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Remark on Division Theorem of Ultradistributions by Fuchsian Differential Operator

By

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Abstract

We announce recent results about division theorem of ultradistributions by a Fuchsian differential operator in the sense of Baouendi-Goulaouic. Details will be appeared in a forthcoming paper.

Introduction

In order to formulate initial value or boundary value problems in the framework of algebraic analysis, so-called division theorem plays a crucial role. In the category of hyperfunctions, this theorem is first proved by Komatsu-Kawai and Schapira in the case of an analytic differential operator under the non-characteristic condition (cf. [5], [10] and [15]), and is extended to the case of systems (cf. [4]). Further, Laurent-Monteiro Fernandes [11] extended this theorem to the cases of a Fuchsian system. Next, if we replace hyperfunctions by distributions, then we can easily prove similar results.

Therefore, we consider the same division problem for Gevrey ultradistributions by Fuchsian differential operators, and shall state our results under the assumption of an irregularity introduced by Tahara [16]. This irregularity is regarded as a counterpart of that in ordinary differential operators.

Further, we give an example that if the assumption of irregularity is not satisfied, then the division theorem does not hold for ultradistributions by using results of Tahara [16].

Details of this article will be appeared in a forthcoming paper [17].

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§ 1. Known Results

First, we shall fix the notation: We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of all the integers, real numbers and complex numbers respectively. Moreover we set $\mathbb{N} := \{n \in \mathbb{Z};\ n \geqslant 1\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{R}_{>0} := \{t \in \mathbb{R};\ r > 0\} \subset \mathbb{R}_{\geqslant 0} := \{t \in \mathbb{R};\ r \geqslant 0\}$.

In this article, all the manifolds are assumed to be paracompact. Let M be a real analytic manifold, and X a complexification of M. We denote by $\mathscr{O}_X$ the *Ring of holomorphic functions*, and by $\mathscr{D}_X$ the *Ring of holomorphic linear differential operators* on X respectively (for $\mathscr{D}$-Module theory, we refer to [2]).

Let N be an analytic hypersurface of M, and Y a complexification of N in X. Since the problem is local, we fix the following coordinates:

$$\begin{array}{ccc} N = \mathbb{R}^n_x \times \{0\} & \hookrightarrow & M = \mathbb{R}^n_x \times \mathbb{R}_t \\ \downarrow & & \downarrow \\ Y = \mathbb{C}^n_z \times \{0\} & \hookrightarrow & X = \mathbb{C}^n_z \times \mathbb{C}_\tau \end{array}$$

We set $\partial_{z_j} := \dfrac{\partial}{\partial z_j}$, $\partial_\tau := \dfrac{\partial}{\partial \tau}$ and so on. Moreover, for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ we set $|\alpha| := \sum\limits_{j=1}^{n} \alpha_j$ and $\partial_z^\alpha := \partial_{z_1}^{\alpha_1} \cdots \partial_{z_n}^{\alpha_n}$.

Let $\Omega \subset M$ be an open set such that $\Omega_0 := \Omega \cap N \neq \emptyset$.

Let $i\colon Y \hookrightarrow X$ be the natural inclusion. For a coherent $\mathscr{D}_X$-Module $\mathscr{M}$ defined on a neighborhood of Y, we denote by $\boldsymbol{D}i^*\mathscr{M}$ the *inverse image* in $\mathscr{D}$-Module theory; that is, $\boldsymbol{D}i^*\mathscr{M} := \mathscr{O}_Y \overset{\boldsymbol{L}}{\underset{\mathscr{O}_X}{\otimes}} \mathscr{M}$. Let $\mathscr{B}_M$ be the sheaf on M of *Sato hyperfunctions*.

Definition 1.1. We say that $\{B_j(z, \tau, \partial_z, \partial_\tau)\}_{j=0}^{m-1}$ is a *Dirichlet system* on Ω if

(1) each B_j is a holomorphic differential operator of order j defined on a complex neighborhood of Ω;
(2) Y is non-characteristic for each $B_j(z, \tau, \partial_z, \partial_\tau)$.

Let P be a holomorphic differential operator of order m defined on a complex neighborhood of Ω. Then the classical division theorem due to Komatsu-Kawai and Schapira states:

Theorem 1.2 (cf. [5], [10], [15]). *Assume that Y is non-characteristic for P. Then*:

(1) *If $u(x, t) \in \Gamma_N(\Omega; \mathscr{B}_M)$ satisfies $Pu = 0$, then $u = 0$.*
(2) *For any $u(x, t) \in \Gamma_N(\Omega; \mathscr{B}_M)$ and Dirichlet system $\{B_j\}_{j=0}^{m-1}$ on Ω, there exist uniquely $v(x, t) \in \Gamma_N(\Omega; \mathscr{B}_M)$ and $u_j(x) \in \Gamma(\widetilde{\Omega}_0; \mathscr{B}_N)$ $(0 \leqslant j \leqslant m-1)$ such that*

$$u(x, t) = Pv(x, t) + \sum_{j=0}^{m-1} B_j(u_{m-j-1}(x) \otimes \delta(t)).$$

If we set $\mathscr{M} := \mathscr{D}_X/\mathscr{D}_X P$, then $\boldsymbol{D}i^*\mathscr{M} \simeq \mathscr{D}_Y^m$, and Theorem 1.2 can be written canonically as

$$\boldsymbol{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_N(\mathscr{B}_M)) \simeq \boldsymbol{R}\mathcal{H}om_{\mathscr{D}_Y}(\boldsymbol{D}i^*\mathscr{M}, \mathscr{B}_N) \otimes or_{N/M}[-1]\,. \tag{1.1}$$

Here, or_N and or_M denote the orientation sheaves of N and M respectively, and set $or_{N/M} := or_N \otimes i^{-1}or_M$; that is, the relative orientation sheaf attached to $N \hookrightarrow M$. We remark that (1.1) also holds for the system case.

