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Diophantine approximation and expansions of real numbers.

By

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§ 1. Introduction

How to describe the best rational approximations to a given real number ? Most "interesting" numbers arise as special values of some smooth function or as "periods" of such ones, others are defined by a suitable expansion and some of them occur in both categories. Of course the diophantine approximation properties can be read on the regular continued fraction (RCF) expansion and the problem becomes really difficult when the number is defined by its expansion to some integer base $b \geq 2$ (or by its β-expansion), since in many cases the approximation given by the simple truncated expansions is not as sharp as required. We explore in this survey different classes of examples before concluding with the dyadic Thue-Morse-Mahler expansion.

§ 2. History

We begin by some historical review on diophantine approximation and its link with transcendency. The first elementary observation on approximation is the following one : if $r = a/b \in \mathbf{Q}$ is any rational number then

$$\left| r - \frac{p}{q} \right| = \left| \frac{a}{b} - \frac{p}{q} \right| = \left| \frac{aq - bp}{bq} \right| \geq \frac{1}{bq}$$

for every rational number $p/q \neq r$ which means that a rational number is very badly approximable by other ones. Now, the first positive result is due to Dirichlet (1842) who established, as a consequence of the pigeonhole principle, that every irrational number can be *approximated at order* 2, namely :

Theorem 2.1 (Dirichlet). *If* $x \notin \mathbf{Q}$, $0 < |x - \frac{p}{q}| < \frac{1}{q^2}$ *for infinitely many* p/q.

Those rational numbers are called "good" or "Dirichlet" approximations and the continued fraction algorithm naturally provides such a sequence; we recall below the classical notations we

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shall use throughout the paper : we denote by $x =: [a_0; a_1, \ldots, a_n, \ldots]$ the RCF expansion of the irrational number x, where $a_0 = [x]$ and the partial quotients a_i are integers ≥ 1 for $i \geq 1$; we also note $p_n/q_n =: [a_0; a_1, \ldots, a_n]$ the n^{th} convergent which happens to satisfy

$$\frac{1}{2q_n q_{n+1}} \leq \left|x - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n q_{n+1}} \leq \frac{1}{a_{n+1} q_n^2} \tag{2.1}$$

A kind of reciprocal to (2.1) will turn out to be very useful later.

At the same period Liouville proved that algebraic numbers cannot be so well approximated since :

Theorem 2.2 (Liouville). *If the algebraic number x has degree d, there exists $C > 0$ such that $|x - \frac{p}{q}| \geq \frac{C}{q^d}$ for every rational number p/q.*

Note that this result is best possible for irrational quadratic numbers. Concerning the other algebraic numbers, Roth (1958), after Thue and Siegel, improved Liouville's theorem in a (almost) final way :

Theorem 2.3 (Roth [31]). *If x is some algebraic number and $\varepsilon > 0$, the inequality $0 < |x - \frac{p}{q}| < \frac{1}{q^{2+\varepsilon}}$ has only finitely many solutions; in other words, if x can be approximated at order $2 + \varepsilon$ for some $\varepsilon > 0$, it must be transcendental.*

More generally, we shall say that x is *approximable at order* α if the inequality $|x - p/q| \leq 1/q^\alpha$ holds for infinitely many rational numbers p/q (i.o. in short).

For some prime number p, let us denote by $|n|_p$ the p-adic absolute value of n. A p-adic version of Roth's theorem appears a few years later, involving p-adic absolute values of the numerators and denominators of the approximants .

Theorem 2.4 (Ridout [30]). *Let x be an irrational number and $\varepsilon > 0$. Given a finite number of prime numbers : $p_1, \ldots, p_k$, the assumption*

$$\prod_{i=1}^{k} |p|_{p_i} \prod_{i=1}^{k} |q|_{p_i} \left|x - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon}} \tag{2.2}$$

for infinitely many rational numbers p/q, implies that x is transcendental.

Observe that this improves Roth's result since $|p|_{p_i}$ and $|q|_{p_i}$ are ≤ 1, and it is all the better as p or q are composite numbers.

§ 3. A first Diophantine classification

§ 3.1. The irrationality exponent

Recall that x is *approximable at order* α if the inequality $|x - p/q| \leq 1/q^\alpha$ holds for infinitely many rational numbers p/q. This leads to a first basic tool in approximation theory, called the irrationality exponent and denoted by ν in this paper :

Definition 3.1. Let x be some irrational number; $\nu(x)$ is the supremum of the real α such that x is approximable at order α.

We are now able to interpret the previous item in terms of ν :

- $\nu(x) \geq 2$ for every irrational number x (Dirichlet).
- $\nu(x) = 2$ for every algebraic number x (Roth).
- There exist transcendental numbers with exponent equal to 2 (in fact a lot as we shall recall), a famous example of which being the Euler basis.

