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On outer components of the homology group of the homological Goldman Lie algebra

By

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Abstract

The group ring of the first homology group of an oriented surface has a natural structure of a Lie algebra. We call it the homological Goldman Lie algebra. The homology group of the homological Goldman Lie algebra is graded by the first homology group of the surface. We conjecture that some components of the homology group, which we call outer components, vanish. We prove that it is affirmative if the degree is smaller than five.

§ 1. Introduction

Let H be a $\mathbb{Z}$-module, i.e., an abelian group, which is not necessarily finitely generated, and $\langle -,- \rangle : H \times H \to \mathbb{Z}$, $(x,y) \mapsto \langle x,y \rangle$, an alternating $\mathbb{Z}$-bilinear form. For example, we consider that H is the first homology group of an oriented surface, and $\langle -,- \rangle$ is the intersection form of the surface. We define a $\mathbb{Z}$-linear map $\mu : H \to \mathrm{Hom}_{\mathbb{Z}}(H,\mathbb{Z})$ by $\mu(x)(y) = \langle x,y \rangle$. Denote by $\mathbb{Q}[H]$ the $\mathbb{Q}$-vector space with basis the set H;

$$\mathbb{Q}[H] := \left\{ \sum_{i=1}^{n} c_i[x_i] \mid n \in \mathbb{N}, c_i \in \mathbb{Q}, x_i \in H \right\},$$

where $[-] : H \to \mathbb{Q}[H]$ is the embedding as basis. Here, we remark that $c[x] \neq [cx]$ for any $c \neq 1$ and $x \in H$. We define a $\mathbb{Q}$-bilinear map $[-,-] : \mathbb{Q}[H] \times \mathbb{Q}[H] \to \mathbb{Q}[H]$ by $[[x],[y]] := \langle x,y \rangle [x+y]$ for $x,y \in H$. It is easy to see that this bilinear map is skew and satisfies the Jacobi identity [G, p.295-p.297]. The Lie algebra $(\mathbb{Q}[H],[-,-])$

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is called the *homological Goldman Lie algebra of* $(H, \langle -,- \rangle)$. The Lie algebra $\mathbb{Q}[H]$ is introduced by Goldman to study the moduli space of $GL_1(\mathbb{R})$-flat bundles over the surface [G, p.295-p.297]. Goldman introduced a more geometric Lie algebra. It is a Lie algebra structure on the free module with basis the set of homotopy classes of free loops on the surface. Goldman used the Lie algebra for study of the space of representations on the fundamental group of the surface [G].

Our purpose is to study the algebraic structure of the homological Goldman Lie algebra. We already determined the ideals [T1], the minimal number of generators [K-K-T] and the second homology group [T2] of the homological Goldman Lie algebra. In general, the homology and cohomology groups of a Lie algebra reflect the algebraic structure of the Lie algebra. For example, a finite dimensional semi-simple Lie algebra is characterized by the Lie algebra cohomology group.

The Lie algebra $\mathbb{Q}[H]$ is a graded Lie algebra of type H [B, Chapter II §11]. Namely, we have $\mathbb{Q}[H] = \bigoplus_{x \in H} \mathbb{Q}[x]$ and $[\mathbb{Q}[x], \mathbb{Q}[y]] \subset \mathbb{Q}[x+y]$. Here $\mathbb{Q}[x]$ is the one-dimensional $\mathbb{Q}$-vector subspace of $\mathbb{Q}[H]$ generated by an element $[x] \in \mathbb{Q}[H]$. Therefore, the homology group of $\mathbb{Q}[H]$ is also graded by the abelian group H. We call $H_p(\mathbb{Q}[H])_{(z)}$ an *inner component* if $z \in \ker\mu$, and $H_p(\mathbb{Q}[H])_{(z)}$ an *outer component* if $z \in H \setminus \ker\mu$. Our main conjecture in this paper is that outer components of the homology group of the homological Goldman Lie algebra vanish. That is,

Conjecture 1.1. For any $p > 0$ and $z \in H \setminus \ker\mu$, we have $H_p(\mathbb{Q}[H])_{(z)} = 0$.

In general, suppose a Lie algebra $\mathfrak{g}$ has an element $X_0 \in \mathfrak{g}$ such that $\mathfrak{g}$ is decomposed into the direct sum of eigenspaces $\mathfrak{g}_{(\lambda)}$ of the action $ad(X_0)$. Here we denote $\mathfrak{g}_{(\lambda)} = \{X \in \mathfrak{g} | [X_0, X] = \lambda X\}$. Then the standard cochain complex $C^*(\mathfrak{g})$ is decomposed into the completed direct sum of eigenspaces $C^p(\mathfrak{g})_{(\lambda)}$ of X_0, $C^*(\mathfrak{g}) = \prod C^*(\mathfrak{g})_{(\lambda)}$. Here we define $C^p(\mathfrak{g})_{(\lambda)} := \{\omega \in C^p(\mathfrak{g}) | \omega(X_1, \ldots, X_p) = 0 \text{ for } X_i \in \mathfrak{g}_{(\lambda_i)}, \lambda_1 + \cdots + \lambda_p \neq \lambda\}$. It is easy to see that the inclusion $C^*(\mathfrak{g})_{(0)} \to C^*(\mathfrak{g})$ induces an isomorphism in the cohomology groups [F, p.44-p.46]. Our conjecture is an analogy of this fact. We will prove that it is affirmative if the degree is smaller than 5. In the paper [T2], we already determine not only outer components but also inner components if the degree is smaller than 3. In the present paper, we prove the conjecture in the case when the degree $p = 3$ and $p = 4$.

§ 2. Decomposition of the homology group

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{Q}$. We define the Lie algebra homology group. Set $C_p(\mathfrak{g}) := \wedge^p \mathfrak{g}$, the pth exterior power of $\mathfrak{g}$. Define the $\mathbb{Q}$-linear map $\partial = \partial_p : C_p(\mathfrak{g}) \to C_{p-1}(\mathfrak{g})$ by setting $\partial_p(X_1 \wedge \cdots \wedge X_p) = \sum_{1 \leq i < j \leq p} (-1)^{i+j} [X_i, X_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_p$ for $X_1, \ldots, X_p \in \mathfrak{g}$. The pair $(C_*(\mathfrak{g}), \partial)$ is a chain complex, that is, $\partial \circ \partial = 0$.

We denote $Z_p(\mathfrak{g}) = \ker \partial_p$ and $B_p(\mathfrak{g}) = \mathrm{im}\partial_{p+1}$. Then we define the pth homology group of $\mathfrak{g}$ by $H_p(\mathfrak{g}) = Z_p(\mathfrak{g})/B_p(\mathfrak{g})$. The graded homology group $H_*(\mathfrak{g}) = H_*(C_*(\mathfrak{g}), \partial)$ is called the *Lie algebra homology group of the Lie algebra* $\mathfrak{g}$. See for example [F][H-S].

Assume that $\mathfrak{g}$ is a graded Lie algebra of type a commutative monoid G. That is, there exist submodules $\mathfrak{g}_{(g)}$ for $g \in G$ with $\mathfrak{g} = \bigoplus_{g\in G} \mathfrak{g}_{(g)}$ and $[\mathfrak{g}_{(g)}, \mathfrak{g}_{(h)}] \subset \mathfrak{g}_{(g+h)}$. For $X \in \mathfrak{g} \setminus \{0\}$, we say X is homogeneous if there exists $g \in G$ with $X \in \mathfrak{g}_{(g)}$, and denote $\deg X = g$. We decompose the homology of the graded Lie algebra $\mathfrak{g}$. Define a subspace $C_p(\mathfrak{g})_{(g)}$ of $C_p(\mathfrak{g})$ by the subspace generated by the set $\{X_1 \wedge \cdots \wedge X_p \in C_p(\mathfrak{g}) \mid X_i \text{ is homogeneous and } \deg X_1 + \cdots + \deg X_p = g\}$. This is well-defined since the condition $\deg X_1 + \cdots + \deg X_p = g$ is independent of permutation of $X_1, \ldots, X_p$. It is clear that $C_*(\mathfrak{g})_{(g)}$ is a subcomplex of $C_p(\mathfrak{g})$, that is, $\partial_p(C_p(\mathfrak{g})_{(g)}) \subset C_{p-1}(\mathfrak{g})_{(g)}$, and $C_p(\mathbb{Q}[H]) = \bigoplus_{z\in H} C_p(\mathbb{Q}[H])_{(z)}$ as chain complex. We define $H_*(\mathfrak{g})_{(g)} = H_*(C_*(\mathfrak{g})_{(g)}, \partial)$. Then we have $H_*(\mathfrak{g}) = \bigoplus_{g\in G} H_*(\mathfrak{g})_{(g)}$.

