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Algebraic transformations of hypergeometric functions arising from theory of Shimura curves

By

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Abstract

In this paper, we will give some examples of algebraic transformations of the ${}_2F_1$-hypergeometric functions. The discovery is achieved by interpreting the ${}_2F_1$-hypergeometric functions as automorphic forms on Shimura curves. Then we obtain identities among hypergeometric functions as identities among automorphic forms on different Shimura curves.

§ 1. Introduction

For real numbers a, b, c with $c \neq 0, -1, -2, \ldots$, the ${}_2F_1$-hypergeometric function (Gaussian hypergeometric function) is defined by the hypergeometric series

$$ {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n $$

for $z \in \mathbb{C}$ with $|z| < 1$, where

$$ (a)_n = \begin{cases} 1, & \text{if } n = 0, \\ a(a+1)\ldots(a+n-1), & \text{if } n \geq 1, \end{cases} $$

is the Pochhammer symbol. The hypergeometric function ${}_2F_1(a, b; c; z)$ is a solution of the differential equation

$$ \theta(\theta + c - 1)F - z(\theta + a)(\theta + b)F = 0, \qquad \theta = z\frac{d}{dz}. $$

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This is a Fuchsian equation on the complex projective line with precisely 3 regular singular points at $z = 0,\ 1,\ \infty$ with local exponents $\{0, 1-c\}$, $\{0, c-a-b\}$, and $\{a, b\}$, respectively.

Using the well-known fact in the classical analysis that a second-order linear ordinary differential equation with three regular singularities at 0, 1, ∞ is completely determined by the local exponents, one can easily deduce Euler's identity

$$ {}_2F_1(a,b;c;z) = (1-z)^{-a}{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right) $$

(among many other similar identities). Since the function $z/(z-1)$ is a rational function of degree 1 of z, we call this identity an algebraic transformation of degree 1 of hypergeometric functions.

The hypergeometric functions also admit algebraic transformations of higher degrees. One of the most famous example is Kummer's quadratic transformation

$$ {}_2F_1\left(2a, 2b; a+b+\frac{1}{2}; z\right) = {}_2F_1\left(a, b; a+b+\frac{1}{2}; 4z(1-z)\right), \tag{1.1} $$

valid for any real numbers a, b with $a+b+1/2 \neq 0, -1, -2, \ldots$. The first quadratic transformations were given by Kummer, and then Goursat gave a complete list of quadratic transformations. In [2], Goursat contributed more than 100 algebraic transformations of degrees $2, 3, 4, 6$. One such example is

$$ {}_2F_1\left(3a, 3a+\frac{1}{2}; 2a+\frac{5}{6}; z\right) = (1+3z)^{-3a}{}_2F_1\left(a, a+\frac{1}{3}; 2a+\frac{5}{6}; \frac{27z(1-z)^2}{(1+3z)^3}\right) $$

of degree 3. In 2009, Vidūnas [7] found many algebraic transformations of degrees $6, 8, 9, 10, 12$, and one of them is

$$ {}_2F_1\left(\frac{5}{42}, \frac{19}{42}; \frac{5}{7}; 27z\right) = f(z)^{-1/28}{}_2F_1\left(\frac{1}{84}, \frac{29}{84}; \frac{6}{7}; -\frac{27g(z)}{4f(z)^3}\right) $$

of degree 10, where

$$ f(z) = 1 - 57z - 1029z^2 + 50421z^3, \qquad g(z) = z^2(1-27z)(3-49z)^7. $$

In a very recent paper, we [6] obtained many new algebraic transformations of hypergeometric functions. For example, one of our favorite identities is

$$ \begin{aligned} &{}_2F_1\left(\frac{1}{20}, \frac{1}{4}; \frac{4}{5}; \frac{64z(1-z-z^2)^5}{(1-z^2)(1+4z-z^2)^5}\right) \\ &\qquad = (1-z^2)^{1/20}(1+4z-z^2)^{1/4}{}_2F_1\left(\frac{3}{10}, \frac{2}{5}; \frac{9}{10}; z^2\right). \end{aligned} $$

The main novelty in [6] is the interpretation of hypergeometric functions as automorphic forms on Shimura curves. As far as we know, this interpretation first appeared in [9].

In this paper, we will give more examples of algebraic transformations of hypergeometric functions to illustrate the role Shimura curves play in proving these identities. We will first review the basic definitions of Shimura curves and their automorphic forms. Then we will prove Kummer's quadratic transformation (1.1) in the cases when the hypergeometric functions are related to automorphic forms on Shimura curves. We then prove 4 identities related to Classes VI and III in Takeuchi's classification of arithmetic triangle groups [4, 5]. We remark that these identities can also be deduced from the results in [6] and some classical algebraic transformations of hypergeometric functions. The purpose of proving these identities is to demonstrate the advantage of using Shimura curves in proving this kind of identities.

§ 2. Preliminaries

In this section, we will review definitions of quaternion algebras, Shimura curves, arithmetic triangle groups, and their relations to hypergeometric functions. Most of the materials in this section are taken from [6, 8]. The fields that we are mainly concerned with are the number fields. In the sequent discussions, therefore, we will always assume that K is a field whose characteristic is not 2.

§ 2.1. Quaternion algebras

A *quaternion algebra* B over a field K is a central simple algebra of dimension 4 over K. Equivalently, a quaternion algebra can be written in the form

$$B = K + Ki + Kj + Kij, \quad i^2 = a,\, j^2 = b,\, ij = -ji,$$

for some nonzero constants a, b in K. In this case, we denote this algebra by $\left(\frac{a,b}{K}\right)$. For example, the Hamilton's quaternions $\mathbb{H} = \left(\frac{-1,-1}{\mathbb{R}}\right)$ and the 2-by-2 matrix algebra $M(2,K)$ are quaternion algebras. The map defined by

$$i \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad j \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

gives an isomorphism between $\left(\frac{1,1}{K}\right)$ and $M(2,K)$.

Notice that every element h in a quaternion algebra satisfies a monic polynomial equation over K of degree at most 2. Therefore, any quaternion algebra B is provided with the unique anti-involution $^-\colon B \longrightarrow B$ satisfying $h + \bar{h} \in K$ for all $h \in B$. Here an anti-involution of B is a K-linear map from B to itself satisfying

$$\overline{ax+by} = a\overline{x} + b\overline{y}, \quad \overline{\overline{x}} = x, \quad \overline{xy} = \overline{y}\,\overline{x}, \qquad \text{for all } a, b \in K,\, x, y \in B.$$

The maps *reduced trace*, and *reduced norm* on B are defined by

$$tr(h) = h + \overline{h}, \quad \text{and} \quad n(h) = h\overline{h},$$

respectively.

If $K = \mathbb{C}$, then up to isomorphisms, there is only one quaternion algebra over $\mathbb{C}$, which is $M(2, \mathbb{C})$. If $K = \mathbb{R}$ or a non-Archimedean local field, then up to isomorphism, there are only two quaternion algebras. One is $M(2, K)$ and the other is the unique division quaternion algebra.

Now assume that K is a number field and that R is its ring of integers. Let v be a place of K and K_v be the completion of K with respect to v. Then the localization $B_v := B \otimes_K K_v$ is a quaternion algebra over K_v. If B_v is isomorphic to $M(2, K_v)$, we say B *splits* at v. If B_v is a division algebra, we say B *ramifies* at v. It is known that the number of ramified places is finite and in fact an even integer. The product of ramified places is called the *discriminant* of the quaternion algebra.

An *order* in B is a finitely generated R-module that is also a ring with unity containing a K-basis for B. An order is *maximal* if it is not properly contained in another order. For instance, the ring $M(2, R)$ is a maximal order in $M(2, K)$; moreover, if R is a principal ideal domain, each maximal order in $M(2, K)$ is conjugate to the maximal order $M(2, R)$ by an element of $\mathrm{GL}(2, K)$. It is known that every order is contained in a maximal order.

