RIMS Kôkyûroku Bessatsu
B5 (2008), 51–60

Studies on Deformation of Fuchsian Systems from the Viewpoint of Rigidity

By

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§ 1. Introduction

We consider a Fuchsian system of differential equations

$$\frac{dY}{dx} = A(x)Y, \tag{1}$$

where $A(x) \in M(n, n; \mathbf{C}(x))$. Monodromy preserving deformation of (1) is a problem of determining the coefficient matrix $A(x)$ so that the monodromy of (1) does not change when the positions of the singular points are varied. The deformation equation becomes a system of differential equations for accessory parameters as functions in the positions of singular points. When the system (1) is free of accessory parameters, the deformation equation becomes trivial and is not interesting; however, in this case we can know many global properties of (1) — the monodromy group, connection coefficients, integral representations of the solutions, etc. In this sense systems free of accessory parameters are interesting, and have been studied for a long time.

Recently there are two remarkable progresses in the study of accessory parameter free systems. One is the work of Katz [14] on rigid local systems, and the other is the work of Yokoyama [18], [8], both of which give algorithms to construct every system free of accessory parameters. The operations in these algorithms can be applied also to systems containing accessory parameters, which will shed a new light to the deformation theory. Moreover the notion of rigidity in the Katz theory will give a useful viewpoint for the study of the deformation theory and the theory of holonomic systems.

In this paper we report two results; one is on a relation among the deformation theory and the rigidity argument, and the other is on a relation among the theory of

Received March 20, 2007, Accepted October 17, 2007.
2000 Mathematics Subject Classification(s): 34M55, 33C65.
Supported by the JSPS grant-in-aid for scientific research B, No. 17340049.
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holonomic systems and the rigidity argument. We also give some problems concerning the rigidity argument, which seem to be interesting.

§ 2. Katz Theory and Deformation Theory

Let $t_1, t_2, \dots, t_p$ be distinct points in $\mathbf{C}$, and $A_1, A_2, \dots, A_p$ $n \times n$-matrices which are independent of the variable x. We consider a Fuchsian system of differential equations

$$\frac{dY}{dx} = \Big(\sum_{j=1}^{p} \frac{A_j}{x - t_j}\Big) Y. \tag{2}$$

Note that generically Fuchsian systems (1) can be transformed into a system of the form (2). The system (2) has regular singular points at $x = t_1, t_2, \dots, t_p, \infty$, and the residue matrix at $x = \infty$ is

$$A_{p+1} := -\sum_{j=1}^{p} A_j.$$

Throughout this paper we assume

(A) for each j, there is no integral difference among distinct eigenvalues of A_j ($j = 1, \dots, p+1$).

Definition 2.1. A tuple $\mathcal{A} = (A_1, \dots, A_p, A_{p+1})$ of $n \times n$-matrices with $A_1 + \cdots + A_p + A_{p+1} = O$ is said to be *rigid* if it is determined by their Jordan canonical forms $C_1, \dots, C_p, C_{p+1}$ uniquely up to isomorphisms. In other words, a tuple $\mathcal{A}$ is rigid if, for any tuple $\mathcal{B} = (B_1, \dots, B_p, B_{p+1})$ with $B_1 + \cdots + B_p + B_{p+1} = O$ satisfying $B_j = D_j A_j D_j^{-1}$ with some non-singular matrix D_j for $j = 1, \dots, p+1$, there exists a non-singular matrix D such that $B_j = D A_j D^{-1}$ for $j = 1, \dots, p+1$.

The *index of rigidity* ι is defined for a tuple $\mathcal{A} = (A_1, \dots, A_p, A_{p+1})$ by

$$\iota = (2 - (p+1))n^2 + \sum_{j=1}^{p+1} \dim Z(A_j),$$

where $Z(A)$ denotes the centralizer of A.

Theorem 2.2 ([14]). *If a tuple $\mathcal{A}$ is irreducible, then we have $\iota \le 2$. In this case the tuple $\mathcal{A}$ is rigid if and only if $\iota = 2$.*

The number $2 - \iota$ can be regarded as the number of the accessory parameters of the corresponding system (2), and hence the number of the unknowns of the deformation equation.

