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Remarks on the Kernel Theorems in Hyperfunctions

By

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Abstract

We give some remarks on the kernel theorems in hyperfunctions. After recalling two types of kernel theorems in hyperfunctions, we study relations between two notions of semicontinuity appearing in the two cases, consider the wave front set condition by comparing it with the case of the kernel theorem in distributions, and study the (singular-)support property for the operators with kernels. We also give a characterization of continuous linear maps between the spaces of real-analytic functions.

§ 1. Introduction

We consider the Euclidian spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ with coordinates $y = (y_1, \ldots, y_m)$ and $x = (x_1, \ldots, x_n)$ respectively, and also consider open subsets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$. We denote by $\mathscr{D}(U)$ the space of differentiable functions on U with compact support endowed with the Schwartz topology, and by $\mathscr{D}'(V)$ the space of distributions on V. The Schwartz kernel theorem in distributions states that the following two conditions are equivalent for a linear map $T \colon \mathscr{D}(U) \to \mathscr{D}'(V)$:

(i) T is continuous.

(ii) T has a distribution kernel $\mathcal{K}(x, y) \in \mathscr{D}'(V \times U)$. By this we mean that the map T can be represented as the following integral operator with kernel $\mathcal{K}$:

$$(Tu)(x) = \int_U \mathcal{K}(x, y)u(y)dy \quad \text{for } u \in \mathscr{D}(U).$$

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Similar situations have been studied in the analytic category in [4], and in particular we gave kernel theorems in hyperfunctions in that paper. The aim of this article is to continue this study and to state some related results referring to kernel theorems.

We first recall the notion of semicontinuity and review the kernel theorems in hyperfunctions introduced in [4]. Then we will give some remarks on the semicontinuity in § 3.1, consider the wave front set condition in § 3.2, study the (singular-)support property in § 3.3, and give a characterization of continuous linear maps in § 3.4.

As for the results in § 3.3, the complete proof will be published elsewhere.

§ 2. Kernel Theorems

We prepare some notations. $\mathscr{O}_{\mathbb{C}^n}$ denotes the sheaf of holomorphic functions on $\mathbb{C}^n$, and $\mathscr{A}_{\mathbb{R}^n} := \mathscr{O}_{\mathbb{C}^n}|_{\mathbb{R}^n}$ the sheaf of real-analytic functions on $\mathbb{R}^n$. The sheaf $\mathscr{B}_{\mathbb{R}^n}$ of Sato's hyperfunctions on $\mathbb{R}^n$ is defined by

$$\mathscr{B}_{\mathbb{R}^n} := \mathcal{H}^n_{\mathbb{R}^n}(\mathscr{O}_{\mathbb{C}^n}) \otimes or_{\mathbb{R}^n/\mathbb{C}^n}.$$

(See [5].) For brevity we often write, for example, $\mathscr{O}$ instead of $\mathscr{O}_{\mathbb{C}^n}$ if there is no risk of confusion. A section of $\mathscr{B}$ is called a hyperfunction.

Hyperfunctions have boundary value representations. In fact, using the notation $G[d] := \{t \in G; |t| < d\}$ for an open convex cone $G \subset \mathbb{R}^n$ and a positive number $d > 0$, we have a natural injective map

$$b_G \colon \varinjlim_{d>0} \mathscr{O}(V + iG[d]) \to \mathscr{B}(V)$$

which is called the boundary value map. Moreover if $\{G_j\}_j$ is a finite family of open convex cones in $\mathbb{R}^n$ whose dual cones $\operatorname{Int} G_j^{\perp}$ form a covering of $\dot{\mathbb{R}}^n = \mathbb{R}^n \setminus \{0\}$, then the map

$$b = b_{\{G_j\}_j} \colon \bigoplus_j \varinjlim_{d>0} \mathscr{O}(V + iG_j[d]) \to \mathscr{B}(V) \tag{2.1}$$

becomes surjective. Note that for a fixed $d > 0$, the map $b_{G[d]} \colon \mathscr{O}(V + iG[d]) \to \mathscr{B}(V)$ is also injective and the map $b = b_{\{G_j[d]\}_j} \colon \bigoplus_j \mathscr{O}(V + iG_j[d]) \to \mathscr{B}(V)$ is also surjective.