§ 2.2. Shimura curves

To define a Shimura curve, we assume that K is a totally real number field and take a quaternion algebra B over K that splits at exactly one infinite place among all infinite places, that is,

$$B \otimes_{\mathbb{Q}} R \simeq M(2, \mathbb{R}) \times \mathbb{H}^{[K:\mathbb{Q}]-1},$$

where $\mathbb{H} = \left(\frac{-1,-1}{\mathbb{R}}\right)$ is Hamilton's quaternion algebra. Then, up to conjugation, there is a unique embedding ι of B into $M(2, \mathbb{R})$.

Let $\mathcal{O}$ be an order of B and $\mathcal{O}^1 = \{\gamma \in \mathcal{O} : n(\gamma) = 1\}$ be the norm-one group of $\mathcal{O}$. Then the image $\Gamma(\mathcal{O})$ of $\mathcal{O}^1$ under the embedding ι is a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$, and hence it acts on the upper half-plane $\mathfrak{H}$ in the usual manner

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \longmapsto \frac{a\tau + b}{c\tau + d}.$$

An element γ of $\Gamma(\mathcal{O})$ is *elliptic*, *parabolic*, or *hyperbolic*, according to whether $|tr(\gamma)| < 2$, $|tr(\gamma)| = 2$, or $|tr(\gamma)| > 2$. The fixed point τ of an elliptic element in $\mathfrak{H}$ is called an *elliptic point* of order n, where n is the order of the isotropy subgroup of τ in

$\Gamma(\mathcal{O})/\{\pm 1\}$. The fixed point of a parabolic element is called a *cusp*. This can appear only when $B = M(2,\mathbb{Q})$.

Note that if $B \neq M(2,\mathbb{Q})$, then the quotient space $\Gamma(\mathcal{O})\backslash\mathfrak{H}$ has a complex structure as a compact Riemann surface; for the matrix algebra $B = \mathrm{M}(2,\mathbb{Q})$, we compactify the Riemann surface $\Gamma(\mathcal{O})\backslash\mathfrak{H}$ by adding cusps. We denote $X(\mathcal{O})$ the quotient space $\Gamma(\mathcal{O})\backslash\mathfrak{H}$, or $\Gamma(\mathcal{O})\backslash(\mathfrak{H}\cup\mathbb{P}^1_{\mathbb{Q}})$ if $B = \mathrm{M}(2,\mathbb{Q})$. This is the so-called *Shimura curve* associated to $\mathcal{O}$. In the case of $B = \mathrm{M}(2,\mathbb{Q})$, the curves $X(\mathcal{O})$ are known as the classical modular curves. In a broader setting, if Γ is any discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ commensurable with $\Gamma(\mathcal{O})$, then the quotient space $\Gamma\backslash\mathfrak{H}$ will also be called a Shimura curve.

Now suppose that the compact Riemann surface $X(\mathcal{O})$ has genus g. Then a classical result says that there exist hyperbolic elements $A_1,\dots,A_g$, $B_1,\dots,B_g$, and elliptic or parabolic elements $C_1,\dots,C_r$ that generate $\Gamma(\mathcal{O})/\{\pm 1\}$ with relations

$$[A_1,B_1]\dots[A_g,B_g]C_1\dots C_r = 1$$

where $[A_i,B_i] = A_iB_iA_i^{-1}B_i^{-1}$ is the commutator of A_i and B_i. We let $(g;e_1,\dots,e_r)$ be the *signature* of $X(\mathcal{O})$, where $e_i=\infty$ if C_i is parabolic and $e_i = n$ is the order of the elliptic point τ fixed by C_i if C_i is elliptic.

Example 2.1.

1. In $B = M(2,\mathbb{Q})$, the corresponding subgroup of $\mathrm{SL}(2,\mathbb{R})$ related to the maximal order $\mathcal{O} = M(2,\mathbb{Z})$ is $\Gamma(\mathcal{O}) = \mathrm{SL}(2,\mathbb{Z})$ and $X(\mathcal{O})$ is the modular curve $X_0(1)$, whose signature is $(0;2,3,\infty)$.

2. Let $\mathcal{O}$ be the order $\mathcal{O} = \mathbb{Z}+\mathbb{Z}i+\mathbb{Z}j+\mathbb{Z}\frac{1+i+j+ij}{2}$ of the quaternion algebra $B = \left(\frac{-1,3}{\mathbb{Q}}\right)$. (Note that B ramifies at 2 and 3.) An embedding $\iota : B \to M(2,\mathbb{R})$ is

$$i \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad j \mapsto \begin{pmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{pmatrix},$$

and

$$\Gamma(\mathcal{O}) = \left\{ \frac{1}{2}\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} : \alpha\overline{\alpha}+\beta\overline{\beta} = 4,\ \alpha\equiv\beta \mod 2,\ \alpha,\beta\in\mathbb{Z}[\sqrt{3}] \right\}.$$

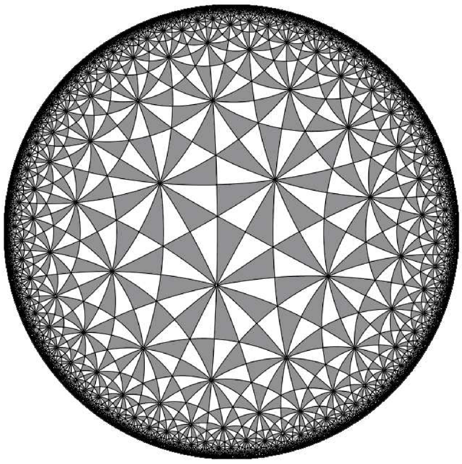
where $\overline{\alpha} = a - b\sqrt{3}$ if $\alpha = a+b\sqrt{3}$ in $\mathbb{Q}(\sqrt{3})$. The signature of $X(\mathcal{O})$ is $(0;2,2,3,3)$.

§ 2.3. Triangle groups

Suppose that a Shimura curve $X(\mathcal{O})$ has signature $(0;e_1,e_2,e_3)$. Then we say the group $\Gamma(\mathcal{O})$ is an *arithmetic triangle group*, and we denote it by $\Gamma(\mathcal{O}) = (e_1,e_2,e_3)$.

The complete lists of all arithmetic triangle groups and their commensurability classes were determined by Takeuchi [4, 5].

If we cut each fundamental domain of an arithmetic triangle group $\Gamma(\mathcal{O})$ into 2 halves in a suitable way, then the fundamental half-domains give a tessellation of the upper half-plane $\mathfrak{H}$ by congruent triangles with internal angles π/e_1, π/e_2, and π/e_3. The following figure shows the tessellation of the unit disc, which is conformally equivalent to $\mathfrak{H}$, by fundamental half-domains of the arithmetic triangle group $(2,3,7)$.



Here each triangle represents a fundamental half-domain. Any combination of a grey triangle with a neighboring white triangle will be a fundamental domain for the triangle group $(2,3,7)$.

In general, for any discrete subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$ such that $\Gamma\backslash\mathfrak{H}$ has finite volume, we can define its signature in the same way. If the signature is $(0;e_1,e_2,e_3)$, then we say Γ is a (hyperbolic) triangle group.

§ 2.4. Automorphic forms on Shimura curves

The definition of an automorphic form on Shimura curves is the same as that of a modular form on classical modular curves.

From now on, for simplicity, we assume that $B \neq M(2,\mathbb{Q})$ so that we do not need to consider cusps. For an integer k, an *automorphic form* of weight k on $\Gamma(\mathcal{O})$ is a holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ such that

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \tag{2.1}$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(\mathcal{O})$ and all $\tau \in \mathfrak{H}$. If f is meromorphic and $k=0$, then f is an *automorphic function.* If the curve $X(\mathcal{O})$ has genus 0, we call an automorphic function a *Hauptmodul* if it generates the field of automorphic functions on $\Gamma(\mathcal{O})$.

