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Differential Euler systems associated to modular forms (1)

By

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Abstract

We construct Euler systems in the space of modular forms associated to symmetric squares of modular forms. The aim of this paper is to state the p-adic Beilinson conjecture associated to symmetric square of modular forms and show an evidence of this conjecture.

§ 1. Introduction.

Cyclotomic units and Beilinson elements satisfy certain norm relations. These norm relations are called the Euler system relations. T. Fukaya, K. Kato and N. Kurokawa construct modular forms satisfying a trace version of Euler system relations associated to the symmetric squares of modular forms.

In sections 2, we will quickly give a review on Eisenstain series $E_N^{k,s}$ satisfying Euler system relations. These are almost the same as the Eisenstein series written in Kato's letter [9].

$$E_N^{k,s}(z_1,z_2) := (\mathrm{Im}(z_1)\mathrm{Im}(z_2))^s \sum_{P,Q\in SL_2(\mathbb{Z})\backslash \Xi_N} j^{k,s}(z_1,z_2;P,Q).$$

Here we put

$$j^{k,s}(z_1,z_2;P,Q) = \det\left(P\begin{pmatrix} z_1 \\ 1\end{pmatrix}, Q\begin{pmatrix} -z_2 \\ 1\end{pmatrix}\right)^k |\det\left(P\begin{pmatrix} z_1 \\ 1\end{pmatrix}, Q\begin{pmatrix} -z_2 \\ 1\end{pmatrix}\right)|^{-2s}. \tag{1.1}$$

Here we define Ξ_N by the following.

(1.2)

$$\Xi_N := \{P,Q \in M_2(\mathbb{Z})\times M_2(\mathbb{Z}) | \det P = \det Q \neq 0, (\det P, N) = 1, Q^{-1}P \in \tilde{\Gamma}_0(N)\}.$$

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Here we put

$$\tilde{\Gamma}_0(N) := \left\{ \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \mid a_1, b_1, c_1, d_1 \in \mathbb{Z}_{(N)}, c_1 \in N\mathbb{Z}_{(N)} \right\}. \tag{1.3}$$

Here $\mathbb{Z}_{(N)} := \{a/b \mid a, b \in \mathbb{Z}, (b, N) = 1\}$. Throughout this paper, we always assume k is an integer and s is a complex number such that $k + 2\mathrm{Re}(s) > 2$ for the convergence of Eisenstein series.

Main result of Section 2 is that this Eisenstein series satisfies the Euler system relation (Theorem 2.2).

In sections 3, we will relate the Eisenstein series studied in sections 2 to special values of L-functions using Shimura's result.

In section 4, using the result of section 3, we will formulate p-adic Beilinson conjecture of two products of modular curves. Roughly speaking, this conjecture predicts the existence of the norm compatible system which is sent to the Eisenstein series defined in section 1 by p-adic regulator.

In section 5, we will give the evidence of the conjecture in section 4.

§ 2. The Eisenstein series satisfying the Euler system relation.

In this section, we introduce the Siegel Eisenstein series satisfying the Euler System relation. First, we need to introduce some notations.

For $z_1, z_2 \in \mathfrak{h}$ ($\mathfrak{h} = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$), we put

$$E_N^{k,s}(z_1, z_2) := (\mathrm{Im}(z_1)\mathrm{Im}(z_2))^s \sum_{P,Q \in SL_2(\mathbb{Z})\backslash \Xi_N} j^{k,s}(z_1, z_2; P, Q).$$

Here $j^{k,s}(z_1, z_2; P, Q)$ and Ξ_N are defined in the introduction. We also put

$$E_{N,\chi}^{k,s} = (\mathrm{Im}(z_1)\mathrm{Im}(z_2))^s \sum_{P,Q \in SL_2(\mathbb{Z})\backslash \Xi_N} \chi(\det(P)) j^{k,s}(z_1, z_2; P, Q) \tag{2.1}$$

for Dirichlet characters χ whose conductor are divisible by N.