Note also that no good topology for the $\mathscr{B}(V)$ is known to exist. For example, $\bigoplus_j \mathscr{O}(V + iG_j[d])$ has a natural Fréchet Schwartz topology, but the topology introduced on $\mathscr{B}(V)$ by the surjective map $\bigoplus_j \mathscr{O}(V + iG_j[d]) \to \mathscr{B}(V)$ is not Hausdorff.

On the other hand, for a compact set $K \subset V$, the space $\mathscr{B}_K(V)$ of hyperfunctions on V supported in K can be identified with the dual space of $\mathscr{A}(K)$, that is,

$$\mathscr{B}_K(V) \simeq \mathscr{A}'(K).$$

Thus $\mathscr{B}_K(V)$, endowed with the strong dual topology becomes an (FS)-space. Similarly the space $\mathscr{B}_c(V)$ of hyperfunctions on V with compact support is endowed with a good topology by

$$\mathscr{B}_c(V) \simeq \mathscr{A}'(V) \simeq \varinjlim_{K \Subset V} \mathscr{A}'(K).$$

Now we consider linear maps

$$T\colon \mathscr{A}'(U)\,(=\mathscr{B}_c(U)) \to \mathscr{B}(V)$$

and

$$T\colon \mathscr{A}(U) \to \mathscr{B}(V),$$

and introduce the notion of semicontinuity for both types of maps.

Definition 2.1. (1) Let $T\colon \mathscr{A}'(U) \to \mathscr{B}(V)$ be a linear map. We say that T is semicontinuous if for any compact set $K \Subset U$ and any relatively compact open set $V' \Subset V$, there exist a finite family $\{G_j\}_j$ of open convex cones in $\mathbb{R}^n$ and a family of continuous linear maps

$$T_j\colon \mathscr{A}'(K) \to \varinjlim_{d>0} \mathscr{O}(V' + iG_j[d]) \tag{2.2}$$

such that

$$(Tu)(x) = \sum_j (T_j u)(x) \quad \text{on } V' \text{ for any } u \in \mathscr{A}'(K).$$

(2) Let $T\colon \mathscr{A}(U) \to \mathscr{B}(V)$ be a linear map. We say that T is semicontinuous if for any relatively compact open set $V' \Subset V$, there exists a finite family $\{G_j\}_j$ of open convex cones in $\mathbb{R}^n$ and a family of continuous linear maps

$$T_j\colon \mathscr{A}(U) \to \varinjlim_{d>0} \mathscr{O}(V' + iG_j[d]) \tag{2.3}$$

such that

$$(Tu)(x) = \sum_j (T_j u)(x) \quad \text{on } V' \text{ for any } u \in \mathscr{A}(U).$$

We define subspaces $\mathscr{B}_{G_j[d]}(V)$ and $\mathscr{B}_{G_j}(V)$ of $\mathscr{B}(V)$ by

$$\mathscr{B}_{G_j[d]}(V) := b_{G_j[d]}(\mathscr{O}(V + iG_j[d])), \tag{2.4}$$

$$\mathscr{B}_{G_j}(V) := \varinjlim_{d>0} \mathscr{B}_{G_j[d]}(V) = b_{G_j}(\varinjlim_{d>0} \mathscr{O}(V + iG_j[d])) \tag{2.5}$$

and identify them with the topological vector spaces $\mathscr{O}(V+iG_j[d])$ and $\varinjlim_{d} \mathscr{O}(V+iG_j[d])$ respectively, since $b_{G_j[d]}$ and b_{G_j} are injective. Under these identifications, the map

$T\colon \mathscr{A}'(U) \to \mathscr{B}(V)$ is semicontinuous if the composition map from $\mathscr{A}'(K) \to \mathscr{B}(V')$ in the following diagram can be factorized through $\bigoplus_j \mathscr{B}_{G_j}(V')$ with a continuous linear map $\bigoplus_j T_j$, for any $K \Subset U$ and $V' \Subset V$.

$$
\begin{tikzcd}
 & & & \bigoplus_j \mathscr{B}_{G_j}(V') \arrow[d, two heads, "b"] \\
\mathscr{A}'(K) \arrow[r, hook] \arrow[urrr, dotted, bend left, "\bigoplus_j T_j"] & \mathscr{A}'(U) \arrow[r, "T"] & \mathscr{B}(V) \arrow[r] & \mathscr{B}(V')
\end{tikzcd}
$$

