RIMS Kôkyûroku Bessatsu
B5 (2008), 7–13

On Invariants of Reiffen's Isolated Singularity

By

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Abstract

A relation between two invariants attached to a family of hypersurface isolated singularities, called Reiffen's singularity, is considered. For the simplest case of Reiffen's singularity, the relation is determined in an explicit manner.

Introduction

The author has studied isolated singularities from the viewpoint of $\mathscr{D}$-modules ([1], [2], [5], [6]). In these studies, we introduce an invariant $\mu_f^{(k)}$ $(k = 0, 1, 2, \dots)$ of the singularity defined as the dimension of the solution space of a holonomic system attached to the dual space of the Milnor algebra. To be more precise, let X be a neighborhood of the origin O of $\mathbb{C}^n$ and f a holomorphic function defining an isolated singularity at O. Let W_f be the dual vector space of the Milnor algebra of the singularity via the Grothendieck local duality. We take an algebraic local cohomology class ω_f which generates W_f over the stalk $\mathscr{O}_{X,O}$ at the origin of the sheaf $\mathscr{O}_X$ of holomorphic functions. Let $\mathcal{A}nn^{(k)}_{\mathscr{D}_{X,O}}(\omega_f)$ be the ideal in the stalk $\mathscr{D}_{X,O}$ at O of the sheaf $\mathscr{D}_X$ of linear partial differential operators generated by annihilating differential operators of ω_f with order smaller than or equal to k. Since the $\mathscr{D}$-module structure of the holonomic system $\mathscr{D}_{X,O}/\mathcal{A}nn^{(k)}_{\mathscr{D}_{X,O}}(\omega_f)$ does not depend on the choice of the generator ω_f of W_f, the dimension of the algebraic local cohomology solution space of the holonomic system can be said to be an invariant of the singularity. So, we denote it by $\mu_f^{(k)}$. When $k = 1$, this invariant $\mu_f^{(1)}$ is directly connected with the theory of the vector field attached to the function f. In addition, $\mu_f^{(1)} = 1$ is a necessary and sufficient condition for the function f to be quasihomogeneous ([4]). In [5], we studied Reiffen's singularity, which

Received May 12, 2007, Accepted June 9, 2007.
2000 Mathematics Subject Classification(s): 32C38, 32S25, 32C37.
Key Words: D-module, Reiffen's singularity, invariant.
Supported by JSPS 17740087.
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was provided in [3] as an example of a hypersurface isolated singularity on which the holomorphic de Rham complex is not exact. We gave $\mu_f^{(1)}$ of Reiffen's singularity f in an explicit manner by using classical invariants the Milnor number and the Tjurina number of f at the origin.

The algebraic local cohomology vector space W_f is also utilized for a computation of b-function. In [7], T. Yano gave an overview of the general theory of b-function. He also computed a vast number of examples of b-function. He defined an invariant $L(f)$ for the function f as the total order of an annihilator $P(s)$ of f^s and illustrated the method for computing b-function for the case where $L(f) = 2$ and 3. However, it seems that there were no investigations into an invariant $L(f)$ in [7].

In this paper, we give a relation between two invariants $\mu_f^{(1)}$ and $L(f)$ of Reiffen's singularity. In Section 1, we give the definition of the invariant $\mu_f^{(1)}$. We give results on $\mu_f^{(1)}$ for Reiffen's singularity. In Section 2, we introduce the definition of $L(f)$, the total order of annihilators of f^s. As an example, we give an explicit form of annihilators $P(s)$ of f^s when $q = 4$ in Section 3.

