(G)$ hold?

1.4. Now let $\mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G)$ denote the full subcategory of $\mathrm{Rep}_{\mathrm{tor}}(G)$ consisted of torsion $\mathbb{Z}_p$-representations of G arising from torsion $(\varphi, \hat{G})$-modules. Then we have inclusions

$$\mathrm{Rep}^{\mathrm{ss}}_{\mathrm{tor}}(G) \subset \mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G) \subset \mathrm{Rep}_{\mathrm{tor}}(G)$$

(cf. [CL2], Theorem 3.1.3). Note that the category $\mathrm{Rep}^{\mathrm{ss}}_{\mathrm{tor}}(G)$ is equivalent to the category of torsion $\mathbb{Z}_p$-representations of G written as a quotient of two representations arising from free $(\varphi, \hat{G})$-modules[1]. Thus the following result implies that all the above inclusions are "equal for the G_∞-case" (we recall the definition of $G_\infty \subset G$ in Section 2):

Proposition 1.5 ([CL2], Proposition 5.6)**.** *Let T be a torsion $\mathbb{Z}_p$-representations of G_∞. Then T is a quotient of two representations arising from free Kisin modules.*

It is a natural question whether the above proposition holds or not after replacing G_∞ and "Kisin modules" with G and "$(\varphi, \hat{G})$-modules", respectively. If it has an affirmative answer, then we know that Question 1.3 also has an affirmative answer. In this article we announce some results on linear algebraic properties of torsion $(\varphi, \hat{G})$-modules and the category $\mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G)$. For example we mention that $\mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G)$ is an abelian category. This fact is really plausible, but we need to establish Theorem 4.13 for a proof. In addition, we announce the Cartier duality theorem for $(\varphi, \hat{G})$-modules in Section 3, and announce the theory of "maximal and minimal models" for $(\varphi, \hat{G})$-modules in Section 4, which classifies $\mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G)$ completely (cf. Corollary 4.15). To

[1]It should be mentioned that it is not known whether any torsion $(\varphi, \hat{G})$-module can be written as a a quotient of two free $(\varphi, \hat{G})$-modules or not.

establish the theory of "maximal and minimal models" for $(\varphi, \hat{G})$-modules, we need to introduce the notion of étale $(\varphi, \hat{G})$-modules (cf. Section 2.2). The theory of étale $(\varphi, \hat{G})$-modules are based on that of étale φ-modules defined by Fontaine [Fo], and étale $(\varphi, \hat{G})$-modules classify representations of G (cf. Proposition 2.22).

Acknowledgements. The author thanks the referee for many helpful comments throughout this paper. This work is supported by the Grant-in-Aid for Young Scientists Start-up.

Convention: For any $\mathbb{Z}$-module M, we always use M_n to denote M/p^nM for a positive integer n. We reserve φ to represent various Frobenius structures and φ_M will denote the Frobenius on M. However, we often drop the subscript if no confusion arises. All representations and actions are assumed to be continuous. Throughout this paper, we fix a prime number $p \geq 2$. For any topological group H, We denote by $\mathrm{Rep}_{\mathbb{Q}_p}(H)$ the category of finite dimensional $\mathbb{Q}_p$-representations of H. We denote by $\mathrm{Rep}_{\mathbb{Z}_p}(H)$ (resp. $\mathrm{Rep}_{\mathrm{tor}}(H)$) the category of $\mathbb{Z}_p$-representations of H, free of finite rank over $\mathbb{Z}_p$ (resp. torsion of finite type over $\mathbb{Z}_p$).

§ 2. On some $(\varphi, \hat{G})$-modules

In this section, after recalling the definition of $(\varphi, \hat{G})$-modules due to Tong Liu ([Li2], Section 2.2), we introduce the notion of étale $(\varphi, \hat{G})$-modules. We define also various rings as follows.

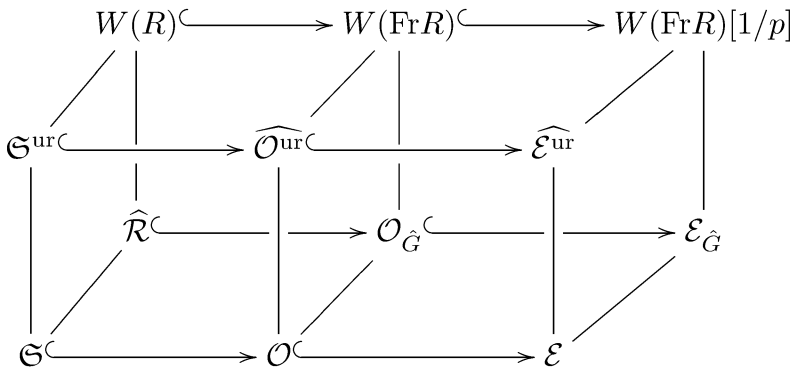


Figure 1. Ring extensions 1

§ 2.1. Liu's $(\varphi, \hat{G})$-modules

2.1. Let k be a perfect field of characteristic $p \geq 2$, $W(k)$ the ring of Witt vectors with coefficients in k, $K_0 = W(k)[1/p]$, K a finite totally ramified extension of K_0, $\mathcal{O}_K$

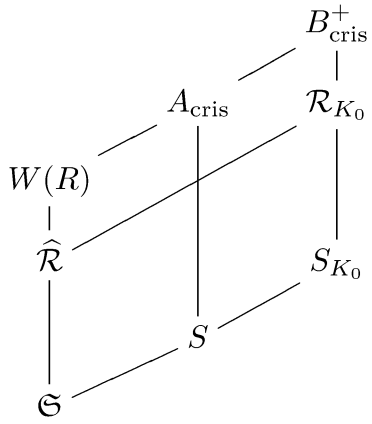


Figure 2. Ring extensions 2

the integer ring of K, $\bar{K}$ a fixed algebraic closure of K and $G = \mathrm{Gal}(\bar{K}/K)$. Throughout this paper, we fix a uniformizer $\pi \in K$ and denote by $E(u)$ its Eisenstein polynomial over K_0. Put $\mathfrak{S} = W(k)[\![u]\!]$. We define a Frobenius endomorphism φ of $\mathfrak{S}$ by $u \mapsto u^p$, extending the Frobenius of $W(k)$. It should be noted that the arguments developed hereafter depends on the choice of π.

A *φ-module over* $\mathfrak{S}$ is an $\mathfrak{S}$-module $\mathfrak{M}$ equipped with a φ-semilinear map $\varphi\colon \mathfrak{M} \to \mathfrak{M}$. A morphism between two φ-modules $(\mathfrak{M}_1, \varphi_1)$ and $(\mathfrak{M}_2, \varphi_2)$ is an $\mathfrak{S}$-linear map $\mathfrak{M}_1 \to \mathfrak{M}_2$ compatible with φ_1 and φ_2. Let r be a non-negative integer. A φ-module $(\mathfrak{M}, \varphi)$ is called *of height* $\leq r$ if $\mathfrak{M}$ is of finite type over $\mathfrak{S}$ and the cokernel of φ^* is annihilated by $E(u)^r$. Here, φ^* stands for the $\mathfrak{S}$-linearization $1 \otimes \varphi\colon \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to \mathfrak{M}$ of φ.

A φ-module $(\mathfrak{M}, \varphi)$ of height $\leq r$ is called a *free Kisin module of height* $\leq r$ if $\mathfrak{M}$ is a free $\mathfrak{S}$-module. On the other hand, a φ-module $(\mathfrak{M}, \varphi)$ of height $\leq r$ is called a *torsion Kisin module of height* $\leq r$ if $\mathfrak{M}$ is u-torsion free and annihilated by some power of p.

We denote by $\mathrm{Mod}^r_{/\mathfrak{S}}$ (resp. $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$) the category of free (resp. torsion) Kisin modules of height $\leq r$. It is known that, for any $\mathfrak{M} \in \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$, there exist $\mathfrak{N}_1 \subset \mathfrak{N}_2$ in $\mathrm{Mod}^r_{/\mathfrak{S}}$ such that $\mathfrak{N}_2/\mathfrak{N}_1$ is isomorphic to $\mathfrak{M}$ ([Li1], Proposition 2.3.2).