Similarly the semicontinuity of a linear map $T\colon \mathscr{A}(U) \to \mathscr{B}(V)$ can be understood as the existence of a continuous linear map $\bigoplus_j T_j$ in the following diagram

$$
\begin{tikzcd}
 & & \bigoplus_j \mathscr{B}_{G_j}(V') \arrow[d, two heads, "b"] \\
\mathscr{A}(U) \arrow[r, "T"] \arrow[urr, dotted, bend left, "\bigoplus_j T_j"] & \mathscr{B}(V) \arrow[r] & \mathscr{B}(V')
\end{tikzcd}
$$

For the case $T\colon \mathscr{A}'(U) \to \mathscr{B}(V)$, we give

Theorem 2.2. *For a linear map $T\colon \mathscr{A}'(U) \to \mathscr{B}(V)$, the following two conditions are equivalent.*

(i) *T is semicontinuous.*

(ii) *There exists a kernel $\mathcal{K}(x, y) \in \mathscr{B}(V \times U)$ such that*

$$
\mathrm{WF_A}\,\mathcal{K} \cap \{(x, y; 0, \eta) \in V \times U \times \mathbb{R}^n \times \mathbb{R}^m; \eta \neq 0\} = \emptyset, \tag{2.6}
$$

and that

$$
(Tu)(x) = \int_U \mathcal{K}(x, y) u(y) dy \quad \text{for any } u \in \mathscr{A}'(U). \tag{2.7}
$$

Here $\mathrm{WF_A}\,\mathcal{K}$ denotes the analytic wave front set of $\mathcal{K}$.

We will refer to the condition (2.6) as to the “the wave front set condition”, and it is equivalent to $\mathcal{K} \in \mathscr{B}_x\mathscr{A}_y(V \times U)$, that is, $\mathcal{K}$ is a hyperfunction with real analytic parameter y. This condition is used in classical analytic microlocal analysis to give a meaning to the product $\mathcal{K}(x, y)u(y)$ in (2.7) for any $u \in \mathscr{A}'(U) \simeq \mathscr{B}_c(U)$.

As for the case $T\colon \mathscr{A}(U) \to \mathscr{B}(V)$, we also give

Theorem 2.3. *For a linear map $T\colon \mathscr{A}(U) \to \mathscr{B}(V)$, the following two conditions are equivalent.*

(i) *T is semicontinuous.*
(ii) *There exists a kernel $\mathcal{K}(x,y) \in \mathscr{B}(V \times U)$ such that*

$$\text{the projection } (V \times U) \cap \operatorname{supp} \mathcal{K} \to V \text{ is proper,} \tag{2.8}$$

and that

$$(Tu)(x) = \int_U \mathcal{K}(x,y)u(y)dy \quad \text{for any } u \in \mathscr{A}(U). \tag{2.9}$$

We shall refer to the condition (2.8) as to "the proper support condition", and it is necessary to give a meaning to the integral in (2.9) for any $u \in \mathscr{A}(U)$ in a standard fashion.

These theorems can be justified by the following remark. If we take an oriented compact analytic manifold V instead of an open set in $\mathbb{R}^n$, then $\mathscr{B}(V)$ is (perhaps not canonically) isomorphic to $\mathscr{A}'(V)$, which is endowed with the strong dual topology. Then the theorems above hold if we replace the condition (i) by the following condition (i)$'$: "T is continuous".

Semicontinuity is thus precisely continuity when V is a compact manifold.

§ 3. Remarks

§ 3.1. Two Kinds of Semicontinuity

In Definition 2.1 (1), we defined the semicontinuity of $T\colon \mathscr{A}'(U) \to \mathscr{B}(V)$ in terms of decomposability into a finite sum of continuous linear maps $T_j : \mathscr{A}'(K) \to \mathscr{B}_{G_j}(V')$, after taking the composition with the inclusion $\mathscr{A}'(K) \hookrightarrow \mathscr{A}'(U)$ and the restriction $\mathscr{B}(V) \to \mathscr{B}(V')$. See (2.2) and (2.5). If we replace them by the existence of $d > 0$ and continuous linear maps

$$\widetilde{T}_j : \mathscr{A}'(K) \to \mathscr{B}_{G_j[d]}(V')$$

with

$$(Tu)(x) = \sum_j (\widetilde{T}_j u)(x) \quad \text{on } V' \text{ for any } u \in \mathscr{A}'(K),$$

