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Variational formulations of the Funk and Apollonian weak metrics on convex sets

By

Sumio YAMADA*

Abstract

On a convex set in a Euclidean space, we consider two asymmetric distance functions, called the Funk and Apollonian weak metrics. We show that both asymmetric metrics have variational characterizations, which in turn tell that the straight lines are the Funk geodesics while the circular arcs are the geodesics for the Apollonian weak metric which meet the boundary perpendicularly. If the convex set is a disc, the metrics lead respectively to the Beltrami-Klein model and the Poincaré model of the hyperbolic plane when the metrics are arithmetically symmetrized. We will show that they have Finsler structures, determined by Minkowski functionals defined on each tangent space.

§ 1. Introduction

We will consider two metrics defined on a convex set Ω, the Funk weak metric and the Apollonian weak metric, both of which are asymmetric. When the convex set Ω is the unit ball in $\mathbb{R}^2$, their symmetrizations are both isometric to the hyperbolic metric, the former identifiable to the Beltrami-Klein model, the latter to the Poincaré model. The geodesics for the former model are the straight line segments connecting a pair of points on the unit circle, and these are also geodesics for the Funk metric. The geodesics for the latter model are the circular arcs which meet the unit circle perpendicularly, and these are geodesics for the Apollonian weak metric. In this sense, even though these two weak metrics are not conformally invariant, they both capture a certain aspect of the hyperbolic geometry realized in the two well-known models.

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Supported in part by the JSPS Grant-in-aid for Scientific Research No.24340009.
*Department of Mathematics, Gakushuin University, Tokyo 171–8588, Japan.
e-mail: `yamada@math.gakushuin.ac.jp`

The goal of this article is to introduce variational formulations of the Funk and Apollonian weak metrics, and to contrast them. In particular, we note that each is formulated as a supremum of logarithmic ratios of lengths of line segments, indexed by the set of supporting hyperplanes of the given convex set. From these variational formulations, one can characterize the distance-realizing curves for the two weak metrics. We also investigate the infinitesimal structure of the Apollonian weak metric, and identify the Minkowski functional providing the metric with a Finsler structure when the metric space is geodesic.

For historical reasons, and in order to be consistent with the existing literature, we use the term "metric" in place of distance function; it does *not* refer to a Riemannian metric. To be more precise, in this article we define a metric on a set X to be a function $\delta : X \times X \to \mathbf{R}_+ \cup \{\infty\}$ satisfying:

1. $\delta(x, x) = 0$ for all x in X,
2. $\delta(x, z) \leq \delta(x, y) + \delta(y, z)$ for all x, y and z in X.

In [11], for example, a function satisfying the above is named *weak metric.* Note that in this definition neither the symmetry $\delta(x, y) = \delta(y, x)$ nor the non-degeneracy $\delta(x, y) = 0 \Rightarrow x = y$ is assumed. A weak metric which is also symmetric, but possibly degenerate, is called a *semi-metric.*

A systematic treatment of the geometry of weak metrics is given in the papers by Papadopoulos-Troyanov [11, 12].

In this article a path $s : [a, b] \to (X, d)$ in a metric space (X, δ) is said to be geodesic when for any $a < t < b$, $\delta(s(a), s(t)) + \delta(s(t), s(b)) = \delta(s(a), s(b))$ is satisfied.

We represent the convex set Ω in $\mathbb{R}^n$ as $\cap_{\pi(b)\in\mathcal{P}} H_{\pi(b)}$ where $H_{\pi(b)}$ is the half space bounded by a supporting hyperplane $\pi(b)$ of Ω at the boundary point b, containing the convex set Ω. The index set $\mathcal{P}$ is the set of all supporting hyperplanes of Ω. That for every boundary point p there exists a supporting hyperplane $\pi(b)$ follows from the definition of the convexity of Ω. In general, there can be more than one supporting hyperplane of Ω at $p \in \partial\Omega$. The index set $\mathcal{P}$ is identified with the set of unit normal vectors to the supporting hyperplanes. It is identified with a subset of S^{n-1}, which is equal to the entire sphere when the convex set is bounded. We denote by $\mathcal{P}(b)$ the set of supporting hyperplanes at $b \in \partial\Omega$.

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§ 2. The Funk metric

We start with a collection of facts known about the Funk metric, which will be contrasted with the geometry of the Apollonian weak metric. Let $d(x, y)$ be the Euclidean distance between the pair of points x and y in Ω. We first define the Funk metric as it first appeared in [4], a weak metric defined on a convex set $\Omega \subset \mathbb{R}^n$.

Definition For a pair of distinct points x and y in Ω, the Funk asymmetric metric is defined by

$$F(x,y) = \log \frac{d(x, b(x,y))}{d(y, b(x,y))}$$

where the point $b(x,y)$ is the intersection of the boundary $\partial\Omega$ and the ray $R(x,y) := \{x + t\xi_{xy} \mid t > 0\}$ from x though y. Here ξ_{xy} is the unit vector along the ray. We define $F(x,x)$ to be 0.

Despite the fact that the Funk metric is not well known in general, we note that the idea of a Funk-type metric is ubiquitous. For example, on a Teichmüller space, the space of conformal structures defined on a topological surface, there are three different Funk-type metrics: the Teichmüller metric [10, 3], Thurston's asymmetric metric [13] and the most recently defined Weil-Petersson-Funk metric [14].

