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Divergence and Resummation in the Normal Form Theory of Vector Fields

By

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Abstract

We present a new approach to the so-called small divisor problem of a singular nonlinear system of partial differential equations from the viewpoint of the WKB analysis. The equations which we study appear in the normal form theory of singular vector fields.

§ 1. Introduction

In the normal form theory of vector fields one often encounters with the divergence caused by the resonance or the small denominators. It is known that a Diophantine condition or the existence of a certain number of first integrals can control the divergence. (cf. [1], [4], [7]). The object of this note is to propose an alternative approach to the problem. Namely, instead of a Diophantine condition or first integrals, we use a WKB solution, a resummation with respect to a certain singular perturbative parameter and an analytic continuation.

Heuristically, we construct a WKB solution in a singular perturbative way, and we make the resummation of divergent WKB solutions even if the Poincaré condition is not verified. By the analytic continuation of a resummed WKB solution with respect to a parameter introduced in the above, we will study the solvability of the original problem in case the divergence of the so-called Poincaré series occurs. This method agrees with the standard argument in the point that the resummed WKB solution is Borel summable if the Poincaré condition of the type (4.11) is verified. In this way, we can rediscover the classical Poincaré series.

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§ 2. Homology Equation with a Parameter

Let $x = {}^t(x_1, \dots, x_n) \in \mathbb{C}^n$, $n \ge 2$ be the variable in $\mathbb{C}^n$, and $\mathbb{R}$ the set of real numbers. Let $\mathbb{Z}_+$ be the set of nonnegative integers. Let $\mathbb{Z}_+^n(k)$ $(k \ge 0)$ be defined by

$$\mathbb{Z}_+^n(k) := \{\gamma = {}^t(\gamma_1, \dots, \gamma_n) \in \mathbb{Z}_+^n; |\gamma| = \gamma_1 + \cdots + \gamma_n \ge k\}.$$

For $\gamma \in \mathbb{Z}_+^n$ we set $x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$. For $k \ge 0$ and $n \ge 1$ we denote the set of formal power series $\sum_{|\eta| \ge k} u_\eta x^\eta$ $(u_\eta \in \mathbb{C}^n)$ by $\mathbb{C}_k^n[[x]]$. We denote the set of vector-valued convergent power series which vanish up to $(k-1)$-th derivatives by $\mathbb{C}_k^n[x]$. Let Λ be an n-square constant matrix. Let L_Λ be the Lie derivative of the linear vector field ${}^t(\Lambda x)\partial_x$, where $\partial_x = \partial/\partial x = {}^t\Big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\Big)$. Namely

$$L_\Lambda v = [\Lambda x, v] = \langle \Lambda x, \partial_x \rangle v - \Lambda v, \quad v = {}^t(v_1, v_2, \dots, v_n), \tag{2.1}$$

where

$$\langle \Lambda x, \partial_x \rangle v = ({}^t(\Lambda x)\partial_x) v = \sum_{j=1}^n (\Lambda x)_j \frac{\partial v}{\partial x_j}, \tag{2.2}$$

with $(\Lambda x)_j$ being the j-th component of Λx. We consider the following system of equations

$$L_\Lambda v = R(v), \tag{2.3}$$

where $v = {}^t(v_1, \dots, v_n)$ is an unknown vector function and $R(x) = {}^t(R_1(x), \dots, R_n(x))$ is a given holomorphic function in some neighborhood of the origin of $\mathbb{C}^n$ such that $R(x) = O(|x|^2)$ when $|x| \to 0$. If we set $v(x) = x + u(x)$, $u(x) = O(|x|^2)$, then we obtain the so-called homology equation

$$L_\Lambda u = R(x + u). \tag{2.4}$$

Remark 1. The equation (2.4) appears as the linearizing equation of the vector field $\mathcal{X} := {}^t(\Lambda y + R(y))\partial_y$, where $R(y) = O(|y|^2)$. Indeed, if the change of the variables $y = v(x)$ linearizes $\mathcal{X}$, then, by setting $X(y) = \Lambda y + R(y)$, we have

$$\mathcal{X} = {}^tX(y)\frac{\partial}{\partial y} = {}^tX(v(x)) {}^t\Big(\frac{\partial x}{\partial y}\Big)\frac{\partial}{\partial x} = {}^tX(v) {}^t\Big(\frac{\partial v}{\partial x}\Big)^{-1}\frac{\partial}{\partial x} = {}^t(\Lambda x)\frac{\partial}{\partial x}. \tag{2.5}$$

It follows that $\Big(\frac{\partial v}{\partial x}\Big)^{-1} X(v) = \Lambda x$. Hence we have (2.4).