A φ-module $(\mathfrak{M}, \varphi)$ is called a *free* (resp. *torsion*) *Kisin module of finite height* if $\mathfrak{M}$ is a free (resp. torsion) Kisin module of height $\leq r$ for some r. We denote by $\mathrm{Mod}^\infty_{/\mathfrak{S}}$ (resp. $\mathrm{Mod}^\infty_{/\mathfrak{S}_\infty}$) the category of free (resp. torsion) Kisin modules of finite height. By definitions, we have

$$\mathrm{Mod}^\infty_{/\mathfrak{S}} = \bigcup_{r\geq 0} \mathrm{Mod}^r_{/\mathfrak{S}}, \quad \mathrm{Mod}^\infty_{/\mathfrak{S}_\infty} = \bigcup_{r\geq 0} \mathrm{Mod}^r_{/\mathfrak{S}_\infty}.$$

2.2. Put $R = \varprojlim \mathcal{O}_{\bar{K}}/p$, where $\mathcal{O}_{\bar{K}}$ is the ring of integers of $\bar{K}$ and the transition maps are given by the p-th power map. Note that the p-th power map is bijective on R. It follows that there exists a unique surjective continuous homomorphism $\theta\colon W(R) \to \widehat{\mathcal{O}}_{\bar{K}}$ which lifts the projection $R \to \mathcal{O}_{\bar{K}}/p$ onto the first component in the inverse limit. Here $\widehat{\mathcal{O}}_{\bar{K}}$ is the p-adic completion of $\mathcal{O}_{\bar{K}}$.

On the other hand, the residue field k is embedded in R by $\lambda \mapsto (\lambda^{1/p^n})_{n\geq 0}$ since k is perfect. The embedding $k \to R$ induces an embedding $W(k) \to W(R)$. Moreover, the embedding $W(k) \to W(R)$ is extended to an embedding $\mathfrak{S} \to W(R)$ by $u \mapsto [\underline{\pi}]$. Here, $[\underline{\pi}]$ is the Teichmüller representative of $\underline{\pi} = (\pi_n)_{n\geq 0} \in R$. It is readily seen that the embedding $\mathfrak{S} \to W(R)$ is compatible with the Frobenius endomorphisms.

Let $\mathcal{O}$ be the p-adic completion of $\mathfrak{S}[1/u]$, which is a discrete valuation ring with uniformizer p and residue field $k((u))$. Denote by $\mathcal{E}$ the field of fractions of $\mathcal{O}$. The inclusion $\mathfrak{S} \hookrightarrow W(R)$ extends to inclusions $\mathcal{O} \hookrightarrow W(\mathrm{Fr}R)$ and $\mathcal{E} \hookrightarrow W(\mathrm{Fr}R)[1/p]$. Here $\mathrm{Fr}R$ is the field of fractions of R. It is not difficult to see that $\mathrm{Fr}R$ is algebraically closed. We denote by $\mathcal{E}^{\mathrm{ur}}$ the maximal unramified field extension of $\mathcal{E}$ in $W(\mathrm{Fr}R)[1/p]$ and $\mathcal{O}^{\mathrm{ur}}$ its integer ring. Let $\widehat{\mathcal{E}^{\mathrm{ur}}}$ be the p-adic completion of $\mathcal{E}^{\mathrm{ur}}$ and $\widehat{\mathcal{O}^{\mathrm{ur}}}$ its integer ring. Put $\mathfrak{S}^{\mathrm{ur}} = \widehat{\mathcal{O}^{\mathrm{ur}}} \cap W(R)$. We regard all these rings as subrings of $W(\mathrm{Fr}R)[1/p]$.

2.3. Let $K_\infty = \cup_{n\geq 0} K(\pi_n)$ and $G_\infty = \mathrm{Gal}(\bar{K}/K_\infty)$. Put $K(\zeta_{p^\infty}) = \cup_{n\geq 0} K(\zeta_{p^n})$ and $K_\infty(\zeta_{p^\infty}) = \cup_{n\geq 0} K_\infty(\zeta_{p^n})$. Put $H_K = \mathrm{Gal}(K_\infty(\zeta_{p^\infty})/K_\infty)$, $H_\infty = \mathrm{Gal}(\bar{K}/K_\infty(\zeta_{p^\infty}))$, $G_{p^\infty} = \mathrm{Gal}(K_\infty(\zeta_{p^\infty})/K(\zeta_{p^\infty}))$ and $\hat{G} = \mathrm{Gal}(K_\infty(\zeta_{p^\infty})/K)$.

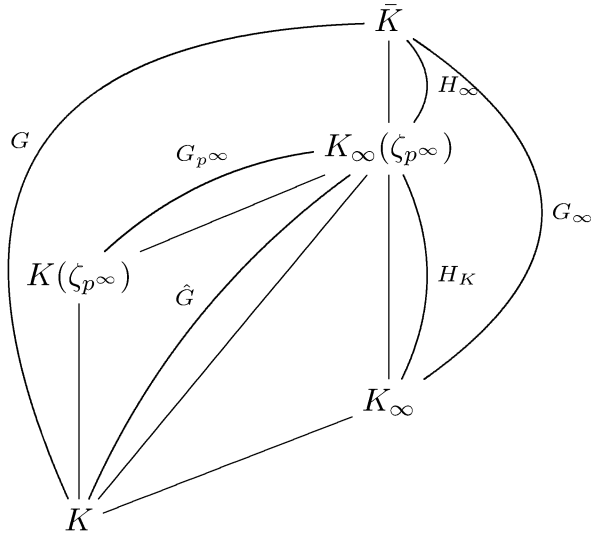


Figure 3. Galois groups of field extensions

Note now that the extension K_∞/K is a strictly APF extension in the sense of [Wi] and that, by the theory of norm fields, G_∞ is naturally isomorphic to the absolute

Galois group of $k((u))$. Thus G_∞ acts on $\mathfrak{S}^{\mathrm{ur}}$, $\widehat{\mathcal{O}^{\mathrm{ur}}}$, $\widehat{\mathcal{E}^{\mathrm{ur}}}$ and $\mathcal{E}^{\mathrm{ur}}/\mathcal{O}^{\mathrm{ur}}$, and fixes the subring $\mathfrak{S} \subset W(R)$.

For a free or torsion Kisin module $\mathfrak{M}$ of finite height, we define a $\mathbb{Z}_p[G_\infty]$-module $T_{\mathfrak{S}}(\mathfrak{M})$ by

$$T_{\mathfrak{S}}(\mathfrak{M}) = \begin{cases} \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}) & \text{if } \mathfrak{M} \text{ is free} \\ \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}_\infty) & \text{if } \mathfrak{M} \text{ is torsion.} \end{cases}$$

Here a G_∞-action on $T_{\mathfrak{S}}(\mathfrak{M})$ is given by $(\sigma.g)(x) = \sigma(g(x))$ for $\sigma \in G_\infty, g \in T_{\mathfrak{S}}(\mathfrak{M}), x \in \mathfrak{M}$.

2.4. Let S be the p-adic completion of $W(k)[u, \frac{E(u)^i}{i!}]_{i\geq 0}$ and endow S with the following structures:

- a continuous φ-semi-linear Frobenius $\varphi\colon S \to S$ defined by $\varphi(u) = u^p$.
- a continuous linear derivation $N\colon S \to S$ defined by $N(u) = -u$.
- a decreasing filtration $(\mathrm{Fil}^i S)_{i\geq 0}$ in S. Here $\mathrm{Fil}^i S$ is the p-adic closure of the ideal generated by the divided powers $\gamma_j(E(u)) = \frac{E(u)^j}{j!}$ for all $j \geq i$.