For a given integer k, the automorphic forms of weight on Γ forms a vector space, denoted by $S_k(\Gamma)$. We can calculate the dimension of $S_k(\Gamma)$ by using the Riemann-Roch formula.

Proposition 2.2 ([3, Theorem 2.23]). *Assume that $B \neq M(2,\mathbb{Q})$. Suppose that the Shimura curve $X(\mathcal{O})$ associated to an order $\mathcal{O}$ in B has signature $(g; e_1, \ldots, e_r)$. Then for even integers k, we have*

$$\dim S_k(\mathcal{O}) = \begin{cases} 0, & \text{if } k < 0, \\ 1, & \text{if } k = 0, \\ g, & \text{if } k = 2, \\ (k-1)(g-1) + \sum_{j=1}^{r} \left\lfloor \frac{k}{2}\left(1 - \frac{1}{e_j}\right) \right\rfloor, & \text{if } k \geq 4. \end{cases}$$

In the case $B \neq M(2,\mathbb{Q})$, there are very few explicit methods to construct automorphic forms on Shimura curves. Very recently, Yang [9] had a breakthrough in such area. When a Shimura curve has genus 0, Yang showed that all automorphisms can be expressed in terms of solutions of the so-called Schwarzian differential equation associated to a Haputmodul. We summarize the results in the following proposition. Note that here we assume that the quaternion algebra is not $M(2,\mathbb{Q})$.

Proposition 2.3 ([9, Theorem 4, Propositions 1 and 6]). *Assume that a Shimura curve X has genus zero with elliptic points $\tau_1, \ldots, \tau_r$ of order $e_1, \ldots, e_r$, respectively. Let $t(\tau)$ be a Hauptmodul of X and set $a_i = t(\tau_i)$, $i = 1, \ldots, r$. For a positive even integer $k \geq 4$, let*

$$d_k = \dim S_k(\mathcal{O}) = 1 - k + \sum_{j=1}^{r} \left\lfloor \frac{k}{2}\left(1 - \frac{1}{e_j}\right) \right\rfloor.$$

Then a basis for the space of automorphic forms of weight k on X is

$$t'(\tau)^{k/2} t(\tau)^j \prod_{\substack{1 \leq j \leq r \\ a_j \neq \infty}} (t(\tau) - a_j)^{-\lfloor k(1-1/e_1)/2 \rfloor}, \quad j = 0, \ldots, d_k - 1.$$

Moreover, the functions $t'(\tau)^{1/2}$ and $\tau t'(\tau)^{1/2}$, as functions of t, satisfy the differential equation

$$f'' + Q(t) f = 0,$$

where

$$Q(t) = \frac{1}{4} \sum_{\substack{1 \leq j \leq r \\ a_j \neq \infty}} \frac{1 - 1/e_j^2}{(t - a_j)^2} + \sum_{\substack{1 \leq j \leq r \\ a_j \neq \infty}} \frac{B_j}{t - a_j}, \quad B_j \in \mathbb{C}.$$

In particular, if $a_j \neq \infty$ for all j, then the constants B_j satisfy

$$\sum_{j=1}^{r} B_j = \sum_{j=1}^{r} \left(a_j B_j + \frac{1}{4}(1 - 1/e_j^2) \right) = \sum_{j=1}^{r} \left(a_j^2 B_j + \frac{1}{2} a_j (1 - 1/e_j^2) \right) = 0;$$

if $a_r = \infty$, *then* B_j *satisfy*

$$\sum_{j=1}^{r-1} B_j = 0, \qquad \sum_{j=1}^{r-1} \left(a_j B_j + \frac{1}{4}(1 - 1/e_j^2) \right) = \frac{1}{4}(1 - 1/e_r^2).$$

Remark.

(1) In [9], the differential equation $f'' + Q(t)f = 0$ is called the *Schwarzian differential equation* associated to t because $Q(t)$ is related to the Schwarzian derivative by the relation

$$2Q(t)t'(\tau)^2 + \{t, \tau\} = 0,$$

where

$$\{t, \tau\} = \frac{t'''(\tau)}{t'(\tau)} - \frac{3}{2} \left(\frac{t''(\tau)}{t'(\tau)} \right)^2$$

is the *Schwarzian derivative.*

(2) In general, in literature [1], if f is a thrice-differentiable function of z, then

$$D(f, z) := -\frac{\{f, z\}}{2f'(z)^2}$$

is called the *automorphic derivative* associated to f and z. In the case f is an automorphic function on a Shimura curve, then $D(f, \tau)$ is also an automorphic function. In particular, if t is a Hauptmodul on a Shimura curve of genus 0, then $Q(t) = D(t, \tau)$ is a rational function of t.

Proposition 2.4. *Automorphic derivatives have the following properties.*

1. $D((az + b)/(cz + d), z) = 0$ *for all* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{C})$.

2. $D(g \circ f, z) = D(g, f(z)) + D(f, z)/(dg/df)^2$.

Proposition 2.5. *Let* $z(\tau)$ *be a Hauptmodul for a Shimura curve* $X(\mathcal{O})$ *of genus* 0. *Let* $R(x) \in \mathbb{C}(x)$ *be the rational function such that the automorphic derivative* $Q(z) = D(z, \tau)$ *is equal to* $R(z)$. *Assume that* γ *is an element of* $\mathrm{SL}(2, \mathbb{R})$ *normalizing the norm-one group of* $\mathcal{O}$ *and let* σ *be the automorphism of* $X(\mathcal{O})$ *induced by* γ. *If* $\sigma : z \mapsto (az + b)/(cz + d)$, *then* $R(x)$ *satisfies*

$$\frac{(ad - bc)^2}{(cx + d)^4} R\left(\frac{ax + b}{cx + d} \right) = R(x).$$

§ 2.5. Hypergeometric functions as automorphic forms on Shimura curves

In the case that the Shimura curve of genus 0 has exactly 3 elliptic points, since the number of singularities of the differential equation is 3, the differential equation is essentially a hypergeometric differential equation. Then one can express the automorphic forms by using ${}_2F_1$-hypergeometric functions.

To be more precise, when a Shimura curve has signature $(0; e_1, e_2, e_3)$, we let τ_1, τ_2, τ_3 be the three elliptic points corresponding to e_1, e_2, e_3. Since X has genus 0, there exists a unique Hauptmodul z that takes values $0, 1, \infty$ at τ_1, τ_2, τ_3, respectively. According to Proposition 2.3, the functions $z'(\tau)^{1/2}$ and $\tau z'(\tau)^{1/2}$, as functions of z, satisfy the differential equation $f'' + Q(z)f = 0$, where

$$Q(z) = \frac{1}{4}\left(\frac{1-1/e_1^2}{z^2} + \frac{1-1/e_2^2}{(z-1)^2}\right) + \frac{B_1}{z} + \frac{B_2}{z-1}$$

with

$$B_2 = \frac{1}{4}\left(-1 + \frac{1}{e_1^2} + \frac{1}{e_2^2} - \frac{1}{e_3^2}\right), \qquad B_1 = -B_2.$$

The local exponents at $0, 1, \infty$ are $\{1/2-1/(2e_1), 1/2+1/(2e_1)\}$, $\{1/2-1/(2e_2), 1/2-1/(2e_2)\}$, and $\{-1/2-1/(2e_3), -1/2+1/(2e_3)\}$, respectively. Therefore, the function $z^{-1/2+1/(2e_1)}(1-z)^{-1/2+1/(2e_2)}z'(\tau)^{1/2}$, as a function of z, satisfies the hypergeometric differential equation

$$\theta(\theta + c - 1)F - z(\theta + a)(\theta + b)F = 0, \qquad \theta = z\frac{d}{dz}$$

with

$$a = \frac{1}{2}\left(1 - \frac{1}{e_1} - \frac{1}{e_2} - \frac{1}{e_3}\right), \qquad b = a + \frac{1}{e_3}, \qquad c = 1 - \frac{1}{e_1}$$

Combining this with Proposition 2.3, we see that every automorphic form on X can be expressed in terms of hypergeometric functions.