The Funk metric defined on a convex set in $\mathbb{R}^n$ is known to be Finsler, and its Minkowski functional can be written down [11, 14] in terms of the shape of the boundary Ω.

Now let π_0 be a supporting hyperplane at $b(x,y)$, namely $\pi_0 \in \mathcal{P}(b(x,y))$. Then note the similarity of the triangles $\triangle(x, \Pi_{\pi_0}(x), b(x,y))$ and $\triangle(y, \Pi_{\pi_0}(y), b(x,y))$, where $\Pi_{\pi_0}(p)$ is the foot of the point p on the hyperplane π_0, or in other words, $\Pi_{\pi_0} : \mathbf{R}^d \to \pi_0$ is the nearest point projection map. This says that

$$\log \frac{d(x, b(x,y))}{d(y, b(x,y))} = \log \frac{d(x, \pi_0)}{d(y, \pi_0)}.$$

Note that by the similarity argument of triangles, the right hand side of the equality is independent of the choice of π_0 in $\mathcal{P}(b(x,y))$.

Using the convexity of Ω, the quantity $F(x,y)$ can be characterized variationally as follows. Define $T(x, \xi, \pi)$ by $\pi \cap \{x + t\xi \mid t > 0\}$ with $\pi \in \mathcal{P}$. Consider the case $\xi = \xi_{xy}$. When the hyperplane supports Ω at $b(x,y)$, we have $T(x, \xi_{xy}, \pi) = b(x,y)$ and otherwise the point $T(x, \xi_{xy}, \pi)$ lies outside Ω. When $\pi \notin \mathcal{P}(b(x,y))$, by the similarity argument between the triangles $\triangle(x, F_\pi(x), T(x, \xi_{xy}, \pi))$ and $\triangle(y, F_\pi(y), T(\xi_{xy}, \pi))$ again we have

$$\frac{d(x,\pi)}{d(y,\pi)} = \frac{d(x, T(x, \xi_{xy}, \pi))}{d(y, T(x, \xi_{xy}, \pi))}.$$

Note that the closest point to x along the ray $R(x,y) = \{x + t\xi_{xy} \mid t > 0\}$ of the form $T(x, \xi_{xy}, \pi)$ is $b(x,y)$. This in turn says that a hyperplane π which supports Ω at $b(x,y)$ maximizes the ratio $d(x, T(x, \xi_{xy}, \pi))/d(y, T(x, \xi_{xy}, \pi))$ among all the elements of $\mathcal{P}$;

$$\log \frac{d(x, b(x,y))}{d(y, b(x,y))} = \sup_{\pi \in \mathcal{P}} \log \frac{d(x,\pi)}{d(y,\pi)}.$$

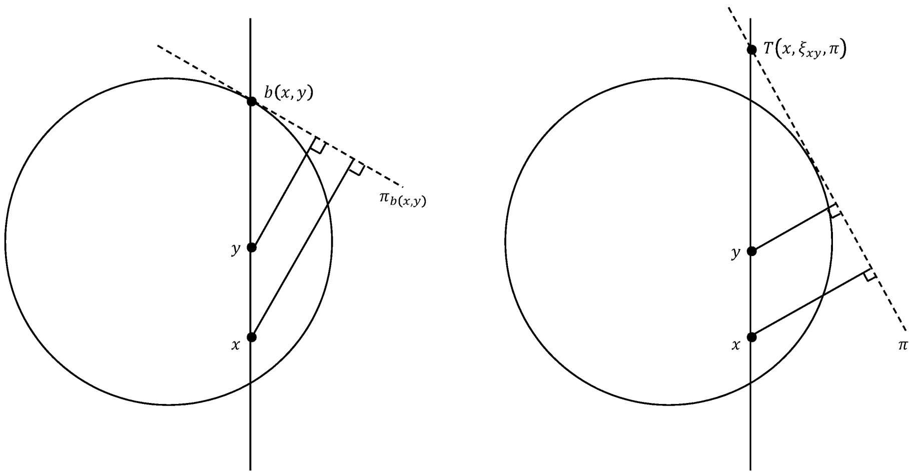

Figure 1. Variations in the pencils of similar triangles

Figure 1 Above is the picture when the convex set Ω is the unit disc.

Hence we have an alternative characterization of the Funk metric [14];

Theorem 1. *The Funk metric on a convex subset $\Omega \subset \mathbf{R}^d$ has the following variational formulation:*

$$F(x,y) = \sup_{\pi \in \mathcal{P}} \log \frac{d(x,\pi)}{d(y,\pi)}.$$

Alternatively it can be written as

$$F(x,y) = \sup_{\pi \in \mathcal{P}} \log \frac{d(x, T(x,\xi_{xy},\pi))}{d(y, T(x,\xi_{xy},\pi))}.$$

With this formulation, one can readily see that $F(x,y)$ satisfies the triangle inequality, for

$$F(x,y) + F(y,z) = \sup_{\pi \in \mathcal{P}} \log \frac{d(x,\pi)}{d(y,\pi)} + \sup_{\pi \in \mathcal{P}} \log \frac{d(y,\pi)}{d(z,\pi)}$$

$$\geq \sup_{\pi \in \mathcal{P}} \Big(\log \frac{d(x,\pi)}{d(y,\pi)} + \log \frac{d(y,\pi)}{d(z,\pi)} \Big) = \sup_{\pi \in \mathcal{P}} \log \frac{d(x,\pi)}{d(z,\pi)} = F(x,z)$$