We consider the homological Goldman Lie algebra $\mathbb{Q}[H]$. The Lie algebra $\mathbb{Q}[H]$ is a graded Lie algebra of type H. By the definition of the grading, we have $\deg[u] = u$ for $u \in H$. Hence we have that $C_p(\mathbb{Q}[H])_{(z)}$ is the $\mathbb{Q}$-vector subspace of $C_p(\mathbb{Q}[H]) = \wedge^p(\mathbb{Q}[H])$ generated by the set

$$\{[u_1] \wedge \cdots \wedge [u_p] \in \wedge^p \mathbb{Q}[H] | u_i \in H, u_1 + \cdots + u_p = z\}.$$

§ 3. For the proof of the conjecture

Let $z \in H \setminus \ker \mu$. Then there exists $y \in H$ with $\langle y, z\rangle \neq 0$. Fix $y \in H$ as above throughout this paper. We try to construct a chain homotopy $\Phi : C_*(\mathbb{Q}[H])_{(z)} \to C_*(\mathbb{Q}[H])_{(z)}$ joining the identity with the zero map. We explain our plan to construct Φ step by step. We define I_k by the $\mathbb{Q}$-ideal of $\wedge\mathbb{Q}[H] = C_*(\mathbb{Q}[H])$ generated by the set

$$\{[a_1 y] \wedge \cdots \wedge [a_k y] \in \wedge^k \mathbb{Q}[H] | a_1, \ldots, a_k \in \mathbb{Z}\} \text{ for } k > 0,$$

and by $I_k = \wedge\mathbb{Q}[H]$ for $k \leq 0$. We introduce the $\mathbb{Q}$-vector subspace $I_{p,k} := C_p(\mathbb{Q}[H])_{(z)} \cap I_k$ and the quotient map $q_{p,k} : C_p(\mathbb{Q}[H])_{(z)} \to C_*(\mathbb{Q}[H])/I_{p,k}$. Then the $\mathbb{Q}$-vector space $I_{p,k}$ is generated by the set

$$\{[a_1 y] \wedge \cdots \wedge [a_k y] \wedge X \in C_p(\mathbb{Q}[H])_{(z)} | a_i \in \mathbb{Z}, X \in C_{p-k}(\mathbb{Q}[H])_{(z-a_1 y-\cdots-a_k y)}\}.$$

We have $C_p(\mathbb{Q}[H])_{(z)} = I_{p,0} \supset I_{p,1} \supset \cdots \supset I_{p,p} = 0$. Our goal is to get a chain homotopy, that is, we want to construct $\mathbb{Q}$-linear maps $\Phi_p : C_p(\mathbb{Q}[H])_{(z)} \to C_{p+1}(\mathbb{Q}[H])_{(z)}$ with $\partial_{p+1} \circ \Phi_p + \Phi_{p-1} \circ \partial_p = id_{C_p(\mathbb{Q}[H])_{(z)}}$. In order to approach the goal step by step, we try to construct $\mathbb{Q}$-linear maps $\Phi_{p,k} : C_p(\mathbb{Q}[H])_{(z)} \to C_{p+1}(\mathbb{Q}[H])_{(z)}$ with $q_{p,k+1} \circ (\partial_{p+1} \circ \Phi_{p,k} + \Phi_{p-1,k} \circ \partial_p) = q_{p,k+1}$. In other words, we construct a "homotopy up to ignoring terms of the form $[a_1 y]\wedge\cdots\wedge[a_{k+1}y]\wedge X$". If we could construct $\Phi_{p,p-1}$, then it is a desired homotopy since $I_{p,p}=0$. We want to construct $\Phi_{p,k}$ inductively on k. Namely, we want to construct $\Psi_{p,k}: C_p(\mathbb{Q}[H])_{(z)} \to C_{p+1}(\mathbb{Q}[H])_{(z)}$ and define $\Phi_{p,k}=\sum_{r=0}^{k}\Psi_{p,r}$ so that they satisfy the following.

(1) $q_{p,k+1}\circ\Psi_{p,k}=0$,
i.e., the image of $\Psi_{p,k}$ consists of terms of the form $[a_1 y]\wedge\cdots\wedge[a_{k+1}y]\wedge X$.

(2) $q_{p,k+1}\circ\partial_{p+1}\circ\Psi_{p,k}=-q_{p,k+1}\circ(\partial_{p+1}\circ\Phi_{p,k-1}+\Phi_{p-1,k-1}\circ\partial_p - id)$, where $\Phi_{p,-1}=0$.

By the two conditions, we have $q_{p,k+1}\circ(\partial_{p+1}\circ\Phi_{p,k}+\Phi_{p-1,k}\circ\partial_p)=q_{p,k+1}$. That is, we achieve the step k if we could construct $\Psi_{p,k}$ with the above conditions. The condition (2) may be inductively computable.

We have already obtained $\Psi_{p,0}$ as follows. We denote by S_p the the symmetric group of degree p, the set of bijections on $\{1,\ldots,p\}$. For $k\in\{1,\ldots,p\}$, we define a map $f_k:\{(u_1,\ldots,u_p)\in H\times\cdots\times H \mid u_1+\cdots+u_p=z\}\to C_{p+1}(\mathbb{Q}[H])_{(z)}$ by

$$f_k(u_1,\ldots,u_p)=\sum_{\sigma\in S_p}\mathrm{sgn}(\sigma)[ky]\wedge[u_{\sigma(1)}-y]\wedge\cdots[u_{\sigma(k)}-y]\wedge[u_{\sigma(k+1)}]\wedge\cdots\wedge[u_{\sigma(p)}],$$

This induces $f_k: C_p(\mathbb{Q}[H])_{(z)}\to C_{p+1}(\mathbb{Q}[H])_{(z)}$ since the map f_k satisfies $f_k(u_{\tau(1)},\ldots, u_{\tau(p)})=\mathrm{sgn}(\tau)f_k(u_1,\ldots,u_p)$ for $\tau\in S_p$. We define $\Psi_{p,0}: C_p(\mathbb{Q}[H])_{(z)}\to C_{p+1}(\mathbb{Q}[H])_{(z)}$ by $\Psi_{p,0}:=\sum_{k=1}^{p}\frac{(-1)^k f_k}{k\langle y,z\rangle k!(p-k)!}$. Then the condition (1) is clear. The map $\Psi_{p,0}$ is a desired one, that is, we have

Proposition 3.1. $q_{p,1}\circ\partial_{p+1}\circ\Psi_{p,0}=q_{p,1}$.

Proof. We prove the proposition together with some heuristic calculation. Take $u_1,\ldots,u_p\in H$ with $u_1+\cdots+u_p=z$. We remark that the term $[u_1]\wedge\cdots\wedge[u_p]$ appears in

$$\sum_{i=1}^{p}\partial_{p+1}([y]\wedge[u_1]\wedge\cdots\wedge[u_{i-1}]\wedge[u_i-y]\wedge[u_{i+1}]\wedge\cdots\wedge[u_p]).$$

In fact, we have

$$\begin{aligned}&\sum_{i=1}^{p}\partial_{p+1}([y]\wedge[u_1]\wedge\cdots\wedge[u_{i-1}]\wedge[u_i-y]\wedge[u_{i+1}]\wedge\cdots\wedge[u_p])\\ \equiv&\sum_{i=1}^{p}\sum_{j=1}^{i-1}(-1)^j\langle y,u_j\rangle[u_j+y]\wedge[u_1]\wedge\cdots\wedge[\hat{u_j}]\wedge\cdots\wedge[u_{i-1}]\\ &\wedge[u_i-y]\wedge[u_{i+1}]\wedge\cdots\wedge[u_p]\\ &+\sum_{i=1}^{p}(-1)^i\langle y,u_i\rangle[u_i]\wedge[u_1]\wedge\cdots\wedge[\hat{u_i}]\wedge\cdots\wedge[u_p]\end{aligned}$$

$$+\sum_{i=1}^{p}\sum_{j=i+1}^{p}(-1)^j\langle y,u_j\rangle[u_j+y]\wedge[u_1]\wedge\cdots\wedge[u_{i-1}]$$
$$\wedge[u_i-y]\wedge[u_{i+1}]\wedge\cdots\wedge[\hat{u_j}]\wedge\cdots\wedge[u_p]$$

and

$$\sum_{i=1}^{p}(-1)^i\langle y,u_i\rangle[u_i]\wedge[u_1]\wedge\cdots\wedge[\hat{u_i}]\wedge\cdots\wedge[u_p]$$
$$=-\sum_{i=1}^{p}\langle y,u_i\rangle[u_1]\wedge\cdots\wedge[u_p]=-\langle y,z\rangle[u_1]\wedge\cdots\wedge[u_p],$$

where we ignore the terms in $I_{p,1}$. The map $S_p\to\{1,\ldots,p\}$, $\sigma\mapsto i=\sigma(1)$, induces the equality $\sum_{i=1}^{g}[y]\wedge[u_1]\wedge\cdots\wedge[u_{i-1}]\wedge[u_i-y]\wedge[u_{i+1}]\wedge\cdots\wedge[u_p]=\frac{(-1)^1f_1(u_1,\ldots,u_p)}{1\cdot 1!(p-1)!}$, where the right hand side is the first term in $\Psi_{p,0}$. The terms of the form

$$\langle y,u_j\rangle[u_j+y]\wedge[u_1]\wedge\cdots\wedge[u_{i-1}]\wedge[u_i-y]\wedge[u_{i+1}]\wedge\cdots\wedge[\hat{u_j}]\wedge\cdots\wedge[u_p]$$

still remain, and appear in

$$\sum_{1\le i<j\le p}\partial[2y]\wedge[u_1]\wedge\cdots\wedge[u_{i-1}]\wedge[u_i-y]\wedge[u_{i+1}]\wedge\cdots\wedge[u_{j-1}]\wedge[u_j-y]\wedge[u_{j+1}]\wedge\cdots\wedge[u_p].$$

The terms involved with the bracket of $[2y]$ and $[u_i-y]$, or $[2y]$ and $[x_j-y]$, are desired ones. The element

$$\sum_{1\le i<j\le p}[2y]\wedge[u_1]\wedge\cdots\wedge[u_{i-1}]\wedge[u_i-y]\wedge[u_{i+1}]\wedge\cdots\wedge[u_{j-1}]\wedge[u_j-y]\wedge[u_{j+1}]\wedge\cdots\wedge[u_p]$$

is the second term in $\Psi_{p,0}$. Inductively, we can construct a desired map $\Psi_{p,0}$. □

Next, we want to construct $\Psi_{p,1}$. We can calculate the right hand side of the condition (2). That is, we have

Proposition 3.2.

$$(\partial_{p+1}\circ\Psi_{p,0}+\Psi_{p-1,0}\circ\partial_p-id)([u_1]\wedge\cdots\wedge[u_p])$$
$$=\sum_{m=1}^{2}\sum_{k=m}^{p+m-2}\sum_{\sigma\in S_p}\frac{(-1)^{k+1}\mathrm{sgn}(\sigma)\langle u_{\sigma(1)}-y,u_{\sigma(2)}-y\rangle}{k\langle y,z\rangle 2!(k-m)!(p-k+m-2)!}[u_{\sigma(1)}+u_{\sigma(2)}-my]$$
$$\wedge[ky]\wedge[u_{\sigma(3)}-y]\wedge\cdots\wedge[u_{\sigma(k-m+2)}-y]\wedge[u_{\sigma(k-m+3)}]\wedge\cdots\wedge[u_{\sigma(p)}].$$