Now we explain the Katz algorithm for constructing rigid tuples. Since a tuple $\mathcal{A} = (A_1, \ldots, A_p, A_{p+1})$ with $A_1 + \cdots + A_p + A_{p+1} = O$ is determined by the first p matrices $(A_1, \ldots, A_p)$, we often use $(A_1, \ldots, A_p)$ instead of $(A_1, \ldots, A_p, A_{p+1})$.

The Katz algorithm consists of two operations — addition and middle convolution. For $\alpha_j \in \mathbf{C}$ $(1 \leq j \leq p)$, the *addition* with parameters $(\alpha_1, \ldots, \alpha_p)$ is defined by

$$(A_1, \ldots, A_p) \mapsto (A_1 + \alpha_1, \ldots, A_p + \alpha_p).$$

For $\lambda \in \mathbf{C}$, the *convolution* with parameter λ is defined by

$$(A_1, \ldots, A_p) \mapsto (G_1, \ldots, G_p),$$

where G_j is a $pn \times pn$-matrix given by

$$G_j = \begin{array}{c} \\ j) \\ \\ \end{array} \begin{pmatrix} & & O & & \\ A_1 & \cdots & A_j + \lambda & \cdots & A_p \\ & & O & & \end{pmatrix}$$

for $1 \leq j \leq p$. We set

$$\mathcal{K} = \begin{pmatrix} \operatorname{Ker} A_1 \\ \vdots \\ \operatorname{Ker} A_p \end{pmatrix}, \qquad \mathcal{L} = \operatorname{Ker}(G_1 + \cdots + G_p).$$

Then it is easy to see that $\mathcal{K}$ and $\mathcal{L}$ become invariant subspaces for the G_j's. Thus we can define $\overline{G}_j$ as the action of G_j on the quotient space $\mathbf{C}^{pn}/(\mathcal{K} + \mathcal{L})$. The *middle convolution* with parameter λ is defined by

$$(A_1, \ldots, A_p) \mapsto (\overline{G}_1, \ldots, \overline{G}_p),$$

These definitions are interpretations from the original ones due to Dettweiler and Reiter [3].

Theorem 2.3 ([14], [3]). *The addition and the middle convolution do not change the index of rigidity.*

The addition and the middle convolution can be regarded as operations for a Fuchsian system (2). The addition with parameters $(\alpha_1, \ldots, \alpha_p)$ gives the Fuchsian system

$$\frac{dZ}{dx} = \Big(\sum_{j=1}^{p} \frac{A_j + \alpha_j}{x - t_j} \Big) Z, \tag{3}$$

and the middle convolution with parameter λ gives

$$\frac{dW}{dx} = \Big(\sum_{j=1}^{p} \frac{\overline{G}_j}{x - t_j}\Big) W. \tag{4}$$

It is immediate to see that the solutions of (2) and (3) are related as

$$Z(x) = Y(x) \prod_{j=1}^{p} (x - t_j)^{\alpha_j}.$$

It is known ([4]) that the solutions of the system (4) can be obtained as Riemann-Liouville transformation of the solutions of (2).

We say that the Fuchsian system (2) is rigid if the tuple $(A_1, \ldots, A_p, A_{p+1})$ of residue matrices is rigid.

Theorem 2.4 ([14], [3]). *Any irreducible rigid Fuchsian system can be obtained from a rank one Fuchsian system by a finite iteration of additions and middle convolutions.*

The monodromy preserving deformation of (2) is described by the Schlesinger system ([16]). Precisely speaking, we have

Theorem 2.5. *There exists a fundamental matrix solution $Y_0(x)$ of (2) such that the monodromy matrices with respect to $Y_0(x)$ are independent of $t_1, \ldots, t_p$, if and only if the Jordan canonical forms of the A_j's are independent of $t_1, \ldots, t_p$ for $1 \le j \le p+1$ and there exists a non-singular matrix P independent of x such that the matrices $A'_j = P A_j P^{-1}$ satisfy*

$$\begin{cases} \dfrac{\partial A'_i}{\partial t_i} = -\displaystyle\sum_{k \neq i} \frac{[A'_i, A'_k]}{t_i - t_k}, \\ \dfrac{\partial A'_j}{\partial t_i} = \dfrac{[A'_i, A'_j]}{t_i - t_j} \qquad (j \neq i). \end{cases} \tag{5}$$

The system (5) can be regarded as a system of differential equations for the entries of the A_j's, and we denote it by (S). Let C_j be a Jordan canonical form which is independent of $t_1, \ldots, t_p$ $(1 \le j \le p+1)$. We denote the condition

$$A_j \sim C_j \qquad (1 \le j \le p+1)$$

by (J). Thus the monodromy preserving deformation is described by two conditions (S) and (J).