Note that the triangle inequality becomes an equality when

$$\sup_{\pi \in \mathcal{P}} \log \frac{d(x,\pi)}{d(y,\pi)} + \sup_{\pi \in \mathcal{P}} \log \frac{d(y,\pi)}{d(z,\pi)} = \sup_{\pi \in \mathcal{P}} \Big(\log \frac{d(x,\pi)}{d(y,\pi)} + \log \frac{d(y,\pi)}{d(z,\pi)} \Big)$$

is satisfied. For this to occur, we only need $\mathcal{P}(b(x,y)) \cap \mathcal{P}(b(y,z)) \neq \emptyset$. Let π_0 be an element in the set $\mathcal{P}(b(x,y)) \cap \mathcal{P}(b(y,z)) \neq \emptyset$. Then the boundary points $b(x,y)$ and $b(y,z)$ share the same supporting hyperplane π_0, and we have

$$\sup_{\pi \in \mathcal{P}} \log \frac{d(x,\pi)}{d(y,\pi)} = \log \frac{d(x,\pi_0)}{d(y,\pi_0)}, \quad \sup_{\pi \in \mathcal{P}} \log \frac{d(y,\pi)}{d(z,\pi)} = \log \frac{d(y,\pi_0)}{d(z,\pi_0)}$$

and

$$\sup_{\pi\in\mathcal{P}} \log \frac{d(x,\pi)}{d(z,\pi)} = \log \frac{d(x,\pi_0)}{d(z,\pi_0)},$$

inducing the equality.

A notable situation where one has $\mathcal{P}(b(x,y)) \cap \mathcal{P}(b(y,z)) \neq \emptyset$ is when x, y and z are collinear, with y lying between x and z. This in turn says that the straight line segment $\overline{xy}$ is a Funk geodesic. Hence the Funk metric is projective, in the sense that the Euclidean line segments in the convex body are also Funk geodesics. On the other hand, when π_0 is in the set $\mathcal{P}(b(x,y)) \cap \mathcal{P}(b(y,z))$ with $b(x,y) \neq b(y,z)$, the concatenation of the line segment $\overline{xy}$ and $\overline{yz}$ is also a Funk geodesic, a situation occurring when the boundary set $\partial\Omega$ contains a Euclidean line segments.

We next consider the complementary situation where $\mathcal{P}(b_1)\cap\mathcal{P}(b_2) = \emptyset$ for any pair of distinct points b_1, b_2 in $\partial\Omega$. Geometrically this characterizes strict convexity of the domain Ω, namely, the case where the boundary $\partial\Omega$ contains no closed line segments. From the preceding argument, it follows that the only way equality in the triangle inequality occurs is when the three points x, y and z are collinear and in that order. Hence for strictly convex domains, the Funk geodesics consist of line segments only, or equivalently, given a pair of points, there is a unique Funk geodesic joining them. We state the discussion above as

Proposition 2. *Given any distinct points x and y in Ω, the straight line segment $\overline{xy}$ is a Funk geodesic. And when Ω is strictly convex, the line segment is the unique geodesic from x to y.*

For more comprehensive treatments, the reader is referred to [11] and [14].

§ 3. The Apollonian weak metric

§ 3.1. Definition and its associated Apollonian geometry

Definition For a pair of distinct points x and y in Ω, the Apollonian weak metric [12] is defined by

$$A(x,y) = \sup_{a\in\partial\Omega} \log \frac{d(x,a)}{d(y,a)}.$$

We define $A(x,x)$ to be 0.

Suppose now that the convex set Ω is bounded. Then the boundary $\partial\Omega$ is compact and there exists some point $a(x,y)$ on $\partial\Omega$ so that

$$A(x,y) = \log \frac{d(x,a(x,y))}{d(y,a(x,y))}.$$

We now describe the point $a(x, y)$ as treated in Kelly's paper [9], and further elaborate on his argument.

For $\lambda > 0$, denote by

$$C_{xy}(\lambda) = \{p \in \Omega \mid \frac{d(x,p)}{d(y,p)} = \lambda\}$$

the circle consisting of points where the ratio of distances to x and y is constant, and equal to λ. As we are interested in taking the supremum of the ratio of the distances, we are concerned with the range $\lambda > 1$. Note then that the intersection of $C_{xy}(\lambda)$ with the ray $R(x, y)$ from x through y is a pair of points I (for In-between x and y) and O (for Outside the interval $\overline{xy}$) so that the collinear points (x, y, I, O) form a harmonic quadruple; namely

$$\frac{d(x,I)}{d(y,I)} = \frac{d(x,O)}{d(y,O)}.$$

Equivalently when I and O satisfy the above equality, they are said to separate x and y harmonically. Note that for each $\lambda > 0$, $C_{xy}(\lambda)$ is a circle whose center lies on the ray $R(x, y)$, and as λ gets larger, $C_{xy}(\lambda)$ becomes smaller, and it eventually converges to y. Note that the appearance of such a pencil of circles is at the origin of the term "Apollonian".