Proposition 2.1. *The sums defining $E_N^{k,s}$ and $E_{N,\chi}^{k,s}$ are absolutely convergent for $k + 2\mathrm{Re}(s) > 2$ and satisfy*

$$E_N^{k,s}\left(\frac{a_1 z_1 + b}{c_1 z_1 + d_1}, \frac{a_2 z_2 + b_2}{c_2 z_2 + d_2}\right) = (c_1 z_1 + d_1)^k (c_2 z_2 + d_2)^k E_N^{k,s}(z_1, z_2)$$

$$E_{N,\chi}^{k,s}\left(\frac{a_1 z_1 + b}{c_1 z_1 + d_1}, \frac{a_2 z_2 + b_2}{c_2 z_2 + d_2}\right) = (c_1 z_1 + d_1)^k (c_2 z_2 + d_2)^k E_{N,\chi}^{k,s}(z_1, z_2)$$

for any $\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right) \in \Gamma_0(N)$.

Proof. For the proof, see [12] (master's thesis of the author). $\square$

In this section, we recall the Euler system relation proved in [12]. We use the following notation. First, let $X_0(N)$ be the modular curve with level N over $\mathbb{Q}$. Let $X_0'(N) = X_0(N) \times_{Spec(\mathbb{Q})} X_0(N)$ and

$$\mathrm{Tr}'_{X_0'(N')/X_0'(N)} = \frac{N\varphi(N)}{N'\varphi(N')} \mathrm{Tr}_{X_0'(N')/X_0'(N)}$$

when N' is divisible by N, where φ means the Euler function and the trace map Tr is defined by

$$\mathrm{Tr}_{X_0'(N')/X_0'(N)}(f) = \sum_{\gamma_1,\gamma_2 \in \Gamma_0(N)/\Gamma_0(N')} f|[\gamma_1,\gamma_2]_{k,s} \tag{2.2}$$

for the function f on $\mathfrak{h} \times \mathfrak{h}$. Here we put

$$f|[\gamma_1,\gamma_2]_{k,s}(z_1,z_2) = (c_1z_1+d_1)^{-k}(c_2z_1+d_2)^{-k} f(\frac{a_1z_1+b_1}{c_1z_1+d_1}, \frac{a_2z_2+b_2}{c_2z_2+d_2})$$

for $\gamma_1 = \left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right), \gamma_2 = \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$.

Theorem 2.2. *(Euler system relation) E_N $(1 \leq N \in \mathbb{Z})$ satisfy the following relations. Let p be a prime. When p divides N, then*

$$\mathrm{Tr}'_{X_0'(Np)/X_0'(N)}(E^{k,s}_{Np}) = E^{k,s}_N \tag{2.3}$$

When p does not divide N, then

$$\begin{aligned}(1-\alpha_p\beta_p \otimes \alpha_p\beta_p)\mathrm{Tr}'_{X_0'(Np)/X_0'(N)}(E^{k,s}_{Np}) = \\ (1-\alpha_p\otimes\alpha_p)(1-\alpha_p\otimes\beta_p)(1-\beta_p\otimes\alpha_p)(1-\beta_p\otimes\beta_p)E^{k,s}_N,\end{aligned} \tag{2.4}$$

where we put formally $\alpha_p + \beta_p = pT_p/p^{k+s}$, $\alpha_p\beta_p = p^{1-k-2s}$ and T_p is the Hecke operator, i.e.

$$\begin{aligned}&(1-\alpha_p\otimes\alpha_p)(1-\alpha_p\otimes\beta_p)(1-\beta_p\otimes\alpha_p)(1-\beta_p\otimes\beta_p) := \\ &1 - T_p'\otimes T_p' + (T_p'^2 - 2p^{1-k-2s})\otimes p^{1-k-2s} + p^{1-k-2s}\otimes T_p'^2 - p^{2(1-k-2s)}T_p'\otimes T_p' + p^{4(1-k-2s)}\end{aligned}$$

with $T_p' = T_p \cdot p^{1-k-s}$, and the action of Hecke operators is defined by

$$((T_p\otimes 1)(f))(z_1,z_2) := 1/p \cdot (\sum_{i=0}^{p-1} f(\frac{z_1+i}{p}, z_2) + p^k f(pz_1,z_2)) \tag{2.5}$$

$$((1\otimes T_p)(f))(z_1,z_2) := 1/p \cdot (\sum_{i=0}^{p-1} f(z_1, \frac{z_2+i}{p})) + p^k f(z_1,pz_2)) \tag{2.6}$$

for the function f on $\mathfrak{h} \times \mathfrak{h}$.