§ 1. Reiffen's Singularity

Let X be a small neighborhood of the origin O of $\mathbb{C}^2$. Let f be a holomorphic function defining an isolated singularity at the origin O of $\mathbb{C}^2$ and $\mathcal{J}$ the Jacobi ideal in $\mathscr{O}_{X,O}$, where $\mathscr{O}_{X,O}$ is the stalk of the sheaf of holomorphic functions on X. Let W_f denote the set in $\mathcal{H}^2_{[O]}(\Omega^2_X)$ of algebraic local cohomology classes annihilated by any germ of functions in $\mathcal{J}$, where Ω^2_X is the sheaf of holomorphic 2-forms on X of the origin O,

$$W_f = \{\eta \in \mathcal{H}^2_{[O]}(\Omega^2_X) \mid g\eta = 0,\ g \in \mathcal{J}\}.$$

By Grothendieck local duality, W_f can be regarded as the dual vector space of $\mathscr{O}_{X,O}/\mathcal{J}$. W_f is generated as an $\mathscr{O}_{X,O}$-module by one algebraic local cohomology class. Let ω_f denote a generator over $\mathscr{O}_{X,O}$ of W_f,

$$W_f = \mathscr{O}_{X,O}\omega_f.$$

Let $\mathcal{L}^{(1)}_{\mathscr{D}_{X,O}}(\omega_f)$ be the set of linear partial differential operators with order at most one that annihilate the cohomology class ω_f, where $\mathscr{D}_{X,O}$ is the stalk at the origin of the sheaf $\mathscr{D}_X$ of the rings of partial differential operators. Let $\mathcal{A}nn^{(1)}_{\mathscr{D}_{X,O}}(\omega_f)$ denote the right ideal in the ring $\mathscr{D}_{X,O}$ generated by $\mathcal{L}^{(1)}_{\mathscr{D}_{X,O}}(\omega_f)$, $\mathcal{A}nn^{(1)}_{\mathscr{D}_{X,O}}(\omega_f) = \mathcal{L}^{(1)}_{\mathscr{D}_{X,O}}(\omega_f)\mathscr{D}_{X,O}$. In [4], we proved that if the singularity in question is not quasihomogeneous, $\mathcal{A}nn^{(1)}_{\mathscr{D}_{X,O}}(\omega_f)$ is a proper subset of the annihilating ideal $\mathcal{A}nn_{\mathscr{D}_{X,O}}(\omega_f)$ in $\mathscr{D}_{X,O}$ of the generator ω_f.

Let $f = z_1^q + z_2^p + z_1 z_2^{p-1}$ with $p, q \in \mathbb{N}$, $q \geq 4$ and $p \geq q+1$. The hypersurface $z_1^q + z_2^p + z_1 z_2^{p-1} = 0$ in $\mathbb{C}^2$ defines a semi-quasihomogeneous singularity of weight (p, q)

with the Milnor number $(p-1)(q-1)$ and the Tjurina number $(p-1)(q-1)-q+3$. This hypersurface is examined in [3] by H.-J. Reiffen as a singularity on which the holomorphic de Rham complex is not exact.

In [5], we study Reiffen's singularity from the viewpoint of $\mathscr{D}$-modules and give the following theorem.

Theorem 1.1 ([5]). *Let $f = z_1^q + z_2^p + z_1 z_2^{p-1}$ with $q \geq 4$ and $p \geq q+1$.*

(1) *W_f is generated by*

$$\omega_f = \Big[\frac{dz}{z_1^{q-1} z_2^{p-1}}\Big] + \sum_{k=1}^{q-2}\Big(-\frac{p-1}{p}\Big)^k \Big[\frac{dz}{z_1^{q-1-k} z_2^{p-1+k}}\Big] - \sum_{k=1}^{q-2}\frac{1}{q}\Big(-\frac{p-1}{p}\Big)^k \Big[\frac{dz}{z_1^{2q-2-k} z_2^{k}}\Big]$$

over $\mathscr{O}_{X,O}$ where $dz = dz_1 \wedge dz_2$.

