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Coupled Painlevé VI Systems in Dimension Four with Affine Weyl Group Symmetry of Type $D_6^{(1)}$, II

By

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Abstract

We give a reformulation of a six-parameter family of coupled Painlevé VI systems with affine Weyl group symmetry of type $D_6^{(1)}$ from the viewpoint of its symmetry and holomorphy properties.

§ 1. Introduction

In [11], [12], we proposed a 6-parameter family of four-dimensional coupled Painlevé VI systems with affine Weyl group symmetry of type $D_6^{(1)}$. This system can be considered as a generalization of the Painlevé VI system. In this paper, from the viewpoint of its symmetry and holomorphy properties we give a reformulation of this system [13] explicitly given by

$$\begin{aligned}
&\frac{dq_1}{dt} = \frac{\partial H}{\partial p_1}, \quad \frac{dp_1}{dt} = -\frac{\partial H}{\partial q_1}, \quad \frac{dq_2}{dt} = \frac{\partial H}{\partial p_2}, \quad \frac{dp_2}{dt} = -\frac{\partial H}{\partial q_2},\\
&\quad H = H_{\mathrm{VI}}(q_1, p_1, \eta, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_3 + \alpha_6)\\
&\qquad + H_{\mathrm{VI}}(q_2, p_2, \eta, t; \alpha_0 + 2\alpha_2 + \alpha_3, \alpha_1 + \alpha_3, \alpha_4, \alpha_5, \alpha_6)\\
&\qquad + \frac{2(q_1 - \eta)q_2\{(q_1 - t)p_1 + \alpha_2\}\{(q_2 - 1)p_2 + \alpha_4\}}{t(t-1)(t-\eta)} \quad (\eta \in \mathbb{C} - \{0, 1\}).
\end{aligned} \tag{1.1}$$

Here q_1, p_1, q_2, p_2 denote unknown complex variables, and $\alpha_0, \alpha_1, \dots, \alpha_6$ are complex parameters satisfying the relation $\alpha_0 + \alpha_1 + 2(\alpha_2 + \alpha_3 + \alpha_4) + \alpha_5 + \alpha_6 = 1$, where the symbol $H_{\mathrm{VI}}(q, p, \eta, t; \beta_0, \beta_1, \beta_2, \beta_3, \beta_4)$ is given in Section 2.

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If we take the limit $\eta \to \infty$, we obtain the Hamiltonian system with well-known Hamiltonian $\widetilde{H}$ (see [11])

$$\begin{aligned} &\frac{dq_1}{dt} = \frac{\partial \widetilde{H}}{\partial p_1}, \quad \frac{dp_1}{dt} = -\frac{\partial \widetilde{H}}{\partial q_1}, \quad \frac{dq_2}{dt} = \frac{\partial \widetilde{H}}{\partial p_2}, \quad \frac{dp_2}{dt} = -\frac{\partial \widetilde{H}}{\partial q_2}, \\ &\widetilde{H} = \widetilde{H}_{\mathrm{VI}}(q_1, p_1, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_3 + \alpha_6) \\ &\qquad + \widetilde{H}_{\mathrm{VI}}(q_2, p_2, t; \alpha_0 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_4, \alpha_5, \alpha_6) \\ &\qquad + \frac{2(q_1 - t)p_1 q_2 \{(q_2 - 1)p_2 + \alpha_4\}}{t(t-1)}, \end{aligned} \tag{1.2}$$

where the symbol $\widetilde{H}_{\mathrm{VI}}$ is also given in Section 2.

Here we review the holomorphy conditions of the system (1.2) (see [11]). Let us consider a polynomial Hamiltonian system with Hamiltonian $H \in \mathbb{C}(t)[q_1, p_1, q_2, p_2]$. We assume that

($A1$) $\deg(H) = 5$ with respect to q_1, p_1, q_2, p_2.
($A2$) This system becomes again a polynomial Hamiltonian system in each coordinate system (x_i, y_i, z_i, w_i) $(i = 0, 2, 3, 4, 5, 6)$:

$$\begin{aligned} &r_0' : x_0 = -((q_1 - t)p_1 - \alpha_0)p_1, \ y_0 = 1/p_1, \ z_0 = q_2, \ w_0 = p_2, \\ &r_2' : x_2 = 1/q_1, \ y_2 = -q_1(q_1 p_1 + \alpha_2), \ z_2 = q_2, \ w_2 = p_2, \\ &r_3' : x_3 = -((q_1 - q_2)p_1 - \alpha_3)p_1, \ y_3 = 1/p_1, \ z_3 = q_2, \ w_3 = p_2 + p_1, \\ &r_4' : x_4 = q_1, \ y_4 = p_1, \ z_4 = 1/q_2, \ w_4 = -q_2(q_2 p_2 + \alpha_4), \\ &r_5' : x_5 = q_1, \ y_5 = p_1, \ z_5 = -((q_2 - 1)p_2 - \alpha_5)p_2, \ w_5 = 1/p_2, \\ &r_6' : x_6 = q_2, \ y_6 = p_1, \ z_6 = -p_2(q_2 p_2 - \alpha_6), \ w_6 = 1/p_2. \end{aligned} \tag{1.3}$$

($A3$) In addition to the assumption ($A2$), the Hamiltonian system in the coordinate r_2 becomes again a polynomial Hamiltonian system in the coordinate system (x_1, y_1, z_1, w_1):

$$r_1' : x_1 = -(x_2 y_2 - \alpha_1)y_2, \ y_1 = 1/y_2, \ z_1 = z_2, \ w_1 = w_2. \tag{1.4}$$

Then such a system coincides with the system (1.2).

In this paper, we make a reformulation to obtain a clear description of invariant divisors, birational symmetries and holomorphy conditions for the system (1.2). Our way is stated as follows:

1. We symmetrize the holomorphy conditions r_i' of the system (1.2).
2. By using these conditions and *polynomiality* of the Hamiltonian, we easily obtain the polynomial Hamiltonian of the system (1.1).

This paper is organized as follows. In Section 2, we give a reformulation of Hamiltonian of P_{VI} and its symmetry and holomorphy. In Section 3, we state our main results for the system of type $D_6^{(1)}$. After we review the notion of accessible singularity in Section 4, we will state the relation between some accessible singularities of the system (1.1) and the holomorphy conditions r_i given in Section 3. After we present a compactification of $\mathbb{C}^4$ which is the phase space of the system (1.1), we will construct its meromorphic solution spaces corresponding to r_i $(i = 1, 2, \dots, 6)$.