Put $S_{K_0} = S[1/p] = K_0 \otimes_{W(k)} S$. The inclusion $\mathfrak{S} \hookrightarrow W(R)$ induces inclusions $\mathfrak{S} \hookrightarrow S \hookrightarrow A_{\mathrm{cris}}$ and $S_{K_0} \hookrightarrow B^+_{\mathrm{cris}}$. We regard all these rings as subrings in B^+_{cris}. Fix a choice of primitive p^i-root of unity ζ_{p^i} for $i \geq 0$ such that $\zeta^p_{p^{i+1}} = \zeta_{p^i}$. Put $\underline{\varepsilon} = (\zeta_{p^i})_{i\geq 0} \in R^\times$ and $t = \log([\underline{\varepsilon}]) \in A_{\mathrm{cris}}$. Denote by $\nu\colon W(R) \to W(\bar{k})$ a unique lift of the projection $R \to \bar{k}$. Since $\nu(\mathrm{Ker}(\theta))$ is contained in the set $pW(\bar{k})$, ν extends to a map $\nu\colon A_{\mathrm{cris}} \to W(\bar{k})$ and $\nu\colon B^+_{\mathrm{cris}} \to W(\bar{k})[1/p]$. For any subring $A \subset B^+_{\mathrm{cris}}$, we put $I_+A = \mathrm{Ker}(\nu \text{ on } B^+_{\mathrm{cris}}) \cap A$. For any integer $n \geq 0$, let $t^{\{n\}} = t^{r(n)}\gamma_{\tilde{q}(n)}(\frac{t^{p-1}}{p})$ where $n = (p-1)\tilde{q}(n) + r(n)$ with $0 \leq r(n) < p-1$ and $\gamma_i(x) = \frac{x^i}{i!}$ is the standard divided power.

We define a subring $\mathcal{R}_{K_0}$ of B^+_{cris} as below:

$$\mathcal{R}_{K_0} = \{\sum_{i=0}^{\infty} f_i t^{\{i\}} \mid f_i \in S_{K_0} \text{ and } f_i \to 0 \text{ as } i \to \infty\}.$$

Put $\widehat{\mathcal{R}} = \mathcal{R}_{K_0} \cap W(R)$ and $I_+ = I_+\widehat{\mathcal{R}}$.

Proposition 2.5 ([Li2], Lemma 2.2.1)**.** (1) *$\widehat{\mathcal{R}}$ (resp. $\mathcal{R}_{K_0}$) is a φ-stable $\mathfrak{S}$-algebra as a subring in $W(R)$ (resp. B^+_{cris}).*
(2) *$\widehat{\mathcal{R}}$ and I_+ (resp. $\mathcal{R}_{K_0}$ and $I_+\mathcal{R}_{K_0}$) are G-stable. The G-action on $\widehat{\mathcal{R}}$ and I_+ (resp. $\mathcal{R}_{K_0}$ and $I_+\mathcal{R}_{K_0}$) factors through $\hat{G}$.*
(3) *There exist natural isomorphisms $\mathcal{R}_{K_0}/I_+\mathcal{R}_{K_0} \simeq K_0$ and $\widehat{\mathcal{R}}/I_+ \simeq S/I_+S \simeq \mathfrak{S}/I_+\mathfrak{S} \simeq W(k)$.*

2.6. Let $\mathfrak{M}$ be a free or torsion Kisin module over $\mathfrak{S}$ of height $\leq r$. We equip $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ with a Frobenius by $\phi_{\widehat{\mathcal{R}}} \otimes \varphi_{\mathfrak{M}}$. It is known that a natural map $\mathfrak{M} \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ is an injection ([CL2], Section 3.1). By this injection, we regard $\mathfrak{M}$ as a $\varphi(\mathfrak{S})$-stable submodule of $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$.

Definition 2.7 ([Li2], Definition 2.2.3 and [CL2], Section 3.1)**.** A *free* (resp. *torsion*) *Liu module* $\hat{\mathfrak{M}}$ *of height* $\leq r$ is a triple $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi, \hat{G})$, where $(\mathfrak{M}, \varphi)$ is a free (resp. torsion) Kisin module $\mathfrak{M}$ of height $\leq r$ and $\hat{G}$ is an $\widehat{\mathcal{R}}$-semilinear $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ satisfying the following conditions:

(1) the $\hat{G}$-action commutes with $\phi_{\widehat{\mathcal{R}}} \otimes \varphi_{\mathfrak{M}}$,

(2) $\mathfrak{M} \subset (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})^{H_K}$,

(3) $\hat{G}$ acts on the $W(k)$-module $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}/I_+(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})$ trivially.

We denote by $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}}$ (resp. $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$) the category of free (resp. torsion) Liu modules of height $\leq r$. By convention, we put

$$\mathrm{Mod}^{\infty,\hat{G}}_{/\mathfrak{S}} = \bigcup_{r\geq 0} \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}}, \quad \mathrm{Mod}^{\infty,\hat{G}}_{/\mathfrak{S}_\infty} = \bigcup_{r\geq 0} \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}.$$

We shall call an object of $\mathrm{Mod}^{\infty,\hat{G}}_{/\mathfrak{S}}$ (resp. $\mathrm{Mod}^{\infty,\hat{G}}_{/\mathfrak{S}_\infty}$) a *free* (resp. *torsion*) *Liu modules of finite height.*

Remark 2.8. The notion of Liu modules was introduced by Tong Liu [Li2] under the name of $(\varphi, \hat{G})$*-modules.*

2.9. For a free or torsion Liu module $\hat{\mathfrak{M}}$ of finite height, we define a $\mathbb{Z}_p[G]$-module $\hat{T}(\hat{\mathfrak{M}})$ by

$$\hat{T}(\hat{\mathfrak{M}}) = \begin{cases} \mathrm{Hom}_{\widehat{\mathcal{R}},\varphi}(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, W(R)) & \text{if } \hat{\mathfrak{M}} \text{ is free} \\ \mathrm{Hom}_{\widehat{\mathcal{R}},\varphi}(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, W(R)_\infty) & \text{if } \hat{\mathfrak{M}} \text{ is torsion.} \end{cases}$$

Here, G acts on $\hat{T}(\hat{\mathfrak{M}})$ by $(\sigma.f)(x) = \sigma(f(\sigma^{-1}(x)))$ for $\sigma \in G,\ f \in \hat{T}(\hat{\mathfrak{M}}),\ x \in \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. We say that a torsion $\mathbb{Z}_p$-representation of G *arises from a Liu module* (*of height* $\leq r$) if it is isomorphic to $\hat{T}(\hat{\mathfrak{M}})$ for some torsion Liu module $\hat{\mathfrak{M}}$ (of height $\leq r$).

It follows from Theorem 3.1.3 (1) of [CL2] that the following diagram is commutative (we write only the torsion case here, but the same result also holds for the free case):

$$\begin{array}{ccc} \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} & \xrightarrow{\hat{T}} & \mathrm{Rep}_{\mathrm{tor}}(G) \\ \Big\downarrow \text{forgetful} & & \Big\downarrow \text{restriction} \\ \mathrm{Mod}^{r}_{/\mathfrak{S}_\infty} & \xrightarrow{T_{\mathfrak{S}}} & \mathrm{Rep}_{\mathrm{tor}}(G_\infty). \end{array}$$

Theorem 2.10. (1) (*[Li2], Theorem 2.3.1*) *The functor $\hat{T}$ induces an anti-equivalence of categories between the category $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}}$ of free Liu modules of height $\leq r$ and the category $\mathrm{Rep}^{\mathrm{ss},r}_{\mathrm{fr}}(G)$ of lattices in semi-stable representations of G with Hodge-Tate weights in $[0,r]$.*
(2) (*[CL2], Theorem 3.1.3*) *Any torsion semi-stable $\mathbb{Z}_p$-representation of G with Hodge-Tate weights in $[0,r]$ arises from a Liu module of height $\leq r$.*

Remark 2.11. (1) Differently from the free case as Theorem 2.10 (1), the functor $\hat{T}\colon \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Rep}_{\mathrm{tor}}(G)$ on the category of torsion Liu modules is not full in general although it is always faithful ([O], Corollary 2.8).
(2) (cf. [Li2], Theorem 3.2.2) If we remove the condition (3) of Definition 2.7, then $\hat{\mathfrak{M}}$ is called a *weak Liu module*. If this is the case, $\hat{T}(\hat{\mathfrak{M}})$ is potentially semi-stable. Moreover, $\hat{T}$ induces a contravariant fully faithful functor from the category of weak Liu modules of height $\leq r$ to the category of G-stable $\mathbb{Z}_p$-lattices in potentially semi-stable representations which are semi-stable over $K(\pi_m)$ and have Hodge-Tate weights in $[0,r]$. Here $m := \mathrm{Max}\{i \geq 0; K_0(\zeta_{p^i}) \subset K\}$.