then we can define a new notion of semicontinuity which is apparently stronger than the original one. But in the present situation, we can establish the equivalence between these notions at the level of each j, as follows. Note that $\mathscr{A}'(K)$ is a Fréchet space, and that $\mathscr{B}_{G_j}(V')$ is an inductive limit of a countable inductive system of Fréchet spaces with continuous injective maps. Moreover $\mathscr{B}_{G_j}(V')$ is Hausdorff. Then any continuous linear map $T_j : \mathscr{A}'(K) \to \mathscr{B}_{G_j}(V')$ can be factorized through $\mathscr{B}_{G_j[d]}(V')$ for some $d > 0$. (See theorem [2, page 198, Chapter 4, Part 1, Section 5, Theorem 1].)

Theorem 3.1. *Let* $\cdots \to X_i \to X_{i+1} \to \cdots$ *be a sequence of Fréchet spaces and continuous linear maps. Denote by* X *the inductive limit of the* X_i*, by* $f_i\colon X_i \to X$ *the natural maps and consider a continuous linear map* $T\colon F \to X$ *where* F *is a Fréchet space. Assume that* X *is Hausdorff. Then there is an index* i^0 *such that* $T(F) \subset f_{i^0}(X_{i^0})$*. Moreover if* f_{i^0} *is injective, then there is a continuous map* $T^0\colon F \to X_{i^0}$ *such that* T *is factorized into* $F \xrightarrow{T^0} X_{i^0} \xrightarrow{f_{i^0}} X$*.*

We can similarly define a new notion of semicontinuity for the case $\mathscr{A}(U) \to \mathscr{B}(V)$ by replacing the existence of T_j in (2.3) in Definition 2.1 (2) by the existence of $d > 0$ and continuous linear maps

$$\widetilde{T}_j\colon \mathscr{A}(U) \to \mathscr{B}_{G_j[d]}(V')$$

with

$$(Tu)(x) = \sum_j (\widetilde{T}_j u)(x) \quad \text{on } V' \text{ for any } u \in \mathscr{A}(U).$$

In this case, a continuous $T_j\colon \mathscr{A}(U) \to \mathscr{B}_{G_j}(V')$ can not in general be factorized "through" $\mathscr{B}_{G_j[d]}(V')$. However, in this case too we can prove that this new semicontinuity is also equivalent to the existence of a kernel $\mathcal{K}$ satisfying (2.8) and (2.9). Thus, the two notions of semicontinuity are equivalent, but if we are given a semicontinuous map $T = \sum_j T_j$ in the sense of Definition 2.1, and if we want to decompose T into $T = \sum_j \widetilde{T}_j$ on V' in the sense of the new semicontinuity, we may need to perform a re-decomposition. For example, consider the inclusion map $i\colon \mathscr{A}(V) \to \mathscr{B}(V)$ and the restriction map $r_G\colon \mathscr{A}(V) \to \mathscr{B}_G(V')$ with an arbitrary fixed $V' \Subset V$ and an arbitrary fixed cone $G \subset \dot{\mathbb{R}}^n$. Then we can easily see that i is semicontinuous and that $i(u) = r_G(u)$ on V' for any $u \in \mathscr{A}(V)$. But there is no $d > 0$ for which any $u \in \mathscr{A}(V)$ can be continued analytically to $V' + iG[d]$. Therefore, we can not have a single representation $i = \widetilde{T}$ on V' with some $\widetilde{T}\colon \mathscr{A}(V) \to \mathscr{B}_{G[d]}(V')$. We can actually obtain a decomposition of type $i = \sum_j \widetilde{T}_j$ on V' with continuous $\widetilde{T}_j\colon \mathscr{A}(V) \to \mathscr{B}_{G_j[d]}(V')$ with some fixed $d > 0$, but the number of cones G_j must be at least $n + 1$.

§ 3.2. The Situations $\mathscr{A}'(U) \to \mathscr{B}(V)$ and $\mathscr{E}'(U) \to \mathscr{D}'(V)$

In Theorem 2.2, a kernel function $\mathcal{K}(x, y)$ which defines a semicontinuous map was characterized by the wave front set condition (2.6). On the other hand, when we consider a similar situation $\mathscr{E}'(U) \to \mathscr{D}'(V)$ in the distribution theory, then it is not difficult to find continuous linear maps which can not be represented by an integral with kernel distribution satisfying a wave front set condition (2.6) with $\mathrm{WF_A}$ replaced by C^∞ wave front set WF.