We introduce the parameter η in (2.3) and (2.4) in a singular perturbative way

$$\eta^{-1}\frac{\partial v}{\partial x}\Lambda x - \Lambda v = R(v), \tag{2.6}$$

$$\eta^{-1}\frac{\partial u}{\partial x}\Lambda x - \Lambda u = R(x+u),\ u = O(|x|^2). \tag{2.7}$$

For the sake of simplicity we consider (2.7) in the following. Moreover, we assume that Λ is put in a Jordan normal form. We note that we do not assume that Λ is semi-simple.

§ 3. WKB Solution

The WKB solution $u_W(x, \eta)$ of (2.7) is the formal power series in η^{-1} of the form

$$u_W(x, \eta) = v_0(x) + \eta^{-1}v_1(x) + \eta^{-2}v_2(x) + \cdots, \quad v_j(x) = O(|x|^2), \tag{3.1}$$

where $v_j(x)$ is holomorphic in some neighborhood of the origin independent of j.

We set

$$\mathcal{L}u := \frac{\partial u}{\partial x}\Lambda x. \tag{3.2}$$

We substitute (3.1) into (2.7) and compare the coefficients of the powers of η^{-1}. We see that the left-hand side of (2.7) is equal to

$$\sum_{\nu=0}^{\infty}(\eta^{-1}\mathcal{L} - \Lambda)v_\nu(x)\eta^{-\nu}. \tag{3.3}$$

On the other hand we have

$$\begin{aligned} R(x+u_W) &= R(x + v_0 + v_1\eta^{-1} + v_2\eta^{-2} + \cdots) \\ &= R(x+v_0) + \eta^{-1}\nabla R(x+v_0)v_1 + O(\eta^{-2}). \end{aligned} \tag{3.4}$$

Comparing the coefficients of $\eta^0 = 1$ and η^{-1} we obtain

$$\Lambda v_0(x) + R(x + v_0(x)) = 0. \tag{3.5}$$

$$\mathcal{L}v_0 = \Lambda v_1 + \nabla R(x+v_0)v_1. \tag{3.6}$$

In order to determine v_0 and v_1 from the above recurrence relations we need a definition.

Definition 3.1. The point x such that

$$\det(\Lambda + \nabla R(x+v_0)) = 0 \tag{3.7}$$

is called the turning point of the equation (2.7).

Let us assume

$$\det \Lambda \neq 0. \tag{3.8}$$

Because $v_0(x) = O(|x|^2)$, $R(x) = O(|x|^2)$, it follows from (3.5) and the implicit function theorem that v_0 is holomorphic in some neighborhood of the origin. In order to determine v_1 from (3.6) we note that the origin $x = 0$ is not a turning point of (2.7).

We can determine v_j inductively. Indeed, we have

$$\mathcal{L}v_{j-1} = \Lambda v_j + \nabla R(x + v_0)v_j + (\text{terms consisting of } v_i,\ i \le j-1). \tag{3.9}$$

Therefore we have proved the following

Theorem 3.2. *The WKB solution* (3.1) *can be uniquely determined as the formal power series of* η^{-1} *with coefficients* $v_j(x)$ *holomorphic in some neighborhood of the origin independent of* j*.*

§ 4. Borel Resummation of the WKB Solution With Respect to a Parameter

We set $V(x,\eta) := \sum_{\nu=1}^{\infty} v_\nu(x)\eta^{-\nu}$. Then the WKB solution is given by $u_W(x,\eta) = v_0(x) + V(x,\eta)$. Hence we may consider the resummation of $V(x,\eta)$.

We define the Borel transform $\widehat{V}(\zeta)$ of $V(x,\eta)$ with respect to η by

$$\widehat{V}(\zeta) := \mathcal{B}(V(x,\cdot))(\zeta) = \sum_{\nu=1}^{\infty} v_\nu(x)\frac{\zeta^{\nu-1}}{(\nu-1)!}. \tag{4.1}$$

Because $v_\nu(x)$ is holomorphic in some neighborhood of $x = 0$ independent of ν, we have the expansion $v_\nu(x) = \sum_\alpha v_{\nu,\alpha}x^\alpha$. Then the right-hand side of (4.1) is equal to

$$\sum_{\nu=1}^{\infty}\Big(\sum_\alpha v_{\nu,\alpha}x^\alpha\Big)\frac{\zeta^{\nu-1}}{(\nu-1)!}. \tag{4.2}$$