Now we define

$$\lambda(x, y) = \inf\{\lambda \mid C_{xy}(\lambda) \subset \Omega\}.$$

The boundary point $a(x, y)$ that maximizes the value of $\log[d(x, a)/d(y, a)]$ is described as the point of tangency between the boundary set $\partial\Omega$ and the circle C_{xy} defined as

$$C_{xy} = C_{xy}(\lambda(x, y)).$$

We observe that the point $a(x, y)$ is a boundary point where the boundary set is at least C^1. This follows from the fact that as one tracks the family of circles $\{C_{xy}(\lambda)\}_\lambda$ starting from the degenerate one $\{y\} = C_{xy}(\infty)$, the first point of contact with the boundary set $\partial\Omega$ is a point which cannot be an isolated extremal point of $\overline{\Omega}$. Otherwise, the boundary set is Lipschitz-continuous ($C^{0,1}$), but not C^1 at $a(x, y)$ and one can shrink the sphere $C_{xy}(\lambda)$ to make λ larger. This in turn says that the set $\mathcal{P}(a(x, y))$ of hyperplanes supporting Ω at $a(x, y)$ consists of a single element, which we write as $\pi_{a(x,y)}$.

Now we note an elementary fact about the cross ratio in the complex plane.

Lemma 3. *Let x, y and z be three distinct points, not collinearly located. Denote by $C_{x,y,z}$ the circle passing through x, y and z. Let $\lambda > 0$ be equal to $d(x, z)/d(y, z)$ and consider the circle $C_{xy}(\lambda)$. Then the circle $C_{xy}(\lambda)$ and $C_{x,y,z}$ meet perpendicularly at two points.*

When x, y and z are collinear, the line through them, which we still denote by $C_{x,y,z}$, cuts through the circle $C_{xy}(\lambda)$ diametrically. By transforming the configuration by an element of $SL(2,\mathbb{R})$, a conformal map, we obtain the picture described in the statement.

§ 3.2. Variational formulation

Now for a pair of points x and y in Ω, and $\pi \in \mathcal{P}$ a supporting hyperplane, there exists a unique circular arc from x through y meeting π perpendicularly at $F_{xy}(\pi)$. Note that as Ω is convex, the foot $F_{xy}(\pi)$ lies outside Ω. In maximizing the value of λ, $F_{xy}(\pi)$ needs to be closest possible to Ω, namely a point of the boundary. **Figure 2** below is the picture when the convex set is the unit disc.

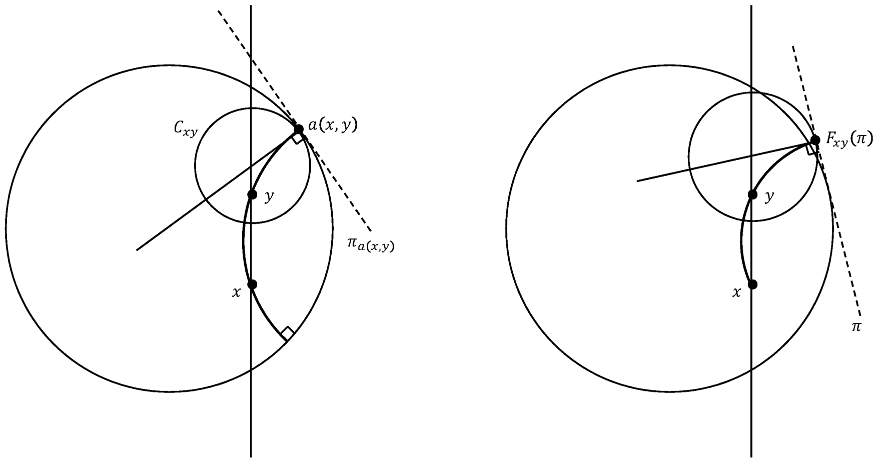

Figure 2. Variations in the pencil of circles

Hence we have the following alternative formulation of the value of the Apollonian weak metric.

Theorem 4. *The Apollonian weak metric $A(x,y)$ has the following representation:*

$$A(x,y) = \sup_{\pi \in \mathcal{P}} \log \frac{d(x, F_{xy}(\pi))}{d(y, F_{xy}(\pi))}.$$

and the supremum is realized by a hyperplane π so that $F_{xy}(\pi)$ is a point of the boundary.

Note that such a hyperplane needs not be unique; consider the situation when the convex set Ω is an ellipsoid $\{(s,t,u) \mid s^2 + t^2 + u^2/4 < 1\}$, and x is the origin, and y is on the positive u-axis.

We note that this expression is similar to the expression for the Funk metric introduced above,

$$F(x,y) = \sup_{\pi \in \mathcal{P}} \log \frac{d(x, T(x, \xi_{xy}, \pi))}{d(y, T(x, \xi_{xy}, \pi))}$$

where the supremum is achieved when the point $T(x, \xi_{xy}, \pi)$ lies on the boundary.