§ 2.2. Étale $(\varphi, \hat{G})$-modules

2.12. Now we introduce the notion of étale $(\varphi,\hat{G})$-modules, which is deeply related with Kisin modules and Liu modules, modifying étale φ-modules defined by Fontaine. For precise information, see [O], Section 5.

2.13. An *étale φ-module over $\mathcal{O}$* is an $\mathcal{O}$-module of finite type M, equipped with a φ-semilinear map $\varphi\colon M \to M$ such that φ^* is bijective. Here, φ^* stands for the $\mathcal{O}$-linearlization $1\otimes\varphi\colon \mathcal{O}\otimes_{\varphi,\mathcal{O}} M \to M$ of φ.

An *étale φ-module over $\mathcal{E}$* is a finite dimensional $\mathcal{E}$-vector space M, equipped with a φ-semilinear map $\varphi\colon M \to M$ such that there exists a φ-stable $\mathcal{O}$-lattice L of M and that L is an étale φ-module over $\mathcal{O}$.

We denote by $\mathbf{\Phi M}_{/\mathcal{O}}$ (resp. $\mathbf{\Phi M}_{/\mathcal{O}_\infty}$) the category of étale φ-modules over $\mathcal{O}$ which are p-torsion free (resp. annihilated by some power of p). We denote by $\mathbf{\Phi M}_{/\mathcal{E}}$ the category of étale φ-modules over $\mathcal{E}$.

2.14. Recall that, as explained in 2.3, G_∞ acts on $\widehat{\mathcal{O}^{\mathrm{ur}}}, \widehat{\mathcal{E}^{\mathrm{ur}}}$ and $\mathcal{E}^{\mathrm{ur}}/\mathcal{O}^{\mathrm{ur}}$.
Let T be a $\mathbb{Z}_p$-representation of G_∞. We put

$$\mathcal{M}(T) = \begin{cases} \mathrm{Hom}_{\mathbb{Z}_p[G_\infty]}(T, \widehat{\mathcal{O}^{\mathrm{ur}}}) & \text{if } T \text{ is free} \\ \mathrm{Hom}_{\mathbb{Z}_p[G_\infty]}(T, \mathcal{E}^{\mathrm{ur}}/\mathcal{O}^{\mathrm{ur}}) & \text{if } T \text{ is annihilated by some power of } p. \end{cases}$$

Moreover, for any $\mathbb{Q}_p$-representation T of G_∞, we put

$$\mathcal{M}(T) = \mathrm{Hom}_{\mathbb{Q}_p[G_\infty]}(T, \widehat{\mathcal{E}^{\mathrm{ur}}}).$$

On the other hand. let M be an étale φ-module over $\mathcal{O}$. We put

$$\mathcal{T}(M) = \begin{cases} \mathrm{Hom}_{\mathcal{O},\varphi}(M, \widehat{\mathcal{O}^{\mathrm{ur}}}) & \text{if } M \text{ is } p\text{-torsion free} \\ \mathrm{Hom}_{\mathcal{O},\varphi}(M, \mathcal{E}^{\mathrm{ur}}/\mathcal{O}^{\mathrm{ur}}) & \text{if } M \text{ is annihilated by some power of } p. \end{cases}$$

Moreover, for any étale φ-module over $\mathcal{E}$, we put

$$\mathcal{T}(M) = \mathrm{Hom}_{\mathcal{E},\varphi}(M, \widehat{\mathcal{E}^{\mathrm{ur}}}).$$

Proposition 2.15 ([Fo], A.1.2.6)**.** *The functor $\mathcal{T}$ induces an anti-equivalence of categories between $\mathbf{\Phi M}_{/\mathcal{O}}$ (resp. $\mathbf{\Phi M}_{/\mathcal{O}_\infty}$, resp. $\mathbf{\Phi M}_{/\mathcal{E}}$) and $\mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)$ (resp. $\mathrm{Rep}_{\mathrm{tor}}(G_\infty)$, resp. $\mathrm{Rep}_{\mathbb{Q}_p}(G_\infty)$). Furthermore, $\mathcal{M}$ is a quasi-inverse functor of $\mathcal{T}$.*

2.16. Let $\mathfrak{M}$ be a Kisin module over $\mathfrak{S}$ of finite height. We shall denote by $\mathfrak{M}[1/u]$ the $\mathcal{O}$-module $\mathcal{O} \otimes_{\mathfrak{S}} \mathfrak{M}$ for short. Then $\mathfrak{M}[1/u]$ is an étale φ-module over $\mathcal{O}$, and there exists a natural isomorphism $T_{\mathfrak{S}}(\mathfrak{M}) \simeq \mathcal{T}(\mathfrak{M}[1/u])$ of $\mathbb{Z}_p[G_\infty]$-modules.

2.17. Next we introduce the notion of étale $(\varphi, \hat{G})$-modules and mention their important properties. We put $\mathcal{O}_{\hat{G}} = W(\mathrm{Fr}R)^{H_\infty}$, which is absolutely unramified and a complete discrete valuation ring with perfect residue field $\mathrm{Fr}R^{H_\infty}$. Put $\mathcal{E}_{\hat{G}} = \mathrm{Fr}\mathcal{O}_{\hat{G}} = \mathcal{O}_{\hat{G}}[1/p]$. By definition, $\varphi_{W(\mathrm{Fr}R)[1/p]}$ is stable on $\mathcal{O}_{\hat{G}}$ and $\mathcal{E}_{\hat{G}}$ which is bijective on themselves. Furthermore, $\hat{G}$ acts on $\mathcal{O}_{\hat{G}}$ and $\mathcal{E}_{\hat{G}}$ continuously. By a natural injection $M \to \mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathcal{O}} M$ (resp. $M \to \mathcal{E}_{\hat{G}} \otimes_{\varphi,\mathcal{E}} M$), we regard M as a sub $\varphi(\mathcal{O})$-module of $\mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathcal{O}} M$ (resp. a sub $\varphi(\mathcal{E})$-module of $\mathcal{E}_{\hat{G}} \otimes_{\varphi,\mathcal{E}} M$).

Definition 2.18 ([O], Definition 5.3)**.** An *étale $(\varphi, \hat{G})$-module $\hat{M}$ over $\mathcal{O}$* is a triple $\hat{M} = (M, \varphi, \hat{G})$, where (M, φ) is an étale φ-module over $\mathcal{O}$ and $\hat{G}$ is a $\hat{G}$-action on $\mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathcal{O}} M$ satisfying the following conditions:

(1) the $\hat{G}$-action commutes with $\varphi_{\mathcal{O}_{\hat{G}}} \otimes \varphi_M$,

(2) $M \subset (\mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathcal{O}} M)^{H_K}$.

If M is free over $\mathcal{O}$ (resp. killed by some power of p), then $\hat{M}$ is called a *free étale $(\varphi, \hat{G})$-module* (resp. a *torsion étale $(\varphi, \hat{G})$-module*). By replacing $\mathcal{O}$ and $\mathcal{O}_{\hat{G}}$ with $\mathcal{E}$ and $\mathcal{E}_{\hat{G}}$, respectively, we define the notion of an *étale $(\varphi, \hat{G})$-module over $\mathcal{E}$*.