Proposition 2.6 ([9, Theorem 9]). *Assume that a Shimura curve X has signature $(0; e_1, e_2, e_3)$. Let $z(\tau)$ be the Hauptmodul of X with values 0, 1, and ∞ at the elliptic points of order e_1, e_2, and e_3, respectively. Let $k \geq 4$ be an even integer. Then a basis for the space of automorphic forms of weight k on X is given by*

$$z^{\{k(1-1/e_1)/2\}}(1-z)^{\{k(1-1/e_2)/2\}}z^j\left({}_2F_1(a, b; c; z) + Cz^{1/e_1}{}_2F_1(a', b', c'; z)\right)^k,$$

$j = 0, \ldots, \lfloor k(1-1/e_1)/2\rfloor + \lfloor k(1-1/e_2)/2\rfloor + \lfloor k(1-1/e_3)/2\rfloor - k$, for some constant C, where for a rational number x, we let $\{x\}$ denote the fractional part of x,

$$a = \frac{1}{2}\left(1 - \frac{1}{e_1} - \frac{1}{e_2} - \frac{1}{e_3}\right), \qquad b = a + \frac{1}{e_3}, \qquad c = 1 - \frac{1}{e_1}$$

and

$$a' = a + \frac{1}{e_1}, \qquad b' = b + \frac{1}{e_1}, \qquad c' = c + \frac{2}{e_1}.$$

§ 3. Algebraic transformations of hypergeometric functions

Suppose that $\Gamma_1 < \Gamma_2$ are two arithmetic triangle groups with Hauptmoduls z_1 and z_2, respectively. Since any automorphic function on Γ_2 is also an automorphic function on Γ_1, we have $z_2 = S(z_1)$ for some $S(x) \in \mathbb{C}(x)$. Likewise, if f_1 and f_2 are two automorphic forms of the same weight k on Γ_1 and Γ_2, respectively, then the ratio f_1/f_2 is an automorphic function on Γ_1 and hence is equal to $R(z_1)$ for some $R(x) \in \mathbb{C}(x)$. After taking the kth roots of the two sides of $f_1/f_2 = R(z_1)$, we obtain an algebraic transformation of hypergeometric function. This explains the existence of Kummer's, Goursat's and Vidūnas' transformations. (Of course, the triangle groups appearing in their transformations may not be arithmetic, but the argument above is still valid.)

More generally, if Γ_1 and Γ_2 are two commensurable arithmetic triangle groups such that the Shimura curve associated to $\Gamma = \Gamma_1 \cap \Gamma_2$ has genus 0. Let z be a Hauptmodul on Γ. Then each of z_1 and z_2 is a rational function of z. Similarly, the ratio f_1/f_2 is also a rational function of z. In view of Theorem 2.6, we can obtain an algebraic transformation of the form

$${}_2F_1\left(a_1, b_1; c_1; S_1(z)\right) = R(z){}_2F_1\left(a_2, b_2; c_2; S_2(z)\right)$$

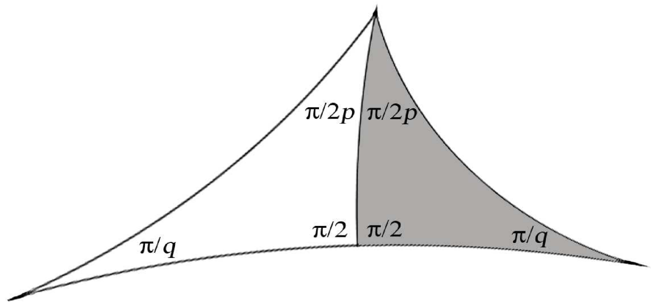
for some rational functions $S_1(z)$ and $S_2(z)$ and some algebraic function $R(z)$. This is the key idea to obtain new algebraic transformations of hypergeometric functions in [6].

§ 3.1. Kummer's quadratic transformation

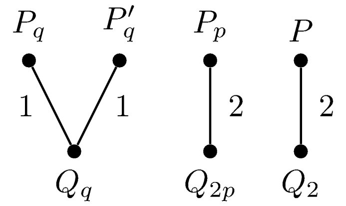
Here, we will use our arguments to prove Kummer's quadratic transformation

$${}_2F_1\left(2a, 2b; a+b+\frac{1}{2}; x\right) = {}_2F_1\left(a, b; a+b+\frac{1}{2}; 4x(1-x)\right).$$

Note that the triangle group (q, q, p) is a subgroup of $(q, 2, 2p)$ of index 2. The (q, q, p)-triangle is decomposed by 2 copies of $(q, 2, 2p)$-triangle.



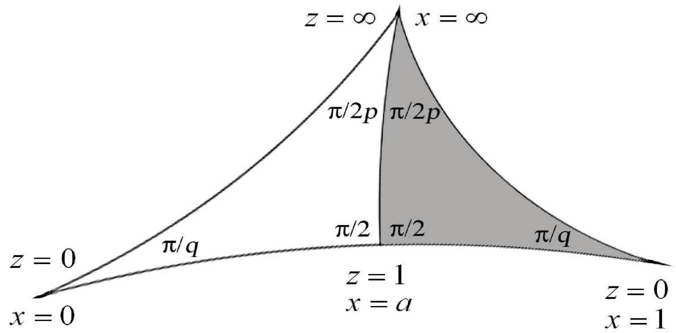
Let x be a Hauptmodul of $\Gamma_1 = (q,q,p)$ and z be a Hauptmodul of $\Gamma_2 = (q,2,2p)$. Label the elliptic points of $X_j = \Gamma_i \setminus \mathfrak{H}$ by P_q, P_q', P_p for X_1 and Q_q, Q_2, Q_{2p} for X_2 such that the ramification data are given by



Here the numbers next to the lines are the ramification indices.

Assume that the values of x and z at these elliptic points are

$$x(P_q) = 0,\ x(P_q') = 1,\ x(P_p) = \infty, \quad \text{and} \quad z(Q_q) = 0,\ z(Q_2) = 1,\ z(Q_{2p}) = \infty,$$



Then the corresponding hypergeometric functions are

$${}_2F_1\left(2\alpha, 2\beta; \alpha+\beta+\frac{1}{2}; x\right), \quad \text{and} \quad {}_2F_1\left(\alpha, \beta; \alpha+\beta+\frac{1}{2}; z\right),$$

where

$$\alpha = \frac{1}{4} - \frac{1}{4p} - \frac{1}{2q}, \quad \beta = \frac{1}{4} + \frac{1}{4p} - \frac{1}{2q}.$$

Also, the ramification data

z	0	1	∞
x	$0, 1$	a, a	∞, ∞

at $z = 0, \infty$ implies $z = ux(1-x)$ for some constant u; the data at $z = 1$ implies $ux(1-x) = 1$ has a repeated root, which shows $u = 4$ and $a = 1/2$. Therefore, the relation between the Hauprmoduls z and x is $z = 4x(1-x)$, and thus the ratio between

$${}_2F_1\left(2\alpha, 2\beta; \alpha+\beta+\frac{1}{2}; x\right), \quad \text{and} \quad {}_2F_1\left(\alpha, \beta; \alpha+\beta+\frac{1}{2}; 4x(1-x)\right).$$

is an algebraic function of x. By considering the analytic behaviors, one can see that they are equal.