The circle $C_{xy}(\lambda(x,y))$ is characterized by having its center at the intersection I of the ray $R(x,y)$ and the line through $F_{xy}(\pi_{xy})$ perpendicular to π_{xy}, with its radius equal to $d(I, F_{xy}(\pi_{xy}))$. Furthermore, the circle $C_{xy}(\lambda(x,y))$ cuts through the ray $R(x,y)$ at two points I and O so that I and O separate x and y harmonically. Also $C_{xy}(\lambda(x,y))$ cuts through the circle $C_{x,y,F_{xy}(\pi_{xy})}$ at two points so that they also separate the points x and y harmonically.

By an argument almost identical to the one for the Funk metric, one sees that $A(x,y)$ satisfies the triangle inequality, for

$$\begin{aligned} A(x,y) + A(y,z) &= \sup_{\pi \in \mathcal{P}} \log \frac{d(x, F_{xy}(\pi))}{d(y, F_{xy}(\pi))} + \sup_{\pi \in \mathcal{P}} \log \frac{d(y, F_{yz}(\pi))}{d(z, F_{yz}(\pi))} \\ &\geq \sup_{\pi \in \mathcal{P}} \Big(\log \frac{d(x, F_{xy}(\pi))}{d(y, F_{xy}(\pi))} + \log \frac{d(y, \pi)}{d(z, F_{yz}(\pi))} \Big) = A(x,z) \end{aligned}$$

The inequality becomes an equality when $\mathcal{P}(F_{xy}(\pi_{xy})) \cap \mathcal{P}(F_{yz}(\pi_{yx})) \neq \emptyset$. Recall the variational formulation of the Funk metric

$$F(x,y) = \sup_{\pi \in \mathcal{P}} \log \frac{d(x, T(x, \xi_{xy}, \pi))}{d(y, T(x, \xi_{xy}, \pi))}$$

where $T(x, \xi_{xy}, \pi)$ is the point where the ray $R(x,y)$ meets the supporting hyperplane π. In particular, since both formulations are about maximizing the value of logarithmic ratio of lengths of two line segments that meet at a boundary point, by imitating the argument of the triangle *equality* for the Funk metric, it follows that the circular arc $C(x,y) \subset C_{x,y,F_{xy}(\pi_{xy})}$ from x to y meeting $\pi_{F_{xy}(\pi_{xy})}$ perpendicularly is an A-geodesic from x to y *provided* that $C(x,y)$ lies inside Ω. We simplify the notation by setting $a(x,y) := F_{xy}(\pi_{xy})$ from now on. We have observed the following:

Proposition 5. *The Apollonian weak metric $A(x,y)$ defined on a convex set Ω is not geodesic in general, namely there may be a pair of points which cannot be connected by a length-minimizing path in Ω. When the portion $C(x,y)$ of the circular arc $C_{x,y,a(x,y)}$ between x and y is entirely contained in Ω, it is a A-geodesic from x to y.*

For an example of a non-geodesic Apollonian weak metric space (Ω, A), consider a very thin convex set Ω, where typically $C_{x,y,a(x,y)}$ spills out of Ω. On the other hand, regardless of the existence of A-geodesics, one can state the condition for A-alignment as follows. First the definition:

Definition Let (X, δ) be a space with a weak metric and x, y and z be three points in X. We say the three points x, y and z (in that order) are δ-aligned if $\delta(x, z) = \delta(x, y) + \delta(y, z)$.

In particular three ordered points x, y and z on a geodesic are aligned. We now have

Proposition 6. *For a pair of points x and z, suppose y lies on $C_{x,z,a(x,z)}$ between x and z. Then the three points are A-aligned.*

In what follows, we call the circular arc $C(x, y) \subset C_{x,y,a(x,y)}$ the A-pseudo-geodesic. When $C(x, y)$ lies entirely in Ω, we call $C(x, y)$ the A-geodesic from x to y.

Note that the existence issue of geodesics did not arise in the case of Funk metric, as the open line segment from x though y to $b(x, y)$ is always entirely contained in the convex set Ω.

§ 3.3. The Finsler structure

Even though the metric space (Ω, A) is not geodesic in general, we can still investigate the possibility of a Finsler structure. We prove the following:

Theorem 7. *Suppose that a pair of points x and y in a convex set Ω are connected by the A-geodesic $C(x, y) \subset C_{x,y,a(x,y)}$; namely $C(x, y)$ is entirely contained in Ω. Then the A-distance from x to y is realized by the following path integral*

$$A(x, y) = \int_a^b p_\Omega(\sigma(t), \sigma'(t))\, dt$$

where $\sigma : [a, b] \to \Omega$ is a C^1 path parameterizing the arc $C(x, y)$ with $\sigma(a) = x$ and $\sigma(b) = y$, and the integrand is the Minkowski functional given by

$$p_\Omega(x, \xi) = \frac{|\xi|}{2 \sup\{r \mid B(x + r\frac{\xi}{|\xi|}, r) \subset \Omega\}}$$

where $x \in \Omega$, $\xi \in T_x\Omega$ and $B(x, r)$ is the Euclidean ball of radius r centered at x.

The expression for the Minkowski functional appeared in the work of P. Hästö [6] where it was shown that linearizing the (symmetric) Apollonian metric d at x and $\xi \in T_x\Omega$, one obtains a linear functional. The functional is symmetric, namely its value for ξ equals that for $-\xi$, as the distance function is symmetric. By taking the asymmetric half of the argument, we obtain a candidate for the Minkowski functional of the Apollonian weak metric. We outline the argument here.