§ 2. Reformulation of P_{VI}-Case

The sixth Painlevé system can be written as the Hamiltonian system (cf. [2], [4])

$$\begin{aligned}
&\frac{dq}{dt} = \frac{\partial H_{\mathrm{VI}}}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H_{\mathrm{VI}}}{\partial q},\\
&t(t-1)(t-\eta)H_{\mathrm{VI}}(q,p,\eta,t;\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4)\\
&\qquad = q(q-1)(q-\eta)(q-t)p^2 + \{\alpha_1(t-\eta)q(q-1) + 2\alpha_2 q(q-1)(q-\eta)\\
&\qquad\quad + \alpha_3(t-1)q(q-\eta) + \alpha_4 t(q-1)(q-\eta)\}p\\
&\qquad\quad + \alpha_2\{(\alpha_1+\alpha_2)(t-\eta) + \alpha_2(q-1) + \alpha_3(t-1) + t\alpha_4\}q\\
&\qquad\qquad\qquad\qquad (\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1, \quad \eta \in \mathbb{C} - \{0,1\}).
\end{aligned} \tag{2.1}$$

The equation for q is given by

$$\begin{aligned}
\frac{d^2q}{dt^2} &= \frac{1}{2}\Big(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t} + \frac{1}{q-\eta}\Big)\Big(\frac{dq}{dt}\Big)^2 - \Big(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t} + \frac{1}{t-\eta}\Big)\frac{dq}{dt}\\
&\quad + \frac{q(q-1)(q-t)(q-\eta)}{t^2(t-1)^2(t-\eta)^2}\Big\{\frac{\alpha_1^2}{2}\frac{\eta(\eta-1)(t-\eta)}{(q-\eta)^2} + \frac{\alpha_4^2}{2}\frac{\eta t}{q^2}\\
&\qquad + \frac{\alpha_3^2}{2}\frac{(\eta-1)(1-t)}{(q-1)^2} + \frac{(1-\alpha_0^2)}{2}\frac{t(t-1)(t-\eta)}{(q-t)^2}\Big\}.
\end{aligned} \tag{2.2}$$

If we take the limit $\eta \to \infty$, we obtain the sixth Painlevé system P_{VI} with well-known Hamiltonian:

$$\begin{aligned}
&\frac{dq}{dt} = \frac{\partial \widetilde{H}_{\mathrm{VI}}}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial \widetilde{H}_{\mathrm{VI}}}{\partial q},\\
&\widetilde{H}_{\mathrm{VI}}(q,p,t;\delta_0,\delta_1,\delta_2,\delta_3,\delta_4)\\
&\quad = \frac{1}{t(t-1)}[p^2(q-t)(q-1)q - \{(\delta_0-1)(q-1)q + \delta_3(q-t)q\\
&\qquad + \delta_4(q-t)(q-1)\}p + \delta_2(\delta_1+\delta_2)q] \quad (\delta_0 + \delta_1 + 2\delta_2 + \delta_3 + \delta_4 = 1),
\end{aligned} \tag{2.3}$$

whose equation for q is given by

$$
\begin{aligned}
\frac{d^2q}{dt^2} &= \frac{1}{2}\Big(\frac{1}{q}+\frac{1}{q-1}+\frac{1}{q-t}\Big)\Big(\frac{dq}{dt}\Big)^2-\Big(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{q-t}\Big)\frac{dq}{dt}\\
&\quad+\frac{q(q-1)(q-t)}{t^2(t-1)^2}\left\{\frac{\alpha_1^2}{2}-\frac{\alpha_4^2}{2}\frac{t}{q^2}-\frac{\alpha_3^2}{2}\frac{(1-t)}{(q-1)^2}+\frac{(1-\alpha_0^2)}{2}\frac{t(t-1)}{(q-t)^2}\right\}.
\end{aligned}
\tag{2.4}
$$

The system (2.1) has extended affine Weyl group symmetry of type $D_4^{(1)}$, whose generators s_i, π_j are given by

$$
\begin{aligned}
&s_0(q,p,t;\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4)=(q,p-\frac{\alpha_0}{q-t},p,t;-\alpha_0,\alpha_1,\alpha_2+\alpha_0,\alpha_3,\alpha_4),\\
&s_1(q,p,t;\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4)=(q,p-\frac{\alpha_1}{q-\eta},t;\alpha_0,-\alpha_1,\alpha_2+\alpha_1,\alpha_3,\alpha_4),\\
&s_2(q,p,t;\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4)\\
&\qquad=(q+\frac{\alpha_2}{p},p,t;\alpha_0+\alpha_2,\alpha_1+\alpha_2,-\alpha_2,\alpha_3+\alpha_2,\alpha_4+\alpha_2),\\
&s_3(q,p,t;\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4)=(q,p-\frac{\alpha_3}{q-1},t;\alpha_0,\alpha_1,\alpha_2+\alpha_3,-\alpha_3,\alpha_4),\\
&s_4(q,p,t;\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4)=(q,p-\frac{\alpha_4}{q},t;\alpha_0,\alpha_1,\alpha_2+\alpha_4,\alpha_3,-\alpha_4),\\
&\pi_1(q,p,t;\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4)=(1-q,-p,1-\eta,1-t;\alpha_0,\alpha_1,\alpha_2,\alpha_4,\alpha_3),\\
&\pi_2(q,p,t;\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4)=(\frac{\eta-q}{\eta-1},(1-\eta)p,\frac{\eta}{\eta-1},\frac{\eta-t}{\eta-1};\alpha_0,\alpha_4,\alpha_2,\alpha_3,\alpha_1),\\
&\pi_3(q,p,t;\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\Big(\frac{(\eta-1)^2(q-t)}{\{\eta(t-2)+1\}q+(\eta-\eta^2-1)t+\eta^2},\\
&\qquad(1-t)p+\frac{(q-1)\{(q-1)p+\alpha_2\}\{\eta(t-2)+1\}}{(\eta-1)^2(t-1)}\\
&\qquad\quad+\frac{(q-t)\{(q-t)p+\alpha_2\}\{\eta(t-2)+1\}}{\eta(t-1)(t-\eta)},\\
&\qquad 1-\eta,\frac{(\eta-1)^2t}{t-\eta t+\eta^2(t-1)};\alpha_4,\alpha_1,\alpha_2,\alpha_3,\alpha_0\Big).
\end{aligned}
\tag{2.5}
$$

Let us consider a polynomial Hamiltonian system with Hamiltonian $H\in\mathbb{C}(t)[q,p]$. We assume that

($A1$) $\deg(H)=6$ with respect to q,p.

($A2$) This system becomes again a polynomial Hamiltonian system in each coordinate r_j $(j=0,1,2,3,4)$:

$$
\begin{aligned}
&r_0: x_0=-((q-t)p-\alpha_0)p,\ y_0=1/p, \quad r_1: x_1=-((q-\eta)p-\alpha_1)p,\ y_1=1/p,\\
&r_2: x_2=1/q,\ y_2=-(qp+\alpha_2)q, \qquad\quad r_3: x_3=-((q-1)p-\alpha_3)p,\ y_3=1/p,\\
&r_4: x_4=-(qp-\alpha_4)p,\ y_4=1/p.
\end{aligned}
\tag{2.6}
$$

Then such a system coincides with the system (2.1).