The Euler basis. The Euler basis e plays an important role in approximation theory, since we do know a lot of things on it. The irrationality and the non-quadraticity are consequences of its classical writing $\sum 1/n!$ (which is nothing but a rising continued fraction expansion). The diophantine properties of e can easily be derived from its RCF expansion (due to Euler himself)

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \ldots]$$

leading to

$$\left|e - \frac{p}{q}\right| \asymp \frac{\operatorname{loglog} q}{q^2 \log q}, \quad \text{for every } p/q. \tag{3.1}$$

Another approach to this result involves the Padé-approximants to the exponential function. With previous notations, $\nu(e) = 2$ and e belongs to this big class of numbers for which the transcendency cannot be established with the aid of Roth's theorem. Simultaneous approximation is thus needed and this is made possible by the fact that e is the value of some G-function. But the adic expansion of e in some integer base remains mysterious and e is conjectured to be *normal* [25].

Note that, except for some numbers related to hypergeometric functions or adhoc constructions, ν is generally not known (since RCF expansion is not). However another example deserves to be quoted : the "Champernowne" number, whose g-adic expansion is

$$0.(1)_g(2)_g \cdots (n)_g \cdots =: M(g)$$

where $(n)_g$ is the pattern given by the g-adic expression of n. By using a tricky identity due to Mahler, M. Amou proved that $\nu(M(g)) = g$ for any $g \geq 2$ ([6]).

§ 3.2. A first classification

Nevertheless, we deduce from the above a rather rough diophantine classification (more elaborate ones are due to Mahler and Koksma [10]). The first class we distinguish consists of those numbers which are approximable at order 2 and not better.

Definition 3.2. The real number x is said to be *badly approximable* (or $x \in \mathbf{Bad}$) if there exists some $C > 0$ such that $|x - \frac{p}{q}| \geq \frac{C}{q^2}$ for every p/q.

There is a very convenient description of **Bad** in terms of the RCF expansion : actually **Bad** consists of the real numbers with *bounded* partial quotients and it contains all quadratic irrational numbers.
The second class we consider consists of so-called very well approximable numbers :

Definition 3.3. The real number x is said to be *very well approximable* (or $x \in \mathcal{W}$) if there exists some $\varepsilon > 0$ such that

$$|x - \frac{p}{q}| \leq \frac{1}{q^{2+\varepsilon}} \ \ i.o. \tag{3.2}$$

Further, we have the Liouville numbers which are approximable at any order.

Definition 3.4. The real number x is a *Liouville* number (or $x \in \mathcal{L}$) if (3.2) holds for every $\varepsilon > 0$.

Observe that $\mathcal{L} \subset \mathcal{W} \subset \mathbf{Bad}^c$. The set $\mathcal{W}$ consists of those numbers satisfying Roth's theorem, thus being transcendental numbers. But **Bad** must contain transcendental numbers too since it is uncountable. The numbers $x \in \mathcal{W} \backslash \mathcal{L}$ are called *diophantine.*
In view to say more on the size of those sets, we make a detour via the metric point of view.

§ 3.3. Khintchin's result

Khintchin considered in turn what we should call the class of numbers approximable at order $q^2\phi(q)$, providing in this way a more subtle classification, and he proved the following Borel-Cantelli like result (where m is the Lebesgue measure) :

Theorem 3.5 (Khintchin). *Let $\phi : \mathbf{N} \mapsto \mathbf{R}^+$ be non-decreasing and consider*

$$\mathcal{K}_\phi = \{x \in [0,1];\ |x - p/q| < 1/q^2\phi(q)\ i.o.\}$$

Then

$$m(\mathcal{K}_\phi) = \begin{cases} 0 \text{ if } \sum 1/q\phi(q) < \infty \\ 1 \text{ if } \sum 1/q\phi(q) = \infty \end{cases}$$

As consequences, we easily obtain that $m(\mathcal{W}) = 0$ by taking $\phi(q) = q^\varepsilon$, $m(\mathbf{Bad}) = 0$ (with $\phi(q) = 1$) and $\nu(x) = 2$ for almost all x since $\nu(x) = 2$ if $x \notin \mathcal{W}$. This shows the limited role plaid by the irrationality exponent.

§ 4. A flavour of dynamics

We focus now on integer base expansions and continued fraction expansion. Mostly one of both expansions is partially unknown or totally unpredictable and we are led to exploit the dynamical structure behind the classical expansions, providing statistical results only.