Proof. For the proof, see [12]. □

§ 3. Relations between $E_N^{k,s}$ and the values of L-function. ($\Gamma_0(N)$ case)

Let $f, g \in S_k(\Gamma_0(N)) \subset S_k(\Gamma(N))$ be normalized Hecke eigenforms. Let

$$\langle E_N^{k,s}(z_1,z_2), f\otimes g\rangle := \int_{(\Gamma_0(N)\backslash\mathfrak{h})^2} E_N^{k,s}(z_1,z_2)\overline{f(z_1)g(z_2)}y_1^{k-2}y_2^{k-2}dz_1dz_2$$

The following theorem holds.

Theorem 3.1. *If $k+2\mathrm{Re}(s)>2$, the following equalities hold. If $f=g$, then*

$$\langle E_{N,\chi}^{k,s}(z_1,z_2), f\otimes g\rangle = \frac{\pi i^{-k}}{(s-1)2^{2s-k}}\cdot\frac{L^{(N)}(k+s-1, f_\chi\otimes f_\chi)}{L^{(N)}(2(k+2s-1),\chi^2)}\cdot\langle f,f\rangle.$$

Here χ is a Dirichlet character whose conductor is divisible by N. If $f\neq g$, then

$$\langle E_{N,\chi}^{k,s}(z_1,z_2), f\otimes g\rangle = 0.$$

Here we put

$$L^{(N)}(s', f_\chi\otimes f_\chi) := \prod_{p\nmid N}\left((1-\chi(p)^2\alpha_p^2p^{-s'})(1-\chi(p)^2\alpha_p\beta_pp^{-s'})^2(1-\chi(p)^2\beta_p^2p^{-s'})\right)^{-1},$$

$$L^{(N)}(s',\chi^2) := \prod_{p\nmid N}(1-\chi^2(p)p^{-s'})^{-1}. \tag{3.1}$$

Here α_p, β_p is determined by the relation $\alpha_p+\beta_p=a_p$, $\alpha_p\beta_p=p^{k-1}$ and a_p is determined by $f=\sum_{n=0}^{\infty}a_nq^n$ with $q=e^{2\pi iz}$.

Proof. For the proof, see [12] (Section 4, Theorem 4.1). This formula is proved by using Euler system relations and self adjoint properties of Hecke operators in my paper. This result is essentially contained in the Shimura's book [19] (Chapter IV, section 22, formula (22.6.6)). □

§ 4. The statement of p-adic Beilinson conjecture of the two products of modular curves.

For positive integers N, n and a prime number p, let F_{N,p^n} be the function field of $X_0(N)\times_{Spec(\mathbb{Q})} X_0(N)\times_{Spec(\mathbb{Q})} Spec(\mathbb{Q}(\zeta_{p^n}))$. According to Theorem 3.1, it is natural to think that there exist elements Z_{N,p^n} $(N = 1, 2, \cdots)$ in $K_3(F_{N,p^n})$ corresponding to $E_N^{k,s}$ and $Z_{N,p^n}(n = 1, 2, \cdots)$ become norm compatible system. Two conjectures follow. We will introduce the following notation.

$$H := \varprojlim_m ((((\mathbb{Z}/p^m\mathbb{Z})[[q_1]])[1/q_1])[[q_2]])[1/q_2])[1/p] \tag{4.1}$$

$$H_n := \varprojlim_m ((((\mathbb{Z}/p^m\mathbb{Z})[[q_1^{1/p^n}]])[1/q_1])[[q_2^{1/p^n}]])[1/q_2])[1/p](\zeta_{p^n}) \tag{4.2}$$

$$H'_n := \varprojlim_m ((((\mathbb{Z}/p^m\mathbb{Z})[[q_1]])[1/q_1])[[q_2]])[1/q_2])[1/p](\zeta_{p^n}) \tag{4.3}$$

$$O_{H_n} := \varprojlim_m ((((\mathbb{Z}/p^m\mathbb{Z})[[q_1^{1/p^n}]])[1/q_1])[[q_2^{1/p^n}]])[1/q_2])(\zeta_{p^n}) \tag{4.4}$$

$$O_{H'_n} := \varprojlim_m ((((\mathbb{Z}/p^m\mathbb{Z})[[q_1]])[1/q_1])[[q_2]])[1/q_2])(\zeta_{p^n}). \tag{4.5}$$