As is easily seen, the entries of the residue matrices of the systems (3) and (4) are functions of the entries of the A_j's. Then the systems (S) for (3) and (4) become systems for the entries of the A_j's. We showed that these (S) coincide with the original (S) for (2):

Theorem 2.6 ([6]). *The system* (S) *is invariant by the addition and the middle convolution.*

On the other hand the condition (J) changes when we operate the addition or the middle convolution. Then, thanks to Theorem 2.6, the addition and the middle convolution give transformations of solutions of the system (S) with distinct parameters. In this sense these operations give Bäcklund transformations for the deformation equation. For example, Okamoto's birational transformation for Painlevé VI is obtained by the middle convolution ([5], [6]).

Painlevé VI is obtained as a deformation equation of the system of rank 2

$$\frac{dY}{dx} = \Big(\frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_2}{x-t}\Big)Y \tag{6}$$

where ([10])

$$A_j = \begin{pmatrix} z_j + \theta_j & -u_j z_j \\ \dfrac{z_j + \theta_j}{u_j} & -z_j \end{pmatrix} \qquad (j = 0, 1, 2),$$

$$-(A_0 + A_1 + A_2) = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}.$$

Also in the works of Boalch [1], Harnad [9] and Mazzocco [15], Painlevé VI appears as a deformation equation of systems of rank 3. We can see that these systems correspond to the system of rank 3 obtained from (6) by operating a middle convolution.

§ 3. Rigidity of Appell's F_4

Appell's hypergeometric series F_4 is a power series in two variables

$$F_4(\alpha, \beta, \gamma, \gamma'; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha, m+n)(\beta, m+n)}{(\gamma, m)(\gamma', n) m! n!} x^m y^n,$$

where $\alpha, \beta, \gamma, \gamma'$ are complex parameters satisfying $\gamma, \gamma' \notin \mathbf{Z}_{\le 0}$, and

$$(\alpha, m) = \frac{\Gamma(\alpha + m)}{\Gamma(\alpha)}.$$

It satisfies a Pfaffian system

$$dZ = \Omega Z, \qquad \Omega = A(x,y)\,dx + B(x,y)\,dy \tag{7}$$

of rank 4, where $A(x, y)$ and $B(x, y)$ are rational functions in x, y (the entries of the unknown vector Z are F_4 and its partial derivatives). The singular locus of (7) is

$$L := \{x = 0\} \cup \{y = 0\} \cup C \cup \{\infty\},$$

where C denotes the quadratic curve

$$(x-y)^2 - 2(x+y) + 1 = 0$$

and ∞ the line at infinity in $\mathbf{P}^2(\mathbf{C})$. It is known ([12]) that the local behaviors of solutions of (7) near L are described by the Riemann scheme

$$\left\{\begin{matrix} x=0 & y=0 & C & \infty \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \alpha \\ 1-\gamma & 1-\gamma' & 0 & \beta \\ 1-\gamma & 1-\gamma' & \gamma+\gamma'-\alpha-\beta-\dfrac{1}{2} & \beta \end{matrix}\right\} \tag{8}$$