§ 4.1. Dynamical systems and expansions

For each "classical" expansion, the shift on digits together with the invariant probability measure of maximal entropy leads to a well-adapted ergodic dynamical system; from the Birkhoff theorem we get mean results on the behaviour of the sequence of digits or patterns. When we restrict the transformation onto a specific compact orbit, the subshift may happen to be uniquely ergodic (with a unique invariant probability measure) enjoying very interesting properties. As is well known, the q-shift $([0,1), \sigma_q, m)$, with $\sigma_q x = qx \mod 1$ and the Lebesgue measure, is ergodic; so is the Gauss dynamical system $([0,1)\backslash\mathbf{Q}, T, \mu)$, where $Tx = \{1/x\}$ and μ has density $\frac{1}{\log 2}\frac{1}{1+x}$ with respect to the Lebesgue measure. In these cases the probability measure is the unique absolutely continuous invariant one (see [9]).

Applying the Birkhoff theorem to the function log, we get a mean estimate on the denominators, a result due to Khintchin and Lévy :

Theorem 4.1 (Lévy ergodic theorem). *For almost all $x \in [0,1)$,*

$$\lim_{n\to\infty} \frac{\log q_n(x)}{n} = \beta := \frac{\pi^2}{12\log 2} \tag{4.1}$$

If the limit (4.1) holds for some given x, say $\beta(x)$, we say that x admits a *Lévy constant*, and x is called a *Lévy number* if x admits β as its Lévy constant; the theorem asserts that almost all x are Lévy numbers. If the subshift $(X = \overline{O(x)}, T)$ happens to be uniquely ergodic for a given x, then x admits a Lévy constant. This is a consequence of the uniform version due to Oxtoby of Birkhoff's theorem.

§ 4.2. Loch's theorem

Actually, any expansion allows to calculate the digits in the other one recursively but no algorithm does exist that exchanges two such expansions; the slightest information on the behaviour in both expansions of some number would thus be welcome... It seems to us that G. Lochs was the first to consider the natural question of how many partial quotients can be computed from the n first digits of x in some adic expansion and he proved the following somewhat surprising result :

Theorem 4.2 (Lochs [27]). *Let $k_n(x)$ be the exact number of first consecutive partial quotients given by the n first decimals of x. Then*

$$\lim_{n\to\infty} \frac{k_n(x)}{n} = \frac{\log 10}{2\beta} \quad a.e..$$

With the following improvement :

Theorem 4.3 (Faivre–Wu [21, 36]). *Assume that the real number x admits a Lévy constant. Then*

$$\lim_{n\to\infty} \frac{k_n(x)}{n} = \frac{\log 10}{2\beta(x)}.$$

Observe that the limit is nothing but the quotient of the entropies $h(\sigma)/h(T)$ (here $\sigma = \sigma_{10}$) as well as the quotient of the Lyapunov exponents $\lambda(\sigma)/\lambda(T)$, shedding a new light on the problem and suggesting different proofs and generalizations ([7, 8, 19]). But the initial proof relies on the previous Lévy's ergodic theorem.

Proof. First of all, by definition of k_n (omitting x) note that

$$0 \le k_1 \le k_2 \le \cdots \quad \text{and} \quad \lim_{n\to\infty} k_n = \infty.$$

We just point out the main idea of the theorem : k_n/n is very close to a Birkhoff mean. Actually, let us fix $x = \sum_{k\ge 1} x_k 10^{-k} \in [0,1)\backslash\mathbf{Q}$, with RCF expansion $[0; a_1, a_2, \ldots]$, and $n \ge 1$. If we put

$$y_n = \sum_{k=1}^{n} \frac{x_k}{10^k} = \frac{[10^n x]}{10^n},$$

then,

$$0 < |x - y_n| < \frac{1}{10^n}. \tag{4.2}$$

Moreover, since x and y_n have the same k_n first partial quotients, we also have

$$y_n = [0; a_1, \ldots, a_{k_n}, b_{k_n+1} \ldots]$$

with $b_{k_n+1} \ne a_{k_n+1}$, and

$$0 < |x - y_n| < \frac{1}{q_{k_n}(q_{k_n} + q_{k_n-1})} \le \frac{1}{q_{k_n}^2}; \tag{4.3}$$

It follows from (4.2), (4.3) and the optimality property of k_n that,

$$\frac{1}{10^n} \le \frac{1}{q_{k_n}^2}$$

and

$$\frac{\log q_{k_n}}{n} \le \frac{\log 10}{2}. \tag{4.4}$$

But more precisely, one can prove that $\log q_{k_n}^2 \sim \log 10^n$: indeed, tricky considerations on continued fractions lead to (see [36])

$$\frac{\log q_{k_n+3}}{n} \ge \frac{\log 10}{2} - \frac{\log 6}{2n}. \tag{4.5}$$