Remark 2.19. Suppose $p > 2$ and fix a topological generator τ of $G_{p^\infty} \simeq \mathbb{Z}_p$. Let $\hat{M}$ be an étale $(\varphi, \hat{G})$-module over $\mathcal{O}$. Then we have a natural τ-action on $\mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} M \simeq \mathcal{O}_{\hat{G}} \otimes_{\varphi^{-1},\mathcal{O}_{\hat{G}}} (\mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathcal{O}} M)$. The étale φ-module M with this τ-action on $\mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} M$ is a *(φ, τ)-module* in the sense of [Ca3].

Denote by $\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}}$ (resp. $\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty}$, resp. $\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{E}}$) the category of free étale $(\varphi, \hat{G})$-modules over $\mathcal{O}$ (resp. the category of torsion étale $(\varphi, \hat{G})$-modules over $\mathcal{O}$, resp. the category of étale $(\varphi, \hat{G})$-modules over $\mathcal{E}$).

Lemma 2.20 ([O], Lemma 5.4). (1) *For any finite free $\mathbb{Z}_p$-representation T of G_∞ (resp. finite torsion $\mathbb{Z}_p$-representation T of G_∞, resp. finite $\mathbb{Q}_p$-representation T of G_∞), the natural map*

$$\mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathcal{O}} \mathcal{M}(T) \to \mathrm{Hom}_{\mathbb{Z}_p[H_\infty]}(T, W(\mathrm{Fr}R))$$
$$(\text{resp. } \mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathcal{O}} \mathcal{M}(T) \to \mathrm{Hom}_{\mathbb{Z}_p[H_\infty]}(T, W(\mathrm{Fr}R)_\infty),$$
$$\text{resp. } \mathcal{E}_{\hat{G}} \otimes_{\varphi,\mathcal{E}} \mathcal{M}(T) \to \mathrm{Hom}_{\mathbb{Q}_p[H_\infty]}(T, W(\mathrm{Fr}R)[1/p]))$$

is an isomorphism.
(2) *For any free étale φ-module M over $\mathcal{O}$ (resp. torsion étale φ-module M over $\mathcal{O}$, resp. étale φ-module M over $\mathcal{E}$), the natural map*

$$\mathcal{T}(M) \to \mathrm{Hom}_{\mathcal{O}_{\hat{G}},\varphi}(\mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathcal{O}} M, W(\mathrm{Fr}R))$$
$$(\text{resp. } \mathcal{T}(M) \to \mathrm{Hom}_{\mathcal{O}_{\hat{G}},\varphi}(\mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathcal{O}} M, W(\mathrm{Fr}R)_\infty),$$
$$\text{resp. } \mathcal{T}(M) \to \mathrm{Hom}_{\mathcal{E}_{\hat{G}},\varphi}(\mathcal{E}_{\hat{G}} \otimes_{\varphi,\mathcal{E}} M, W(\mathrm{Fr}R)[1/p]))$$

is an isomorphism.

2.21. For any $T \in \mathrm{Rep}_{\mathbb{Z}_p}(G)$ or $T \in \mathrm{Rep}_{\mathrm{tor}}(G)$ (resp. $T \in \mathrm{Rep}_{\mathbb{Q}_p}(G_\infty)$), we define an étale $(\varphi, \hat{G})$-module $\hat{\mathcal{M}}(T)$ as follows: $\hat{\mathcal{M}}(T) = \mathcal{M}(T)$ as an étale φ-module and a $\hat{G}$-action on $\mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathcal{O}} \mathcal{M}(T)$ (resp. $\mathcal{E}_{\hat{G}} \otimes_{\varphi,\mathcal{E}} \mathcal{M}(T)$) is naturally defined via Lemma 2.20 (1).

For any $\hat{M} \in \mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}}$, $\hat{M} \in \mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty}$ or $\hat{M} \in \mathbf{\Phi M}^{\hat{G}}_{/\mathcal{E}}$ we define a representation $\hat{\mathcal{T}}(\hat{M})$ of G as follows: $\hat{\mathcal{T}}(\hat{M}) = \mathcal{T}(M)$ as a G_∞-representation and naturally extend its G_∞-action to G via Lemma 2.20 (2).

Combining Proposition 2.15 and Lemma 2.20, we can verify without difficulty the following fundamental proposition.

Proposition 2.22 ([O], Proposition 5.5). *The contravariant functor $\hat{\mathcal{T}}$ is an anti-equivalence of categories between $\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}}$ (resp. $\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty}$, resp. $\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{E}}$) and $\mathrm{Rep}_{\mathbb{Z}_p}(G)$ (resp. $\mathrm{Rep}_{\mathrm{tor}}(G)$, resp. $\mathrm{Rep}_{\mathbb{Q}_p}(G)$). Furthermore, $\hat{\mathcal{M}}$ is a quasi-inverse of $\hat{\mathcal{T}}$.*

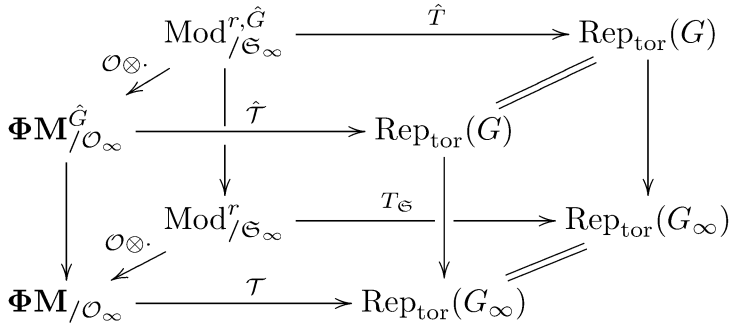
It follows from the construction of $\hat{\mathcal{T}}$ that the following diagram is commutative (we write only the torsion case here, but the same results also hold for other cases):

$$\begin{array}{ccc} \mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty} & \xrightarrow{\hat{\mathcal{T}}} & \mathrm{Rep}_{\mathrm{tor}}(G) \\ \Big\downarrow \text{forgetful} & & \Big\downarrow \text{restriction} \\ \mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty} & \xrightarrow{\mathcal{T}} & \mathrm{Rep}_{\mathrm{tor}}(G_\infty). \end{array}$$

2.23. Let $\hat{\mathfrak{M}}$ be a free (resp. torsion) Liu module of finite height. Extending the $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ to $\mathcal{O}_{\hat{G}} \otimes_{\widehat{\mathcal{R}}} (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}) \simeq \mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathcal{O}} \mathfrak{M}[1/u]$ by a natural way,

we obtain an object of $\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}}$ (resp. $\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty}$). We shall denote by $\widehat{\mathfrak{M}[1/u]}$ the triple $(\mathcal{O} \otimes_{\mathfrak{S}} \mathfrak{M}, \varphi, \hat{G})$ for short. Note that there exists a canonical isomorphism $\hat{T}(\hat{\mathfrak{M}}) \simeq \hat{\mathcal{T}}(\widehat{\mathfrak{M}[1/u]})$ of $\mathbb{Z}_p$-representations of G.

2.24. Summarizing the above, we have the following commutative diagram:



Here two vertical arrows in the left side are forgetful functors and those in the right side are restriction functors. An analogous commutative diagram exists also for the free case.

Remark 2.25. To solve Question 1.3, we have to check "differences" (if exists) between the following three categories:

$$\mathrm{Rep}^{\mathrm{ss}}_{\mathrm{tor}}(G) \subset \mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G) \subset \mathrm{Rep}_{\mathrm{tor}}(G),$$

By Proposition 2.22, the equality $\mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G) = \mathrm{Rep}_{\mathrm{tor}}(G)$ holds if and only if, for any $\hat{M} \in \mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty}$, there exists a Liu submodule $\hat{\mathfrak{M}}$ of finite height of $\hat{M}$ such that $\widehat{\mathfrak{M}[1/u]} = \hat{M}$.

§ 3. Cartier duality theorem

In this section, we give a brief summary of the Cartier duality for Liu modules. Throughout this section, we assume $r < \infty$.