The phase space of the system (2.1) (resp. (2.3)) can be characterized by the rational surface of type $D_4^{(1)}$ (see [6], [8], [9]). Figure 1 denotes the accessible singular points and the resolution process for each system.

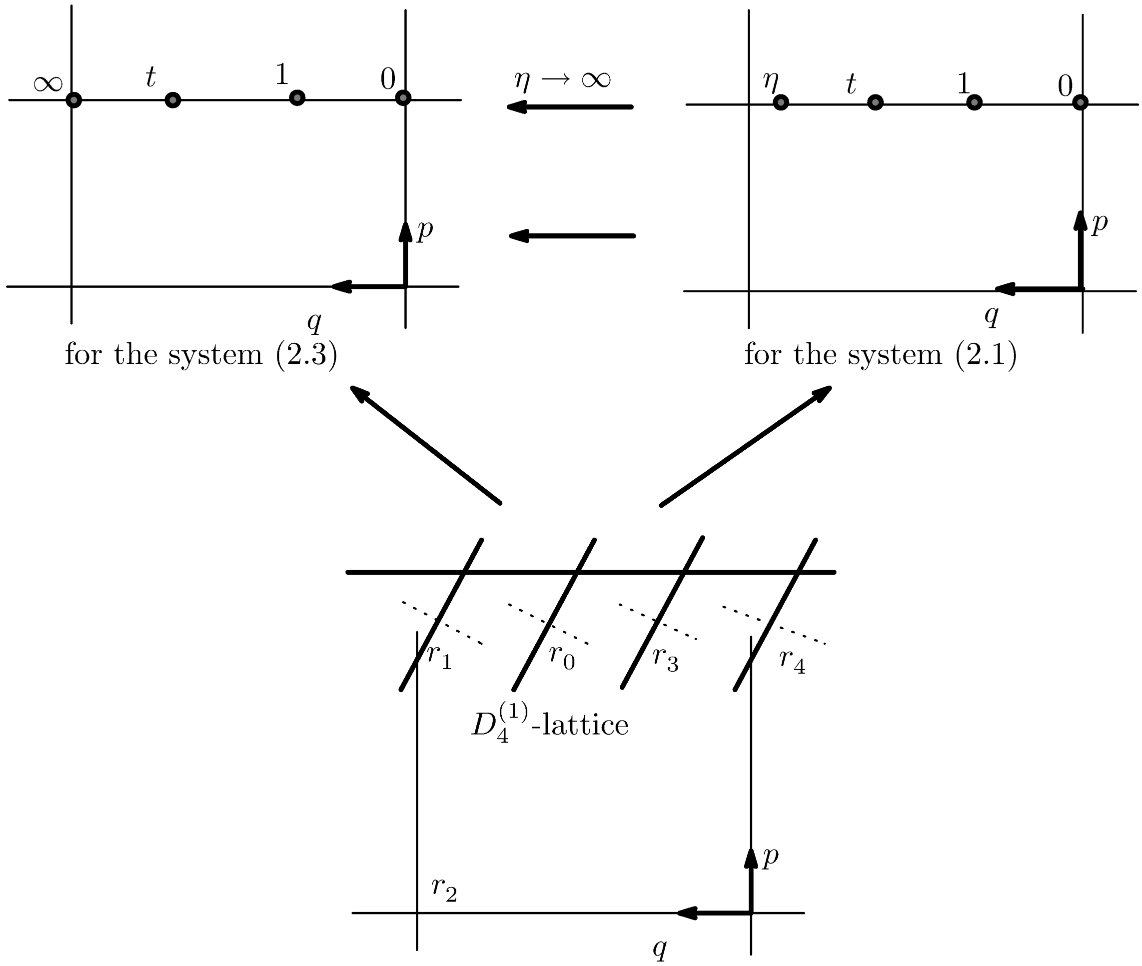

Figure 1. Each figure denotes the Hirzebruch surface. Each bullet denotes the accessible singular point of each system. It is well-known that each point can be resolved by blowing-up at two times (see [6], [8], [9]). By these transformations, we obtain the rational surface of type $D_4^{(1)}$ for each system.

We remark that the system (2.1) has the following invariant divisors:

parameter's relation	invariant divisors
$\alpha_0 = 0$	$f_0 := q - t$
$\alpha_1 = 0$	$f_1 := q - \eta$
$\alpha_2 = 0$	$f_2 := p$
$\alpha_3 = 0$	$f_3 := q - 1$
$\alpha_4 = 0$	$f_4 := q$

§3. The Case of Type $D_6^{(1)}$

Theorem 3.1. *The system* (1.1) *admits extended affine Weyl group symmetry of type* $D_6^{(1)}$ *as the group of its Bäcklund transformations, whose generators* s_i, π_j *are explicitly given as follows*: *with the notation* $(*) := (q_1, p_1, q_2, p_2, \eta, t; \alpha_0, \alpha_1, \dots, \alpha_6)$,