Example 3.2. We define a linear map $T\colon \mathscr{E}'(\mathbb{R}) \to \mathscr{D}'(\mathbb{R})$ by

$$(Tu)(x) := \mathcal{F}^{-1}[2\xi Y(\xi)\widehat{u}(-\xi^2)](x) \quad \text{for } u(y) \in \mathscr{E}'(\mathbb{R}), \tag{3.1}$$

where $\widehat{u}$ is a Fourier transform of u, $\mathcal{F}^{-1}$ denotes the Fourier inverse transformation, and $Y(\xi)$ is a Heaviside function. Then we can prove that T is continuous and that T has no kernel with (C^∞-)wave front set property.

The continuity of T can be shown as follows. We can easily see that T is the transpose of a map $S\colon \mathscr{D}(\mathbb{R}) \to \mathscr{E}(\mathbb{R})$ given by

$$(S\varphi)(y) := \mathcal{F}^{-1}[Y(\eta)\widehat{\varphi}(-\sqrt{\eta})](y) \quad \text{for } \varphi(x) \in \mathscr{D}(\mathbb{R}),$$

and the map S satisfies the estimate

$$\begin{aligned}\|\partial_y^j(S\varphi)\|_{L^\infty(\mathbb{R})} &\le \|\eta^j\widehat{\varphi}(-\sqrt{\eta})\|_{L^1(0,\infty)} = \|\xi^{2j+1}\widehat{\varphi}(\xi)\|_{L^1(-\infty,0)} \\ &\le c\|\xi^{2j+1}(1+\xi^2)\widehat{\varphi}(\xi)\|_{L^\infty(\mathbb{R})} \le c(\|\partial_x^{2j+1}\varphi\|_{L^1(\mathbb{R})} + \|\partial_x^{2j+3}\varphi\|_{L^1(\mathbb{R})}),\end{aligned}$$

with some constant c.

Assume that T has a kernel $\mathcal{K}(x,y) \in \mathscr{D}'(\mathbb{R}\times\mathbb{R})$ satisfying

$$\begin{aligned}&\operatorname{WF}\mathcal{K} \cap \{(x,y;0,\eta); \eta \neq 0\} = \emptyset, \\ &(Tu)(x) = \int_{\mathbb{R}} \mathcal{K}(x,y)u(y)dy \quad \text{for any } u \in \mathscr{E}'(\mathbb{R}).\end{aligned} \tag{3.2}$$

Then $\mathcal{K}$ must coincide with the unique kernel of the composition map

$$\mathscr{D}(\mathbb{R}) \hookrightarrow \mathscr{E}'(\mathbb{R}) \xrightarrow{T} \mathscr{D}'(\mathbb{R}),$$

which exists by means of the classical Schwartz kernel theorem. In this situation, for any $u(y) \in \mathscr{D}(\mathbb{R})$ and $\varphi(x) \in \mathscr{D}(\mathbb{R})$, we have

$$\begin{aligned}\langle \mathcal{K}(x,y), \varphi(x)\otimes u(y)\rangle_{(x,y)} &= \langle (Tu)(x), \varphi(x)\rangle_x = \langle 2\xi Y(\xi)\widehat{u}(-\xi^2), (\mathcal{F}^{-1}\varphi)(\xi)\rangle_\xi \\ &= \langle 2\xi Y(\xi)\delta(\eta-\xi^2), (\mathcal{F}^{-1}\varphi)(\xi)\cdot\widehat{u}(-\eta)\rangle_{(\xi,\eta)} \\ &= 2\pi\langle \mathcal{F}^{-1}[2\xi Y(\xi)\delta(\eta-\xi^2)](x,y), \varphi(x)\otimes u(y)\rangle_{(x,y)},\end{aligned}$$

which implies $\mathcal{K} = 2\pi\mathcal{F}^{-1}[2\xi Y(\xi)\delta(\eta-\xi^2)]$. If we actually calculate the defining holomorphic function $F(z,w)$ of $\mathcal{K}$ as a hyperfunction, we get the properties

- $F \in \mathscr{O}(\{\operatorname{Im} w > 0\})$. Thus, $\operatorname{WF_A}\mathcal{K} \subset \{\xi = 0, \eta \ge 0\}$.
- $F(0,w) = -\dfrac{1}{2\pi i w}$ for $\operatorname{Im} w > 0$, which implies $\mathcal{K}(0,y) = -\dfrac{1}{2\pi i}\cdot\dfrac{1}{y+i0}$. Thus, $\mathcal{K}$ is not C^∞ in any neighborhood of the origin.