§ 3.1. Cartier duality theorem for Kisin modules

3.1. The results in this subsection are due to Liu, see Section 3.1 of [Li1] for more details.

We put $\mathfrak{S}_\infty = \mathfrak{S}[1/p]/\mathfrak{S}$. Let $\mathfrak{M}$ be a Kisin module of height $\leq r$, and put

$$\mathfrak{M}^\vee = \begin{cases} \mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S}) & \text{if } \mathfrak{M} \text{ is free} \\ \mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S}_\infty) & \text{if } \mathfrak{M} \text{ is torsion.} \end{cases}$$

Then we have natural pairings

$$\langle \cdot, \cdot \rangle \colon \mathfrak{M} \times \mathfrak{M}^\vee \to \mathfrak{S} \quad \text{if } \mathfrak{M} \text{ is free} \qquad (*)$$

and

$$\langle \cdot, \cdot \rangle \colon \mathfrak{M} \times \mathfrak{M}^\vee \to \mathfrak{S}_\infty \quad \text{if } \mathfrak{M} \text{ is torsion.} \qquad (**)$$

The Frobenius $\varphi^\vee_{\mathfrak{M}}$ on $\mathfrak{M}^\vee$ is defined to be

$$\langle \varphi_{\mathfrak{M}}(x), \varphi^\vee_{\mathfrak{M}}(y) \rangle = c_0^{-r} E(u)^r \varphi(\langle x, y \rangle) \quad \text{for } x \in \mathfrak{M}, y \in \mathfrak{M}^\vee.$$

Here, pc_0 is the constant coefficient of $E(u)$.

Theorem 3.2 ([Li1], Section 3.1)**.** *Let $\mathfrak{M}$ be a free (resp. torsion) Kisin module of height $\leq r$.*
(1) *$\mathfrak{M}^\vee$ is a free (resp. torsion) Kisin module of height $\leq r$.*
(2) *The correspondence $\mathfrak{M} \mapsto \mathfrak{M}^\vee$ gives rise to an anti-equivalence on the category of free (resp. torsion) Kisin modules of height $\leq r$, and a natural map $\mathfrak{M} \to (\mathfrak{M}^\vee)^\vee$ is an isomorphism.*
(3) *The paring $(*)$ (resp. $(**)$) is perfect. Thus we have a canonical isomorphism $T_{\mathfrak{S}}(\mathfrak{M}^\vee) \simeq T^\vee_{\mathfrak{S}}(\mathfrak{M})(r)$ of $\mathbb{Z}_p[G_\infty]$-modules.*
(4) *Taking the duals preserves a short exact sequence of free (resp. torsion) Kisin modules of height $\leq r$.*

§ 3.2. Cartier duality theorem for Liu modules

3.3. Now we explain the Cartier duality for Liu modules. The Cartier duality for Liu modules is based on that of Kisin modules. In fact, as explained in the below, underlying Kisin modules of Cartier duals of Liu modules are Cartier duals of Kisin modules. See Section 3 of [O] for more details.

For simplicity, we suppose $p > 2$ in this subsection. We put $\widehat{\mathcal{R}}_\infty = \widehat{\mathcal{R}}[1/p]/\widehat{\mathcal{R}}$. Let $\hat{\mathfrak{M}}$ be a Liu module of height $\leq r$, and put

$$(\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M})^\vee = \begin{cases} \mathrm{Hom}_{\mathfrak{S}}(\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}, \widehat{\mathcal{R}}) & \text{if } \mathfrak{M} \text{ is free} \\ \mathrm{Hom}_{\mathfrak{S}}(\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}, \widehat{\mathcal{R}}_\infty) & \text{if } \mathfrak{M} \text{ is torsion.} \end{cases}$$

Then we can show that a natural map $\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^\vee \to (\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M})^\vee$ is an isomorphism ([O], Lemma 3.5). Consider natural parings

$$\langle \cdot, \cdot \rangle \colon (\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}) \times (\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M})^\vee \to \widehat{\mathcal{R}} \quad \text{if } \mathfrak{M} \text{ is free} \qquad (\sharp)$$

and

$$\langle \cdot, \cdot \rangle \colon (\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}) \times (\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M})^\vee \to \widehat{\mathcal{R}}_\infty \quad \text{if } \mathfrak{M} \text{ is torsion.} \qquad (\sharp\sharp)$$

Fixing a topological generator τ of G_{p^∞}, we can define a $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}^\vee = (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})^\vee$ such that

$$\langle \tau(x), \tau(y) \rangle = \hat{c}^r(\langle x, y \rangle) \quad \text{for } x \in \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, y \in (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})^\vee.$$

Here,

$$\hat{c} = \prod_{n=1}^{\infty} \varphi^n\left(\frac{E(u)}{\tau(E(u))}\right) \in \widehat{\mathcal{R}}^\times.$$

Let $\hat{\mathfrak{M}}$ be a Liu module of height $\leq r$. Denote by $\hat{\mathfrak{M}}^\vee$ a triple $\hat{\mathfrak{M}}^\vee = (\mathfrak{M}^\vee, \varphi^\vee, \hat{G})$, where $(\mathfrak{M}^\vee, \varphi^\vee)$ is the Cartier dual of $(\mathfrak{M}, \varphi)$ and $\hat{G}$ is a $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}^\vee$ as above.

Theorem 3.4 ([O], Theorem 3.9)**.** *Let $\hat{\mathfrak{M}}$ be a free (resp. torsion) Liu module of height $\leq r$.*
(1) *$\hat{\mathfrak{M}}^\vee$ is a free (resp. torsion) Liu module of height $\leq r$.*
(2) *The correspondence $\hat{\mathfrak{M}} \mapsto \hat{\mathfrak{M}}^\vee$ gives rise to an anti-equivalence on the category of free (resp. torsion) Liu modules of height $\leq r$, and a natural map $\hat{\mathfrak{M}} \to (\hat{\mathfrak{M}}^\vee)^\vee$ is an isomorphism.*
(3) *The paring ($\sharp$) (resp. ($\sharp\sharp$)) is perfect. Thus we have a canonical isomorphism $\hat{T}(\hat{\mathfrak{M}}^\vee) \simeq \hat{T}^\vee(\hat{\mathfrak{M}})(r)$ of $\mathbb{Z}_p[G]$-modules.*
(4) *Taking the duals preserves a short exact sequence of free (resp. torsion) Liu modules of height $\leq r$.*

For a proof of Theorem 3.4, we have only to observe is "compatibilities" between the Cartier duality for Kisin modules, an $\widehat{\mathcal{R}}$-structure and a $\hat{G}$-action on Liu modules. One of the most serious problem is that we have only a few information for the ring $\widehat{\mathcal{R}}$. However, fortunately, we do not need explicit calculations in the proof but we only need some ring theoretic properties for $\widehat{\mathcal{R}}$ (cf. the property that $\widehat{\mathcal{R}}/I_+ \simeq W(k)$ is p-torsion free).

§ 4. Maximal and minimal theory

In this section, we define an abelian full subcategory $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ (resp. $\mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$) of $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$, whose objects are called *maximal* (resp. *minimal*), such that the functor $\hat{T}$ restricted on $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ (resp. $\mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$) is fully faithful and its essential image coincides with that of $\hat{T}\colon \mathrm{Mod}^{\infty,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Rep}_{\mathrm{tor}}(G)$. As one of by-products, we obtain the fact that the category of torsion $\mathbb{Z}_p$-representations of G arising from Liu modules is an abelian category (cf. Corollary 4.15). Note that, in general, the category $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ is not abelian and $\hat{T}\colon \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Rep}_{\mathrm{tor}}(G)$ is not fully faithful.