It follows that

$$60 q_{k_n+3}^2 \ge 10^{n+1} \ge q_{k_{n+1}}^2 \ge 64 q_{k_{n+1}-6}^2$$

by trivial estimates. Therefore $k_{n+1} \le k_n + 8$. The very important fact is that all theses inequalities hold for *every* underlying irrational number x. Thus, $(k_n(x))$ has bounded gaps uniformly with respect to x, and the following has been established :

$$\frac{\log q_{k_n}(x)}{n} \to \frac{\log 10}{2} \text{ for every irrational } x. \tag{4.6}$$

Now, (4.6) together with Lévy's theorem gives

$$\frac{k_n(x)}{n} = \frac{k_n(x)}{\log q_{k_n(x)}} \frac{\log q_{k_n(x)}}{n} \to \frac{\log 10}{2\beta} \quad a.e.$$

and in the same way

$$\frac{k_n(x)}{n} \to \frac{\log 10}{2\beta(x)}$$

if the number x admits a Lévy constant. $\square$

Question : If $n_k(x)$ is the number of first consecutive decimals given by the k first partial quotients of x, do we have $\lim_{k\to\infty} \dfrac{n_k(x)}{k} = 2\beta/\log 10$ a.e. ?

§ 5. First examples of explicit expansions

There are however several examples of real numbers defined by their expansion in some integer base and whose continued fraction expansion is known. We present some (overlapping) classes of expansions with more or less explicit approximation properties.

§ 5.1. Lacunary numbers

The transcendency of numbers $\sum_{n\geq 1} 2^{-c_n}$ for an increasing sequence (c_n) of positive integers satisfying $\limsup_{n\to\infty} c_{n+1}/c_n = \infty$ goes back to Liouville. Mahler proved in [28] that the number $\sum_{n\geq 1} 2^{-2^n}$ must in turn be transcendental. Following J. Shallit, we consider more generally what we call *lacunary numbers*, mainly numbers with a specific lacunary adic expansion

$$x = \sum_{n\geq 1} u^{-c_n}, \quad u \text{ integer } \geq 2,$$

where $c_{n+1}/c_n \geq 2$ for n large enough. Shallit proved that the RCF expansion of any such number can be given explicitely and that good rational approximations can be produced by truncation.

Theorem 5.1 (Shallit [32]). *The transcendental number* $\displaystyle\sum_{n=0}^{\infty} \frac{1}{u^{2^n}}$ *is in* **Bad**, *its partial quotients being bounded by* $u+2$ *if* $u \geq 3$ *and by* 6 *if* $u = 2$.

Proof. The proof is based on the following identity, which is known as "folding lemma" :

Lemma 5.2. *Let* $\frac{P}{Q} = [0; a_1, \ldots, a_N]$, $a_N \geq 2$*; then, for every* $x \geq 1$,

$$\frac{P}{Q} + \frac{1}{xQ^2} = [0; \underbrace{a_1, \ldots, a_{N-1}}, a_N, x-1, 1, a_N - 1, \underbrace{a_{N-1}, \ldots, a_1}] \tag{5.1}$$

(so that $\frac{P}{Q} + \frac{1}{Q^2} = [0; \underbrace{a_1, \ldots, a_{N-1}}, a_N + 1, a_N - 1, \underbrace{a_{N-1}, \ldots, a_1}]$*).*

For the theorem now, let us fix u, $g = \sum_{k=0}^{\infty} \frac{1}{u^{2^k}}$ and $g_n = \sum_{k=0}^{n} \frac{1}{u^{2^k}}$ the truncated sum. First computations give

$$g_0 = \frac{1}{u}, \quad g_1 = \frac{1}{u} + \frac{1}{u^2} = [0; u-1, u+1],$$

$$g_2 = [0; u-1, u+2, u, u-1].$$

Putting $g_n = \frac{P_n}{Q_n} = [0; a_1, \ldots, a_N]$ with $\frac{P_n}{Q_n}$ irreducible, then $Q_n = u^{2^n}$ so that $g_{n+1} = g_n + \frac{1}{u^{2^{n+1}}}$. By the folding lemma we have for $n \geq 1$

$$\frac{P_{n+1}}{Q_{n+1}} = \frac{P_n}{Q_n} + \frac{1}{Q_n^2} = [0; \underbrace{a_1, \ldots, a_{N-1}}, a_N + 1, a_N - 1, \underbrace{a_{N-1}, \ldots, a_1}]. \tag{5.2}$$

The assertion follows by induction. □

The following consequence of (5.2) is interesting in view of Lochs' theorem :

Corollary 5.3 (Shallit). *The rational number g_n has the same first $2^n - 1$ partial quotients as g.*

In the general case, a similar description holds and certain partial quotients occur with regularity; moreover ([11])

Corollary 5.4 (Bugeaud). *The irrationality exponent of a lacunary number is given by*

$$\nu(x) = \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

Another consequence of (5.1) is the following, which has been used by Korobov ([24]) : every number of the form $\sum_k \frac{1}{A_k}$ where the integers $A_k \geq 2$ are such that A_k^2 divides A_{k+1}, has a predictable pattern as its RCF expansion.