Proof. We prove the proposition only for $p=3$. We have

$$\Psi_{2,0}\circ\partial_3([u_1]\wedge[u_2]\wedge[u_3])$$

$$
\begin{aligned}
&=\Psi_{2,0}\left(\frac{-1}{2!1!}\sum_{\tau\in S_3}\mathrm{sgn}(\tau)\langle u_{\tau(1)},u_{\tau(2)}\rangle[u_{\tau(1)}+u_{\tau(2)}]\wedge[u_{\tau(3)}]\right)\\
&=\frac{1}{2!}\sum_{\tau\in S_3}\mathrm{sgn}(\tau)\frac{\langle u_{\tau(1)},u_{\tau(2)}\rangle}{\langle y,z\rangle}\big(\\
&\quad [y]\wedge[u_{\tau(1)}+u_{\tau(2)}-y]\wedge[u_{\tau(3)}]+[y]\wedge[u_{\tau(1)}+u_{\tau(2)}]\wedge[u_{\tau(3)}-y]\\
&\qquad -\frac{1}{2}[2y]\wedge[u_{\tau(1)}+u_{\tau(2)}-y]\wedge[u_{\tau(3)}-y]\big)
\end{aligned}
$$

and

$$
\begin{aligned}
&\partial_4\circ\Psi_{3,0}([u_1]\wedge[u_2]\wedge[u_3])\\
&=\partial_4\big(-\frac{1}{\langle y,z\rangle 1!2!}\sum_{\sigma\in S_3}\mathrm{sgn}(\sigma)[y]\wedge[u_{\sigma(1)}-y]\wedge[u_{\sigma(2)}]\wedge[u_{\sigma(3)}]\\
&\quad +\frac{1}{2\langle y,z\rangle 2!1!}\sum_{\sigma\in S_3}\mathrm{sgn}(\sigma)[2y]\wedge[u_{\sigma(1)}-y]\wedge[u_{\sigma(2)}-y]\wedge[u_{\sigma(3)}]\\
&\quad -\frac{1}{3\langle y,z\rangle}[3y]\wedge[u_1-y]\wedge[u_2-y]\wedge[u_3-y]\big)\\
&=-\frac{1}{\langle y,z\rangle 2!}\sum_{\sigma\in S_3}\mathrm{sgn}(\sigma)\big(-\langle y,u_{\sigma(1)}\rangle[u_{\sigma(1)}]\wedge[u_{\sigma(2)}]\wedge[u_{\sigma(3)}]\\
&\qquad +\langle y,u_{\sigma(2)}\rangle[u_{\sigma(2)}+y]\wedge[u_{\sigma(1)}-y]\wedge[u_{\sigma(3)}]\\
\Big(\begin{pmatrix}1\,2\,3\\2\,3\,1\end{pmatrix}\Big)\quad&\qquad -\langle y,u_{\sigma(3)}\rangle[u_{\sigma(3)}+y]\wedge[u_{\sigma(1)}-y]\wedge[u_{\sigma(2)}]\\
&\qquad -\langle u_{\sigma(1)}-y,u_{\sigma(2)}\rangle[u_{\sigma(1)}+u_{\sigma(2)}-y]\wedge[y]\wedge[u_{\sigma(3)}]\\
\Big(\begin{pmatrix}1\,2\,3\\2\,3\,1\end{pmatrix}\Big)\quad&\qquad +\langle u_{\sigma(1)}-y,u_{\sigma(3)}\rangle[u_{\sigma(1)}+u_{\sigma(3)}-y]\wedge[y]\wedge[u_{\sigma(2)}]\\
\Big(\begin{pmatrix}1\,2\,3\\3\,1\,2\end{pmatrix}\Big)\quad&\qquad -\langle u_{\sigma(2)},u_{\sigma(3)}\rangle[u_{\sigma(2)}+u_{\sigma(3)}]\wedge[y]\wedge[u_{\sigma(1)}-y]\big)\\
&\quad +\frac{1}{2\langle y,z\rangle 2!}\sum_{\sigma\in S_3}\mathrm{sgn}(\sigma)\big(-2\langle y,u_{\sigma(1)}\rangle[u_{\sigma(1)}+y]\wedge[u_{\sigma(2)}-y]\wedge[u_{\sigma(3)}]\\
&\qquad +2\langle y,u_{\sigma(2)}\rangle[u_{\sigma(2)}+y]\wedge[u_{\sigma(1)}-y]\wedge[u_{\sigma(3)}]\\
\Big(\begin{pmatrix}1\,2\,3\\2\,3\,1\end{pmatrix}\Big)\quad&\qquad -2\langle y,u_{\sigma(3)}\rangle[u_{\sigma(3)}+2y]\wedge[u_{\sigma(1)}-y]\wedge[u_{\sigma(2)}-y]\\
&\qquad -\langle u_{\sigma(1)}-y,u_{\sigma(2)}-y\rangle[u_{\sigma(1)}+u_{\sigma(2)}-2y]\wedge[2y]\wedge[u_{\sigma(3)}]
\end{aligned}
$$

$$\left(\begin{pmatrix}1\,2\,3\\2\,3\,1\end{pmatrix}\right) \qquad + \langle u_{\sigma(1)} - y, u_{\sigma(3)}\rangle [u_{\sigma(1)} + u_{\sigma(3)} - y] \wedge [2y] \wedge [u_{\sigma(2)} - y]$$

$$\left(\begin{pmatrix}1\,2\,3\\3\,1\,2\end{pmatrix}\right) \qquad - \langle u_{\sigma(2)} - y, u_{\sigma(3)}\rangle [u_{\sigma(2)} + u_{\sigma(3)} - y] \wedge [2y] \wedge [u_{\sigma(1)} - y]\Big)$$

$$+ \frac{1}{\langle y, z\rangle 2!} \sum_{\sigma\in S_3} \mathrm{sgn}(\sigma)\langle y, u_{\sigma(1)}\rangle [u_{\sigma(1)} + 2y] \wedge [u_{\sigma(2)} - y] \wedge [u_{\sigma(3)} - y]$$

$$+ \frac{1}{3\langle y, z\rangle 2!} \sum_{\tau\in S_3} \mathrm{sgn}(\tau)\langle u_{\tau(1)} - y, u_{\tau(2)} - y\rangle$$

$$[u_{\tau(1)} + u_{\tau(2)} - 2y] \wedge [3y] \wedge [u_{\tau(3)} - y]$$

$$=[u_1] \wedge [u_2] \wedge [u_3]$$

$$+ \sum_{\sigma\in S_3} \frac{\langle u_{\sigma(1)} - y, u_{\sigma(2)}\rangle + \langle u_{\sigma(1)}, u_{\sigma(2)} - y\rangle}{\mathrm{sgn}(\sigma)\langle y, z\rangle 2!}$$

$$[u_{\sigma(1)} + u_{\sigma(2)} - y] \wedge [y] \wedge [u_{\sigma(3)}]$$

$$+ \sum_{\sigma\in S_3} \frac{\langle u_{\sigma(1)}, u_{\sigma(2)}\rangle}{\mathrm{sgn}(\sigma)\langle y, z\rangle 2!} [u_{\sigma(1)} + u_{\sigma(2)}] \wedge [y] \wedge [u_{\sigma(3)} - y]$$

$$- \sum_{\sigma\in S_3} \frac{\langle u_{\sigma(1)} - y, u_{\sigma(2)} - y\rangle}{\mathrm{sgn}(\sigma) 2\langle y, z\rangle 2!} [u_{\sigma(1)} + u_{\sigma(2)} - 2y] \wedge [2y] \wedge [u_{\sigma(3)}]$$

$$- \sum_{\sigma\in S_3} \frac{\langle u_{\sigma(1)} - y, u_{\sigma(2)}\rangle + \langle u_{\sigma(1)}, u_{\sigma(2)} - y\rangle}{\mathrm{sgn}(\sigma) 2\langle y, z\rangle 2!}$$

$$[u_{\sigma(1)} + u_{\sigma(2)} - y] \wedge [2y] \wedge [u_{\sigma(3)} - y]$$

$$+ \sum_{\tau\in S_3} \frac{\langle u_{\tau(1)} - y, u_{\tau(2)} - y\rangle}{\mathrm{sgn}(\tau) 3\langle y, z\rangle 2!} [u_{\tau(1)} + u_{\tau(2)} - 2y] \wedge [3y] \wedge [u_{\tau(3)} - y].$$

Here each $\left(\begin{pmatrix}1\,2\,3\\2\,3\,1\end{pmatrix}\right)$ and $\left(\begin{pmatrix}1\,2\,3\\3\,1\,2\end{pmatrix}\right)$ means a tag for the attached term, and we replace σ by $\sigma\circ\theta$ at the term attached tag $\theta \in S_3$ to obtain another similar term. We remark that the correspondence $S_3 \ni \sigma \mapsto \sigma\circ\theta \in S_3$ is a bijection. Since $\langle u-y, v\rangle + \langle u, v-y\rangle - \langle u, v\rangle = \langle u - y, v - y\rangle$, we have

$$(\partial_4 \circ \Psi_{3,0} + \Psi_{2,0} \circ \partial_3)([u_1] \wedge [u_2] \wedge [u_3])$$

$$=[u_1] \wedge [u_2] \wedge [u_3]$$

$$+ \sum_{\sigma\in S_3} \frac{\langle u_{\sigma(1)} - y, u_{\sigma(2)} - y\rangle}{\mathrm{sgn}(\sigma)\langle y, z\rangle 2!} [u_{\sigma(1)} + u_{\sigma(2)} - y] \wedge [y] \wedge [u_{\sigma(3)}]$$

$$- \sum_{\sigma\in S_3} \frac{\langle u_{\sigma(1)} - y, u_{\sigma(2)} - y\rangle}{\mathrm{sgn}(\sigma) 2\langle y, z\rangle 2!} [u_{\sigma(1)} + u_{\sigma(2)} - y] \wedge [2y] \wedge [u_{\sigma(3)} - y]$$

$$- \sum_{\sigma\in S_3} \frac{\langle u_{\sigma(1)}-y, u_{\sigma(2)}-y\rangle}{\mathrm{sgn}(\sigma)2\langle y,z\rangle 2!}[u_{\sigma(1)}+u_{\sigma(2)}-2y]\wedge[2y]\wedge[u_{\sigma(3)}]$$
$$+ \sum_{\sigma\in S_3} \frac{\langle u_{\sigma(1)}-y, u_{\sigma(2)}-y\rangle}{\mathrm{sgn}(\sigma)3\langle y,z\rangle 2!}[u_{\sigma(1)}+u_{\sigma(2)}-2y]\wedge[3y]\wedge[u_{\sigma(3)}-y].$$