If the right-hand side absolutely converges, then we can change the order of the summation

$$\mathcal{B}(V(x,\cdot))(\zeta) = \sum_\alpha\sum_\nu v_{\nu,\alpha}\frac{\zeta^{\nu-1}}{(\nu-1)!}x^\alpha. \tag{4.3}$$

We define the Borel-Laplace resum $V_W(x,\eta)$ by

$$V_W(x,\eta) := \sum_\alpha L\Big(\sum_{\nu=1}^{\infty} v_{\nu,\alpha}\frac{\zeta^{\nu-1}}{(\nu-1)!}\Big)x^\alpha, \tag{4.4}$$

where L is the Laplace transform given by

$$Lf(\eta) := \int_0^\infty e^{-\zeta\eta}f(\zeta)d\zeta, \tag{4.5}$$

where we assume the suitable growth condition on f. Finally we define the Borel-Laplace resummation $U_W(x,\eta)$ of the WKB solution $u_W(x,\eta)$ by

$$U_W(x,\eta) := v_0(x) + V_W(x,\eta). \tag{4.6}$$

Let λ_j $(j = 1, 2, \ldots, n)$ be the eigenvalues of Λ counted with multiplicity. We say that the Poincaré condition is satisfied if the convex hull of λ_j $(j = 1, 2, \ldots, n)$ in the complex plane does not contain the origin. We say that $\eta \in \mathbb{C}$ is a resonance if there exist k, $1 \le k \le n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n(2)$ such that

$$\sum_{j=1}^{n} \lambda_j \alpha_j - \eta\lambda_k = 0. \tag{4.7}$$

Let ξ, $0 \le \xi \le 2\pi$ and $\theta > 0$. We define the sector $S_{\xi,\theta}$ by

$$S_{\xi,\theta} := \{\eta \in \mathbb{C}; |\arg\eta - \xi| < \theta/2\}. \tag{4.8}$$

Then we have

Theorem 4.1. *Suppose that either the Poincaré condition or the one*

$$\exists \tau_0, 0 \le \tau_0 \le \pi, e^{-\sqrt{-1}\tau_0}\lambda_j \in \mathbb{R}\setminus\{0\}, j = 1, 2, \ldots, n \tag{4.9}$$

is satisfied. Then there exist ξ, $\theta > 0$ and a neighborhood Ω of $x = 0$ such that $U_W(x,\eta)$ is holomorphic in $(x,\eta) \in \Omega \times S_{\xi,\theta}$ and solves (2.7).

The WKB solution $u_W(x,\eta)$ is a G^2-asymptotic expansion of $U_W(x,\eta)$ in $\Omega \times S_{\xi,\theta}$ when $\eta \to \infty$, $\eta \in S_{\xi,\theta}$. Namely, for every $N \ge 0$ and $R > 0$, there exist $C > 0$ and $K > 0$ such that

$$|U_W(x,\eta) - \sum_{\nu=0}^{N} \eta^{-\nu} v_\nu(x)| \le \frac{CK^N N!}{|\eta|^{N+1}}, \quad \forall (x,\eta) \in \Omega \times S_{\xi,\theta},\ |\eta| \ge R. \tag{4.10}$$

Remark 2. This theorem is valid for those equations with small denominators as well as with infinite resonances.

Theorem 4.2. *Suppose that*

$$|\arg\lambda_j| < \frac{\pi}{4}, \quad j = 1, 2, \ldots, n. \tag{4.11}$$

Then there exist ξ, $\theta > \pi$ and a neighborhood Ω of $x = 0$ such that $U_W(x,\eta)$ is holomorphic in $\Omega \times S_{\xi,\theta}$, and it is a unique solution of (2.7). *The function $U_W(x,\eta)$ is the Borel sum of $u_W(x,\eta)$.*

The proof of the former half of Theorem 4.1 is given in [6]. The complete proofs of the theorems will be published elsewhere.

§ 5. Analytic Continuation of the Resummed WKB Solution

In this section we study the solvability of the equation (2.7) with $\eta = 1$ by the analytic continuation of the resummed WKB solution with respect to η. We assume that $\eta = 1$ is not a resonance. First we study the analytic continuation of $U_W(x, \eta)$ in case the Poincaré condition is verified.