§ 5.2. Sturmian numbers

The most famous example of adic expansion whose continued fraction expansion is known, is certainly the following one : $\sum_{k=1}^{\infty} x_k 2^{-k} := 0.1011010110\cdots$ where (x_k) is the *Fibonacci word* on $\{0,1\}$ i.e. the fixed point of the morphism ζ on $\{0,1\}$ defined by $\zeta(0) = 1$, $\zeta(1) = 10$ and starting by 1. Indeed, this striking identity appears in [17] :

Theorem 5.5 (Davison [32]). *Let θ denote the golden number. Then*

$$x = 0.1011010110\cdots = [0; 2^0, 2^1, 2^1, 2^2, 2^3, 2^5, \ldots, 2^{f_{n-2}}, \ldots] \tag{5.3}$$

where the partial quotients are the Fibonacci numbers, defined by $f_0 = 0, f_1 = 1$, and $f_k = f_{k-1} + f_{k-2}$ for $k \geq 2$. Thus x is diophantine and $\nu(x) = 1 + \theta$.

It follows that x is very well approximable thus transcendental.
Recall that a *sturmian word* is an infinite word on a two-letters alphabet with minimal complexity ($p(n) = n+1$) and a sturmian number admits a sturmian word on $\{0,1\}$ as its binary expansion. Remind that for some infinite word u, the complexity : $p(n) = \mathrm{card}\{u_k u_{k+1} \cdots u_{k+n-1}, \ \ k \geq 1\}$ (See [5] for more details). Davison's result admits various generalizations, first of all to characteristic sturmian numbers. This extends to the closed orbit and leads to a precise approximation rate ([29, 14]). More precisely,

Proposition 5.6. *Let x be a sturmian number. Then*

$$\nu(x) = 1 + limsup_{n\to\infty}[a_n; a_{n-1}, a_{n-2}, \ldots, a_1]$$

where $\alpha = [0; a_1, \ldots]$ is the slope of the word (i.e. the frequency of 1).

Note that the transcendency of sturmian numbers has earlier been proved by Ferenczi-Mauduit 1997 by using Ridout's theorem ([22]). It follows from their work that the complexity p of the adic expansion of every irrational algebraic number satisfies $\liminf_{n\to\infty}(p(n) - n) = \infty$. Tamura has obtained a generalization of identity (5.3) to three (and more) dimensions ([33, 34]). The Fibonacci sequence is replaced by a linear recurrence of degree 3, the continued fraction algorithm is replaced by a Jacobi-Perron type algorithm, and the morphism ζ is replaced by a similar morphism on three letters.

§ 5.3. Automatic and morphic numbers

Another class of interesting numbers relative to expansions and approximation consists of some automatic and morphic numbers ([5, 29]).

Definition 5.7. The real number x is a *morphic* number if, for some $b \geq 2$, the b-ary expansion of x is the fixed point of a morphism on the alphabet $\{0, 1, \ldots, b-1\}$, or the image of such a word under a letter-to-letter mapping.

Definition 5.8. Such an x is said to be *automatic* if the involved morphism is uniform (of constant length).

Morphic words have very low complexity, in fact p is sublinear for most of them. In [16] Cobham claimed that automatic irrational numbers should be transcendental and this has been proved by Adamczewski and Bugeaud as a consequence of the following, which makes use of a method based on repetition of blocks combined with Schmidt's subspace theorem.

Theorem 5.9 (Adamczewski & Bugeaud [1])**.** *The complexity of the adic expansion of some irrational algebraic number satisfies* $\liminf_{n\to\infty}(p(n)/n) = \infty$.

The rate of approximation had to be precised for such numbers. Adamczewski and Cassaigne [3] proved that an automatic number x cannot be Liouville ($\nu(x) < \infty$) while diophantine numbers may appear with morphic numbers as already observed with the Fibonacci case. Adamczewski and Rivoal [4] gave upper bounds for the irrationality exponent of classical automatic numbers but these bounds are probably not best possible. The following questions seem relevant (see [13] for partial answers) :

Questions:

1. Is the irrationality exponent of a morphic number always algebraic ?

2. Is the irrationality exponent of an automatic number always rational ?

3. Is there some link between the irrationality exponent and the Perron-Frobenius eigenvalue of the morphism ?

§ 5.4. Cantorian numbers

A famous class of specific adic expansions is given by the classical Cantor set

Definition 5.10 (Cantor's set).

$$K = \{\sum_{n\geq 1} \frac{2\varepsilon_n}{3^n}, \ \ \varepsilon_n \in \{0,1\}.\}$$

Many open questions involve the cantorian numbers. Mahler has conjectured that these numbers are either rational or transcendental and he rised the following weaker question :
How close can irrational elements of Cantor's set be approximated by rational numbers ? which is exactly our purpose!
An interesting approach, based on the classical proof of Lagrange's theorem, has been performed by M. Keane in view to exclude the quadratic irrational numbers from K. But the proof has to be completed ([23]).