From these properties, we can show that $\mathrm{WF}\,\mathcal{K} \ni (0,0;0,1)$, which contradicts the C^∞ wave front set condition (3.2).

Note that the canonical inclusion map $\mathscr{D}'(V) \hookrightarrow \mathscr{B}(V)$ satisfies a similar condition to Definition 2.1 (2); that is, for any $V' \Subset V$, there exist finite number of open convex cones G_j's with $\bigcup_j \mathrm{Int}\, G_j^\perp = \dot{\mathbb{R}}$ and continuous maps $T_j\colon \mathscr{D}'(V) \to \mathscr{B}_{G_j}(V')$, which make the following diagram commute.

$$\begin{tikzcd} & & \bigoplus_j \mathscr{B}_{G_j}(V') \arrow[d, two heads, "b"] \\ \mathscr{D}'(V) \arrow[urr, dotted, bend left, "\bigoplus_j T_j"] \arrow[r, "T"] & \mathscr{B}(V) \arrow[r] & \mathscr{B}(V') \end{tikzcd}$$

This can be shown in the following way. Take a function $\phi \in C_0^\infty(V)$ satisfying $\phi \equiv 1$ on V'. For $u \in \mathscr{D}'(V)$, we can calculate a family $\{F_j\}_j$ of defining functions of ϕu, using twisted Radon transforms associated with a decomposition of $\dot{\mathbb{R}}$. If we choose the decomposition of $\dot{\mathbb{R}}$ suitably (depending of $\{G_j\}_j$), then each F_j belongs to $\mathscr{O}(V' + iG_j[d])$ with some $d > 0$, and the correspondence $u \mapsto F_j$ defines a linear continuous map $T_j\colon \mathscr{D}'(V) \to \mathscr{B}_{G_j}(V')$. Since $\phi \equiv 1$ on V', we have $\sum_j b(F_j) = \phi u = u$.

§ 3.3. Uniform Estimates for Supports and Singular-Supports

Consider a semicontinuous map $T\colon \mathscr{A}'(U) \to \mathscr{B}(V)$ with kernel $\mathcal{K}$. We denote by p (resp. q) the projection from the product space $V \times U$ to its first (resp. second) component.

$$\begin{tikzcd} V & V \times U \arrow[l, "p"'] \arrow[r, "q"] & U \\ & \operatorname{supp}\mathcal{K} \arrow[ul, "p'=p|_{\operatorname{supp}\mathcal{K}}"] \arrow[u, hook] \arrow[ur, "q'=q|_{\operatorname{supp}\mathcal{K}}"'] & \end{tikzcd}$$

Assume that the map $q' = q|_{\operatorname{supp}\mathcal{K}}\colon \operatorname{supp}\mathcal{K} \to U$ is proper. Then $q^{-1}(\operatorname{supp} u) \cap \operatorname{supp}\mathcal{K}$ is compact for any $u \in \mathscr{A}'(U)$ since it coincides $q'^{-1}(\operatorname{supp} u)$. Therefore $\operatorname{supp} Tu$ is also compact. Note that if $K \Subset U$ is a fixed compact subset, then $\operatorname{supp} Tu$ are estimated uniformly in all u satisfying $\operatorname{supp} u \subset K$, as

$$\operatorname{supp} Tu \subset p(q^{-1}(K) \cap \operatorname{supp}\mathcal{K}) \quad \bigl(= p'(q'^{-1}(K))\bigr).$$

Here we consider a converse:

Theorem 3.3. *Let $T\colon \mathscr{A}'(U) \to \mathscr{B}(V)$ be a semicontinuous map with kernel $\mathcal{K} \in \mathscr{B}(V \times U)$ satisfying* (2.6).