(2) *The algebraic local cohomology solution space*

$$\mathcal{H}om_{\mathscr{D}_{X,O}}(\mathscr{D}_{X,O}/\mathcal{A}nn^{(1)}_{\mathscr{D}_{X,O}}(\omega_f), \mathcal{H}^2_{[O]}(\Omega^2_X))$$

is spanned by ω_f and

$$\sum_{k=0}^{s}\Big(-\frac{p-1}{p}\Big)^k \Big[\frac{dz}{z_1^{1+k} z_2^{1+s-k}}\Big], \quad s = 0, 1, \ldots, q-4.$$

The dimension of the algebraic local cohomology solution space

$$\mathcal{H}om_{\mathscr{D}_{X,O}}(\mathscr{D}_{X,O}/\mathcal{A}nn^{(1)}_{\mathscr{D}_{X,O}}(\omega_f), \mathcal{H}^2_{[O]}(\Omega^2_X))$$

does not depend on the choice of the generator ω_f. Let $\mu_f^{(1)}$ denote the dimension of the solution space for $\mathcal{A}nn^{(1)}_{\mathscr{D}_{X,O}}(\omega_f)$,

$$\mu_f^{(1)} = \dim \mathcal{H}om_{\mathscr{D}_{X,O}}(\mathscr{D}_{X,O}/\mathcal{A}nn^{(1)}_{\mathscr{D}_{X,O}}(\omega_f), \mathcal{H}^2_{[O]}(\Omega^2_X)).$$

The result (2) in the theorem above implies that $\mu_f^{(1)} = q-2$. In other words, we have the following result.

Corollary 1.2 ([5]). *Let $f = z_1^q + z_2^p + z_1 z_2^{p-1}$ with $q \geq 4$ and $p \geq q+1$. Then*

$$\mu_f^{(1)} = \mu - \tau + 1,$$

where $\mu = \dim \mathscr{O}_{X,O}/\mathcal{J}$ is the Milnor number and $\tau = \dim \mathscr{O}_{X,O}/(f, \mathcal{J})$ is the Tjurina number.

For a method to compute cohomology classes and annihilators in the case of general isolated hypersurface singularities, we refer to [2] and [6].

§ 2. Yano's Invariant $L(f)$

A b-function associated with a function f is defined as polynomials $b \in \mathbb{C}[s]$ in s satisfying

$$P f^{s+1} = b f^s \tag{2.1}$$

for some linear partial differential operator $P(s) = \sum s^j P_j(z,\partial) \in \mathscr{D}_{X,O}[s]$. For a given operator $P(s) = \sum s^j P_j(z,\partial) \in \mathscr{D}_{X,O}[s]$, $\max_j (j + \operatorname{ord} P_j(z,\partial))$ is called the total order of $P(s)$ and denoted by $\operatorname{ord}^T P(s)$. Set

$$\mathcal{J}(s) = \{P(s) \in \mathscr{D}_X[s] \mid P(s) f^s = 0\}.$$

There exists an operator of the form

$$P(s) = \sum_{j=0}^{\ell} s^{\ell-j} P_j(z,\partial) \tag{2.2}$$

in $\mathcal{J}(s)$ such that $\operatorname{ord}^T P = \ell$ and $P_0(z,\partial) = 1$. We denote by $L(f)$ the minimum of $\operatorname{ord}^T P(s)$ for $P(s) \in \mathcal{J}(s)$ of the form specified as (2.1) and (2.2), which measures non-quasihomogeneity of f. Especially, $L(f) = 1$ is a necessary and sufficient condition for the function to be quasihomogeneous. In [7], T. Yano developed a general theory of b-function and gave various examples of b-function. He introduced the number $L(f)$ and investigated a method to determine b-functions for f being isolated singularities with $L(f) = 2$ and $L(f) = 3$.

For Reiffen's singularity, we investigate the number $L(f)$ as follows.