In order to obtain a candidate for the linear functional on $T_x\Omega$, first note the following equality

$$\lim_{t\to 0^+} \frac{A(x, x + t\xi)}{t} = \sup_{a\in\partial\Omega} \lim_{t\to 0^+} \frac{|\xi|}{t} \log\left(\frac{d(x, a)}{d(x + t\xi/|\xi|, a)}\right) = \sup_{a\in\partial\Omega} \frac{\cos\theta}{d(x, a)} |\xi|$$

where the second equality follows from expanding the Euclidean inner product

$$d(x+t\xi,a)^2 = d(x,a)^2 + t^2|\xi|^2 - 2d(x,a)|\xi|t\cos\theta$$

with θ the angle between ξ and the vector $x-a$. We note that the last expression above

$$\sup_{a\in\partial\Omega} \frac{\cos\theta}{d(x,a)}|\xi|$$

is a convex functional defined on $T_x\Omega$, as for each fixed $a \in \partial\Omega$, the functional $|\xi|\cos\theta/d(x,a)$ is linear in ξ, in particular convex in ξ, and taking supremum of convex functionals results in a convex functional.

Now let $\rho(a)$ be the radius of the spheres through x and a with center on the line $\{x+t\xi\}$. An elementary geometric argument shows $\cos\theta/d(x,a) = 1/2\rho(a)$. Taking the supremum of $1/2\rho(a)$ over $a \in \partial\Omega$, we obtain the desired expression

$$p_\Omega(x,\xi) = \frac{|\xi|}{2\sup\{r \mid B(x + r\frac{\xi}{|\xi|}, r) \subset \Omega\}}$$

where the value of the supremum is achieved at $a(x)$ when the ball $B(x + r\frac{\xi}{|\xi|}, r)$ is inscribed in Ω and tangent to $\partial\Omega$ at $a(x)$.

Actually one can see the validity of the expression for $p_\Omega(x,\xi)$ by looking at the picture below (**Figure 3**), which is the limiting case of the left picture in **Figure 2** when y approaches to x.

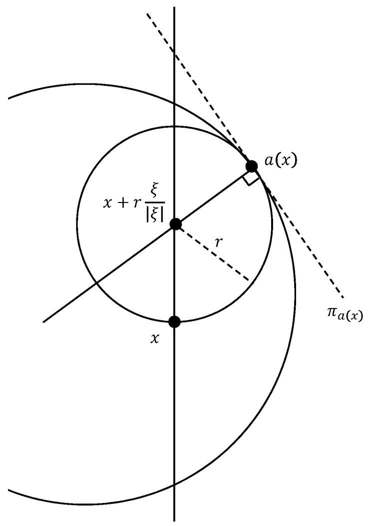

Figure 3. Geometry of the Minkowski functional

To see that the infimum of the lengths of all piecewise C^1 curves connecting x and y in Ω with respect to this Minkowski functional p_Ω indeed amounts to the value $A(x,y)$, we use the following observation.

Proposition 8 (Monotonicity). *For a pair of convex sets Ω and $\tilde{\Omega}$ with $\Omega \subset \tilde{\Omega}$, the corresponding Apollonian weak metrics A_Ω and $A_{\tilde{\Omega}}$ have the following inequality:*

$$A_\Omega(x,y) \geq A_{\tilde{\Omega}}(x,y)$$

for x, y in Ω.

This inequality is a direct consequence of the variational formulation (Theorem 4) of $A(x,y)$.

Consider the situation where x and y are distinct points in Ω, and the circular arc $C(x,y)$ lies entirely in Ω. Now let $\tilde{\Omega}$ be the half plane whose boundary is the supporting hyperplane $\pi_{a(x,y)}$ of Ω at $a(x,y)$. Then from the inclusion $\Omega \subset \tilde{\Omega}$, we know that $A_\Omega(x,y) \geq A_{\tilde{\Omega}}(x,y)$.

On one hand, note that we know that the value of the $A_{\tilde{\Omega}}$-distance from x to y;

$$A_{\tilde{\Omega}}(x,y) = \text{length}_{A_{\tilde{\Omega}}} C(x,y) = \int_0^1 P_{\tilde{\Omega}}(\sigma(t), \sigma'(t))\, dt = \log \frac{|x - a(x,y)|}{|y - a(x,y)|}$$

where $C(x,y)$ is the part of the circular arc $C_{x,y,a(x,y)}$ and $\sigma : [0,1] \to C(x,y)$ is a monotone parameterization of $C(x,y)$ with $\sigma(0) = x$ and $\sigma(1) = y$. The last equality follows from the fact that

$$\frac{d}{dt} \log \frac{|x - a(x,y)|}{|\sigma(t) - a(x,y)|} = P_{\tilde{\Omega}}(\sigma(t), \sigma'(t)).$$