Therefore we have the proposition for $p=3$. We can prove the proposition similarly by direct calculation even if $p\neq 3$. □

Our next goal is to construct $\Psi_{p,1}$. However, for $p\geq 5$, we have not yet succeeded. For example, we consider

$$\begin{aligned}
&\Psi_{p,1}([u_1]\wedge\cdots\wedge[u_p])\\
=&\sum_{m=1}^{2}\sum_{1\leq\ell<k\leq p}\sum_{a=0}^{1}\sum_{b=0}^{\ell-a}\sum_{\sigma\in S_p}\\
&\frac{(-1)^{\ell+k+1}\mathrm{sgn}(\sigma)\langle u_{\sigma(1)}-y,u_{\sigma(2)}-y\rangle}{\ell k\langle y,z\rangle^2 2!(k-m-b)!(\ell-b-a)!(p-k+m-2-\ell+b+a)!}\\
&[\ell y]\wedge[ky]\wedge[u_{\sigma(1)}+u_{\sigma(2)}-(m+a)y]\wedge[u_{\sigma(3)}-2y]\wedge\cdots\wedge[u_{\sigma(\ell+2)}-2y]\\
&\wedge[u_{\sigma(\ell+3)}-y]\wedge\cdots[u_{\sigma(k+\ell-m-b-a+2)}-y]\wedge[u_{\sigma(k+\ell-m-b-a+3)}]\wedge\cdots\wedge[u_{\sigma(p)}].
\end{aligned}$$

Then many terms vanish, but the terms of the form $[u_i-(n-2)y]\wedge[ny]\wedge X$ and $[u_i-ny]\wedge[ny]\wedge X$ remain. We do not know whether we should correct the scalar, add some terms or calculate a totally different expression. For $p<5$, we succeeded in constructing $\Psi_{p,k}$. We describe it in the next section. In the rest of this section, we describe related matters.

We have

$$\ker(q_{p+1,k-1}\circ\partial_p : C_p(\mathbb{Q}[H])_{(z)}\to C_{p-1}(\mathbb{Q}[H])_{(z)}/I_{p-1,k-1})\supset I_{p,k}=\ker q_{p,k}.$$

Hence we have the induced map $\partial_p : C_p(\mathbb{Q}[H])_{(z)}/I_{p,k}\to C_{p-1}(\mathbb{Q}[H])_{(z)}/I_{p-1,k-1}$. This means that we have chain complexes $(C_*(\mathbb{Q}[H])_{(z)}/I_{*,k-*},\partial)$ for $k\in\mathbb{Z}$. This chain complex is bounded since $I_{p,k}=C_p(\mathbb{Q}[H])_{(z)}$ for $k\leq 0$. The natural projection $C_p(\mathbb{Q}[H])_{(z)}/I_{p,k+1}\to C_p(\mathbb{Q}[H])_{(z)}/I_{p,k}$ induces the chain map $(C_*(\mathbb{Q}[H])_{(z)}/I_{*,k+1-*},\partial)\to(C_*(\mathbb{Q}[H])_{(z)}/I_{*,k-*},\partial)$.

We have the center $\mathfrak{z}(\mathbb{Q}[H])=\mathbb{Q}[\ker\mu]$ and the derived Lie subalgebra $[\mathbb{Q}[H],\mathbb{Q}[H]]=\mathbb{Q}[H\setminus\ker\mu]$ [T1]. Hence we have that the projection

$$\varpi:\mathbb{Q}[H]=\mathbb{Q}[\ker\mu]\oplus\mathbb{Q}[H\setminus\ker\mu]\to\mathbb{Q}[H\setminus\ker\mu]$$

is a surjective Lie algebra homomorphism. The inclusion $\mathbb{Q}[H\setminus\ker\mu]\to\mathbb{Q}[H]$ is a section of ϖ. In particular, the induced map $H_p(\varpi):H_p(\mathbb{Q}[H])\to H_p(\mathbb{Q}[H\setminus\ker\mu])$

is a surjective homomorphism. By restriction, we have a surjective homomorphism $H_p(\varpi)_{(v)} : H_p(\mathbb{Q}[H])_{(v)} \to H_p(\mathbb{Q}[H \setminus \ker\mu])_{(v)}$. Hence $H_p(\mathbb{Q}[H])_{(v)} = 0$ implies $H_p(\mathbb{Q}[H \setminus \ker\mu])_{(v)} = 0$.

Proposition 3.3. *If $H_q(\mathbb{Q}[H \setminus \ker\mu])_{(u)} = 0$ for $u \in H \setminus \ker\mu$ and $q < p$, then the homomorphism $H_p(\varpi)_{(v)} : H_p(\mathbb{Q}[H])_{(v)} \to H_p(\mathbb{Q}[H \setminus \ker\mu])_{(v)}$ is an isomorphism for $v \in H \setminus \ker\mu$.*

Proof. By the Künneth formula, we have the isomorphism

$$\begin{aligned} H_p(\mathbb{Q}[H])_{(v)} \to & H_p(\mathbb{Q}[\ker\mu])_{(v)} \oplus H_p(\mathbb{Q}[H \setminus \ker\mu])_{(v)} \\ & \oplus \bigoplus_{q=1}^{p-1} \bigoplus_{u \in H} H_{p-q}(\mathbb{Q}[\ker\mu])_{(v-u)} \otimes H_q(\mathbb{Q}[H \setminus \ker\mu])_{(u)}. \end{aligned}$$

If $u \in \ker\mu, r \in \mathbb{N}$, then we have $H_r(\mathbb{Q}[\ker\mu])_{(v-u)} = 0$ since $v - u \in H \setminus \ker\mu$. If $u \in H \setminus \ker\mu$, then $H_q(\mathbb{Q}[H \setminus \ker\mu])_{(u)} = 0$ for $q < p$ by the assumption of the proposition. Hence we have the isomorphism $H_p(\mathbb{Q}[H])_{(v)} \to H_p(\mathbb{Q}[H \setminus \ker\mu])$. This map is $H_p(\varpi)$. This completes the proof of the proposition. □

We already have $H_q(\mathbb{Q}[H \setminus \ker\mu])_{(u)} = 0$ for $u \in H \setminus \ker\mu$ and $q < 5$ in the next section, then we have $H_5(\varpi)_{(v)} : H_5(\mathbb{Q}[H])_{(v)} \to H_5(\mathbb{Q}[H \setminus \ker\mu])_{(v)}$ is an isomorphism for $v \in H \setminus \ker\mu$.

§ 4. Main theorem

We define a two-cochain $\eta_y \in C^2(\mathbb{Q}[H])$ by $\eta_y([u] \wedge [v]) = \langle u-y, v-y\rangle$ for $u, v \in H$.

Theorem 4.1. *For $p < 5$ and $z \in H \setminus \ker\mu$, we have $H_p(\mathbb{Q}[H])_{(z)} = 0$.*

Proof. Take $y \in H$ with $\langle y, z\rangle \neq 0$. First we prove the theorem for $p = 1, 2$. We already defined $\Phi_1 = \Psi_{1,0} : C_1(\mathbb{Q}[H])_{(z)} = \mathbb{Q}[z] \to C_2(\mathbb{Q}[H])_{(z)}$ by $\Phi_1([z]) = \frac{-1}{\langle y,z\rangle}[y] \wedge [z-y]$. Then we have $\partial_2 \circ \Phi_1 = id_{C_3(\mathbb{Q}[H])_{(z)}}$ by Proposition 3.1 since $I_{1,1} = 0$. We define $\Psi_{2,k} : C_2(\mathbb{Q}[H])_{(z)} \to C_3(\mathbb{Q}[H])_{(z)}$ by

$$\begin{aligned} & \Psi_{2,0}([u_1] \wedge [u_2]) \\ = & -\frac{1}{\langle y, z\rangle}[y] \wedge \big([u_1 - y] \wedge [u_2] + [u_1] \wedge [u_2 - y]\big) \\ & + \frac{1}{2\langle y, z\rangle}[2y] \wedge [u_1] \wedge [u_2] \end{aligned}$$

and

$$\Psi_{2,1}([u_1]\wedge[u_2])=\frac{\eta_y([u_1]\wedge[u_2])}{2\langle y,z\rangle}[y]\wedge[2y]\wedge[z-3y].$$

Then we have $\partial_3\circ\Phi_2+\Phi_1\circ\partial_2=id_{C_2(\mathbb{Q}[H])_{(z)}}$, where we define $\Phi_2=\Psi_{2,0}+\Psi_{2,1}$. This is the proof in the paper [T2].