Let us consider the monodromy representation of (7). It is known that the fundamental group of $\mathbf{P}^2(\mathbf{C}) \setminus L$ has the presentation

$$\begin{aligned} \pi_1(\mathbf{P}^2(\mathbf{C}) \setminus L, P_0) = \langle \mu_1, \mu_2, \mu_3, \mu_4 \mid \mu_1\mu_2 = \mu_2\mu_1,\ (\mu_1\mu_3)^2 = (\mu_3\mu_1)^2, \\ (\mu_2\mu_3)^2 = (\mu_3\mu_2)^2,\ \mu_4\mu_3\mu_2\mu_3\mu_1 = 1\rangle, \end{aligned} \tag{9}$$

where μ_1, μ_2, μ_3 and μ_4 are loops each of which encircles $x = 0$, $y = 0$, C and ∞, respectively, once in the positive direction (cf. [11]). Then, if we denote the monodromy matrix corresponding to μ_j by M_j $(1 \le j \le 4)$, from (8) and (9) we obtain

(10)

$$M_1 \sim \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & e^{2\pi\sqrt{-1}(1-\gamma)} & \\ & & & e^{2\pi\sqrt{-1}(1-\gamma)} \end{pmatrix}, \quad M_2 \sim \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & e^{2\pi\sqrt{-1}(1-\gamma')} & \\ & & & e^{2\pi\sqrt{-1}(1-\gamma')} \end{pmatrix},$$

$$M_3 \sim \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & e^{2\pi\sqrt{-1}(\gamma+\gamma'-\alpha-\beta-1/2)} \end{pmatrix}, \quad M_4 \sim \begin{pmatrix} e^{2\pi\sqrt{-1}\alpha} & & & \\ & e^{2\pi\sqrt{-1}\alpha} & & \\ & & e^{2\pi\sqrt{-1}\beta} & \\ & & & e^{2\pi\sqrt{-1}\beta} \end{pmatrix},$$

$$\begin{aligned} &M_1M_2 = M_2M_1, \quad (M_1M_3)^2 = (M_3M_1)^2, \quad (M_2M_3)^2 = (M_3M_2)^2, \\ &M_4M_3M_2M_3M_1 = I. \end{aligned}$$

We showed that the condition (10) determines the tuple (M_1, M_2, M_3, M_4) of the monodromy matrices uniquely up to simultaneous similar transformations:

Theorem 3.1 ([7]). *For generic values of parameters $\alpha, \beta, \gamma, \gamma'$, the condition* (10) *determines the tuple (M_1, M_2, M_3, M_4) uniquely up to simultaneous similar transformations.*

The result above tells that the monodromy representation of the system (7) defines a physically rigid local system on $\mathbf{P}^2(\mathbf{C}) \setminus L$.

It is interesting to compare this result with rigidities of sections of the system (7). If we restrict the system (7) to the line y =constant, we get the system of ordinary differential equations in x

$$\frac{dZ}{dx} = A(x,y)Z. \tag{11}$$

The Riemann scheme of (11) is

$$\left\{\begin{matrix} x=0 & x=y-2\sqrt{y}+1 & x=y+2\sqrt{y}+1 & x=\infty \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \alpha-\gamma'+1 \\ -\gamma & 0 & 0 & \beta \\ -\gamma & \gamma+\gamma'-\alpha-\beta-\dfrac{5}{2} & \gamma+\gamma'-\alpha-\beta-\dfrac{5}{2} & \beta-\gamma'+1 \end{matrix}\right\}, \tag{12}$$

from which we obtain the index of rigidity

$$\iota = (-2)\cdot 4^2 + (2^2+2^2) + (3^2+1^2) + (3^2+1^2) + (1^2+1^2+1^2+1^2) = 0.$$

This implies that the system (11) is not rigid, and has 2 accessory parameters. In the last section we will discuss on the deformation of systems corresponding the Riemann scheme (12).

The line y =constant is not generic, since it goes through the intersection point of two singular lines $\{y=0\}$ and ∞. If we consider the restriction of (7) on a generic line, we get a system with 6 accessory parameters ([7]). Thus the Pfaffian system (7) gives several systems with distinct indexes of rigidity.

§ 4. Problems

Looking at the above results, we notice several natural problems.