Whence the more modest questions : How are elements of K distributed inside the diophantine classes we introduced previously ? And first of all, does there exist $x \in \mathbf{Bad} \cap K$? $x \in \mathcal{L} \cap K$? $x \in (\mathcal{W} \backslash \mathcal{L}) \cap K$?

• Existence of Liouville numbers in K can easily be proved : it is sufficient to consider $x = \sum_{n=1}^{\infty} \frac{2}{3^{n!}}$.
• Existence of badly approximable elements of K has already been established. Indeed the number $x = 2\sum_{k=0}^{\infty} \frac{1}{3^{2^k}}$ belongs to $\mathbf{Bad} \cap K$ as seen in a previous item, since $\mathbf{Bad}$ is trivially stable under multiplication by 2.
• This process has been generalized by Levesley-Salp-Velani ([26]) then by Y. Bugeaud ([11]) who construct in this way elements in K with prescribed irrationality exponent :

Theorem 5.11 (Bugeaud [11]). *Let τ be a real number ≥ 2. For every $\lambda > 0$, there exists an explicit n_0 such that*

$$x = 2\sum_{n\geq n_0} 3^{-[\lambda\tau^n]} \in K$$

has τ as its irrationality exponent.

This proves that there exist very well approximable numbers in K which are not Liouville. Moreover, it clearly follows from the above constructions that $\mathbf{Bad}\cap K$ and $\mathcal{L}\cap K$ are uncountable. What can we say on the size of those sets ? We are back to the metric point of view but, now, without the help of the Lebesgue measure since $m(K) = 0$.

Let us consider the "canonical" probability measure on K,

$$\mu = *_{n=1}^{\infty} \frac{1}{2}(\delta_0 + \delta_{\frac{2}{3^n}}),$$

that is the unique probability measure of maximal entropy on K. This measure enjoys very nice properties and we shall need the following :

Lemma 5.12. *For any interval I,*

$$\mu(\varepsilon I) \leq C\varepsilon^{\alpha}\mu(I) \quad \textit{for every } 0 < \varepsilon \leq 1 \tag{5.4}$$

where $\alpha := \log 2/\log 3 = \dim(K)$.

(Actually, μ is the restriction to K of the α-Hausdorff measure). If I is any interval centered in $(0,1)$ and $\varepsilon > 0$, we denote by εI the interval of length $\varepsilon|I|$ with the same center as I.
In addition, the dynamical system (K, σ_3, μ) is ergodic. Thus, $\mu(\mathbf{Bad} \cap K)$ and $\mu(\mathcal{W} \cap K)$ must be equal to either 0 or 1 since $\mathbf{Bad} \cap K$ and $\mathcal{W} \cap K$ are clearly 3-invariant subsets of K. The following results are very meaningful, proving that, from the diophantine point of view, the triadic Cantor set behaves like the whole real line.

Theorem 5.13 (B.Weiss [35])**.** *We have* $\mu(\mathcal{W} \cap K) = 0$.

Theorem 5.14 (Einsiedler & all [20])**.** *We have* $\mu(\mathbf{Bad} \cap K) = 0$.

Proof of Theorem 5.13. We reproduce here the nice proof of B. Weiss' result. Actually he observed that the easy direction in the proof of Khintchin's theorem still holds for certain auto-similar measures, including the Cantor-Lebesgue measure μ. Theorem 5.13 is an easy consequence of the next one by considering $\phi(q) = q^{\varepsilon}$, $\varepsilon > 0$.

Theorem 5.15. *Let $\phi : (0, \infty) \to \mathbf{R}^+$ be such that $\sum_q \frac{1}{q\phi(q)^{\alpha}} < \infty$ and $\psi(q) := 1/q\phi(q)$ non increasing. Then $\mu(\mathcal{K}_\phi) = 0$.*

Proof. Let us recall our notation :

$$\mathcal{K}_\phi = \{x,\ |x - p/q| \leq \frac{1}{q^2\phi(q)} \quad i.o.\}.$$

By our assumption on the monotonicity, we may restrict the definition to the irreducible rational numbers p/q, indeed, infinitely many such rational numbers do appear in $\mathcal{K}_\phi$. For convenience, we write $B(x,r) :=]x - r, x + r[$ and we follow the classical proof of Khintchin's theorem :

$$\mathcal{K}_\phi \subset \limsup_q E_q$$

where $E_q := E_q(\phi) = \cup_{p,\ p/q \in (0,1)} B(\frac{p}{q}, \psi(q)/q)$; we are just led to prove that

$$\Sigma := \sum_{\substack{p \in \mathbf{Z}, q \in \mathbf{N} \\ (p,q)=1}} \mu(B(\frac{p}{q}, \psi(q)/q)) < \infty.$$