Next we prove the theorem for $p=3$. We define $\Psi_{3,k}:C_3(\mathbb{Q}[H])_{(z)}\to C_4(\mathbb{Q}[H])_{(z)}$ by

$$\begin{aligned}
&\Psi_{3,0}([u_1]\wedge[u_2]\wedge[u_3])\\
=&-\frac{1}{\langle y,z\rangle}[y]\wedge\big([u_1-y]\wedge[u_2]\wedge[u_3]\\
&\quad+[u_1]\wedge[u_2-y]\wedge[u_3]+[u_1]\wedge[u_2]\wedge[u_3-y]\big)\\
&+\frac{1}{2\langle y,z\rangle}[2y]\wedge\big([u_1-y]\wedge[u_2-y]\wedge[u_3]\\
&\quad+[u_1-y]\wedge[u_2]\wedge[u_3-y]+[u_1]\wedge[u_2-]\wedge[u_3-y]\big)\\
&-\frac{1}{3\langle y,z\rangle}[3y]\wedge[u_1-y]\wedge[u_2-y]\wedge[u_3-y],
\end{aligned}$$

$$\begin{aligned}
&\Psi_{3,1}([u_1]\wedge[u_2]\wedge[u_3])\\
=&\frac{-1}{3\cdot 1\langle y,z\rangle^2 2}\sum_{\sigma\in S_3}(\mathrm{sgn}\sigma)\eta_y([u_{\sigma(1)}]\wedge[u_{\sigma(2)}])\big(\\
&[y]\wedge[3y]\wedge[u_{\sigma(1)}+u_{\sigma(2)}-3y]\wedge[u_{\sigma(3)}-y]\\
&+[y]\wedge[3y]\wedge[u_{\sigma(1)}+u_{\sigma(2)}-2y]\wedge[u_{\sigma(3)}-2y]\big)\\
&+\frac{1}{3\cdot 2\langle y,z\rangle^2 2}\sum_{\sigma\in S_3}(\mathrm{sgn}\sigma)\eta_y([u_{\sigma(1)}]\wedge[u_{\sigma(2)}])\\
&[2y]\wedge[3y]\wedge[u_{\sigma(1)}+u_{\sigma(2)}-3y]\wedge[u_{\sigma(3)}-2y]\\
&+\frac{1}{2\cdot 1\langle y,z\rangle^2 2}\sum_{\sigma\in S_3}(\mathrm{sgn}\sigma)\eta_y([u_{\sigma(1)}]\wedge[u_{\sigma(2)}])\big(\\
&[y]\wedge[2y]\wedge[u_{\sigma(1)}+u_{\sigma(2)}-3y]\wedge[u_{\sigma(3)}]\\
&+[y]\wedge[2y]\wedge[u_{\sigma(1)}+u_{\sigma(2)}-2y]\wedge[u_{\sigma(3)}-y]\\
&+[y]\wedge[2y]\wedge[u_{\sigma(1)}+u_{\sigma(2)}-y]\wedge[u_{\sigma(3)}-2y]\big),
\end{aligned}$$

and

$$\Psi_{3,2}([u_1]\wedge[u_2]\wedge[u_3])$$

$$= \frac{1}{3 \cdot 2\langle y, z\rangle^2} d\eta_y([u_1] \wedge [u_2] \wedge [u_3])[y] \wedge [2y] \wedge [3y] \wedge [z - 6y].$$

We define $\Phi_3 : C_3(\mathbb{Q}[H])_{(z)} \to C_4(\mathbb{Q}[H])_{(z)}$ by $\Phi_3 = \Psi_{3,0} + \Psi_{3,1} + \Psi_{3,2}$. Then we have

(*) $$\partial_4 \circ \Phi_3 + \Phi_2 \circ \partial_3 = id_{C_3(\mathbb{Q}[H])_{(z)}}.$$

Proof of (*). For $u_1, u_2, u_3 \in H$, the value

$$\sum_{\tau \in S_3} \mathrm{sgn}(\tau)\langle u_{\tau(1)} - ay, u_{\tau(2)} - ay\rangle\langle u_{\tau(1)} + u_{\tau(2)} - by, u_{\tau(3)} - cy\rangle$$

often appears at the coefficient. We remark that $\mathrm{sgn}(\tau)\langle u_{\tau(1)} - ay, u_{\tau(2)} - ay\rangle\langle u_{\tau(1)} + u_{\tau(2)} - by, u_{\tau(3)} - cy\rangle$ does not change if we replace τ by $\tau \circ \begin{pmatrix} 1\,2\,3 \\ 2\,1\,3 \end{pmatrix}$. We have

$$\begin{aligned}
&\langle u_{\tau(1)} - ay, u_{\tau(2)} - ay\rangle\langle u_{\tau(1)} + u_{\tau(2)} - by, u_{\tau(3)} - cy\rangle \\
=&(\langle u_{\tau(1)}, u_{\tau(2)}\rangle\langle u_{\tau(2)}, u_{\tau(3)}\rangle - \langle u_{\tau(3)}, u_{\tau(1)}\rangle) \\
&+ ac(\langle y, u_{\tau(1)}\rangle^2 - \langle y, u_{\tau(2)}\rangle^2) + ab(\langle y, u_{\tau(2)}\rangle\langle y, u_{\tau(3)}\rangle - \langle y, u_{\tau(3)}\rangle\langle y, u_{\tau(1)}\rangle) \\
&+ \langle y, u_{\tau(1)}\rangle(c\langle u_{\tau(1)}, u_{\tau(2)}\rangle + a\langle u_{\tau(2)}, u_{\tau(3)}\rangle - a\langle u_{\tau(3)}, u_{\tau(1)}\rangle) \\
&+ \langle y, u_{\tau(2)}\rangle(c\langle u_{\tau(1)}, u_{\tau(2)}\rangle - a\langle u_{\tau(2)}, u_{\tau(3)}\rangle + a\langle u_{\tau(3)}, u_{\tau(1)}\rangle) \\
&+ \langle y, u_{\tau(3)}\rangle(-b\langle u_{\tau(1)}, u_{\tau(2)}\rangle).
\end{aligned}$$

Hence we have the formula

$$\begin{aligned}
&\sum_{\tau \in S_3} \mathrm{sgn}(\tau)\langle u_{\tau(1)} - ay, u_{\tau(2)} - ay\rangle\langle u_{\tau(1)} + u_{\tau(2)} - by, u_{\tau(3)} - cy\rangle \\
=&2\langle y, u_1\rangle((2a - b)\langle u_2, u_3\rangle + (c - a)\langle u_3, u_1\rangle + (c - a)\langle u_1, u_2\rangle) \\
&+ 2\langle y, u_2\rangle((2a - b)\langle u_3, u_1\rangle + (c - a)\langle u_1, u_2\rangle + (c - a)\langle u_2, u_3\rangle) \\
&+ 2\langle y, u_3\rangle((2a - b)\langle u_1, u_2\rangle + (c - a)\langle u_2, u_3\rangle + (c - a)\langle u_3, u_1\rangle).
\end{aligned} \quad (**)$$

In particular, we can calculate $-d\eta_y([u_1] \wedge [u_2] \wedge [u_3])$ in the case $a = 0$ and $b = c = 1$. If $2a - b = -(c - a)$, i.e., $a - b + c = 0$, then the value equals $(2a - b)d\eta_y([u_1] \wedge [u_2] \wedge [u_3])$. In particular, we have $d\eta_y([u_1] \wedge [u_2] \wedge [u_3]) = d\eta_y([u_1 - y] \wedge [u_2 - y] \wedge [u_3 - y])$.

We have

$$\begin{aligned}
&q_{p,2} \circ \partial_4 \circ \Psi_{3,1}([u_1] \wedge [u_2] \wedge [u_3]) \\
= &- \sum_{\sigma \in S_3} \frac{\mathrm{sgn}(\sigma)\eta_y([u_{\sigma(1)}] \wedge [u_{\sigma(2)}])}{3\langle y, z\rangle^2 2} (\\
&\langle y, u_{\sigma(1)} + u_{\sigma(2)}\rangle[u_{\sigma(1)} + u_{\sigma(2)} - 2y] \wedge [3y] \wedge [u_{\sigma(3)} - y]
\end{aligned}$$