Theorem 2.6 asserts that the deformation equation is an invariant under the addition and the middle convolution. The converse assertion is not known: Are two Fuchsian systems connected by the additions and the middle convolutions when they have the same deformation equation? We know the index of rigidity is also an invariant under the same operations (Theorem 2.3). What is the difference of these invariants? More precisely, the index of rigidity gives the number of the unknowns of the deformation equation, so that, if two systems have the same deformation equations, the indexes of rigidity of the systems necessarily coincide. Do two Fuchsian systems have the same deformation equation when they have the same index of rigidity? If the answer is affirmative, the deformation equation of any Fuchsian system becomes a Garnier system, because any index of rigidity is realized by a Fuchsian system of rank 2. This seems incorrect, and we'd like to pose the problem: Describe the difference of the two invariants — the index of rigidity and the deformation equation.

A result in Section 3 gives an interesting example to the last problem. The section (11) of the Pfaffian system (7) on y =constant has the Riemann scheme (12) and the index of rigidity 0. Moreover it contains 4 parameters $\alpha, \beta, \gamma, \gamma'$. On the other hand, the system (6) which yields Painlevé VI has also the index of rigidity 0 and contains 4 parameters $\theta_0, \theta_1, \theta_2, \kappa_1$. Note that κ_2 is determined by the relation $\theta_0+\theta_1+\theta_2+\kappa_1+\kappa_2 = 0$. The former system is of rank 4, while the latter is of rank 2; however, we can operate the addition and the middle convolution to the latter to obtain the rank 4 system

$$\frac{dY}{dx} = \Big(\frac{B_0}{x} + \frac{B_1}{x-1} + \frac{B_2}{x-t}\Big)Y \tag{13}$$

with the same deformation equation thanks to Theorem 2.6. The Riemann scheme of (13) is

$$\left\{ \begin{array}{cccc} x=0 & x=1 & x=t & x=\infty \\ 0 & 0 & 0 & -\lambda \\ 0 & 0 & 0 & -\lambda \\ \alpha_0+\lambda & 0 & 0 & \kappa_1-\alpha_0-\lambda \\ \theta_0+\alpha_0+\lambda & \theta_1+\lambda & \theta_2+\lambda & \kappa_2-\alpha_0-\lambda \end{array} \right\}. \tag{14}$$

The index of rigidity are calculated by the spectral types of residue matrices. For the Riemann scheme (12), the spectral types are

$$(2,2),\ (3,1),\ (3,1),\ (1,1,1,1),$$

and for (14),

$$(2,1,1),\ (3,1),\ (3,1),\ (2,1,1).$$

Thus the types are different, while both give the same index of rigidity. Are these systems connected by the additions and the middle convolutions? A finer invariant may be given by using the spectral types of residue matrices. This will be a fundamental problem for the deformation theory and the theory of Fuchsian systems.

We have not yet treated Yokoyama's algorithm. Similar problems will be posed for this algorithm. It is also interesting to study the difference of Katz' one and Yokoyama's one. The difference will appear in constructing non-rigid Fuchsian systems.

Theorem 3.1 asserts the rigidity of the monodromy representation of the Pfaffian system (7). However, it seems that the notion of rigidity for local systems over $\mathbf{P}^n(\mathbf{C})\backslash S$, S being a hypersurface, has not yet been established in general. The point for the definition will be the definition of *local monodromies*. In particular, if an irreducible divisor of S has a singularity, it is not clear whether we can define the local monodromy for the divisor. Zariski's example [19, Section 8] may explain the difficulty.

We can regard the system (11) as a non-rigid system which can be prolonged into a rigid Pfaffian system. The solutions of (11) have integral representation coming from one for F_4 ([11], [17]), and we can calculate the monodromy. This suggests us that the prolongability may correspond to the computability of the monodromy.

Another interesting example is given by Kato [13]. He starts from the invariant polynomials for the complex reflection group G_{336}, and constructed a linear ordinary differential equation of the third order whose monodromy group coincides with G_{336}. This system can be prolonged into a Pfaffian system, which turns out to be equivalent to the system obtained by Boulanger [2]. Also in this case the equation is prolongable and the monodromy group is computable. Moreover the Pfaffian system thus obtained is so interesting that it does not come from power series nor integral expression of solutions.

Thus we believe that the rigidity viewpoint will be useful for the study of Fuchsian systems.

Acknowledgement

The author expresses his sincere gratitude to his collaborators Galina Filipuk and Youichi Ueno, and to Professors Mitsuo Kato and Joichi Kaneko for their useful suggestions and the exciting discussions.

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