§ 4.1. Maximal and minimal Kisin modules

4.1. Let r be a non-negative integer or $r = \infty$. Let $\mathfrak{M} \in \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$. We denote by $F^r_{\mathfrak{S}}(\mathfrak{M}[1/u])$ the partially ordered set (by inclusion) of $\mathfrak{N} \in \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ which is a torsion Kisin submodule of the étale φ-module $\mathfrak{M}[1/u]$ such that $\mathfrak{N}[1/u] = \mathfrak{M}[1/u]$. It is verified that $F^r_{\mathfrak{S}}(\mathfrak{M}[1/u])$ has a maximum (cf. Corollary 3.2.6, [CL1]), denoted by $\mathrm{Max}^r(\mathfrak{M})$. It is also verified that, if $r < \infty$, $F^r_{\mathfrak{S}}(\mathfrak{M}[1/u])$ has a minimum (cf. Corollary 3.2.6, *loc. cit.*), denoted by $\mathrm{Min}^r(\mathfrak{M})$. By definition, we have $\mathrm{Max}^r(\mathrm{Max}^r(\mathfrak{M})) = \mathrm{Max}^r(\mathfrak{M})$ and $\mathrm{Min}^r(\mathrm{Min}^r(\mathfrak{M})) = \mathrm{Min}^r(\mathfrak{M})$.

Let $\iota^{\mathfrak{M}}_{\max} : \mathfrak{M} \to \mathrm{Max}^r(\mathfrak{M})$ and $\iota^{\mathfrak{M}}_{\min} : \mathrm{Min}^r(\mathfrak{M}) \to \mathfrak{M}$ denote the inclusions.

Theorem 4.2 ([CL1], Section 3.3 and 3.4)**.** *Under the assumptions of 4.1, we have the following assertions.*
(1) *The morphisms* $T_{\mathfrak{S}}(\iota^{\mathfrak{M}}_{\max})$ *and* $T_{\mathfrak{S}}(\iota^{\mathfrak{M}}_{\min})$ *are bijective.*
(2) *Let* $f : \mathfrak{M} \to \mathfrak{M}'$ (*resp.* $f : \mathfrak{M}' \to \mathfrak{M}$) *be a morphism of* $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$. *If* $T_{\mathfrak{S}}(f)$ *is bijective, then there exists a unique morphism* $g : \mathfrak{M}' \to \mathrm{Max}^r(\mathfrak{M})$ (*resp.* $g : \mathrm{Min}^r(\mathfrak{M}) \to \mathfrak{M}'$) *such that* $g \circ f = \iota^{\mathfrak{M}}_{\max}$ ($f \circ g = \iota^{\mathfrak{M}}_{\min}$).

4.3. It follows from the theorem that $\mathfrak{M} \mapsto \mathrm{Max}^r(\mathfrak{M})$ (resp. $\mathfrak{M} \mapsto \mathrm{Min}^r(\mathfrak{M})$) gives rise to a functor $\mathrm{Max}^r : \mathrm{Mod}^r_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ (resp. $\mathrm{Min}^r : \mathrm{Mod}^r_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$). Denote by $\mathrm{Max}^r_{/\mathfrak{S}_\infty}$ (resp. $\mathrm{Min}^r_{/\mathfrak{S}_\infty}$) the essential image of Max^r (resp. Min^r). By definition, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Mod}^r_{/\mathfrak{S}_\infty} & \xrightarrow{\quad T_{\mathfrak{S}} \quad} & \mathrm{Rep}_{\mathrm{tor}}(G_\infty) \\ & {\scriptstyle \mathrm{Max}^r} \searrow \quad \nearrow {\scriptstyle T_{\mathfrak{S}}} & \\ & \mathrm{Max}^r_{/\mathfrak{S}_\infty}. & \end{array}$$

(We have the same commutative diagram after replacing Max with Min in the above.)

The following is a summary of main results in Section 3.3 and 3.4 of [CL1].

Theorem 4.4. *Under the assumptions of 4.1, we have the following assertions.*
(1) *The categories* $\mathrm{Max}^r_{/\mathfrak{S}_\infty}$ *and* $\mathrm{Min}^r_{/\mathfrak{S}_\infty}$ *are abelian.*
(2) *The functor* $\mathrm{Max}^r : \mathrm{Mod}^r_{/\mathfrak{S}_\infty} \to \mathrm{Max}^r_{/\mathfrak{S}_\infty}$ ($\mathrm{Min}^r : \mathrm{Mod}^r_{/\mathfrak{S}_\infty} \to \mathrm{Min}^r_{/\mathfrak{S}_\infty}$) *is exact and left adjoint* (*resp. right adjoint*) *to the inclusion functor* $\mathrm{Max}^r_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ (*resp.* $\mathrm{Min}^r_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$).
(3) *The restriction of* $T_{\mathfrak{S}}$ *on* $\mathrm{Max}^r_{/\mathfrak{S}_\infty}$ *and* $\mathrm{Min}^r_{/\mathfrak{S}_\infty}$ *are exact and fully faithful, respectively.*
(4) *Taking the Cartier duals gives rise to an anti-equivalence of abelian categories between* $\mathrm{Max}^r_{/\mathfrak{S}_\infty}$ *and* $\mathrm{Min}^r_{/\mathfrak{S}_\infty}$.
(5) *If* $er < p - 1$, *then* $\mathrm{Max}^r_{/\mathfrak{S}_\infty} = \mathrm{Min}^r_{/\mathfrak{S}_\infty} = \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ *and* $\mathrm{Max}^r = \mathrm{Min}^r$.

Definition 4.5 ([CL1], Definition 3.3.1 and 3.4.1)**.** Let $\mathfrak{M}$ be a torsion Kisin module of height $\leq r$. We say that $\mathfrak{M}$ is *maximal* (resp. *minimal*) if $\iota_{\max}^{\mathfrak{M}}$ (resp. $\iota_{\min}^{\mathfrak{M}}$) is bijective. Note that this definition depends on the choice of r.

4.6. In the case where $r = 1$, the category $\mathrm{Mod}^{r}_{/\mathfrak{S}_\infty}$ is dually equivalent to the category of finite flat group schemes over $\mathcal{O}_K$ killed by a power of p. Under this equivalence, the functor Min^r (resp. Max^r) corresponds to the maximal (resp. minimal) models defined by Raynaud in [Ra].

Combining Proposition 1.5 and Theorem 4.4, we see that torsion $\mathbb{Z}_p$-representations of G_∞ are completely classified by maximal Kisin modules of finite height:

Corollary 4.7. *The functor* $T_{\mathfrak{S}} \colon \mathrm{Mod}^{\infty}_{/\mathfrak{S}_\infty} \to \mathrm{Rep}_{\mathrm{tor}}(G_\infty)$ *induces the equivalence of abelian categories between the category* $\mathrm{Max}^{\infty}_{/\mathfrak{S}_\infty}$ *of maximal torsion Kisin modules of finite height and the category* $\mathrm{Rep}_{\mathrm{tor}}(G_\infty)$ *of torsion* $\mathbb{Z}_p$*-representations of* G_∞*.*

§ 4.2. Maximal and minimal Liu modules

4.8. The same results on maximal and minimal theory for Kisin modules as in the previous subsection holds for Liu modules. The proofs of maximal and minimal theory for Liu modules in [O] are similar to those of Kisin modules in [CL1], but we need some careful considerations for $\hat{G}$-actions on Liu modules. When we see $\hat{G}$-actions on Liu modules, we need to use the ring $\widehat{\mathcal{R}}$, however the structure of this ring is complicated. On the other hand, the definitions of étale $(\varphi, \hat{G})$-modules need only simple rings, which is one of the advantage of étale $(\varphi, \hat{G})$-modules. Results appearing in this subsection is based on this advantage.

4.9. Let r be a non-negative integer or $r = \infty$. Let $\hat{\mathfrak{M}} \in \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$. We denote by $F^{r,\hat{G}}_{\mathfrak{S}}(\widehat{\mathfrak{M}[1/u]})$ the partially ordered set (by inclusion) of $\hat{\mathfrak{N}} \in \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ which is a Liu submodule of the étale $(\varphi, \hat{G})$-module $\widehat{\mathfrak{M}[1/u]}$ such that $\widehat{\mathfrak{N}[1/u]} = \widehat{\mathfrak{M}[1/u]}$. It is verified that $F^{r,\hat{G}}_{\mathfrak{S}}(\widehat{\mathfrak{M}[1/u]})$ has a maximum, denoted by $\mathrm{Max}^r(\hat{\mathfrak{M}})$. It is also verified that, if $r < \infty$, $F^{r,\hat{G}}_{\mathfrak{S}}(\widehat{\mathfrak{M}[1/u]})$ has a minimum, denoted by $\mathrm{Min}^r(\hat{\mathfrak{M}})$. By definition, we have $\mathrm{Max}^r(\mathrm{Max}^r(\hat{\mathfrak{M}})) = \mathrm{Max}^r(\hat{\mathfrak{M}})$ and $\mathrm{Min}^r(\mathrm{Min}^r(\hat{\mathfrak{M}})) = \mathrm{Min}^r(\hat{\mathfrak{M}})$.