$$
\begin{aligned}
&+ \langle y, u_{\sigma(3)}\rangle [u_{\sigma(1)} + u_{\sigma(2)} - 3y] \wedge [3y] \wedge [u_{\sigma(3)}]\\
&- 3\langle y, u_{\sigma(1)} + u_{\sigma(2)}\rangle [u_{\sigma(1)} + u_{\sigma(2)}] \wedge [y] \wedge [u_{\sigma(3)} - y]\\
&- 3\langle y, u_{\sigma(3)}\rangle [u_{\sigma(1)} + u_{\sigma(2)} - 3y] \wedge [y] \wedge [u_{\sigma(3)} + 2y]\\
&+ \langle y, u_{\sigma(1)} + u_{\sigma(2)}\rangle [u_{\sigma(1)} + u_{\sigma(2)} - y] \wedge [3y] \wedge [u_{\sigma(3)} - 2y]\\
&+ \langle y, u_{\sigma(3)}\rangle [u_{\sigma(1)} + u_{\sigma(2)} - 2y] \wedge [3y] \wedge [u_{\sigma(3)} - y]\\
&- 3\langle y, u_{\sigma(1)} + u_{\sigma(2)}\rangle [u_{\sigma(1)} + u_{\sigma(2)} + y] \wedge [y] \wedge [u_{\sigma(3)} - 2y]\\
&- 3\langle y, u_{\sigma(3)}\rangle [u_{\sigma(1)} + u_{\sigma(2)} - 2y] \wedge [y] \wedge [u_{\sigma(3)} + y]\big)\\
&+ \sum_{\sigma\in S_3} \frac{\operatorname{sgn}(\sigma)\eta_y([u_{\sigma(1)}] \wedge [u_{\sigma(2)}])}{3\cdot 2\langle y, z\rangle^2 2}\big(\\
&2\langle y, u_{\sigma(1)} + u_{\sigma(2)}\rangle [u_{\sigma(1)} + u_{\sigma(2)} - y] \wedge [3y] \wedge [u_{\sigma(3)} - 2y]\\
&+ 2\langle y, u_{\sigma(3)}\rangle [u_{\sigma(1)} + u_{\sigma(2)} - 3y] \wedge [3y] \wedge [u_{\sigma(3)}]\\
&- 3\langle y, u_{\sigma(1)} + u_{\sigma(2)}\rangle [u_{\sigma(1)} + u_{\sigma(2)}] \wedge [2y] \wedge [u_{\sigma(3)} - 2y]\\
&- 3\langle y, u_{\sigma(3)}\rangle [u_{\sigma(1)} + u_{\sigma(2)} - 3y] \wedge [2y] \wedge [u_{\sigma(3)} + y]\big)\\
&+ \sum_{\sigma\in S_3} \frac{\operatorname{sgn}(\sigma)\eta_y([u_{\sigma(1)}] \wedge [u_{\sigma(2)}])}{2\cdot 1\langle y, z\rangle^2 2}\big(\\
&\langle y, u_{\sigma(1)} + u_{\sigma(2)}\rangle [u_{\sigma(1)} + u_{\sigma(2)} - 2y] \wedge [2y] \wedge [u_{\sigma(3)}]\\
&+ \langle y, u_{\sigma(3)}\rangle [u_{\sigma(1)} + u_{\sigma(2)} - 3y] \wedge [2y] \wedge [u_{\sigma(3)} + y]\\
&- 2\langle y, u_{\sigma(1)} + u_{\sigma(2)}\rangle [u_{\sigma(1)} + u_{\sigma(2)} - y] \wedge [y] \wedge [u_{\sigma(3)}]\\
&- 2\langle y, u_{\sigma(3)}\rangle [u_{\sigma(1)} + u_{\sigma(2)} - 3y] \wedge [y] \wedge [u_{\sigma(3)} + 2y]\\
&+ \langle y, u_{\sigma(1)} + u_{\sigma(2)}\rangle [u_{\sigma(1)} + u_{\sigma(2)} - y] \wedge [2y] \wedge [u_{\sigma(3)} - y]\\
&+ \langle y, u_{\sigma(3)}\rangle [u_{\sigma(1)} + u_{\sigma(2)} - 2y] \wedge [2y] \wedge [u_{\sigma(3)}]\\
&- 2\langle y, u_{\sigma(1)} + u_{\sigma(2)}\rangle [u_{\sigma(1)} + u_{\sigma(2)}] \wedge [y] \wedge [u_{\sigma(3)} - y]\\
&- 2\langle y, u_{\sigma(3)}\rangle [u_{\sigma(1)} + u_{\sigma(2)} - 2y] \wedge [y] \wedge [u_{\sigma(3)} + y]\\
&+ \langle y, u_{\sigma(1)} + u_{\sigma(2)}\rangle [u_{\sigma(1)} + u_{\sigma(2)}] \wedge [2y] \wedge [u_{\sigma(3)} - 2y]\\
&+ \langle y, u_{\sigma(3)}\rangle [u_{\sigma(1)} + u_{\sigma(2)} - y] \wedge [2y] \wedge [u_{\sigma(3)} - y]\\
&- 2\langle y, u_{\sigma(1)} + u_{\sigma(2)}\rangle [u_{\sigma(1)} + u_{\sigma(2)} + y] \wedge [y] \wedge [u_{\sigma(3)} - 2y]\\
&- 2\langle y, u_{\sigma(3)}\rangle [u_{\sigma(1)} + u_{\sigma(2)} - y] \wedge [y] \wedge [u_{\sigma(3)}]\big)\\
=& - \sum_{\sigma\in S_3} \frac{\operatorname{sgn}(\sigma)\eta_u([u_{\sigma(1)}] \wedge [u_{\sigma(2)}])}{3\langle y, z\rangle 2}[u_{\sigma(1)} + u_{\sigma(2)} - 2y] \wedge [3y] \wedge [u_{\sigma(3)} - y]\\
&+ \sum_{\sigma\in S_3} \frac{\operatorname{sgn}(\sigma)\eta_u([u_{\sigma(1)}] \wedge [u_{\sigma(2)}])}{2\langle y, z\rangle 2}\big(\\
&[u_{\sigma(1)} + u_{\sigma(2)} - 2y] \wedge [2y] \wedge [u_{\sigma(3)}] + [u_{\sigma(1)} + u_{\sigma(2)} - 1y] \wedge [2y] \wedge [u_{\sigma(3)} - y]\big)
\end{aligned}
$$

$$- \sum_{\sigma \in S} \frac{\mathrm{sgn}(\sigma)\eta_u([u_{\sigma(1)}] \wedge [u_{\sigma(2)}])}{\langle y, z\rangle 2}[u_{\sigma(1)} + u_{\sigma(2)} - y] \wedge [y] \wedge [u_{\sigma(3)}]$$
$$= - q_{3,2} \circ (\partial_4 \circ \Phi_{3,0} + \Phi_{2,0} \circ \partial_3 - id)([u_1] \wedge [u_2] \wedge [u_2]).$$

The last equation follows from Proposition 3.2. If we calculate the ignored terms by the formula (**), we have

$$(\partial_4 \circ \Phi_{3,0} + \Phi_{2,0} \circ \partial_3 - id + \partial_4 \circ \Psi_{3,1})([u_1] \wedge [u_2] \wedge [u_2])$$
$$= \sum_{\sigma \in S_3} \frac{\eta_y([u_{\sigma(1)}] \wedge [u_{\sigma(2)}])\langle u_{\sigma(1)} + u_{\sigma(2)} - 3y, u_{\sigma(3)} - y\rangle}{3 \cdot 1\mathrm{sgn}(\sigma)\langle y, z\rangle^2 2}[y] \wedge [3y] \wedge [z - 4y]$$
$$+ \sum_{\sigma \in S_3} \frac{\eta_y([u_{\sigma(1)}] \wedge [u_{\sigma(2)}])\langle u_{\sigma(1)} + u_{\sigma(2)} - 2y, u_{\sigma(3)} - 2y\rangle}{3 \cdot 1\mathrm{sgn}(\sigma)\langle y, z\rangle^2 2}[y] \wedge [3y] \wedge [z - 4y]$$
$$- \sum_{\sigma \in S_3} \frac{\eta_y([u_{\sigma(1)}] \wedge [u_{\sigma(2)}])\langle u_{\sigma(1)} + u_{\sigma(2)} - 3y, u_{\sigma(3)} - 2y\rangle}{3 \cdot 2\mathrm{sgn}(\sigma)\langle y, z\rangle^2 2}[2y] \wedge [3y] \wedge [z - 4y]$$
$$- \sum_{\sigma \in S_3} \frac{\eta_y([u_{\sigma(1)}] \wedge [u_{\sigma(2)}])\langle u_{\sigma(1)} + u_{\sigma(2)} - 3y, u_{\sigma(3)}\rangle}{2 \cdot 1\mathrm{sgn}(\sigma)\langle y, z\rangle^2 2}[y] \wedge [2y] \wedge [z - 4y]$$
$$- \sum_{\sigma \in S_3} \frac{\eta_y([u_{\sigma(1)}] \wedge [u_{\sigma(2)}])\langle u_{\sigma(1)} + u_{\sigma(2)} - 2y, u_{\sigma(3)} - y\rangle}{2 \cdot 1\mathrm{sgn}(\sigma)\langle y, z\rangle^2 2}[y] \wedge [2y] \wedge [z - 4y]$$
$$- \sum_{\sigma \in S_3} \frac{\eta_y([u_{\sigma(1)}] \wedge [u_{\sigma(2)}])\langle u_{\sigma(1)} + u_{\sigma(2)} - y, u_{\sigma(3)} - 2y\rangle}{2 \cdot 1\mathrm{sgn}(\sigma)\langle y, z\rangle^2 2}[y] \wedge [2y] \wedge [z - 4y]$$
$$= \frac{d\eta_y([u_1] \wedge [u_2] \wedge [u_2])}{\langle y, z\rangle}(-\frac{1}{3}[y] \wedge [3y] \wedge [z - 4y] + \frac{1}{6}[2y] \wedge [3y] \wedge [z - 5y]).$$

On the other hand, we have

$$\Psi_{2,1} \circ \partial_3([u_1] \wedge [u_2] \wedge [u_3])$$
$$= \Psi_{2,1}\left(\frac{-1}{2!1!}\sum_{\tau \in S_3} \mathrm{sgn}(\tau)\langle u_{\tau(1)}, u_{\tau(2)}\rangle [u_{\tau(1)} + u_{\tau(2)}] \wedge [u_{\tau(3)}]\right)$$
$$= - \sum_{\tau \in S_3} \frac{\langle u_{\tau(1)}, u_{\tau(2)}\rangle \eta_y([u_{\tau(1)} + u_{\tau(2)}] \wedge [u_{\tau(3)}])}{2\mathrm{sgn}(\tau)\langle y, z\rangle 2}[y] \wedge [2y] \wedge [z - 3y]$$
$$= \frac{d\eta_y([u_1] \wedge [u_2] \wedge [u_3])}{2\langle y, z\rangle}[y] \wedge [2y] \wedge [z - 3y].$$

Hence we have

$$(\partial_4 \circ \Phi_{3,1} + \Phi_{2,1} \circ \partial_3 - id)([u_1] \wedge [u_2] \wedge [u_2])$$
$$= \frac{d\eta_y([u_1] \wedge [u_2] \wedge [u_2])}{\langle y, z\rangle}\big(\frac{1}{2}[y] \wedge [2y] \wedge [z - 3y]$$

$$
\begin{aligned}
&-\frac{1}{3}[y]\wedge[3y]\wedge[z-4y]+\frac{1}{6}[2y]\wedge[3y]\wedge[z-5y]\big)\\
=&-\frac{d\eta_y([u_1]\wedge[u_2]\wedge[u_2])}{3\cdot 2\cdot 1\langle y,z\rangle^2}\partial_4([y]\wedge[2y]\wedge[3y]\wedge[z-6y])\\
=&-\partial_4\circ\Psi_{3,2}([u_1]\wedge[u_2]\wedge[u_2]).
\end{aligned}
$$

This is the condition (2) at page 3 for $p=3$ and $k=2$. Therefore, we have $\partial_4\circ\Phi_3+\Phi_2\circ\partial_3=id_{C_3(\mathbb{Q}[H])_{(z)}}$. □

The proposition follows from this when $p=3$.