$$s_0 \colon (*) \mapsto (q_1, p_1 - \frac{\alpha_0}{q_1 - t}, q_2, p_2, \eta, t; -\alpha_0, \alpha_1, \alpha_2 + \alpha_0, \alpha_3, \alpha_4, \alpha_5, \alpha_6),$$
$$s_1 \colon (*) \mapsto (q_1, p_1 - \frac{\alpha_1}{q_1 - \eta}, q_2, p_2, \eta, t; \alpha_0, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6),$$
$$s_2 \colon (*) \mapsto (q_1 + \frac{\alpha_2}{p_1}, p_1, q_2, p_2, \eta, t; \alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4, \alpha_5, \alpha_6),$$
$$s_3 \colon (*) \mapsto (q_1, p_1 - \frac{\alpha_3}{q_1 - q_2}, q_2, p_2 + \frac{\alpha_3}{q_1 - q_2}, \eta, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3, \alpha_5, \alpha_6),$$
$$s_4 \colon (*) \mapsto (q_1, p_1, q_2 + \frac{\alpha_4}{p_2}, p_2, \eta, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3 + \alpha_4, -\alpha_4, \alpha_5 + \alpha_4, \alpha_6 + \alpha_4),$$
$$s_5 \colon (*) \mapsto (q_1, p_1, q_2, p_2 - \frac{\alpha_5}{q_2 - 1}, \eta, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 + \alpha_5, -\alpha_5, \alpha_6),$$
$$s_6 \colon (*) \mapsto (q_1, p_1, q_2, p_2 - \frac{\alpha_6}{q_2}, \eta, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 + \alpha_6, \alpha_5, -\alpha_6),$$
$$\begin{aligned}\pi_1 \colon (*) \\ \mapsto \Big(&\frac{(t-1)q_1}{t - q_1 - \eta t + \eta t q_1}, \frac{(-t + q_1 + \eta t - \eta t q_1)(tp_1 - q_1 p_1 - \alpha_2 - \eta t p_1 + \eta t q_1 p_1 + \alpha_2 \eta t)}{t(t-1)(\eta - 1)}, \\ &\frac{(t-1)q_2}{t - q_2 - \eta t + \eta t q_2}, \frac{(-t + q_2 + \eta t - \eta t q_2)(tp_2 - q_2 p_2 - \alpha_4 - \eta t p_2 + \eta t q_2 p_2 + \alpha_4 \eta t)}{t(t-1)(\eta - 1)}, \\ &\frac{1}{\eta}, \frac{\eta(t-1)}{t - \eta - \eta t + \eta^2 t}; \alpha_1, \alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\Big),\end{aligned}$$
$$\pi_2 \colon (*) \mapsto (1 - q_1, -p_1, 1 - q_2, -p_2, 1 - \eta, 1 - t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_5),$$
$$\begin{aligned}\pi_3 \colon (*) \mapsto \Big(&\frac{t(q_2 - \eta)}{t(q_2 - \eta) + \eta^2(t - q_2)}, \\ &\frac{(t(q_2 - \eta) + \eta^2(t - q_2))(t(q_2 - \eta)p_2 + \alpha_4(t - \eta^2) + \eta^2(t - q_2)p_2)}{t\eta^2(t - \eta)}, \\ &\frac{t(q_1 - \eta)}{t(q_1 - \eta) + \eta^2(t - q_1)}, \\ &\frac{(t(q_1 - \eta) + \eta^2(t - q_1))(t(q_1 - \eta)p_1 + \alpha_2(t - \eta^2) + \eta^2(t - q_1)p_1)}{t\eta^2(t - \eta)}, \\ &-\frac{1}{\eta - 1}, -\frac{(\eta - 1)t}{t - \eta t + \eta^2(t - 1)}; \alpha_5, \alpha_6, \alpha_4, \alpha_3, \alpha_2, \alpha_0, \alpha_1\Big).\end{aligned}$$

We note that these transformations s_i, π_j are birational and symplectic.

Theorem 3.2. *Consider a polynomial Hamiltonian system with Hamiltonian* $H \in \mathbb{C}(t)[q_1, p_1, q_2, p_2]$. *Assume that*

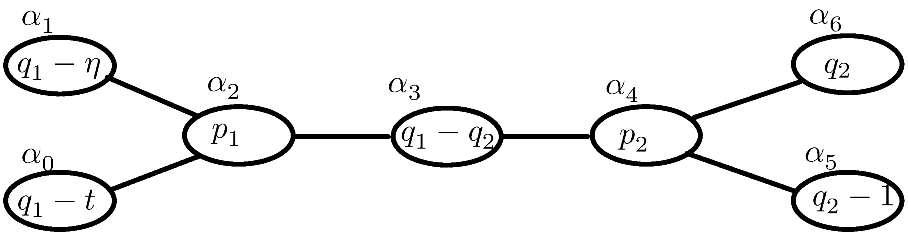

Figure 2. This figure denotes Dynkin diagram of type $D_6^{(1)}$.

$(A1)$ $\deg(H)=6$ *with respect to* q_1,p_1,q_2,p_2.

$(A2)$ *This system becomes again a polynomial Hamiltonian system in each coordinate system* (x_i,y_i,z_i,w_i) $(i=0,1,\dots,6)$:

$$\begin{aligned}
&r_0: x_0=-((q_1-t)p_1-\alpha_0)p_1,\ y_0=1/p_1,\ z_0=q_2,\ w_0=p_2,\\
&r_1: x_1=-((q_1-\eta)p_1-\alpha_1)p_1,\ y_1=1/p_1,\ z_1=q_2,\ w_1=p_2 \quad (\eta\in\mathbb{C}-\{0,1\}),\\
&r_2: x_2=1/q_1,\ y_2=-q_1(q_1p_1+\alpha_2),\ z_2=q_2,\ w_2=p_2,\\
&r_3: x_3=-((q_1-q_2)p_1-\alpha_3)p_1,\ y_3=1/p_1,\ z_3=q_2,\ w_3=p_2+p_1,\\
&r_4: x_4=q_1,\ y_4=p_1,\ z_4=1/q_2,\ w_4=-q_2(q_2p_2+\alpha_4),\\
&r_5: x_5=q_1,\ y_5=p_1,\ z_5=-((q_2-1)p_2-\alpha_5)p_2,\ w_5=1/p_2,\\
&r_6: x_6=q_1,\ y_6=p_1,\ z_6=-p_2(q_2p_2-\alpha_6),\ w_6=1/p_2.
\end{aligned} \tag{3.1}$$

Then such a system coincides with the system (1.1).

The proof is similar to [10].

Proposition 3.3. *The system* (1.1) *has the following invariant divisors*:

parameter's relation	*invariant divisors*
$\alpha_0=0$	$f_0:=q_1-t$
$\alpha_1=0$	$f_1:=q_1-\eta$
$\alpha_2=0$	$f_2:=p_1$
$\alpha_3=0$	$f_3:=q_1-q_2$
$\alpha_4=0$	$f_4:=p_2$
$\alpha_5=0$	$f_5:=q_2-1$
$\alpha_6=0$	$f_6:=q_2$

§ 4. Accessible Singularities

Let us review the notion of accessible singularity. Let B be a connected open domain in $\mathbb{C}$ and $\pi\colon \mathcal{W}\to B$ a smooth proper holomorphic map. We assume that

$\mathcal{H} \subset \mathcal{W}$ is a normal crossing divisor which is flat over B. Let us consider a rational vector field $\widetilde{v}$ on $\mathcal{W}$ satisfying the condition

$$\widetilde{v} \in H^0(\mathcal{W}; \Theta_{\mathcal{W}}(-\log \mathcal{H})(\mathcal{H})).$$

Fixing $t_0 \in B$ and $P \in \mathcal{W}_{t_0}$, we can take a local coordinate system $(x_1, x_2, \ldots, x_n)$ of $\mathcal{W}_{t_0}$ centered at P such that $\mathcal{H}_{\mathrm{smooth}}$ can be defined by the local equation $x_1 = 0$. Since $\widetilde{v} \in H^0(\mathcal{W}; \Theta_{\mathcal{W}}(-\log \mathcal{H})(\mathcal{H}))$, we can write down the vector field $\widetilde{v}$ near $P = (0, 0, \ldots, 0, t_0)$ as follows:

$$\widetilde{v} = \frac{\partial}{\partial t} + a_1 \frac{\partial}{\partial x_1} + \frac{a_2}{x_1} \frac{\partial}{\partial x_2} + \cdots + \frac{a_n}{x_1} \frac{\partial}{\partial x_n}. \tag{4.1}$$

This vector field defines the following system of differential equations

$$\frac{dx_1}{dt} = a_1(x_1, \ldots, x_n, t), \quad \frac{dx_2}{dt} = \frac{a_2(x_1, \ldots, x_n, t)}{x_1}, \ldots, \frac{dx_n}{dt} = \frac{a_n(x_1, \ldots, x_n, t)}{x_1}. \tag{4.2}$$

Here $a_i(x_1, x_2, \ldots, x_n, t)$, $i = 1, 2, \ldots, n$, are holomorphic functions defined near $P = (0, \ldots, 0, t_0)$.