Using the monotonicity of A_Ω, we have a lower bound for the A-distance:

$$A_\Omega(x,y) \geq \log \frac{|x - a(x,y)|}{|y - a(x,y)|}.$$

On the other hand, we calculate the A_Ω-length of the arc $C(x,y) \subset C_{x,y,a(x,y)}$ parameterized by $\sigma(t)$, which is the unique A_Ω-geodesic from x to y. As seen above we know the primitive function of the Minkowski functional along σ explicitly:

$$\frac{d}{d\tau} \log \frac{|\sigma(t) - a(x,y)|}{|\sigma(t+\tau) - a(x,y)|}\bigg|_{\tau=0} = P_\Omega(\sigma(t), \sigma'(t)),$$

which follows from the observation that $a(x,y) = a(\sigma(t), \sigma(t'))$ for all $0 \leq t < t' \leq 1$. Hence the A_Ω-length of $C(x,y)$ is given as

$$\int_0^1 p_\Omega(\sigma(t), \sigma'(t))\, dt = \log \frac{|x - a(x,y)|}{|y - a(x,y)|},$$

in turn implying the inequality

$$A_\Omega(x,y) \leq \frac{|x - a(x,y)|}{|y - a(x,y)|}.$$

By combining these observations together, we have shown that for a pair of points x and y whose A_Ω-pseudo-geodesic $C(x,y)$ lies entirely in Ω, the Apollonian weak distance $A_\Omega(x,y)$ is realized as a Finsler structure

$$\inf_{\gamma\in\Gamma_{x,y}} \int_\gamma p_\Omega(\gamma(t),\gamma'(t))\ dt$$

where $\Gamma_{x,y}$ is the set of piecewise C^1 paths in Ω joining x and y.

§ 4. Symmetrization of the weak metrics

§ 4.1. Hilbert metric

Given an open convex body Ω in a Euclidean space, Hilbert in 1895 ([5]) proposed a natural metric $H(x,y)$, now called the Hilbert metric, defined on Ω, as the logarithm of the (unsigned/absolute) cross ratio of a quadruple, $x, y, b(x,y)$ and $b(y,x)$, where $b(x,y)$ is where the ray from x through y hits the boundary of Ω.

The logarithm of the cross ratio indeed defines a metric, which is Finslerian and projective. A Finsler structure on a Euclidean space determines a norm on each tangent space, and the norm itself is called the Minkowski functional. A metric is said to be projective when Euclidean straight lines are geodesic. The unit disc with its Hilbert metric $H(x,y)$ is a prominent example; it is Klein's model for the hyperbolic plane.

The Finsler structure of the Hilbert metric is known ([14]) to be determined by the Minkowski functional $p_\Omega(x,\xi)$ as

$$p_\Omega(x,\xi) = \sup_{\pi\in\mathcal{P}} \frac{\langle \xi,\eta_\pi\rangle}{d(x,\pi)} + \sup_{\pi\in\mathcal{P}} \frac{\langle -\xi,\eta_\pi\rangle}{d(x,\pi)}$$

where $\eta_\pi \in T_x\Omega$ is the unit vector perpendicular to the supporting hyperplane π directed toward π.

As the value $H(x,y)$ can be written as

$$\log\frac{d(x,b(x,y))d(y,b(y,x))}{d(y,b(x,y))d(x,b(y,x))} = \log\frac{d(x,b(x,y))}{d(y,b(x,y))} + \log\frac{d(y,b(y,x))}{d(x,b(y,x))},$$

Funk [4] looked at the first term of the right hand side above as a metric, even though it is asymmetric, which has been called the Funk metric. The reader is referred to [11, 12] where the historical and technical backgrounds are presented comprehensively.

§ 4.2. Apollonian/Barbilian metric

In 1934, D. Barbilian [1] introduced a metric, which is the arithmetic symmetrization of the Apollonian weak metric $A(x,y)$. Namely the symmetric metric is defined as

$$\mathrm{Ap}(x,y) = \sup_{a\in\partial\Omega} \log\frac{d(x,a)}{d(y,a)} + \sup_{a\in\partial\Omega} \log\frac{d(y,a)}{d(x,a)}$$

for a pair of distinct points x and y. This metric was defined independently in 1995 by A. Beardon [2] who named it as the Apollonian semi-metric. It is called semi-metric, for the distance function may not separate distinct points.

Having obtained the Minkowski functional for the Apollonian weak metric $A(x,y)$ in the previous section, we have the Minkowski functional for the Apollonian metric $\mathrm{Ap}(x,y)$:

$$p_\Omega(x,\xi) = \frac{|\xi|}{2\sup\{r \mid B(x+r\frac{\xi}{|\xi|},r)\subset\Omega\}} + \frac{|\xi|}{2\sup\{r \mid B(x-r\frac{\xi}{|\xi|},r)\subset\Omega\}},$$

making the symmetric metric Ap Finsler when the metric space (Ω,Ap) is geodesic. As mentioned above, the expression for $p_\Omega(x,\xi)$ appeared in [6].

Now we ask:

Question Find a necessary and sufficient condition for the convex set Ω so that its Apollonian metric is Riemannian, namely the Minkowski functional $p_\Omega(x,\xi)$ corresponds to a positive definite bilinear form defined on $T_x\Omega$.

The analogous question for the Hilbert metric has been answered by D. Kay [8]: *A Hilbert geometry is Riemannian if and only if it is hyperbolic, namely the convex set Ω is an ellipse.* It is tempting to conjecture that the situation for the Apollonian metric is the same, namely the metric space is Riemannian if and only if the convex set is an ellipse.