Last we prove the theorem for $p=4$. Define $\Phi_4 : C_4(\mathbb{Q}[H])_{(z)}\to C_5(\mathbb{Q}[H])_{(z)}$ by $\Phi_4=\Psi_{4,0}+\Psi_{4,1}+\Psi_{4,2}+\Psi_{4,3}$, where we define

$$
\begin{aligned}
&\Psi_{4,1}([u_1]\wedge[u_2]\wedge[u_3]\wedge[u_4])\\
=&\frac{1}{4\cdot 1\langle y,z\rangle^2}\sum_{\sigma\in S_{2,0,2}}(\mathrm{sgn}\sigma)\eta_y([u_{\sigma(1)}]\wedge[u_{\sigma(2)}])\\
&[y]\wedge[4y]\wedge[u_{\sigma(1)}+u_{\sigma(2)}-3y]\wedge[u_{\sigma(3)}-y]\wedge[u_{\sigma(4)}-y]\\
&-\frac{1}{4\cdot 2\langle y,z\rangle^2}\sum_{\sigma\in S_{2,1,1}}(\mathrm{sgn}\sigma)\eta_y([u_{\sigma(1)}]\wedge[u_{\sigma(2)}])\\
&[2y]\wedge[4y]\wedge[u_{\sigma(1)}+u_{\sigma(2)}-3y]\wedge[u_{\sigma(3)}-2y]\wedge[u_{\sigma(4)}-y]\\
&+\frac{1}{4\cdot 3\langle y,z\rangle^2}\sum_{\sigma\in S_{2,2,0}}(\mathrm{sgn}\sigma)\eta_y([u_{\sigma(1)}]\wedge[u_{\sigma(2)}])\\
&[3y]\wedge[4y]\wedge[u_{\sigma(1)}+u_{\sigma(2)}-3y]\wedge[u_{\sigma(3)}-2y]\wedge[u_{\sigma(4)}-2y]\\
&+\frac{1}{4\cdot 1\langle y,z\rangle^2}\sum_{\sigma\in S_{2,1,1}}(\mathrm{sgn}\sigma)\eta_y([u_{\sigma(1)}]\wedge[u_{\sigma(2)}])\\
&[y]\wedge[4y]\wedge[u_{\sigma(1)}+u_{\sigma(2)}-2y]\wedge[u_{\sigma(3)}-2y]\wedge[u_{\sigma(4)}-y]\\
&-\frac{1}{4\cdot 2\langle y,z\rangle^2}\sum_{\sigma\in S_{2,2,0}}(\mathrm{sgn}\sigma)\eta_y([u_{\sigma(1)}]\wedge[u_{\sigma(2)}])\\
&[2y]\wedge[4y]\wedge[u_{\sigma(1)}+u_{\sigma(2)}-2y]\wedge[u_{\sigma(3)}-2y]\wedge[u_{\sigma(4)}-2y]\\
&-\frac{1}{3\cdot 1\langle y,z\rangle^2}\sum_{\sigma\in S_{2,1,1}}(\mathrm{sgn}\sigma)\eta_y([u_{\sigma(1)}]\wedge[u_{\sigma(2)}])\\
&[y]\wedge[3y]\wedge[u_{\sigma(1)}+u_{\sigma(2)}-3y]\wedge[u_{\sigma(3)}-y]\wedge[u_{\sigma(4)}]\\
&-\frac{1}{3\cdot 1\langle y,z\rangle^2}\sum_{\sigma\in S_{2,1,1}}(\mathrm{sgn}\sigma)\eta_y([u_{\sigma(1)}]\wedge[u_{\sigma(2)}])\\
&[y]\wedge[3y]\wedge[u_{\sigma(1)}+u_{\sigma(2)}-2y]\wedge[u_{\sigma(3)}-2y]\wedge[u_{\sigma(4)}]\\
&-\frac{2}{3\cdot 1\langle y,z\rangle^2}\sum_{\sigma\in S_{2,2}}(\mathrm{sgn}\sigma)\eta_y([u_{\sigma(1)}]\wedge[u_{\sigma(2)}])
\end{aligned}
$$

$$
\begin{aligned}
&[y]\wedge[3y]\wedge[u_{\sigma(1)}+u_{\sigma(2)}-2y]\wedge[u_{\sigma(3)}-y]\wedge[u_{\sigma(4)}-y]\\
&-\frac{1}{3\cdot 1\langle y,z\rangle^2}\sum_{\sigma\in S_{2,1,1}}(\mathrm{sgn}\sigma)\eta_y([u_{\sigma(1)}]\wedge[u_{\sigma(2)}])\\
&[y]\wedge[3y]\wedge[u_{\sigma(1)}+u_{\sigma(2)}-y]\wedge[u_{\sigma(3)}-2y]\wedge[u_{\sigma(4)}-y]\\
&+\frac{1}{3\cdot 2\langle y,z\rangle^2}\sum_{\sigma\in S_{2,1,1}}(\mathrm{sgn}\sigma)\eta_y([u_{\sigma(1)}]\wedge[u_{\sigma(2)}])\\
&[2y]\wedge[3y]\wedge[u_{\sigma(1)}+u_{\sigma(2)}-3y]\wedge[u_{\sigma(3)}-2y]\wedge[u_{\sigma(4)}]\\
&+\frac{1}{3\cdot 2\langle y,z\rangle^2}\sum_{\sigma\in S_{2,2}}(\mathrm{sgn}\sigma)\eta_y([u_{\sigma(1)}]\wedge[u_{\sigma(2)}])\\
&[2y]\wedge[3y]\wedge[u_{\sigma(1)}+u_{\sigma(2)}-3y]\wedge[u_{\sigma(3)}-y]\wedge[u_{\sigma(4)}-y]\\
&+\frac{1}{3\cdot 2\langle y,z\rangle^2}\sum_{\sigma\in S_{2,1,1}}(\mathrm{sgn}\sigma)\eta_y([u_{\sigma(1)}]\wedge[u_{\sigma(2)}])\\
&[2y]\wedge[3y]\wedge[u_{\sigma(1)}+u_{\sigma(2)}-2y]\wedge[u_{\sigma(3)}-2y]\wedge[u_{\sigma(4)}-y]\\
&+\frac{1}{3\cdot 2\langle y,z\rangle^2}\sum_{\sigma\in S_{2,1,1}}(\mathrm{sgn}\sigma)\eta_y([u_{\sigma(1)}]\wedge[u_{\sigma(2)}])\\
&[2y]\wedge[3y]\wedge[u_{\sigma(1)}+u_{\sigma(2)}-y]\wedge[u_{\sigma(3)}-2y]\wedge[u_{\sigma(4)}-2y]\\
&+\frac{1}{2\cdot 1\langle y,z\rangle^2}\sum_{\sigma\in S_{2,2}}(\mathrm{sgn}\sigma)\eta_y([u_{\sigma(1)}]\wedge[u_{\sigma(2)}])\\
&[y]\wedge[2y]\wedge[u_{\sigma(1)}+u_{\sigma(2)}-3y]\wedge[u_{\sigma(3)}]\wedge[u_{\sigma(4)}]\\
&+\frac{1}{2\cdot 1\langle y,z\rangle^2}\sum_{\sigma\in S_{2,1,1}}(\mathrm{sgn}\sigma)\eta_y([u_{\sigma(1)}]\wedge[u_{\sigma(2)}])\\
&[y]\wedge[2y]\wedge[u_{\sigma(1)}+u_{\sigma(2)}-2y]\wedge[u_{\sigma(3)}-y]\wedge[u_{\sigma(4)}]\\
&+\frac{1}{2\cdot 1\langle y,z\rangle^2}\sum_{\sigma\in S_{2,1,1}}(\mathrm{sgn}\sigma)\eta_y([u_{\sigma(1)}]\wedge[u_{\sigma(2)}])\\
&[y]\wedge[2y]\wedge[u_{\sigma(1)}+u_{\sigma(2)}-y]\wedge[u_{\sigma(3)}-2y]\wedge[u_{\sigma(4)}]\\
&+\frac{1}{2\cdot 1\langle y,z\rangle^2}\sum_{\sigma\in S_{2,2}}(\mathrm{sgn}\sigma)\eta_y([u_{\sigma(1)}]\wedge[u_{\sigma(2)}])\\
&[y]\wedge[2y]\wedge[u_{\sigma(1)}+u_{\sigma(2)}-y]\wedge[u_{\sigma(3)}-y]\wedge[u_{\sigma(4)}-y],
\end{aligned}
$$

$$
\begin{aligned}
&\Psi_{4,2}([u_1]\wedge[u_2]\wedge[u_3]\wedge[u_4])\\
=&\frac{1}{4\cdot 3\cdot 1\langle y,z\rangle^3}\sum_{\sigma\in S_{3,1}}(\mathrm{sgn}\sigma)d\eta_y([u_{\sigma(1)}]\wedge[u_{\sigma(2)}]\wedge[u_{\sigma(3)}])\big(\\
&[y]\wedge[3y]\wedge[4y]\wedge[u_{\sigma(1)}+u_{\sigma(2)}+u_{\sigma(3)}-6y]\wedge[u_{\sigma(4)}-2y]
\end{aligned}
$$