(1) *Assume that* $T(\mathscr{A}'(U)) \subset \mathscr{B}_c(V)$ $(\simeq \mathscr{A}'(V))$, *that is,* Tu *has compact support for any* $u \in \mathscr{A}'(U)$. *Then* $q|_{\operatorname{supp}\mathcal{K}} \colon \operatorname{supp}\mathcal{K} \to U$ *is proper and* T *is continuous as a map from* $\mathscr{A}'(U) \to \mathscr{A}'(V)$.

(2) *Assume that the analytic singular support of* Tu *is compact for any* $u \in \mathscr{A}'(U)$. *Then* $q|_{\operatorname{singsupp}\mathcal{K}} \colon \operatorname{singsupp}\mathcal{K} \to U$ *is proper. Here* $\operatorname{singsupp}\mathcal{K}$ *denotes the analytic singular support of* $\mathcal{K}$.

Assertion (1) follows from (2) since we have a uniqueness result for kernels. Also note that in Theorem 3.3, we have assumed only the compactness of $\operatorname{supp} Tu$ for each $u \in \mathscr{A}'(U)$ and have not assumed the uniformity of $\operatorname{supp} Tu$ in u, but the conclusion gives us the uniformity.

In the proof of (2), we use the following two propositions.

Proposition 3.4. *Let* $\mathcal{K}(x, y)$ *be a kernel with the wave front set condition* (2.6) *defined on* $V \times U$, *that is,* $\mathcal{K}(x, y) \in \mathscr{B}_x\mathscr{A}_y(V \times U)$. *Then, we can find an elliptic differential operator* $P(\partial_x)$ *of infinite order with constant coefficients in the* x *variables, a kernel* $\mathcal{K}'(x, y) \in \mathscr{B}_x\mathscr{A}_y(V \times U) \cap C^\infty(V \times U)$, *and an analytic function* $\mathcal{K}''(x, y) \in \mathscr{A}(V \times U)$, *such that*

$$\mathcal{K} = P(\partial_x)\mathcal{K}' + \mathcal{K}''.$$

Proposition 3.5. *The conclusion* (2) *in the theorem 3.3 holds for* $\mathcal{K}(x, y) \in \mathscr{B}_x\mathscr{A}_y(V \times U) \cap C^\infty(V \times U)$ *satisfying* (2.6).

§ 3.4. Linear Maps Between the Spaces of Real-Analytic Functions

Here we consider a linear map $T \colon \mathscr{A}(U) \to \mathscr{A}(V)$ and study the continuity of T and the semicontinuity of the composition map of T and the inclusion map $i \colon \mathscr{A}(V) \hookrightarrow \mathscr{B}(V)$. We give

Proposition 3.6. *The linear map* $i \circ T \colon \mathscr{A}(U) \to \mathscr{B}(V)$ *is semicontinuous if and only if* $T \colon \mathscr{A}(U) \to \mathscr{A}(V)$ *is continuous.*

Corollary 3.7. *For a linear map* $T \colon \mathscr{A}(U) \to \mathscr{A}(V)$, *the following two conditions are equivalent.*

(i) T *is continuous.*

(ii) *There exists a kernel hyperfunction* $\mathcal{K}(x, y) \in \mathscr{B}(V \times U)$ *with real analytic parameter* x (*i.e.* $\mathcal{K} \in \mathscr{A}_x\mathscr{B}_y(V \times U)$), *satisfying the proper support condition* (2.8) *such that*

$$(Tu)(x) = \int_U \mathcal{K}(x, y)u(y)dy \quad \text{for any } u \in \mathscr{A}(U). \tag{3.3}$$

For the proof, we use the kernel theorem 2.3 and the following result of Kaneko [3] (for proofs see Kaneko (loc. cit.) and also [1]).

Theorem 3.8. *Let* $\mathcal{K} \in \mathscr{B}(V \times U)$ *be a kernel satisfying* (2.8) *and consider the operator* $T\colon \mathscr{A}(U) \to \mathscr{B}(V)$ *given by* $Tu = \int_U \mathcal{K}(x,y)u(y)dy$. *Assume that* Tu *is real analytic on* V *for any* $u \in \mathscr{A}(U)$. *Then* $\mathcal{K}$ *has* x *as a real analytic parameter, that is,* $\mathcal{K}$ *satisfies*

$$\mathrm{WF_A}\,\mathcal{K} \cap \{(x,y,\xi,0) \in V \times U \times \mathbb{R}^n \times \mathbb{R}^m; \xi \neq 0\} = \emptyset.$$

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