Definition 4.1. With the notation above, assume that the rational vector field $\widetilde{v}$ on $\mathcal{W}$ satisfies the condition

$$\widetilde{v} \in H^0(\mathcal{W}; \Theta_{\mathcal{W}}(-\log \mathcal{H})(\mathcal{H})). \tag{A}$$

We say that $\widetilde{v}$ has an *accessible singularity* at $P = (0, 0, \ldots, 0, t_0)$ if

$$x_1 = 0 \text{ and } a_i(0, 0, \ldots, 0, t_0) = 0 \quad \text{for every} \quad i, \; 2 \le i \le n.$$

If $P \in \mathcal{H}_{\mathrm{smooth}}$ is not an accessible singularity, all solutions of the ordinary differential equation passing through P are vertical solutions, that is, the solutions are contained in the fiber $\mathcal{W}_{t_0}$ over $t = t_0$. If $P \in \mathcal{H}_{\mathrm{smooth}}$ is an accessible singularity, there may be a solution of (4.2) which passes through P and goes into the interior $\mathcal{W} - \mathcal{H}$ of $\mathcal{W}$.

Here we review the notion of *local index*. Let v be an algebraic vector field with an accessible singular point $\overrightarrow{p} = (0, 0, \ldots, 0)$ and $(x_1, x_2, \ldots, x_n)$ a coordinate system in a neighborhood centered at $\overrightarrow{p}$. Assume that the system associated with v near $\overrightarrow{p}$ can be written as

$$
(4.3)\quad \frac{d}{dt}Q\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
= \frac{1}{x_1}\left\{Q\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}Q^{-1}\cdot Q\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} x_1 f_1(x_1,x_2,\dots,x_n,t) \\ f_2(x_1,x_2,\dots,x_n,t) \\ \vdots \\ f_n(x_1,x_2,\dots,x_n,t) \end{pmatrix}\right\},
$$

$$
(f_i \in \mathbb{C}(t)[x_1,\dots,x_n],\ Q \in GL(n,\mathbb{C}(t)), a_i \in \mathbb{C}(t))
$$

where $f_1(x_1,x_2,\dots,x_n,t)$ is a polynomial which vanishes at $\overrightarrow{p}$ and $f_i(x_1,x_2,\dots,x_n,t)$, $i=2,3,\dots,n$ are polynomials of order at least 2 in $x_1,x_2,\dots,x_n$. We call ordered set of the eigenvalues $(a_1,a_2,\dots,a_n)$ *local index* at $\overrightarrow{p}$.

We remark that we are interested in the case where

$$
(4.4)\qquad (1, a_2/a_1, \dots, a_n/a_1) \in \mathbb{Z}^n.
$$

These properties suggest the possibilities that a_1 is the residue of the formal Laurent series:

$$
(4.5)\qquad y_1(t) = \frac{a_1}{(t-t_0)} + b_1 + b_2(t-t_0) + \cdots + b_n(t-t_0)^{n-1} + \cdots \quad (b_i \in \mathbb{C}),
$$

and the ratio $(a_2/a_1,\dots,a_n/a_1)$ is resonance data of the formal Laurent series of each $y_i(t)$ $(i=2,\dots,n)$, where $(y_1,\dots,y_n)$ is original coordinate system satisfying

$$
(x_1,\dots,x_n) = (f_1(y_1,\dots,y_n),\dots,f_n(y_1,\dots,y_n)),\ f_i(y_1,\dots,y_n) \in \mathbb{C}(t)(y_1,\dots,y_n).
$$

Example 4.2. For the Noumi-Yamada system of type $A_4^{(1)}$, its local index can be defined at each accessible singular point (cf. [15]).

§ 5. On Some Hamiltonian Structures of the System (1.1)

In this section, we will give the holomorphy conditions r_i $(i=0,1,\dots,6)$ by resolving some accessible singular loci of the system (1.1). Each of them contains a 3-parameter family of meromorphic solutions.

In order to consider the singularity analysis for the system (1.1), as a compactification of $\mathbb{C}^4$ which is the phase space of the system (1.1), first we take a 4-dimensional projective space $\mathbb{P}^4$. In this space the rational vector field $\tilde{v}$ associated with the system (1.1) satisfies the condition:

$$
\tilde{v} \in H^0(\mathbb{P}^4, \Theta_{\mathbb{P}^4}(-\log H)(3H)),
$$

where H denotes the boundary divisor $H \cong \mathbb{P}^3$. To calculate its accessible singularities, we must replace the compactification of $\mathbb{C}^4$ with the condition (A) given in Section 4. We present a complex manifold $\mathcal{S}$ obtained by gluing twelve copies $U_j \cong \mathbb{C}^4 \ni (X_j, Y_j, Z_j, W_j)$, $j = 0, 1, \dots, 11$:

$$U_j \times B = \mathbb{C}^4 \times B \ni (X_j, Y_j, Z_j, W_j, t) \ (j = 0, 1, \dots, 11)$$

via the following birational transformations:

$$\begin{aligned}
&0)\ X_0 = q_1, && Y_0 = p_1, && Z_0 = q_2, && W_0 = p_2,\\
&1)\ X_1 = 1/q_1, && Y_1 = -(q_1p_1 + \alpha_2)q_1, && Z_1 = q_2, && W_1 = p_2,\\
&2)\ X_2 = q_1, && Y_2 = p_1, && Z_2 = 1/q_2, && W_2 = -(q_2p_2 + \alpha_4)q_2,\\
&3)\ X_3 = q_1, && Y_3 = 1/p_1, && Z_3 = q_2, && W_3 = p_2/p_1,\\
&4)\ X_4 = q_1, && Y_4 = p_1/p_2, && Z_4 = q_2, && W_4 = 1/p_2,\\
&5)\ X_5 = 1/q_1, && Y_5 = -(q_1p_1 + \alpha_2)q_1, && Z_5 = 1/q_2, && W_5 = -(q_2p_2 + \alpha_4)q_2,\\
(5.1)\quad &6)\ X_6 = 1/q_1, && Y_6 = -\frac{1}{(q_1p_1 + \alpha_2)q_1}, && Z_6 = q_2, && W_6 = -\frac{p_2}{(q_1p_1 + \alpha_2)q_1},\\
&7)\ X_7 = 1/q_1, && Y_7 = -\frac{(q_1p_1 + \alpha_2)q_1}{p_2}, && Z_7 = q_2, && W_7 = 1/p_2,\\
&8)\ X_8 = 1/q_1, && Y_8 = -\frac{1}{(q_1p_1 + \alpha_2)q_1}, && Z_8 = 1/q_2, && W_8 = \frac{(q_2p_2 + \alpha_4)q_2}{(q_1p_1 + \alpha_2)q_1},\\
&9)\ X_9 = 1/q_1, && Y_9 = \frac{(q_1p_1 + \alpha_2)q_1}{(q_2p_2 + \alpha_4)q_2}, && Z_9 = 1/q_2, && W_9 = -\frac{1}{(q_2p_2 + \alpha_4)q_2},\\
&10)\ X_{10} = q_1, && Y_{10} = 1/p_1, && Z_{10} = 1/q_2, && W_{10} = -\frac{(q_2p_2 + \alpha_4)q_2}{p_1},\\
&11)\ X_{11} = q_1, && Y_{11} = -\frac{p_1}{(q_2p_2 + \alpha_4)q_2}, && Z_{11} = 1/q_2, && W_{11} = -\frac{1}{(q_2p_2 + \alpha_4)q_2}.
\end{aligned}$$

We note that the transformation

$$\pi\colon (q_1, p_1, q_2, p_2; \alpha_2, \alpha_4) \mapsto (q_2, p_2, q_1, p_1; \alpha_4, \alpha_2) \tag{5.2}$$

is an automorphism of $\mathcal{S}$.

The restriction $\{(q_1, p_1, q_2, p_2) \mid q_2 = p_2 = 0\}$ (resp. $\{(q_1, p_1, q_2, p_2) \mid q_1 = p_1 = 0\}$) of this manifold $\mathcal{S}$ is a Hirzebruch surface respectively. We remark that this generalization of the Hirzebruch surface is different from the one given by H. Kimura (see [3]).

The canonical divisor $K_{\mathcal{S}}$ of $\mathcal{S}$ is given by

$$\begin{aligned}
K_{\mathcal{S}} = -3\mathcal{H} = &\bigcup_{i \in \{3,6,8,10\}} \{(X_i, Y_i, Z_i, W_i) \in U_i \mid Y_i = 0\}\\
&\cup \bigcup_{j \in \{4,7,9,11\}} \{(X_j, Y_j, Z_j, W_j) \in U_j \mid W_j = 0\},
\end{aligned} \tag{5.3}$$

and satisfies the following relations:

$$(5.4)\qquad \begin{cases} dX_j \wedge dY_j \wedge dZ_j \wedge dW_j = dq_1 \wedge dp_1 \wedge dq_2 \wedge dp_2 \quad (j = 1, 2, 5), \\ dX_3 \wedge dY_3 \wedge dZ_3 \wedge dW_3 = -\dfrac{1}{p_1^3} dq_1 \wedge dp_1 \wedge dq_2 \wedge dp_2, \\ dX_6 \wedge dY_6 \wedge dZ_6 \wedge dW_6 = -\dfrac{1}{Y_1^3} dX_1 \wedge dY_1 \wedge dZ_1 \wedge dW_1, \\ dX_8 \wedge dY_8 \wedge dZ_8 \wedge dW_8 = -\dfrac{1}{Y_5^3} dX_5 \wedge dY_5 \wedge dZ_5 \wedge dW_5. \end{cases}$$

It is easy to see that each patching data (X_i, Y_i, Z_i, W_i) $(i = 1, 2, 5)$ is birational and symplectic, moreover the system (1.1) becomes again a *polynomial* Hamiltonian system in each coordinate system.

Proposition 5.1. *After a series of explicit blowing-ups and blowing-downs of* $\mathbb{P}^4$*, we obtain the smooth projective* 4*-fold* $\mathcal{S}$ *and a birational morphism* $\varphi \colon \mathcal{S} \cdots \to \mathbb{P}^4$.

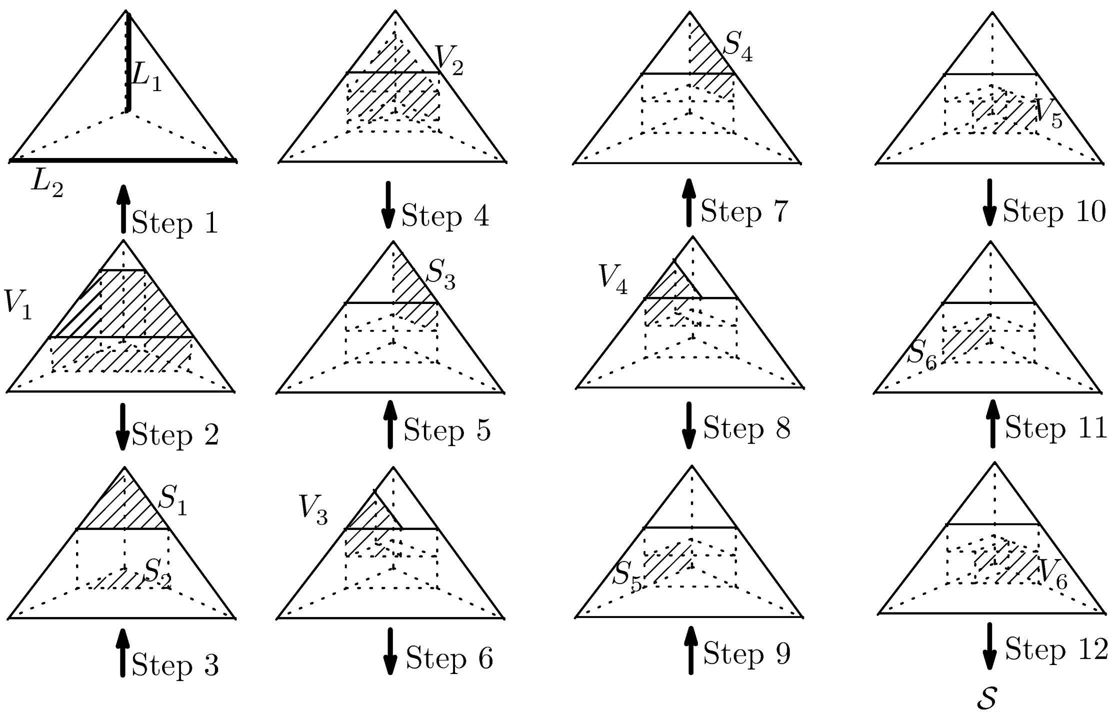

Figure 3. This figure denotes the steps which are needed to obtain the 4-fold $\mathcal{S}$. The first figure denotes the boundary divisor $\mathbb{P}^3$ in $\mathbb{P}^4$. Up arrow denotes blowing-up, and down arrow denotes blowing-down. Each step is explained in the below summary.