Remark. Here, we give another way to determine the value $\alpha = x(P)$. Let G be the group of all symmetries of the tessellation of the hyperbolic plane by the (q,q,p)-triangles and G_0 be the subgroup generated by the reflections across the edges of (q,q,p)-triangles. Then the factor group G/G_0 is of order 2. Since the group relation $\Gamma_1 < \Gamma_2$ admits the decomposition, the triangle group $\Gamma_2 = (q,2,2p)$ corresponds to the group G/G_0. Therefore, any element of Γ_2 not in Γ_1 induces an automorphim of order 2 on the curve X_2. Such an automorphism must fix the points P, P_p and permute the elliptic points P_q, P'_q. In terms of the Hauptmodul x, such an automorphism is given by

$$\sigma : x \mapsto 1 - x$$

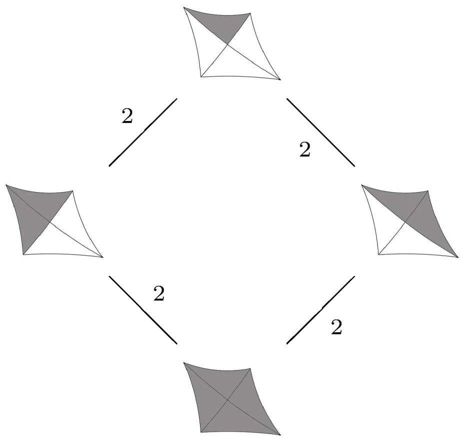
which implies that $x(P) = 1/2$.

§ 3.2. Automorphic forms on arithmetic triangle groups in Takeuchi's class II and the associate algebraic transformations

Let us take Takeuchi's Class II of commensurable arithmetic triangle groups as an example, which comes from the quaternion algebra over $\mathbb{Q}$ with discriminant 6. This is a sub-diagram of the subgroup diagram of Class II.

$$\begin{tikzcd}
& (2,4,6) \arrow[dl, dash, "2"'] \arrow[dr, dash, "2"] & \\
(2,6,6) \arrow[dr, dash, "2"'] & & (3,4,4) \arrow[dl, dash, "2"] \\
& (2,2,3,3) &
\end{tikzcd}$$

The node $(2,2,3,3)$ in the diagram means that the related curve X, obtained by $\Gamma = (2,6,6) \cap (3,4,4)$, has signature $(0;2,2,3,3)$. The relations of these subgroups admit the Coxeter decompositions of a quadrilateral polygon that is symmetric with respect to both the diagonals as shown below.



Associated to groups $(3,4,4)$, $(2,6,6)$ and Γ, we have the identities

$$ {}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{3}{4};\frac{z^2}{4(z-1)}\right) = (1-z)^{1/12}{}_2F_1\left(\frac{1}{12},\frac{1}{4};\frac{1}{2};z(2-z)\right). $$

and

$$ \sqrt{2}{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{5}{4};\frac{z^2}{4(z-1)}\right) = (1-z)^{1/3}(2-z)^{1/2}{}_2F_1\left(\frac{7}{12},\frac{3}{4};\frac{3}{2};z(2-z)\right). $$

Moreover, we can express all automorphic forms on Γ in terms of hypergeometric functions. (The algebraic transformation associated to the pair of groups $(2,4,6)$, $(3,4,4)$, and the pair of $(2,4,6)$, $(2,6,6)$ are Kummer's quadratic transformations, so we skip the associated transformations here.)

Let the Hauptmoduls be denoted by

$(2,3,3,2)$	$(4,4,3)$	$(2,6,6)$
z	u	t

where for (e_1,e_2,e_3), we choose the uniformizers in a way such that the values at the vertices e_1, e_2, e_3 are 0, 1, and ∞, respectively. For $(2,2,3,3)$, we assume that z takes values 0 at one of the elliptic point of order 2 and values 1 and ∞ the two elliptic points of order 3, respectively. Then from the ramification data, we have the relations

$$ u = \frac{z^2}{4(z-1)} \quad \text{and} \quad t = z(2-z). $$

The Hilbert-Poincaré series for X is

$$ \begin{aligned} \sum_{k\geq 0} \dim S_k(\Gamma)x^k &= 1 + x^4 + x^6 + x^8 + x^{10} + 3x^{12} + \cdots + 5x^{24} + \cdots \\ &= \frac{1+x^{12}}{(1-x^4)(1-x^6)} \end{aligned} $$

which means that there are automorphic forms f_4, f_6, f_{12} of wight 4, 6, and 12 that generate the graded ring of automorphic forms. Moreover, there exists a linear relation among f_4^6, f_6^4, f_{12}^2, $f_4^3 f_6^2$, $f_4^3 f_{12}$, and $f_6^2 f_{12}$.

According to Proposition 2.2, we can find the dimensions of $S_k(\Gamma)$ on $\Gamma_1 = (3,4,4)$ and $\Gamma_2 = (2,6,6)$ are

$$\dim S_6(\Gamma_1) = 1, \quad \dim S_8(\Gamma_1) = 1, \quad \dim S_{12}(\Gamma_1) = 1$$
$$\dim S_6(\Gamma_2) = 0, \quad \dim S_8(\Gamma_2) = 1, \quad \dim S_{12}(\Gamma_2) = 2$$

Moreover, the space $S_6(\Gamma_1)$ can be spanned by

$$F_6(u) = u^{1/4}(1-u)^{1/4}\left({}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{3}{4};u\right) + C_1 u^{1/4}{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{5}{4};u\right)\right)^6,$$

for some constant C_1, the space $S_8(\Gamma_1)$ can be spanned by

$$F_8(u) = \left({}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{3}{4};u\right) + C_1 u^{1/4}{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{5}{4};u\right)\right)^8,$$

and $F_6(u)^2$ spans the automorphic forms of wight 12 on Γ_1. Similarly, on Γ_2, the sets $\{G_8(t)\}$ and $\{G_{12,1}(t), G_{12,2}(t)\}$ span the spaces of automorphic forms of weight 8 and 12, respectively, where

$$G_8(t) = (1-t)^{1/3}\left({}_2F_1\left(\frac{1}{12},\frac{1}{4};\frac{1}{2};t\right) + C_2 t^{1/2}{}_2F_1\left(\frac{7}{12},\frac{3}{4};\frac{3}{2};t\right)\right)^8,$$

$$G_{12,1}(t) = \left({}_2F_1\left(\frac{1}{12},\frac{1}{4};\frac{1}{2};t\right) + C_2 t^{1/2}{}_2F_1\left(\frac{7}{12},\frac{3}{4};\frac{3}{2};t\right)\right)^{12},$$

and

$$G_{12,1}(t) = t\left({}_2F_1\left(\frac{1}{12},\frac{1}{4};\frac{1}{2};t\right) + C_2 t^{1/2}{}_2F_1\left(\frac{7}{12},\frac{3}{4};\frac{3}{2};t\right)\right)^{12},$$

for some $C_2 \in \mathbb{C}$.