Note that H_n is a complete discrete valuation field and the valuation ring is O'_{H_n}, whose residue field is $(\mathbb{F}_p[[t_1^{1/p^n}]][1/t_1])[[t_2^{1/p^n}]][1/t_2]$. Note that H'_n is a complete discrete valuation field and the valuation ring is $O_{H'_n}$, whose residue field is $(\mathbb{F}_p[[t_1]][1/t_1])[[t_2]][1/t_2]$.

Let $\hat{K}_3(O_H[[\epsilon - 1]])$ be defined by the following.

$$\hat{K}_3(O_H[[\epsilon - 1]]) := \varprojlim_m K_3(O_H[[\epsilon - 1]])/U^{(r)}K_3(O_H[[\epsilon - 1]]) \tag{4.6}$$

$$\hat{K}_3(O_{H'_n}) := \varprojlim_m K_3(O_{H'_n})/U^{(r)}K_3(O_{H'_n}) \tag{4.7}$$

Here $U^{(r)}K_3(O_H[[\epsilon-1]])$ is the subgroup of $K_3(O_H[[\epsilon-1]])$ generated by the symbols of the form $\{1+(p, \epsilon-1)^r O_H[[\epsilon-1]], O_H[[\epsilon-1]]^\times, O_H[[\epsilon-1]]^\times\}$, and $U^{(r)}K_3(O_{H'_n})$ is the subgroup of $K_3(O_{H'_n})$ generated by the symbols of the form $\{1+p^r O_{H'_n}, O_{H'_n}^\times, O_{H'_n}^\times\}$.

§ 4.1. Conjecture.

Let E_{k,N,p^n} be defined as follows.

$$E_{k,N,p^n} = \frac{1}{\varphi(p^n)} \sum_{\substack{\chi:(\mathbb{Z}/lcm(N,p^n)\mathbb{Z})\to\mathbb{C}^\times \\ \chi(1+p^n\mathbb{Z})=1}} \frac{L^{(N)}(2(k-1),\chi^2)}{L^{(N)}(k-1,\chi)} E_{N,\chi}^{k,0} \tag{4.8}$$

Let E'_{N,p^n} be the "E_{2,N,p^n}" in the sense of Gross and Keatings (See [5] for details).

Proposition 4.1. *All Fourier coefficients of E_{k,N,p^n} and E'_{N,p^n} are in $\mathbb{Q}(\zeta_{p^n})$, and E_{k,N,p^n} and E'_{N,p^n} become trace compatible system. i.e.*

$$\mathrm{Tr}_{\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}(\zeta_{p^m})}(E_{k,N,p^n}) = E_{k,N,p^m} \tag{4.9}$$

$$\mathrm{Tr}_{\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}(\zeta_{p^m})}(E'_{N,p^n}) = E'_{N,p^m} \tag{4.10}$$

Proof. For the proof, see [12] (Section 11). In [12], this result is proved by direct computation of Fourier coefficients. Note that Shimura essentially proved this result in more general setting in the paper [17] (Section 11). □

It is convenient to define the following notations.
(4.11)

$$\mathrm{reg} : \hat{K}_3(O_H[[\epsilon-1]]) \overset{dlog}{\to} \hat{\Omega}^3_{O_H[[\epsilon-1]]} \overset{(s_n/p^{3n})_n}{\to} \varprojlim_n H_n \frac{dq_1}{q_1} \wedge \frac{dq_2}{q_2} \overset{\mathrm{Tr}}{\to} \varprojlim_n H'_n \frac{dq_1}{q_1} \wedge \frac{dq_2}{q_2},$$