We decompose the sum over dyadic blocks in view to make use of property (5.4) :

$$\Sigma = \sum_{Q=0}^{\infty} \sum_{\substack{q \in [2^Q, 2^{Q+1}[\\ (p,q)=1}} \ldots$$

This is made possible by the disjointness property of the finite set of intervals $B(\frac{p}{q}, 2^{-2(Q+2)})$ when q runs along $[2^Q, 2^{Q+1}[$ and p/q are in irreducible form.
Now observe that $\psi(q)/q$ becomes smaller than 1 for q (or Q) large enough, so that, by (5.4) with $I = B(\frac{p}{q}, 2^{-2(Q+2)})$ and $2^Q \leq q < 2^{Q+1}$,

$$\mu(B(\frac{p}{q}, \frac{\psi(q)}{q})) \leq C(2^{2(Q+2)}\frac{\psi(q)}{q})^\alpha \mu(B(\frac{p}{q}, 2^{-2(Q+2)})).$$

We deduce from the above

$$\Sigma \leq C\sum_{Q=0}^{\infty} 2^{\alpha(2Q-Q)}\psi(2^Q)^\alpha \Big(\sum_{\substack{2^Q \leq q < 2^{Q+1} \\ (p,q)=1}} \mu(B(\frac{p}{q}, 2^{-2(Q+2)}))\Big)$$

$$\leq C\sum_{Q=0}^{\infty} 2^Q (2^Q)^{\alpha-1}\psi(2^Q)^\alpha = C\sum_{Q=0}^{\infty} \frac{1}{\phi(2^Q)^\alpha},$$

since $\sum_{\substack{2^Q \leq q < 2^{Q+1} \\ (p,q)=1}} \mu(B(\frac{p}{q}, 2^{-2(Q+2)})) = \mu(\cup B(\frac{p}{q}, 2^{-2(Q+2)})) \leq 1$. The positive sequence $u_n = 1/n\phi(n)^\alpha$ is non-increasing and the series $\sum_k 2^k u_{2^k}$ converges as soon as $\sum_n u_n$ converges too which ends the proof. □

□

The second quoted theorem appears as a consequence of some much more involved results ([20]). It seems likely that the geometrical properties of the Cantor-Lebesgue measure could be used to provide a direct proof.

§ 6. The Thue-Morse-Mahler number

We focus in this last section on the emblematic Thue-Morse-Mahler number. Let

$$\mathbf{t} = t_1 t_2 t_3 \ldots = 0110100110010110100101100110100110010110\ldots$$

denote the Thue–Morse word on $\{0,1\}$, that is, the fixed point starting by 0 of the morphism τ defined by $\tau(0) = 01$ and $\tau(1) = 10$.

Let $b \geq 2$ be an integer. In a fundamental paper, Mahler [28] established that the Thue–Morse–Mahler number

$$\xi_{\mathbf{t},b} = \sum_{k\geq 1} \frac{t_k}{b^k} = \frac{1}{b^2} + \frac{1}{b^3} + \frac{1}{b^5} + \frac{1}{b^8} + \frac{1}{b^9} + \ldots$$

is transcendental and Dekking has proposed an alternative proof ([18, 5]). It is proved in [12] that the irrationality exponent of $\xi_{\mathbf{t},b}$ is equal to 2, by using Padé-approximants to the generating function of this sequence. This result shows that the transcendency of $\xi_{\mathbf{t},b}$ cannot be proved by applying Roth's theorem. We focus on the so-called Thue–Morse constant $\xi_{\mathbf{t},2}$, which we simply denote by ξ. The open Problem 9 on page 403 of [5] asks whether it has bounded partial quotients. Observe that the fact that the irrationality exponent of ξ equals 2 prevents its sequence of partial quotients to increase too rapidly to infinity. However, there are uncountably many real numbers having irrationality exponent equal to 2 and whose sequence of partial quotients is increasing!

A computation shows that

$$\xi = [0; 2, 2, 2, 1, 4, 3, 5, 2, 1, 4, 2, 1, 5, 44, 1, 4, 1, \ldots] = 0.412454\ldots$$

and the determination of the first thousands of partial quotients suggests that ξ has unbounded partial quotients. But notice the difference between ξ and the sturmian numbers whose sequence of partial quotients is increasing rapidly.