Let us summarize the steps which are needed to obtain the 4-fold $\mathcal{S}$.

1. Blow up along two curves $L_1 \cong \mathbb{P}^1$ and $L_2 \cong \mathbb{P}^1$.

2. Blow down the 3-fold $V_1 \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

3. Blow up along two surfaces $S_1 \cong \mathbb{P}^2$ and $S_2 \cong \mathbb{P}^2$.

4. Blow down the 3-fold $V_2 \cong \mathbb{P}^2 \times \mathbb{P}^1$.

5. Blow up along the surface $S_3 \cong \mathbb{P}^1 \times \mathbb{P}^1$.

6. Blow down the 3-fold $V_3 \cong \mathbb{P}^2 \times \mathbb{P}^1$.

7. Blow up along the surface $S_4 \cong \mathbb{P}^1 \times \mathbb{P}^1$.

8. Blow down the 3-fold $V_4 \cong \mathbb{P}^2 \times \mathbb{P}^1$.

9. Blow up along the surface $S_5 \cong \mathbb{P}^1 \times \mathbb{P}^1$.

10. Blow down the 3-fold $V_5 \cong \mathbb{P}^2 \times \mathbb{P}^1$.

11. Blow up along the surface $S_6 \cong \mathbb{P}^1 \times \mathbb{P}^1$.

12. Blow down the 3-fold $V_6 \cong \mathbb{P}^2 \times \mathbb{P}^1$.

It is easy to see that this rational vector field $\tilde{v}$ satisfies the condition:

$$\tilde{v} \in H^0(\mathcal{S}; \Theta_{\mathcal{S}}(-\log \mathcal{H})(\mathcal{H})). \tag{5.5}$$

The following lemma shows that this rational vector field $\tilde{v}$ has five accessible singular loci on the boundary divisor $\mathcal{H} \times \{t\} \subset \mathcal{S} \times \{t\}$ for each $t \in B$.

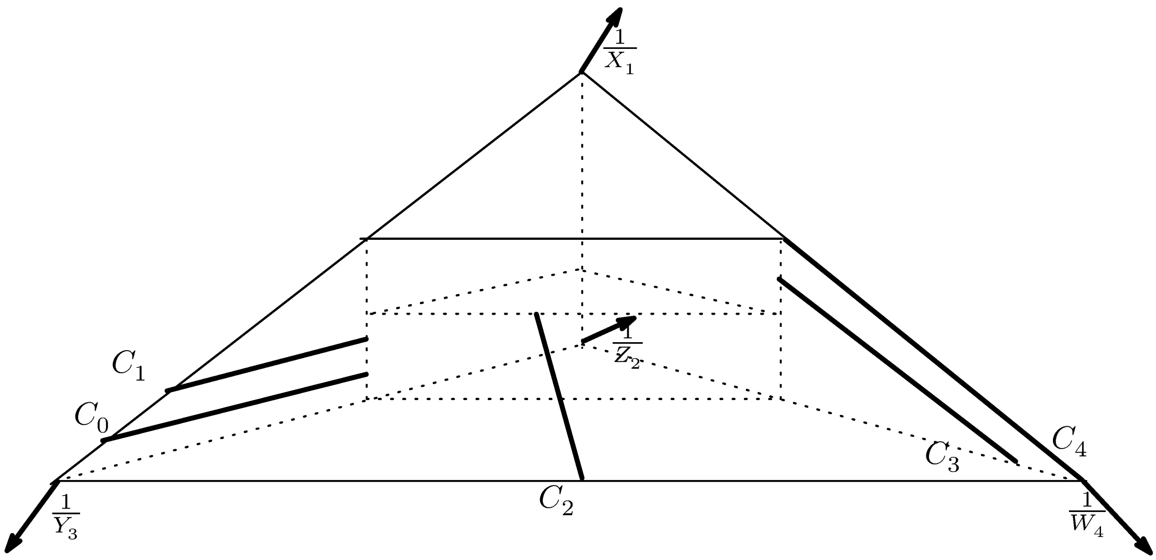

Figure 4. This figure denotes the boundary divisor $\mathcal{H}$ of $\mathcal{S}$. This divisor is covered by eight affine spaces $U_3 \cup U_4 \cup U_6 \cup U_7 \cup \cdots \cup U_{11}$. The bold lines C_i $(i = 0, 1, \ldots, 4)$ in $\mathcal{H}$ denote the accessible singular loci of the system (1.1) (see Lemma 5.2).

Lemma 5.2. *The rational vector field $\tilde{v}$ has the following accessible singular loci*:

$$
\begin{cases}
C_0 = \{(X_3, Y_3, Z_3, W_3) \mid X_3 = t, Y_3 = W_3 = 0\}, \\
C_1 = \{(X_3, Y_3, Z_3, W_3) \mid X_3 = \eta, Y_3 = W_3 = 0\}, \\
C_2 = \{(X_3, Y_3, Z_3, W_3) \mid X_3 = Z_3, Y_3 = 0, W_3 = -1\}, \\
C_3 = \{(X_4, Y_4, Z_4, W_4) \mid Y_4 = W_4 = 0, Z_4 = 1\}, \\
C_4 = \{(X_4, Y_4, Z_4, W_4) \mid Y_4 = Z_4 = W_4 = 0\}.
\end{cases}
\tag{5.6}
$$

This lemma can be proven by a direct calculation. □

Next let us calculate its local index at each point of C_i.

Singular locus	Singular point	Type of local index
C_0	$(X_3, Y_3, Z_3, W_3) = (t, 0, a, 0)$	$(2, 1, 0, 1)$
C_1	$(X_3, Y_3, Z_3, W_3) = (\eta, 0, a, 0)$	$(2, 1, 0, 1)$
C_2	$(X_4, Y_4, Z_4, W_4) = (a, -1, a, 0)$	$(0, 1, 2, 1)$
C_3	$(X_4, Y_4, Z_4, W_4) = (a, 0, 1, 0)$	$(0, 1, 2, 1)$
C_4	$(X_4, Y_4, Z_4, W_4) = (a, 0, 0, 0)$	$(0, 1, 2, 1)$

Here $a \in \mathbb{C}$.