Substituting $u = z^2/4(z-1)$ and $t = z(2-z)$ into $F_6(u)$, $F_8(u)$, $F_6(u)^2$, $G_8(t)$, $G_{12,1}(t)$, and $G_{12,2}(t)$, they become automorphic forms on Γ. Also, the space $S_6(\Gamma)$ is equal to the space spanned by $F_6\left(z^2/(4z-4)\right)$, and the automorphic form

$$F_8\left(z^2/(4z-4)\right) = CG_8\left(z(2-z)\right), \quad C \in \mathbb{C}$$

is a basis of $S_8(\Gamma)$. Comparing the behaviors of these functions, we can find that the constant C is equal to 1, and $C_2 = \frac{(-1)^{1/4}C_1}{2}$. Thus, by taking 8th roots of the two sides, we can get the algebraic transformation

$${}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{3}{4};\frac{z^2}{4(z-1)}\right) = (1-z(2-z))^{1/24}\,{}_2F_1\left(\frac{1}{12},\frac{1}{4};\frac{1}{2};z(2-z)\right).$$

$$\sqrt{2}{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{5}{4};\frac{z^2}{4(z-1)}\right)=(1-z)^{1/3}(2-z)^{1/2}{}_2F_1\left(\frac{7}{12},\frac{3}{4};\frac{3}{2};z(2-z)\right).$$

Observe that since the $\dim S_4(\Gamma)=\dim S_8(\Gamma)=1$, if $S_4(\Gamma)$ is spanned by some automorphic form f_4 then f_4^2 spans $S_8(\Gamma)$, which can be also spanned by $F_8\left(z^2/(4z-4)\right)$. So we can choose

$$f_4=\left({}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{3}{4};u\right)+C_1u^{1/4}{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{5}{4};u\right)\right)^4,$$

and we can find the set

$$\left\{F_4^3\left(z^2/(4z-4)\right),G_{12,1}\left(2z-z^2\right),F_6^2\left(z^2/(4z-4)\right)\right\}$$

forms a basis of $S_{12}(\Gamma)$. (We remark that $4F_6^2\left(z^2/(4z-4)\right)=iG_{12,2}\left(2z-z^2\right)$.)

As a conclusion, the graded ring of automorphic forms on Γ can be generated by the following functions

$$f_4=\left({}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{3}{4};\frac{z^2}{4z-4}\right)+C_1\frac{z^2}{4z-4}^{1/4}{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{5}{4};\frac{z^2}{4z-4)}\right)\right)^4,$$

$$f_6=\left(\frac{z^2}{4(z-1)}\right)^{1/4}\left(\frac{(z-2)^2}{4(1-z)}\right)^{1/4}$$

$$\left({}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{3}{4};\frac{z^2}{4z-4}\right)+C_1\left(\frac{z^2}{4z-4)}\right)^{1/4}{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{5}{4};\frac{z^2}{4z-4}\right)\right)^6,$$

$$f_{12}=\left({}_2F_1\left(\frac{1}{12},\frac{1}{4};\frac{1}{2};z(2-z)\right)+\frac{(-1)^{1/4}C_1}{2}(z(2-z))^{1/2}{}_2F_1\left(\frac{7}{12},\frac{3}{4};\frac{3}{2};z(2-z)\right)\right)^{12},$$

with the relation $f_4^6-4if_6^2f_{12}-f_{12}^2=0$.

§ 4. Algebraic transformations associated to Class VI

According to [6], the subgroup diagram for Takeuchi's Class VI is

$$\begin{tikzcd}
(2,4,5) \arrow[d, dash, "2"'] \arrow[dr, dash, "6"] & & (2,4,10) \arrow[dl, dash, "2"'] \arrow[d, dash, "2"] \\
(2,5,5) \arrow[dr, dash, "6"] & (4,4,5) \arrow[d, dash, "2"'] & (2,10,10) \arrow[dl, dash, "2"'] \arrow[d, dash, "2"] \\
& (2,2,5,5) \arrow[d, dash, "2"'] & (5,10,10) \arrow[dl, dash, "2"] \\
& (5,5,5,5) &
\end{tikzcd}$$

Theorem 4.1.

(1) Associated to the groups $(2,5,5)$, $(4,4,5)$, *and* $(2,2,5,5)$ *are the identities*

$$
\begin{aligned}
&{}_2F_1\left(\frac{3}{20},\frac{2}{5};\frac{4}{5};\frac{-4z}{(1-z)^2}\right)\\
&\qquad=\frac{(1-z)^{3/10}(2-z/\alpha)^{1/8}}{(2-3z\alpha+2z^2)^{1/8}(1-2z\alpha)^{1/8}}\,{}_2F_1\left(\frac{1}{20},\frac{1}{4};\frac{4}{5};\frac{-z(2\alpha-z)^5}{(1-2z\alpha)^5}\right),\\
&2\,{}_2F_1\left(\frac{7}{20},\frac{3}{5};\frac{6}{5};\frac{-4z}{(z-1)^2}\right)\\
&\qquad=\frac{(1-z)^{7/10}(2-z/\alpha)^{9/8}}{(2-3\alpha z+2z^2)^{1/8}(1-2z\alpha)^{9/8}}\,{}_2F_1\left(\frac{1}{4},\frac{9}{20};\frac{6}{5};\frac{-z(2\alpha-z)^5}{(1-2z\alpha)^5}\right),
\end{aligned}
$$

where α *is a root of* x^2+1.

(2) The following equalities are obtained by the groups $(2,10,10)$, $(2,5,5)$, *and* $(2,2,5,5)$.

$$
\begin{aligned}
&{}_2F_1\left(\frac{3}{20},\frac{1}{4};\frac{9}{10};z^2\right)\\
&\qquad=\frac{(2-z/\alpha)^{1/8}}{(2-3\alpha z+2z^2)^{1/8}(1-2z\alpha)^{1/8}}\,{}_2F_1\left(\frac{1}{20},\frac{1}{4};\frac{4}{5};\frac{-z(2\alpha-z)^5}{(1-2z\alpha)^5}\right),\\
&2\,{}_2F_1\left(\frac{1}{4},\frac{7}{20};\frac{11}{10};z^2\right)\\
&\qquad=\frac{(2-z/\alpha)^{9/8}}{(2-3\alpha z+2z^2)^{1/8}(1-2z\alpha)^{9/8}}\,{}_2F_1\left(\frac{1}{4},\frac{9}{20};\frac{6}{5};\frac{-z(2\alpha-z)^5}{(1-2z\alpha)^5}\right),
\end{aligned}
$$

where α *is a root of* x^2+1.

(3) From the groups $(2,10,10)$, $(4,4,5)$, *and* $(2,2,5,5)$, *we can obtain the algebraic transformations*

$$
{}_2F_1\left(\frac{3}{20},\frac{2}{5};\frac{4}{5};\frac{-4z}{(1-z)^2}\right)=(1-z)^{3/10}\,{}_2F_1\left(\frac{3}{20},\frac{1}{4};\frac{9}{10};z^2\right),
$$

$$
{}_2F_1\left(\frac{7}{20},\frac{3}{5};\frac{6}{5};\frac{-4z}{(z-1)^2}\right)=(1-z)^{7/10}\,{}_2F_1\left(\frac{1}{4},\frac{7}{20};\frac{11}{10};z^2\right).
$$

Proof. Now we let $\Gamma_1=(2,5,5)$, $\Gamma_2=(4,4,5)$, $\Gamma_3=(2,10,10)$, $\Gamma_4=(2,2,5,5)$, and X_1, X_2, X_3, X_4 be the related Shimura curves. Let the Hauptmoduls be denoted by

$(5,2,5)$	$(5,4,4)$	$(10,2,10)$	$(5,2,5,2)$
z_1	z_2	z_3	t

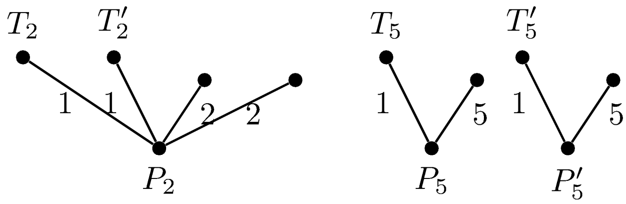
where for (e_1, e_2, e_3), we choose the uniformizers in a way such that the values at the vertices e_1, e_2, e_3 are 0, 1, and ∞, respectively. For $(5, 2, 5, 2)$, we assume that t takes values 1 at one of the elliptic point of order 2 and values 0 and ∞ the two elliptic points of order 5, respectively.

Then t takes value -1 at the other elliptic point of order 2, and the relations

$$z_3 = t^2 \quad \text{and} \quad z_2 = \frac{-4t}{(1-t)^2}$$

can be easily determined.