With very elementary technics (apart a type Roth theorem), we can prove that

Theorem 6.1 ([15]). *1. ξ is transcendental.*
2. The irrationality exponent of ξ is less than $5/2$ (actually equals 2).
3. The sequence of its partial quotients does not increase to infinity. More precisely, ξ has infinitely many partial quotients equal to 4 or 5. Furthermore, there are infinitely many pairs of consecutive partial quotients both less than or equal to 5.

Proof. Throughout the proof, for $n \geq 0$, we denote by

$$\mathcal{F}_n := 2^{2^n} + 1$$

the n-th Fermat number. Although we ignore the formula of the nth denominator, we are able to determine a subsequence of the (q_n) by a simple application of Legendre's theorem :

Lemma 6.2. *For any $n \geq 1$ the integers $\mathcal{F}_n$ and $2^{2\cdot 2^n}\mathcal{F}_n$ are denominators of convergents to ξ.*

We turn back to the theorem :
1. It follows from the identification of $(2^{2\cdot 2^n}\mathcal{F}_n)$ as a subsequence of the sequence of denominators of the convergents to ξ that the transcendency of ξ is an immediate consequence of the Ridout theorem that we recalled in the historical item. For our purpose, choosing the prime number 2 and $q := q_n = 2^{2\cdot 2^n}\mathcal{F}_n$ in (2.2), we get

$$|q_n|_2 \times |\xi - \frac{p_n}{q_n}| \leq \frac{1}{2^{2\cdot 2^n} q_n^2} = \frac{1}{\mathcal{F}_n^2 2^{6\cdot 2^n}} \leq \frac{1}{q_n^{2+1/2}}$$

which is sufficient for applying Ridout's theorem, and ξ is transcendental.

2. Recall that the irrationality exponent of some irrational number x is

$$\nu(x) = \inf\{\alpha,\ |x - \frac{p}{q}| < \frac{1}{q^\alpha} \text{ infinitely often.}\}$$

Since we have constructed a sufficiently 'dense' sequence of not so bad rational approximations, we can get a bound for its irrationality exponent. This is the object of the next lemma [4].

Lemma 6.3. *Let x be a real number. Assume that there exist real numbers $\theta \geq 1$, $\varepsilon > 0$ and a sequence (p_n/q_n) of rational numbers such that :*

(i) $\limsup \dfrac{\log q_{n+1}}{\log q_n} \le \theta$,

(ii) $|x - \dfrac{p_n}{q_n}| \le \dfrac{1}{q_n^{1+\varepsilon}}$ *for* $n \ge 1$,

then the irrationality exponent of x *is at most equal to* $1 + \max(\theta, \theta/\varepsilon)$.

In view of both lemmas, we consider the sequence of denominators of the form $\mathcal{F}_n$ as well as $2^{2\cdot 2^n}\mathcal{F}_n$, ranged by increasing order, that we rename (Q_n) and the sequence of the so-associated convergents (P_n/Q_n). Since

$$\mathcal{F}_{n+1} < 2^{2\cdot 2^n}\mathcal{F}_n < \mathcal{F}_{n+2}$$

it is easily checked that

$$\limsup \frac{\log Q_{n+1}}{\log Q_n} \le 3/2$$

and so $\nu(\xi) \le 5/2$.

3. The sequence (r_n) of convergents with denominators $\mathcal{F}_n$ is easily seen to satisfy

$$0.175(\frac{1}{2^{2^n}} + \frac{1}{2^{2^{n+1}}}) \le \xi - r_n \le 0.1751\frac{1}{2^{2^n} - 1}. \tag{6.1}$$

Also, for $\ell \ge 1$, we have (see e.g. [10]),

$$|q_\ell \xi - p_\ell| = \frac{1}{q_\ell} \cdot \frac{1}{[a_{\ell+1}; a_{\ell+2}, \ldots] + [0; a_\ell, a_{\ell-1}, \ldots]}, \tag{6.2}$$

where $(p_\ell/q_\ell)_{\ell \ge 0}$ denotes the sequence of convergents to ξ. We deduce from (6.1) and (6.2), that

$$[a_{\ell+1}; a_{\ell+2}, \ldots] + [0; a_\ell, a_{\ell-1}, \ldots] \asymp 1/0.175 \asymp 5.71, \tag{6.3}$$

for arbitrarily large integers ℓ. Let ℓ be a positive integer for which (6.3) holds. If $a_\ell \ge 2$, then $a_{\ell+1}$ must be equal to 5 and $a_{\ell+2}$ is at most 4. If $a_\ell = 1$, then $a_{\ell+1}$ equals 4 or 5.

Remark : By similar arguments, we can prove in addition that ξ has infinitely many partial quotients greater than or equal to 50. But the main question remains open. □

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