Example 5.3. Let us take the coordinate system (x, y, z, w) centered at the point $(X_3, Y_3, Z_3, W_3) = (t, 0, 0, 0)$. The system (1.1) is rewritten as follows:

$$
\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \frac{1}{y}\left\{\begin{pmatrix} 2\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} + \cdots\right\}
$$

satisfying (4.3). In this case, the local index is $(2, 1, 0, 1)$. This suggests the possibilities that $b_0 = 1$ is the residue of the formal Laurent series:

$$
y(t) = \frac{1}{(t - t_0)} + b_1 + b_2(t - t_0) + \cdots + b_n(t - t_0)^{n-1} + \cdots \quad (b_i \in \mathbb{C}),
\tag{5.7}
$$

and the ratio $(\frac{2}{1}, \frac{0}{1}, \frac{1}{1}) = (2, 0, 1)$ is resonance data of the formal Laurent series of $(x(t), z(t), w(t))$ respectively. There exists a 3-parameter family of meromorphic solutions which passes through $(X_3, Y_3, Z_3, W_3) = (t_0, 0, 0, 0)$.

Example 5.4. Let us take the coordinate system (x, y, z, w) centered at the point $(X_4, Y_4, Z_4, W_4) = (0, -1, 0, 0)$. The system (1.1) is rewritten as follows:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \frac{1}{w}\left\{\frac{\eta}{(t-1)(t-\eta)}\begin{pmatrix} 2 & 0 & -2 & 0 \\ -2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} + \cdots\right\}$$

satisfying (4.3). To the system above, we make the linear transformation

$$\begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 2 & 1 & -2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

to arrive at

$$\frac{d}{dt}\begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix} = \frac{1}{W}\left\{\frac{\eta}{(t-1)(t-\eta)}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix} + \cdots\right\}.$$

Proposition 5.5. *If we resolve the accessible singular loci given in Lemma* 5.2 *by blowing-ups, then we can obtain the canonical coordinates* $r_j (j = 0, 1, 3, 5, 6)$.

Proof. By the following steps, we can resolve the accessible singular locus C_4.

Step 1: We blow up along the curve C_4:

$$X_4{}^{(1)} = X_4 \ , \quad Y_4{}^{(1)} = \frac{Y_4}{W_4} \ , \quad Z_4{}^{(1)} = \frac{Z_4}{W_4} \ , \quad W_4{}^{(1)} = W_4.$$

Step 2: We blow up along the surface $\{(X_4{}^{(1)}, Y_4{}^{(1)}, Z_4{}^{(1)}, W_4{}^{(1)}) \mid Z_4{}^{(1)} - \alpha_6 = W_4{}^{(1)} = 0\}$:

$$X_4{}^{(2)} = X_4{}^{(1)} \ , \quad Y_4{}^{(2)} = Y_4{}^{(1)} \ , \quad Z_4{}^{(2)} = \frac{Z_4{}^{(1)} - \alpha_6}{W_4{}^{(1)}} \ , \quad W_4{}^{(2)} = W_4{}^{(1)}.$$

Thus we have resolved the accessible singular locus C_4.

By choosing a new coordinate system as

$$(x_6, y_6, z_6, w_6) = (X_4{}^{(2)}, Y_4{}^{(2)}, -Z_4{}^{(2)}, W_4{}^{(2)}),$$

we can obtain the coordinate r_6.

By the following steps, we can resolve the accessible singular locus C_2.

Step 1: We blow up along the curve C_2:

$$X_5{}^{(1)}=\frac{X_3-Z_3}{Y_3}\ ,\quad Y_5{}^{(1)}=Y_3\ ,\quad Z_5{}^{(1)}=Z_3\ ,\quad W_5{}^{(1)}=\frac{W_3+1}{Y_3}.$$

Step 2: We blow up along the surface $\{(X_5{}^{(1)},Y_5{}^{(1)},Z_5{}^{(1)},W_5{}^{(1)})\mid X_5{}^{(1)}-\alpha_3=Y_5{}^{(1)}=0\}$:

$$X_5{}^{(2)}=\frac{X_5{}^{(1)}-\alpha_3}{Y_5{}^{(1)}}\ ,\quad Y_5{}^{(2)}=Y_5{}^{(1)}\ ,\quad Z_5{}^{(2)}=Z_5{}^{(1)}\ ,\quad W_5{}^{(2)}=W_5{}^{(1)}.$$

Thus we have resolved the accessible singular locus C_2.

By choosing a new coordinate system as

$$(x_3,y_3,z_3,w_3)=(-X_5{}^{(2)},Y_5{}^{(2)},Z_5{}^{(2)},W_5{}^{(2)}),$$

we can obtain the coordinate r_3.

For the remaining accessible singular locus, the proof is similar.

Collecting all the cases, we have obtained the canonical coordinate systems (x_j,y_j,z_j,w_j) $(j=0,1,3,5,6)$, which proves Proposition 5.5. □

We remark that each coordinate system contains a three-parameter family of meromorphic solutions of (1.1) as the initial conditions.

The difference between r_i and r_i' is only the case of $i=1$. The relation between r_1 and r_1' can be explained by the one for the accessible singularities C_1 and C_∞ given by

$$\begin{aligned}
C_1&=\{(X_6,Y_6,Z_6,W_6)\mid X_6=\frac{1}{\eta},Y_3=W_3=0\}\\
&\quad\cup\{(X_8,Y_8,Z_8,W_8)\mid X_8=\frac{1}{\eta},Y_8=W_8=0\},\\
C_\infty&=\{(X_6,Y_6,Z_6,W_6)\mid X_6=Y_3=W_3=0\}\\
&\quad\cup\{(X_8,Y_8,Z_8,W_8)\mid X_8=Y_8=W_8=0\}.
\end{aligned}\tag{5.8}$$

As $\eta\to\infty$, C_1 tends to C_∞. The resolution of C_∞ is the same way given in Proof of Proposition 5.5.

Proposition 5.6. *After a series of explicit blowing-ups given in Proposition 5.5, we obtain the smooth projective 4-fold $\widetilde{\mathcal{S}}$ and a morphism $\varphi\colon\widetilde{\mathcal{S}}\to\mathcal{S}$. Its canonical divisor $K_{\widetilde{\mathcal{S}}}$ of $\widetilde{\mathcal{S}}$ is given by*

$$K_{\widetilde{\mathcal{S}}}=-3\widetilde{\mathcal{H}}-\sum_{i=0}^{4}\mathcal{E}_i,\tag{5.9}$$

where the symbol $\widetilde{\mathcal{H}}$ denotes the proper transform of $\mathcal{H}$ by φ and $\mathcal{E}_i$ denote the exceptional divisors obtained by Step 1 (see Proof of Proposition 5.5).

We note that $\widetilde{\mathcal{S}}$ is its phase space including the meromorphic solution spaces corresponding to r_i. It is still an open question whether we will construct the phase space parametrized all meromorphic solutions including holomorphic solutions.

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