Here we put

$$\hat{\Omega}^3_{O_H[[\epsilon-1]]} := O_H[[\epsilon-1]]\frac{dq_1}{q_1} \wedge \frac{dq_2}{q_2} \wedge \frac{d\epsilon}{\epsilon}, \tag{4.12}$$

and s_n is defined as follows. If $z = \sum_{i,j,k\geq 0} a_{i,j,k} q_1^i q_2^j (\epsilon-1)^k \in O_H[[\epsilon-1]]$, $s_n(z\frac{dq_1}{q_1}\wedge\frac{dq_2}{q_2}\wedge\frac{d\epsilon}{\epsilon})$ is defined by

$$s_n(z\frac{dq_1}{q_1} \wedge \frac{dq_2}{q_2} \wedge \frac{d\epsilon}{\epsilon}) = s'_n(z)\frac{dq_1}{q_1} \wedge \frac{dq_2}{q_2} \in O_{H'_n}\frac{dq_1}{q_1} \wedge \frac{dq_2}{q_2}. \tag{4.13}$$

Here we put

$$s'_n(z) := \sum_{i,j,k\geq 0} a_{i,j,k} q_1^{i/p^n} q_2^{j/p^n} (\zeta_{p^n}-1)^k \tag{4.14}$$

Let $i : \varprojlim_n K_3(F_{N,p^n}) \to \varprojlim_n K_3(H'_n)$ be the map induced by the inclusion map $F_{N,p^n} \to H'_n$. Let $pr : \varprojlim_n K_3(H_n) \to \varprojlim_n K_3(H'_n)$ be the inverse limit of norm map of K-group. We use the following theorem to state the p-adic Beilinson conjecture.

Theorem 4.2. *There exists an isomorphisms.*

$$f : \hat{K}_3(O_H[[\epsilon-1]])^{N_\varphi=1} \to \varprojlim_n \hat{K}_3(O_{H_n}). \tag{4.15}$$

Here f is defined by $f(a,b,c) = \{s'_n(a), s'_n(b), s'_n(c)\}$ for $a,b,c \in O_H[[\epsilon-1]]$. Here N_φ is the Coleman norm operator introduced in [15] and φ is the ring endomorphism which sends q_1, q_2 and ϵ to q_1^p, q_2^p and ϵ^p, respectively.

Proof. For the proof, see [15] (Sarah's paper). □

We state the conjecture which we call p-adic Beilinson conjecture.

Conjecture 4.3. There exist elements Z_{N,p^n} in $K_3(F_{N,p^n})$ for all natural number n, Z_N in $\hat{K}_3(O_H[[\epsilon-1]])$ and constants C_N which depends only on N such that $\mathrm{reg}(Z_N) = (C_N E'_{N,p^n})_n$, $i(Z_{N,p^n}) = pr(f(Z_N))$.

§ 5. Sketch of the proof of p-adic local Beilinson conjecture

In this section, we will give the evidence of Conjecture 4.3. The following two conjectures follow from Conjecture 4.3. We call Conjecture 5.2 the p-adic local Beilinson conjecture in this paper. We plan to give the proofs of the following Conjectures 5.1 and 5.2 in [13].

Conjecture 5.1. For any natural number N, there exist an element Z'_N in $\hat{\Omega}^3_{O_H[[\epsilon-1]]}$ and a natural number C_N such that $s_n(Z'_N) = C_N E'_{N,p^n}$ for all natural number n.

We will give a sketch of the proof, which will be written in the forthcoming paper [13] precisely. Roughly speaking, $(C_N E'_{N,p^n})_n$ is in the image of the map s_n if and only if the symmetric square of modular form has p-adic L-function which is "integral". Note that the existence of the p-adic L-functions is proved by Schmidt in special case and Hida's book [6] (Chapter 5.3.6, p.296) in general situation.

Conjecture 5.2. For any natural number N, there exist an element Z_N in $\hat{K}_3(O_H[[\epsilon-1]])$ and a natural number C_N such that $\mathrm{reg}(Z_N) = C_N(E'_{N,p^n})_n$.

We will give a sketch of the proof of Conjecture 5.2 which will be written in the forthcoming paper [13] precisely. In [12], we determined the image of reg completely, using [11]. Using this, Conjecture 5.1 and trace compatibility of $(C_N E'_{N,p^n})_n$ imply the above conjecture.

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