Incidentally we mention here that there is another symmetrization of $A(x,y)$;

$$\mathrm{Ap}_{1/2}(x,y) = \max\Big(\sup_{a\in\partial\Omega}\log\frac{d(x,a)}{d(y,a)},\sup_{a\in\partial\Omega}\log\frac{d(y,a)}{d(x,a)}\Big)$$

which is equal to

$$\sup_{a\in\partial\Omega}\Big|\log\frac{d(x,a)}{d(y,a)}\Big|$$

which was introduced by Hästo-Lindén [7], and is named the half-Apollonian semi-metric. One should note that in the literature of these metrics, the domain Ω is not assumed to be convex; an assumption we have adhered to in this article for the sake of comparison to the Funk metric.

§ 4.3. Comparison when Ω is the unit disc

Much of the interest in those various weak metrics and their symmetrizations arises partly from the fact that they are all closely related to the hyperbolic metric, when the convex domain Ω is the unit disc or the upper half plane. We list some known facts about the Hilbert and Apollonian/Barbilian metrics.

1. For the unit disc $\mathbb{D}^2 = \{z \in \mathbb{C} \mid |z| < 1\}$, the Hilbert metric gives the Beltrami-Klein model of the hyperbolic plane, where the geodesics are the straight line segments, and the distance function $H(x, y)$ is the logarithm of the cross ratio between the four ordered points $b(y, x), x, y, b(x, y)$; namely
$$H(x,y) = \log \frac{|x - b(x,y)||y - b(y,x)|}{|y - b(x,y)||x - b(y,x)|}$$

2. For the unit disc $\mathbb{D}^2$, the Apollonian weak metric has the expression ([12])
$$A(x,y) = \log \Big(\frac{|x-y| + |x\overline{y} - 1|}{|1 - |y|^2|} \Big)$$
and the Apollonian metric $\mathrm{Ap}(x, y)$ is the hyperbolic metric, with the expression
$$\mathrm{Ap}(x,y) = \log \Big(\frac{|1 - x\overline{y}| + |x-y|}{|1 - x\overline{y}| - |x-y|} \Big)$$
where x and y are the complex coordinates of $\mathbb{C}$.

3. For the upper half plane $\mathbb{U} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$, the Apollonian weak metric is given ([3]) by
$$A(x,y) = \log \Big(\frac{|x - \overline{y}| + |x-y|}{|x - \overline{y}|} \Big)$$
and the Apollonian metric is the hyperbolic metric on $\mathbb{U}$, with the expression ([3])
$$\mathrm{Ap}(x,y) = \log \Big(\frac{|x - \overline{y}| + |x-y|}{|x - \overline{y}| - |x-y|} \Big)$$

We consider the geodesics for these models of hyperbolic plane. As the unit disc is strictly convex, given a pair of points, the Euclidean line segments connecting them is the unique F-geodesics in both ways;
$$F(x,y) = F(x,t) + F(t,y) \text{ and } F(y,x) = F(y,t) + F(t,x)$$
for any point t on the line segment $\overline{xy}$. This in turn says that the line segments are the geodesics for the Hilbert metric so that we have $H(x, y) = H(x, t) + H(t, y)$. This is the Beltrami-Klein model of the hyperbolic plane.

On the other hand, from the discussion from the previous section, the A-geodesics on the unit disc are the circular arcs which meet the unit circle perpendicularly. This fact is also proved in [12]. Note that these arcs always exist in $\mathbb{D}^2$, making the metric space $(\mathbb{D}^2, A)$ geodesic. In particular given any pair of points x and y, there always exists a unique circular arc $C_{x,y,a(x,y)}$ so that the portion of arc between x and y becomes the A-geodesic from x and y, as well as the A-geodesic from y to x; namely
$$A(x,y) = A(x,t) + A(t,y) \text{ and } A(y,x) = A(y,t) + A(t,x)$$

for any point t on the circular arc between x and y. This in turn says that the circular arcs are geodesics for the Apollonian metric so that we have $\mathrm{Ap}(x,y) = \mathrm{Ap}(x,t) + \mathrm{Ap}(t,y)$. This is the Poincaré model of the hyperbolic plane.

Between these two models for the hyperbolic plane, there is a canonical correspondence; the following map Φ sends the unit disc to itself, and is an isometry from the Beltrami-Klein disc model $(\mathbb{D}^2, H)$ to the Poincare disc model $(\mathbb{D}, \mathrm{Ap})$

$$\Phi : z \mapsto \frac{z}{1+\sqrt{1-|z|^2}}.$$

Geometrically, this map Φ is described as follows: start with a point z on the unit disc centered at the origin of the xy-plane, and project it down to the southern hemisphere $\{x^2+y^2+z^2=1, z<0\}$ vertically, call the point $\phi(z)$. The point $\Phi(z)$ is where the line through the north pole $(0,0,1)$ and $\phi(z)$ and the xy-plane meet. By this map, the straight edge whose endpoints are two points P and Q on the unit circle is sent to the circular arc that meets the unit circle at P and Q perpendicularly, sending the H-geodesics to the Ap-geodesics, and at the same time preserving the hyperbolic distance $H(z_1,z_2) = \mathrm{Ap}(\Phi(z_1), \Phi(z_2))$. We also mention that the map Φ is not an isometry between $(\mathbb{D}^2, A_{\mathbb{D}^2})$ and $(\mathbb{U}, A_{\mathbb{U}})$.

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