If we label the elliptic points of X_j by P_2, P_5, P_5' for X_1, and T_2, T_2', T_5, T_5' for X_4 such that the ramifications data are given by



then from the ramification data

z_1	0	1	∞
t	$0, a^{\times 5}$	$\pm 1, b^{\times 2}, c^{\times 2}$	$\infty, d^{\times 5}$

for some unknowns a, b, c, $d \in \mathbb{C}$, we can find

$$z_1 = \frac{t(2\alpha - t)^5}{(2t\alpha - 1)^5}, \quad \text{with } \alpha^2 + 1 = 0.$$

By Proposition 2.2, we have

$$\dim S_8(\Gamma_1) = \dim S_8(\Gamma_3) = 1, \dim S_8(\Gamma_2) = 2 \quad \text{and} \quad \dim S_8(\Gamma_4) = 3.$$

Therefore, the space $S_8(\Gamma_1)$ is spanned by

$$F_1 = z_1^{1/5}\left({}_2F_1\left(\frac{1}{20}, \frac{1}{4}; \frac{4}{5}; z_1\right) + C_1 z_1^{1/5}{}_2F_1\left(\frac{1}{4}, \frac{9}{20}; \frac{6}{5}; z_1\right)\right)^8 \tag{4.1}$$

for some constant C_1; the function

$$F_2 = z_2^{1/5}\left({}_2F_1\left(\frac{3}{20}, \frac{2}{5}; \frac{4}{5}; z_2\right) + C_2 z_2^{1/5}{}_2F_1\left(\frac{7}{20}, \frac{3}{5}; \frac{6}{5}; z_2\right)\right)^8 \tag{4.2}$$

is an automorphic form of weight 8 on Γ_2, for some constant C_2, and the space $S_8(\Gamma_3)$ is spanned by

$$F_3 = z_3^{3/5}\left({}_2F_1\left(\frac{3}{20}, \frac{1}{4}; \frac{9}{10}; z_3\right) + C_3 z_3^{1/10}{}_2F_1\left(\frac{1}{4}; \frac{7}{20}; \frac{11}{10}; z_3\right)\right)^8, \tag{4.3}$$

for some constant C_3. To get a basis for $S_8(\Gamma_4)$, we need to work out the Schwarzian differential equation associated to t.

By Theorem 2.3, the function $t'(\tau)$, as a function of t, satisfies

$$\frac{d^2}{dt^2}f + Q(t)f = 0,$$

where

$$Q(t) = \frac{1}{4}\left(\frac{24}{25t^2} + \frac{3}{4(1-t)^2} + \frac{3}{4(1+t)^2} + \frac{B_1}{t} + \frac{B_2}{t-1} + \frac{B_3}{t+1}\right)$$

for some complex numbers satisfying

$$B_1 + B_2 + B_3 = 0, \quad B_2 - B_3 + \frac{3}{2} = 0. \tag{4.4}$$

To determine the all the values of B_1, B_2, and B_3, we need another condition. Here, we use the automorphism of X_4 coming from the normal subgroup relation $\Gamma_4 \lhd \Gamma_3$. Let γ be an element of Γ_3 not in Γ_4. Then such an element γ leads to the relation

$$t(\gamma\tau) = -t(\tau).$$

Now by Proposition 2.4, we have

$$D(-t(\tau),\tau) = D(t(\gamma\tau),\tau) = D(t(\gamma\tau),\gamma\tau) + D(\gamma\tau,\tau)/(d\gamma\tau/d\tau)^2 = Q(t(\gamma\tau));$$

on the other hand, we also have, by the same proposition,

$$D(-t(\tau),\tau) = D(-t(\tau),t(\tau)) + D(t(\tau),\tau)/(-1)^2 = Q(t(\tau))$$

Thus, we have

$$Q(t) = Q(-t). \tag{4.5}$$

The condition (4.4) and the identity (4.5) give us

$$B_1 = 0,\ B_2 = -\frac{3}{4},\ B_3 = \frac{3}{4},$$

and hence

$$Q(t) = \frac{6}{25t^2} + \frac{3}{16}\left(\frac{1}{(1-t)^2} + \frac{1}{(1+t)^2} - \frac{1}{1-t} + \frac{1}{1+t}\right).$$

Then if we let

$$f_1 = t^{2/5}\left(1 - \frac{5}{24}t^2 - \frac{215}{2432}t^4 - \frac{91015}{1692672}t^6 - \frac{2047105}{54165504}t^8 - \frac{1529715}{53215232}t^{10} - \cdots\right),$$
$$f_2 = t^{3/5}\left(1 - \frac{15}{88}t^2 - \frac{115}{408}t^4 - \frac{18145}{349184}t^6 - \frac{17144865}{458129408}t^8 - \frac{105957295}{3665035264}t^{10} - \cdots\right).$$

be a basis for the solution space of the Schwarzian differential equation $d^2f/dt^2 + Q(t)f = 0$, then by Theorem 2.3, a basis for the space $S_8(\Gamma_4)$ is given by

$$\{g,\ tg,\ t^2g\}, \qquad g = \frac{(f_1 + Cf_2)^8}{t^3(1-t)^2(1+t)^2},$$

for some constant C.

After substituting $z_3 = t^2$, $z_2 = -4t/(1-t)^2$, and $z_1 = -t(2\alpha - t)^5/(1-2t\alpha)^5$ into (4.1), (4.2) and (4.3), respectively, by comparing their t-series, we find

$$\begin{aligned} F_1 =& \alpha^{1/5}(-2 + 3\alpha t - 2t^2)g, \\ F_2 =& (-4)^{1/5}\left(1 - 2t + t^2\right)g, \\ F_3 =& tg. \end{aligned} \tag{4.6}$$

Simplifying the relation

$$tF_1 = \alpha^{1/5}(-2 + 3\alpha t - 2t^2)F_3,$$

$$(-4)^{1/5}\left(1 - 2t + t^2\right)F_1 = \alpha^{1/5}(-2 + 3\alpha t - 2t^2)F_2,$$

and

$$tF_2 = (-4)^{1/5}\left(1 - 2t + t^2\right)F_3$$

together with

$$\frac{C_1}{C_2} = -\frac{1}{2}\left(\frac{\alpha}{4}\right)^{1/5}, \ \frac{C_1}{C_3} = \frac{\alpha^{1/5}}{2},$$

we can get the identities. $\square$

§ 5. Algebraic transformations associated to Class III

In this section, we consider some algebraic transformations associated to the Class III. Here is a sub-diagram of the subgroup diagram.

$$\begin{tikzcd}
(2,6,8) \arrow[d, dash, "2"'] \arrow[dr, dash, "2"] & & (2,3,8) \arrow[dl, dash, "10"'] \arrow[d, dash, "2"] \\
(4,6,6) \arrow[dr, dash, "2"] & (3,8,8) \arrow[d, dash, "2"'] & (3,3,4) \arrow[dl, dash, "10"'] \\
& (3,4,3,4) &
\end{tikzcd}$$

Theorem 5.1.