Theorem 2.1. *Let $f = z_1^4 + z_2^p + z_1 z_2^{p-1}$ with $p \in \mathbb{N}$ and $p \geq 5$. Then*

$$L(f) = 2.$$

Proof. In the next section, we give annihilators $\mathcal{J}(s)$ of f^s for $f = z_1^4 + z_2^p + z_1 z_2^{p-1}$ in an explicit manner. One finds $P(s) \in \mathcal{J}(s)$ with $L(f) = 2$. □

For general cases of Reiffen's singularity, we find the following:
For $f = z_1^q + z_2^p + z_1 z_2^{p-1}$ with p, $q \in \mathbb{N}$, $q \geq 4$ and $p \geq q + 1$,

$$L(f) = \mu_f^{(1)}$$

holds.

§ 3. Annihilators of f^s for $f = z_1^4 + z_2^p + z_1 z_2^{p-1}$

Let us illustrate the case where $q = 4$, $f = z_1^4 + z_2^p + z_1 z_2^{p-1}$ with $p \geq 5$. A basis of W_f is given by $3(p-1)$ algebraic local cohomology classes, $\Big[\dfrac{dz}{z_1^{\ell_1} z_2^{\ell_2}}\Big]$ with $1 \leq \ell_1 \leq 3$ and $1 \leq \ell_2 \leq p-2$, $\Big[\dfrac{dz}{z_1 z_2^{p-1}}\Big]$ and the algebraic local cohomology classes of the form

$$\Big[\frac{dz}{z_1^2 z_2^{p-1}}\Big] + \Big(-\frac{p-1}{p}\Big)^k \Big[\frac{dz}{z_1 z_2^p}\Big] - \frac{1}{4}\Big(-\frac{p-1}{p}\Big)\Big[\frac{dz}{z_1^4 z_2}\Big]$$

and

$$\Big[\frac{dz}{z_1^3 z_2^{p-1}}\Big] + \Big(-\frac{p-1}{p}\Big)\Big[\frac{dz}{z_1^2 z_2^p}\Big] + \Big(-\frac{p-1}{p}\Big)^2 \Big[\frac{dz}{z_1 z_2^{p+1}}\Big] \\ - \frac{1}{4}\Big(-\frac{p-1}{p}\Big)\Big[\frac{dz}{z_1^5 z_2}\Big] - \frac{1}{4}\Big(-\frac{p-1}{p}\Big)^2\Big[\frac{dz}{z_1^4 z_2^2}\Big]$$

where $dz = dz_1 \wedge dz_2$. The last one is a generator of W_f over $\mathscr{O}_{X,O}$ and thus denote it by ω_f:

$$\omega_f = \Big[\frac{dz}{z_1^3 z_2^{p-1}}\Big] + \Big(-\frac{p-1}{p}\Big)\Big[\frac{dz}{z_1^2 z_2^p}\Big] + \Big(-\frac{p-1}{p}\Big)^2 \Big[\frac{dz}{z_1 z_2^{p+1}}\Big] \\ - \frac{1}{4}\Big(-\frac{p-1}{p}\Big)\Big[\frac{dz}{z_1^5 z_2}\Big] - \frac{1}{4}\Big(-\frac{p-1}{p}\Big)^2\Big[\frac{dz}{z_1^4 z_2^2}\Big].$$

$Ann^{(1)}_{\mathscr{D}_{X,O}}(\omega)$ is generated by partial derivatives $4z_1^3 + z_2^{p-1}$ and $pz_2^{p-1} + (p-1)z_1 z_2^{p-2}$ of f and first order differential operators with the first order part

$$(z_1 + \frac{p}{p-1} z_2) z_2 \frac{\partial}{\partial z_2}, \qquad z_1^3 \frac{\partial}{\partial z_2}, \qquad z_2^3 \frac{\partial}{\partial z_2}, \qquad z_1(z_1 + \frac{p}{p-1} z_2)\frac{\partial}{\partial z_1},$$

$$z_1 z_2 \frac{\partial}{\partial z_1} + \Big(-\frac{1}{p}\frac{p-4}{p-2}\Big(-\frac{p-1}{p}\Big) z_1^2 + \frac{2}{p-2} z_2^2\Big)\frac{\partial}{\partial z_2},$$

$$z_2^{p-2}\frac{\partial}{\partial z_1} + \Big(\frac{12}{(p-1)(p-2)} z_1^2 - \frac{12}{p-2}\Big(-\frac{p}{p-1}\Big)^2 z_2^2\Big)\frac{\partial}{\partial z_2}.$$