Remark 4.10. The author does not know whether $\mathrm{Max}^r(\mathfrak{M})$ (resp. $\mathrm{Min}^r(\mathfrak{M})$) coincides with the underlying Kisin module of $\mathrm{Max}^r(\hat{\mathfrak{M}})$ (resp. $\mathrm{Min}^r(\hat{\mathfrak{M}})$) or not. If $er < p - 1$, then they are coincide since the set $F^r_{\mathfrak{S}}(\mathfrak{M}[1/u])$ contains only one element $\mathfrak{M}$ (see Remark after Corollary 3.2.6 of [CL1]).

Let $\iota_{\max}^{\hat{\mathfrak{M}}} \colon \hat{\mathfrak{M}} \to \mathrm{Max}^r(\hat{\mathfrak{M}})$ and $\iota_{\min}^{\hat{\mathfrak{M}}} \colon \mathrm{Min}^r(\hat{\mathfrak{M}}) \to \hat{\mathfrak{M}}$ denote the inclusions.

Theorem 4.11 ([O], Proposition 5.18)**.** *Under the assumptions of 4.8, we have the following assertions.*
(1) *The morphisms* $\hat{T}(\iota^{\hat{\mathfrak{M}}}_{\max})$ *and* $\hat{T}(\iota^{\hat{\mathfrak{M}}}_{\min})$ *are bijective.*
(2) *Let* $f\colon \hat{\mathfrak{M}} \to \hat{\mathfrak{M}}'$ *(resp.* $f\colon \hat{\mathfrak{M}}' \to \hat{\mathfrak{M}}$ *) be a morphism of* $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$*. If* $\hat{T}(f)$ *is bijective, then there exists a unique morphism* $g\colon \hat{\mathfrak{M}}' \to \mathrm{Max}^r(\hat{\mathfrak{M}})$ *(resp.* $g\colon \mathrm{Min}^r(\hat{\mathfrak{M}}) \to \hat{\mathfrak{M}}'$*) such that* $g \circ f = \iota^{\hat{\mathfrak{M}}}_{\max}$ $(f \circ g = \iota^{\hat{\mathfrak{M}}}_{\min})$.

4.12. It follows from the theorem that $\hat{\mathfrak{M}} \mapsto \mathrm{Max}^r(\hat{\mathfrak{M}})$ (resp. $\hat{\mathfrak{M}} \mapsto \mathrm{Min}^r(\hat{\mathfrak{M}})$) gives rise to a functor $\mathrm{Max}^r\colon \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ (resp. $\mathrm{Min}^r\colon \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$). Denote by $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ (resp. $\mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$) the essential image of Max^r (resp. Min^r). As well as the previous subsection, we see that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} & \xrightarrow{\quad\hat{T}\quad} & \mathrm{Rep}_{\mathrm{tor}}(G) \\ & {\scriptstyle \mathrm{Max}^r} \searrow \quad \nearrow {\scriptstyle \hat{T}} & \\ & \mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}. & \end{array}$$

(We have the same commutative diagram after replacing Max with Min in the above.)

The following is a summary of main results in Section 5.5 and 5.6 of [O].

Theorem 4.13. *Under the assumptions of 4.8, we have the following assertions.*
(1) *The categories* $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *and* $\mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *are abelian.*
(2) *The functor* $\mathrm{Max}^r\colon \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ $(\mathrm{Min}^r\colon \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty})$ *is exact and left adjoint (resp. right adjoint) to the inclusion functor* $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *(resp.* $\mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$*).*
(3) *The restriction of* $\hat{T}$ *on* $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *and* $\mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *are exact and fully faithful, respectively.*
(4) *Taking the Cartier duals gives rise to an anti-equivalence of abelian categories between* $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *and* $\mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$.
(5) *If* $er < p-1$*, then* $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty} = \mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty} = \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *and* $\mathrm{Max}^r = \mathrm{Min}^r$.

Definition 4.14 ([O], Definition 5.15)**.** Let $\hat{\mathfrak{M}}$ be a torsion Liu module of height $\leq r$. We say that $\mathfrak{M}$ is *maximal* (resp. *minimal*) if $\iota^{\hat{\mathfrak{M}}}_{\max}$ (resp. $\iota^{\hat{\mathfrak{M}}}_{\min}$) is bijective. Note that this definition depends on the choice of r.

Corollary 4.15 ([O], Corollary 1.3)**.** *The functor* $\hat{T}\colon \mathrm{Mod}^{\infty,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Rep}_{\mathrm{tor}}(G)$ *induces the equivalence of abelian categories between the category* $\mathrm{Max}^{\infty,\hat{G}}_{/\mathfrak{S}_\infty}$ *of maximal torsion Liu modules of finite height and the category* $\mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G)$ *of torsion* $\mathbb{Z}_p$*-representations of* G *arising from Liu modules.*

Corollary 4.16 ([O], Corollary 1.4). *Suppose $er < p-1$. Then the category $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ is abelian and artinian. Furthermore, the functor $\hat{T}\colon \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Rep}_{\mathrm{tor}}(G)$ is exact and fully faithful, and its essential image is stable under taking a subquotient.*

Such a result on torsion Breuil modules has been proved by X. Caruso (cf. [Ca2], Théorème 1.0.4).

By the Cartier duality theorem (cf. Theorem 3.4) and Theorem 4.13, it is not difficult to check the following:

Theorem 4.17 ([O], Theorem 1.1). *The category $\mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G)$ is an abelian full subcategory of $\mathrm{Rep}_{\mathrm{tor}}(G)$ which is stable under taking a subquotient, $\oplus$, $\otimes$ and a dual.*

References

[Ca1] Xavier Caruso, Conjecture de l'inértié modre de Serre, Thesis, Université de Paris-Sud (2005).

[Ca2] Xavier Caruso, Représentations semi-stables de torsion dans le case $er < p-1$, J. Reine Angew. Math., **594** (2006), 35–92.

[Ca3] Xavier Caruso, Representations galoisiennes p-adiques et (ϕ, τ)-modules, to appear in Duke Math. J.

[CL1] Xavier Caruso and Tong Liu, Quasi-semi-stable representations, Bull. Soc. Math. France, **137** (2009), 185–223.

[CL2] Xavier Caruso and Tong Liu, Some bounds for ramification of p^n-torsion semi-stable representations, J. Algebra, **325** (2011), 70–96.

[Fo] Jean-Marc Fontaine, Représentations p-adiques des corps locaux. I, The Grothendieck Festschrift, Vol. II, Progr. Math., **87**, Birkhäuser, Boston, (2008), 61–88.

[Li1] Tong Liu, Torsion p-adic Galois representations and a conjecture of Fontaine, Ann. Sci. École Norm. Supér., (4) **40** (2007) 633–674.

[Li2] Tong Liu, A note on lattices in semi-stable representations, Math. Ann., **346** (2010), 117–138.

[Na] Kentaro Nakamura, Deformations of trianguline B-pairs and Zariski density of two dimensional crystalline representations, available at http://arxiv.org/abs/1006.4891

[O] Yoshiyasu Ozeki, Torsion representations arising from $(\varphi, \hat{G})$-modules, J. Number Theory, **133** (2013), 3810–3861.

[Ra] Michel Raynaud, Schémas en groupes de type $(p, \ldots, p)$, Bull. Soc. Math., France, **102** (1974), 241–280.

[Wi] Jean-Pierre Wintenberger, Le corps des normes de certaines extensions infinies de corps locaux; applications, Ann. Sci. École Norm. Supér., (4) **16** (1983), 59–89.