$$
\begin{aligned}
&+ [y] \wedge [3y] \wedge [4y] \wedge [u_{\sigma(1)} + u_{\sigma(2)} + u_{\sigma(3)} - 5y] \wedge [u_{\sigma(4)} - 3y]\big) \\
&- \frac{1}{4 \cdot 3 \cdot 2 \langle y, z\rangle^3} \sum_{\sigma \in S_{3,1}} (\mathrm{sgn}\sigma) d\eta_y([u_{\sigma(1)}] \wedge [u_{\sigma(2)}] \wedge [u_{\sigma(3)}])\big(\\
&[2y] \wedge [3y] \wedge [4y] \wedge [u_{\sigma(1)} + u_{\sigma(2)} + u_{\sigma(3)} - 6y] \wedge [u_{\sigma(4)} - 3y] \\
&+ \frac{1}{4 \cdot 3 \cdot 1 \langle y, z\rangle^3} \sum_{\sigma \in S_{2,2}} (\mathrm{sgn}\sigma) \eta_y([u_{\sigma(1)}] \wedge [u_{\sigma(2)}]) \eta_{2y}([u_{\sigma(3)}] \wedge [u_{\sigma(4)}])\big(\\
&[y] \wedge [3y] \wedge [4y] \wedge [u_{\sigma(1)} + u_{\sigma(2)} - 3y] \wedge [u_{\sigma(3)} + u_{\sigma(4)} - 5y] \\
&+ [y] \wedge [3y] \wedge [4y] \wedge [u_{\sigma(1)} + u_{\sigma(2)} - 4y] \wedge [u_{\sigma(3)} + u_{\sigma(4)} - 4y]\big) \\
&- \frac{1}{4 \cdot 3 \cdot 2 \langle y, z\rangle^3} \sum_{\sigma \in S_{2,2}} (\mathrm{sgn}\sigma) \eta_y([u_{\sigma(1)}] \wedge [u_{\sigma(2)}]) \eta_{2y}([u_{\sigma(3)}] \wedge [u_{\sigma(4)}]) \\
&[2y] \wedge [3y] \wedge [4y] \wedge [u_{\sigma(1)} + u_{\sigma(2)} - 4y] \wedge [u_{\sigma(3)} + u_{\sigma(4)} - 5y] \\
&- \frac{1}{4 \cdot 2 \cdot 1 \langle y, z\rangle^3} \sum_{\sigma \in S_{3,1}} (\mathrm{sgn}\sigma) d\eta_y([u_{\sigma(1)}] \wedge [u_{\sigma(2)}] \wedge [u_{\sigma(3)}])\big(\\
&[y] \wedge [2y] \wedge [4y] \wedge [u_{\sigma(1)} + u_{\sigma(2)} + u_{\sigma(3)} - 6y] \wedge [u_{\sigma(4)} - y] \\
&+ [y] \wedge [2y] \wedge [4y] \wedge [u_{\sigma(1)} + u_{\sigma(2)} + u_{\sigma(3)} - 5y] \wedge [u_{\sigma(4)} - 2y] \\
&+ [y] \wedge [2y] \wedge [4y] \wedge [u_{\sigma(1)} + u_{\sigma(2)} + u_{\sigma(3)} - 4y] \wedge [u_{\sigma(4)} - 3y]\big) \\
&- \frac{1}{4 \cdot 2 \cdot 1 \langle y, z\rangle^3} \sum_{\sigma \in S_{2,2}} (\mathrm{sgn}\sigma) \eta_y([u_{\sigma(1)}] \wedge [u_{\sigma(2)}]) \eta_{2y}([u_{\sigma(3)}] \wedge [u_{\sigma(4)}])\big(\\
&[y] \wedge [2y] \wedge [4y] \wedge [u_{\sigma(1)} + u_{\sigma(2)} - 4y] \wedge [u_{\sigma(3)} + u_{\sigma(4)} - 3y] \\
&+ [y] \wedge [2y] \wedge [4y] \wedge [u_{\sigma(1)} + u_{\sigma(2)} - 3y] \wedge [u_{\sigma(3)} + u_{\sigma(4)} - 4y] \\
&+ [y] \wedge [2y] \wedge [4y] \wedge [u_{\sigma(1)} + u_{\sigma(2)} - 2y] \wedge [u_{\sigma(3)} + u_{\sigma(4)} - 5y]\big) \\
&+ \frac{1}{3 \cdot 2 \cdot 1 \langle y, z\rangle^3} \sum_{\sigma \in S_{3,1}} (\mathrm{sgn}\sigma) d\eta_y([u_{\sigma(1)}] \wedge [u_{\sigma(2)}] \wedge [u_{\sigma(3)}])\big(\\
&[y] \wedge [2y] \wedge [3y] \wedge [u_{\sigma(1)} + u_{\sigma(2)} + u_{\sigma(3)} - 6y] \wedge [u_{\sigma(4)}] \\
&+ [y] \wedge [2y] \wedge [3y] \wedge [u_{\sigma(1)} + u_{\sigma(2)} + u_{\sigma(3)} - 5y] \wedge [u_{\sigma(4)} - y] \\
&+ [y] \wedge [2y] \wedge [3y] \wedge [u_{\sigma(1)} + u_{\sigma(2)} + u_{\sigma(3)} - 4y] \wedge [u_{\sigma(4)} - 2y] \\
&+ [y] \wedge [2y] \wedge [3y] \wedge [u_{\sigma(1)} + u_{\sigma(2)} + u_{\sigma(3)} - 3y] \wedge [u_{\sigma(4)} - 3y]\big) \\
&+ \frac{1}{3 \cdot 2 \cdot 1 \langle y, z\rangle^3} \sum_{\sigma \in S_{2,2}} (\mathrm{sgn}\sigma) \eta_y([u_{\sigma(1)}] \wedge [u_{\sigma(2)}]) \eta_{2y}([u_{\sigma(3)}] \wedge [u_{\sigma(4)}])\big(\\
&[y] \wedge [2y] \wedge [3y] \wedge [u_{\sigma(1)} + u_{\sigma(2)} - 4y] \wedge [u_{\sigma(3)} + u_{\sigma(4)} - 2y] \\
&[y] \wedge [2y] \wedge [3y] \wedge [u_{\sigma(1)} + u_{\sigma(2)} - 3y] \wedge [u_{\sigma(3)} + u_{\sigma(4)} - 3y] \\
&[y] \wedge [2y] \wedge [3y] \wedge [u_{\sigma(1)} + u_{\sigma(2)} - 2y] \wedge [u_{\sigma(3)} + u_{\sigma(4)} - 4y]
\end{aligned}
$$

$$[y] \wedge [2y] \wedge [3y] \wedge [u_{\sigma(1)} + u_{\sigma(2)} - y] \wedge [u_{\sigma(3)} + u_{\sigma(4)} - 5y]\big),$$

and

$$\begin{aligned}
&\Psi_{4,3}([u_1] \wedge [u_2] \wedge [u_3] \wedge [u_4]) \\
= & -\frac{1}{3 \cdot 2 \cdot 1 \langle y, z\rangle^4} \sum_{\sigma \in S_{3,1}} (\mathrm{sgn}\sigma) \langle u_{\sigma(1)} + u_{\sigma(2)} + u_{\sigma(3)} - 6y, u_{\sigma(4)} - 3y\rangle \\
& d\eta_y([u_{\sigma(1)}] \wedge [u_{\sigma(2)}] \wedge [u_{\sigma(3)}])[y] \wedge [2y] \wedge [3y] \wedge [4y] \wedge [z - 10y] \\
& -\frac{1}{3 \cdot 2 \cdot 1 \langle y, z\rangle^4} \sum_{\sigma \in S_{2,2}} (\mathrm{sgn}\sigma) \langle u_{\sigma(1)} + u_{\sigma(2)} - 4y, u_{\sigma(3)} + u_{\sigma(4)} - 5y\rangle \\
& \eta_y([u_{\sigma(1)}] \wedge [u_{\sigma(2)}])\eta_{2y}([u_{\sigma(3)}] \wedge [u_{\sigma(4)}])[y] \wedge [2y] \wedge [3y] \wedge [4y] \wedge [z - 10y] \\
& (=0).
\end{aligned}$$

The way we check the condition (2) for $p = 4$ and $k \leq 2$ is similar to that for $p = 3$. When we check the condition (2) for $p = 3$ and $k = 3$, the following values appear in our computation. For $u_1, u_2, u_3, u_4 \in H$,

$$\begin{aligned}
& \sum_{\tau \in S_{3,1}} \mathrm{sgn}(\tau) d\eta_y([u_{\tau(1)}] \wedge [u_{\tau(2)}] \wedge [u_{\tau(3)}]) \\
& \qquad \langle u_{\tau(1)} + u_{\tau(2)} + u_{\tau(3)} - ay, u_{\tau(4)} - by\rangle \\
= & \sum_{\sigma \in S_{1,1,2}} \mathrm{sgn}(\sigma) \langle y, u_{\sigma(1)}\rangle \langle u_{\sigma(1)}, u_{\sigma(2)}\rangle \\
& \qquad (\langle u_{\sigma(2)}, u_{\sigma(3)}\rangle - \langle u_{\sigma(2)}, u_{\sigma(4)}\rangle - \langle u_{\sigma(3)}, u_{\sigma(4)}\rangle) \\
& + b \sum_{\sigma \in S_{1,3}} \mathrm{sgn}(\sigma) \langle y, u_{\sigma(1)}\rangle^2 (\langle u_{\sigma(2)}, u_{\sigma(3)}\rangle + \langle u_{\sigma(3)}, u_{\sigma(4)}\rangle + \langle u_{\sigma(4)}, u_{\sigma(2)}\rangle) \\
& + a \sum_{\sigma \in S_{2,2}} \mathrm{sgn}(\sigma) \langle y, u_{\sigma(1)}\rangle \langle y, u_{\sigma(2)}\rangle \langle u_{\sigma(1)} - u_{\sigma(2)}, u_{\sigma(3)} - u_{\sigma(4)}\rangle
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{\tau \in S_{2,2}} \mathrm{sgn}(\tau) \eta_y([u_{\tau(1)}] \wedge [u_{\tau(2)}]) \eta_{2y}([u_{\tau(3)}] \wedge [u_{\tau(4)}]) \\
& \qquad \langle u_{\tau(1)} + u_{\tau(2)} - a'y, u_{\tau(3)} + u_{\tau(4)} - b'y\rangle \\
= & -\sum_{\sigma \in S_{1,1,2}} \mathrm{sgn}(\sigma) \langle y, u_{\sigma(1)}\rangle \langle u_{\sigma(1)}, u_{\sigma(2)}\rangle \\
& \qquad (\langle u_{\sigma(2)}, u_{\sigma(3)}\rangle - \langle u_{\sigma(2)}, u_{\sigma(4)}\rangle + (a' - b')\langle u_{\sigma(3)}, u_{\sigma(4)}\rangle) \\
& + (b' - 2a') \sum_{\sigma \in S_{1,3}} \mathrm{sgn}(\sigma) \langle y, u_{\sigma(1)}\rangle^2
\end{aligned}$$

$$(\langle u_{\sigma(2)}, u_{\sigma(3)}\rangle + \langle u_{\sigma(3)}, u_{\sigma(4)}\rangle + \langle u_{\sigma(4)}, u_{\sigma(2)}\rangle)$$
$$+(a'-2b')\sum_{\sigma\in S_{2,2}} \mathrm{sgn}(\sigma)\langle y, u_{\sigma(1)}\rangle\langle y, u_{\sigma(2)}\rangle\langle u_{\sigma(1)}-u_{\sigma(2)}, u_{\sigma(3)}-u_{\sigma(4)}\rangle$$

Then we have $\partial_5 \circ \Phi_4 + \Phi_3 \circ \partial_4 = id_{C_4(\mathbb{Q}[H])_{(z)}}$. Hence we have $H_4(\mathbb{Q}[H])_{(z)} = 0$. □

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