Theorem 5.1. *Suppose that the Poincaré condition is satisfied and that $\eta = 1$ is not a resonance. Then the WKB solution $U_W(x, \eta)$ can be analytically continued along any path on $\mathbb{C}$ which avoids resonances as a single-valued holomorphic function of η up to $\eta = 1$. The analytic continuation of $U_W(x, \eta)$ to $\eta = 1$ coincides with the classical Poincaré series solution.*

Sketch of Proof. First we note that the resummed WKB solution coincides with the Poincaré series if η is in some sector and x is in some neighborhood of the origin. On the other hand, the Poincaré series is an infinite sum of negative powers of $\langle \lambda, \alpha \rangle - \eta\lambda_k$ $(1 \le k \le n,\ \alpha \in \mathbb{Z}_+^n(2))$ whose coefficients are polynomials of x. Because the series converges when x is in some neighborhood of the origin and η is in a bounded open set containing 1 whose closure is contained in the complement of the resonances, we can make the analytic continuation. Hence the theorem follows. □

Next we study the analytic continuation in case the Poincaré condition is not verified. We assume that there exists $n_s \in \mathbb{Z}$, $1 \le n_s \le n$ such that

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_{n_s} < 0 < \lambda_{n_s+1} \le \cdots \le \lambda_n. \tag{5.1}$$

In the following we assume that Λ is put in a Jordan normal form for the sake of simplicity. Let $e_j = {}^t(0, \dots, 1, \cdots, 0)$ be the j-th unit vector. Let J_0 be defined by

$$J_0 := \{j; 1 \le j \le n,\ e_j \text{ is an eigenvector of } \Lambda\}. \tag{5.2}$$

We note that if $\mathcal{X}$ is semi-simple, then we can take $J_0 = \{1, 2, \dots, n\}$. For a small number $r_0 > 0$ we define

$$S := \prod_{j \in J_0 \text{ and } j \le n_s} \{z \in \mathbb{C}; z \in S_{0,\theta}, |z| < r_0\} \times \prod_{j > n_s \text{ or } j \notin J_0} \{z \in \mathbb{C}; z \in S_{0,\theta}\}. \tag{5.3}$$

Let $\alpha_s = (\alpha_1^s, \alpha_2^s, \dots, \alpha_n^s) \in \mathbb{Z}_+^n$ be such that $\alpha_j^s = 0$ if $j \notin J_0$ or $j > n_s$ and $\langle \Lambda, \alpha_s \rangle - \lambda_j < 0$, $j = 1, 2, \dots, n$.

Let $\widetilde{O}(X)$ be the set of holomorphic functions in an open set X. We define $\mathcal{O}(X)$ the n-product of $\widetilde{O}(X)$, namely $\mathcal{O}(X) := \widetilde{O}(X) \times \cdots \times \widetilde{O}(X)$. Let Σ_0 be a neighborhood of $\overline{S}$, where $\overline{S}$ is the closure of S, and $R \in \mathcal{O}(\Sigma_0)$. We assume

$$R(x) = x^{\alpha_s} \tilde{R}(x), \quad \widetilde{R}(x) \in \mathcal{O}(\Sigma_0), \tag{5.4}$$

$$\sup_{x\in\Sigma_0}(|\widetilde{R}(x)| + |\nabla\widetilde{R}(x)|) < \varepsilon, \tag{5.5}$$

with $\varepsilon > 0$ chosen later.

Example 5.2. Let $K > 0$ be a small constant, and let $C_j > 0$ $(j = 1, 2, \ldots, n)$ and $0 < \theta < \pi/2$. We define $S := S_{0,\theta} \times \cdots \times S_{0,\theta}$. Let $R(x)$ be given by $R(x) = Kx^{\alpha_s}\exp(-\sum_1^n C_j x_j)$. Then (5.4) and (5.5) are satisfied if we take $K > 0$ sufficiently small depending on ε.

We have

Theorem 5.3. *There exist $\varepsilon > 0$ and a neighborhood Ω of $\eta = 1$ such that, for every $R \in \mathcal{O}(\Sigma_0)$ satisfying the above conditions, there exists a solution $u_S(x,\eta)$ of* (2.7) *which is holomorphic in $(x,\eta) \in S \times \Omega$.*

Remark 3. We shall remark about the relation of the resummed WKB solution $U_W(x,\eta)$ and the solution $u_S(x,\eta)$. If $\widetilde{R}(x)$ depends only on the stable variable $x_1, \ldots, x_{n_s}$, then we have $u_S(x,\eta) = U_W(x,\eta)$ if $(x,\eta) \in S \times \Omega$. Namely, $U_W(x,\eta)$ can be analytically continued up to $\eta = 1$ if $x \in S$. It is an open problem whether the assertion holds without assuming that $\widetilde{R}(x)$ depends only on the stable variables. We will discuss the problem in a future paper.

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