(1) Corresponding to the pair of $(4,6,6)$ *and* $(3,3,4)$ *are the following identities*

$$S^{1/8}{}_2F_1\left(\frac{5}{24},\frac{3}{8};\frac{3}{4};\frac{4t}{(t+1)^2}\right)=(1+t)^{3/8}{}_2F_1\left(\frac{1}{24},\frac{3}{8};\frac{3}{4};\frac{(28+16\beta)tR^4}{(1+t)S^3}\right),$$
$$S^{7/8}{}_2F_1\left(\frac{11}{24},\frac{5}{8};\frac{5}{4};\frac{4t}{(t+1)^2}\right)=R(1+t)^{5/8}{}_2F_1\left(\frac{7}{24},\frac{5}{8};\frac{5}{4};\frac{(28+16\beta)tR^4}{(1+t)S^3}\right),$$

(2) Corresponding to the pair of $(3,8,8)$ *and* $(3,3,4)$ *are the following identities*

$${}_2F_1\left(\frac{1}{24},\frac{3}{8};\frac{3}{4};\frac{(28+16\beta)tR^4}{(1+t)S^3}\right)=S^{1/8}(1+t)^{1/24}{}_2F_1\left(\frac{5}{24},\frac{1}{3};\frac{7}{8};t^2\right),$$
$$R\,{}_2F_1\left(\frac{7}{24},\frac{5}{8};\frac{5}{4};\frac{(28+16\beta)tR^4}{(1+t)S^3}\right)=S^{7/8}(1+t)^{7/24}{}_2F_1\left(\frac{1}{3},\frac{11}{24};\frac{9}{8};t^2\right),$$

(3) Associated to the groups $(3,8,8)$, $(4,6,6)$ *are the following identities*

$${}_2F_1\left(\frac{5}{24},\frac{3}{8};\frac{3}{4};\frac{4t}{(1+t)^2}\right)=(1+t)^{5/12}{}_2F_1\left(\frac{5}{24},\frac{1}{3};\frac{7}{8};t^2\right),$$
$${}_2F_1\left(\frac{11}{24},\frac{5}{8};\frac{5}{4};\frac{4t}{(1+t)^2}\right)=(1+t)^{11/12}{}_2F_1\left(\frac{1}{3},\frac{11}{24};\frac{9}{8};t^2\right),$$

where

$$R=1+\frac{-17+56\beta}{81}t^2,\quad S=1+\frac{13+8\beta}{3}t-\frac{25+32\beta}{9}t^2+\frac{17-56\beta}{81}t^3,$$

and β *is a root of* x^2+2.

Proof. Let $\Gamma_1=(4,6,6)$, $\Gamma_2=(3,8,8)$, $\Gamma_3=(3,3,4)$, $\Gamma=(3,4,3,4)$, and the Hauptmoduls be denoted by

$(4,6,6)$	$(8,3,8)$	$(4,3,3)$	$(4,3,4,3)$
z_1	z_2	z_3	t

where for (e_1,e_2,e_3), we choose the Hauptmoduls such that the values at the vertices e_1, e_2, e_3 are 0, 1, and ∞, respectively. For $(3,4,3,4)$, we assume that t takes value 1 at one of the elliptic point of order 3 and values 0 and ∞ the two elliptic points of order 4, respectively.

Then we can find that t takes value -1 at the other elliptic point of order 3, and the relations between these Hauptmoduls are

$$z_1=\frac{4t}{(1+t)^2},\quad z_2=t^2,\quad z_3=\frac{4(7+4\beta)t\left(1+\frac{-17+56\beta}{81}t^2\right)^4}{(t+1)\left(1+\frac{13+8\beta}{3}t-\frac{25+32\beta}{9}t^2+\frac{17-56\beta}{81}t^3\right)^3}. \tag{5.1}$$

(For more detail, please see Lemma 7 in [6].)

By Proposition 2.2, we have

$$\dim S_6(\Gamma_1) = \dim S_6(\Gamma_2) = \dim S_6(\Gamma_3) = 1 \quad \text{and} \quad \dim S_6(\Gamma) = 3.$$

Therefore, the space $S_6(\Gamma_1)$ is spanned by

$$F_1 = z_1^{1/4}(1-z_1)^{1/2}\left({}_2F_1\left(\frac{5}{24},\frac{3}{8};\frac{3}{4};z_1\right) + C_1 z_1^{1/4}{}_2F_1\left(\frac{11}{24},\frac{5}{8};\frac{5}{4};z_1\right)\right)^6 \tag{5.2}$$

for some constant C_1; the space $S_6(\Gamma_2)$ is spanned by

$$F_2 = z_2^{5/8}\left({}_2F_1\left(\frac{5}{24},\frac{1}{3};\frac{7}{8};z_2\right) + C_2 z_2^{1/8}{}_2F_1\left(\frac{1}{3},\frac{11}{24};\frac{9}{8};z_2\right)\right)^6 \tag{5.3}$$

for some constant C_2, and the space $S_6(\Gamma_3)$ is spanned by

$$F_3 = z_3^{1/4}\left({}_2F_1\left(\frac{1}{24},\frac{3}{8};\frac{3}{4};z_3\right) + C_3 z_3^{1/4}{}_2F_1\left(\frac{7}{24},\frac{5}{8};\frac{5}{4};z_3\right)\right)^6 \tag{5.4}$$

for some constant C_3.

By Theorem 2.3, a basis for the space $S_6(\Gamma)$ is

$$\{g,\ tg,\ t^2 g\}, \qquad g = \frac{(f_1 + Cf_2)^6}{t^2(1-t)^2(1+t)^2},$$

for some constant C, where $\{f_1, f_2\}$ is a basis for the solution space of the Schwarzian differential equation $d^2f/dt^2 + Q(t)f = 0$ associate to t.

By Theorem 2.3, the rational function $Q(t)$ must be

$$Q(t) = \frac{1}{4}\left(\frac{15}{16t^2} + \frac{8}{9(1-t)^2} + \frac{8}{9(1+t)^2} + \frac{B_1}{t} + \frac{B_2}{t-1} + \frac{B_3}{t+1}\right)$$

satisfying

$$B_1 + B_2 + B_3 = 0, \quad B_2 - B_3 + \frac{16}{9} = 0. \tag{5.5}$$

Note that for any element γ of Γ_2 not Γ, we have the equality

$$t(\gamma\tau) = -t(\tau).$$

Now by Proposition 2.4, we have

$$D(-t(\tau),\tau) = D(t(\gamma\tau),\tau) = D(t(\gamma\tau),\gamma\tau) + D(\gamma\tau,\tau)/(d\gamma\tau/d\tau)^2 = Q(t(\gamma\tau)),$$

$$D(-t(\tau),\tau) = D(-t(\tau),t(\tau)) + D(t(\tau),\tau)/(-1)^2 = Q(t(\tau)),$$

and thus

$$Q(t) = Q(-t). \tag{5.6}$$

Therefore, from the information (5.5) and (5.6), we can get

$$Q(t) = \frac{15}{64t^2} + \frac{2}{9}\left(\frac{1}{(1-t)^2} + \frac{1}{(1+t)^2} - \frac{1}{t-1} + \frac{1}{t+1}\right).$$

Here, we choose a basis for the solution space of the Schwarzian differential equation $d^2f/dt^2 + Q(t)f = 0$ with t-series

$$f_1 = t^{5/8}\left(1 - \frac{16}{81}t^2 - \frac{1168}{12393}t^4 - \frac{99568}{1673055}t^6 - \frac{1922128}{45172485}t^8 - \frac{32018768}{980508645}t^{10} - \cdots\right),$$
$$f_2 = t^{3/8}\left(1 - \frac{16}{63}t^2 - \frac{176}{1701}t^4 - \frac{65008}{1056321}t^6 - \frac{1792496}{42101937}t^8 - \frac{254491952}{7957266093}t^{10} - \cdots\right).$$

After substituting (5.1) into (5.2), (5.3) and (5.4), one has

$$\begin{aligned} C^6F_1 &= \sqrt{2}(1-t^2)g, \\ C^6F_2 &= tg, \\ C^6F_3 &= \sqrt{2}(7+4\beta)^{1/4}\left(1 + \frac{(-17+56\beta)}{81}t^2\right)g. \end{aligned} \tag{5.7}$$

Simplifying the relations

$$(7+4\beta)^{1/4}\left(1 + \frac{-17+56\beta}{81}t^2\right)F_1 = (1-t^2)F_3,$$

$$tF_3 = \sqrt{2}(7+4\beta)^{1/4}\left(1 + \frac{-17+56\beta}{81}t^2\right)F_2,$$

and

$$tF_1 = \sqrt{2}(1-t^2)F_2,$$

we can get the identities described in the theorems. □

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