The algebraic local cohomology solution space

$$\mathcal{H}om_{\mathscr{D}_{X,O}}(\mathscr{D}_{X,O}/\mathcal{A}nn^{(1)}_{\mathscr{D}_{X,O}}(\omega_f), \mathcal{H}^2_{[O]}(\Omega^2_X))$$

is spanned by ω_f and the delta function $\Big[\dfrac{dz}{z_1 z_2}\Big]$.

$\mathcal{J}(s)$ is generated by the following four operators.

- $((p-1)z_1z_2^{p-2}+pz_2^{p-1})\dfrac{\partial}{\partial z_1}-(4z_1^3+z_2^{p-1})\dfrac{\partial}{\partial z_2}$,

- $$4((p-1)^3z_2^{p-3}-4p^3z_2)z_1+(p-4)(p-1)^2z_2^{p-2})\frac{\partial}{\partial z_1}$$
$$+(-4(p-1)(p-4)z_1^2+4p(p-4)z_1z_2+3(p-1)^2z_2^{p-2}-4^2p^2z_2^2)\frac{\partial}{\partial z_2}$$
$$-4(p-1)^3z_2^{p-3}s+4^2p^3z_2s,$$

- $((p-1)z_1^2+pz_1z_2)\dfrac{\partial}{\partial z_1}+(4z_1z_2+4z_2^2)\dfrac{\partial}{\partial z_2}-4(p-1)z_1s-4pz_2s$,

- $$((p-1)^3z_2^{p-4}-4p^3)s^2+\Big(-\frac{(p-4)p^3}{p-1}z_2\frac{\partial}{\partial z_1}+\frac{3(p-4)p^2}{p-1}z_2\frac{\partial}{\partial z_2}$$
$$+\frac{(p-1)^2(7p-16)}{4}z_2^{p-4}-\frac{p^2(4p^2-7p+12)}{p-1}\Big)s$$
$$+\Big((-\frac{(p-1)^3}{4^2}z_2^{p-4}+\frac{p^3}{4})z_1^2+\frac{(p-4)p^3}{4(p-1)}z_1z_2+\frac{(p-4)(p-1)(p+4)}{4^2}z_2^{p-2}\Big)\frac{\partial^2}{\partial z_1^2}$$
$$+\Big(-\frac{(p-4)(p+4)}{4}z_1^2+(-\frac{3(p-1)^2}{8}z_2^{p-3}+(p+4)pz_2)z_1$$
$$-\frac{(p-4)(p-1)}{2}z_2^{p-2}+\frac{(p-4)p^2}{p-1}z_2^2\Big)\frac{\partial^2}{\partial z_1\partial z_2}$$
$$+\Big(\frac{3(p-4)}{2}z_1^2-\frac{(p-4)(5p+4)}{4(p-1)}z_1z_2-\frac{9(p-1)}{4^2}z_2^{p-2}+\frac{(p+8)p}{p-1}z_2^2\Big)\frac{\partial^2}{\partial z_2^2}$$
$$+\Big((-\frac{(p-1)^2(8p-17)}{16}z_2^{p-4}+\frac{p^2(5p^2-8p+12)}{4(p-1)})z_1$$
$$-\frac{3(p-1)(p-2)(p-4)}{8}z_2^{p-3}+\frac{p^3(p-4)}{4(p-1)}z_1\Big)\frac{\partial}{\partial z_1}$$
$$+(-\frac{3p(p-4)(p+2)}{4(p-1)}z_1-\frac{3(p-1)(7p-13)}{16}z_2^{p-3}+\frac{p(15p^2-16p+64)}{4(p-1)}z_2)\frac{\partial}{\partial z_2}.$$

The total order of the last one is $q-2=2$.

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