Remark 1.3. By using (1.1), non-characteristic boundary value problem can be formulated as follows: First, recall an exact sequence

$$0 \to \Gamma_N(\mathscr{B}_M) \to \Gamma_{M_+}(\mathscr{B}_M)\big|_N \to \Gamma_{\Omega_+}(\mathscr{B}_M)\big|_N \otimes or_{N/M} \to 0.$$

Here, $\Omega_+ := \{(x,t) \in M; t > 0\} \subset M_+ := \Omega_+ \cup N$. Hence there is a distinguished triangle

$$\begin{aligned}\boldsymbol{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_N(\mathscr{B}_M)) &\to \boldsymbol{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{M_+}(\mathscr{B}_M))\big|_N \\ &\to \boldsymbol{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{\Omega_+}(\mathscr{B}_M))\big|_N \otimes or_{N/M} \xrightarrow{+1}.\end{aligned}$$

Taking cohomologies and using (1.1), we obtain the *boundary value morphism*:

$$\mathrm{b}_+ \colon \mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{\Omega_+}(\mathscr{B}_M))\big|_N \to \mathcal{H}om_{\mathscr{D}_Y}(\boldsymbol{D}i^*\mathscr{M}, \mathscr{B}_N) \simeq \mathscr{B}_N^{\oplus m}.$$

Let $\mathscr{D}_{X\leftarrow Y}$ be the transfer $(\mathscr{D}_X, \mathscr{D}_Y)$ bi-Module associated with $i\colon Y \hookrightarrow X$. Then

$$\boldsymbol{D}i^!\mathscr{M} := \boldsymbol{R}\mathcal{H}om_{\mathscr{D}_Y}(\boldsymbol{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_{X\leftarrow Y}), \mathscr{D}_Y)[-1]$$

is called the *extraordinary inverse image* of $\mathscr{M}$ in $\mathscr{D}$-Module theory. Note that if Y is non-characteristic for $\mathscr{M}$, then by a result of [14] we have

$$\boldsymbol{D}i^!\mathscr{M} \simeq \boldsymbol{D}i^*\mathscr{M}. \tag{1.2}$$

Next, we extend (1.1) to Fuchsian Modules in the sense of Laurent-Monteiro Fernandes [11].

Theorem 1.4 ([11], [12]). *Let $\mathscr{M}$ be a Fuchsian $\mathscr{D}_X$-Module along Y in the sense of Laurent-Monteiro Fernandes* [11]. *Then:*

(1) *All the cohomologies of $\boldsymbol{D}i^*\mathscr{M}$ and $\boldsymbol{D}i^!\mathscr{M}$ are coherent $\mathscr{D}_Y$-Modules.*
(2) *There exist following isomorphisms:*

$$\boldsymbol{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_{X\leftarrow Y})[1] \simeq \boldsymbol{R}\mathcal{H}om_{\mathscr{D}_Y}(\boldsymbol{D}i^!\mathscr{M}, \mathscr{D}_Y),$$

$$\boldsymbol{R}\Gamma_Y \boldsymbol{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)[2] \simeq \boldsymbol{R}\mathcal{H}om_{\mathscr{D}_Y}(\boldsymbol{D}i^!\mathscr{M}, \mathscr{O}_Y). \tag{1.3}$$

Applying the functor $\boldsymbol{R}\Gamma_N(*) \otimes or_M$ to (1.3), we obtain the following isomorphism as a generalization of (1.1) (see (1.2)):

$$\boldsymbol{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_N(\mathscr{B}_M)) \simeq \boldsymbol{R}\mathcal{H}om_{\mathscr{D}_Y}(\boldsymbol{D}i^!\mathscr{M}, \mathscr{B}_N) \otimes or_{N/M}[-1]\,. \tag{1.4}$$

We refer (1.4) as a *division isomorphism.*

Remark 1.5. Let $\mathscr{F}$ be a $\mathscr{D}_Y$-Module. Since $\mathscr{D}_{X\leftarrow Y}$ is flat as a right $\mathscr{D}_Y$-Module, by Theorem 1.4 we see

$$\begin{aligned}
&\boldsymbol{R}\mathcal{H}om_{\mathscr{D}_Y}(\boldsymbol{D}i^!\mathscr{M}, \mathscr{F})[-1] \simeq \boldsymbol{R}\mathcal{H}om_{\mathscr{D}_Y}(\boldsymbol{D}i^!\mathscr{M}, \mathscr{D}_Y) \overset{\boldsymbol{L}}{\underset{\mathscr{D}_Y}{\otimes}} \mathscr{F}[-1] \\
&\simeq \boldsymbol{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_{X\leftarrow Y}) \overset{\boldsymbol{L}}{\underset{\mathscr{D}_Y}{\otimes}} \mathscr{F} \simeq \boldsymbol{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_{X\leftarrow Y} \overset{\boldsymbol{L}}{\underset{\mathscr{D}_Y}{\otimes}} \mathscr{F}) \\
&\simeq \boldsymbol{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_{X\leftarrow Y} \underset{\mathscr{D}_Y}{\otimes} \mathscr{F}).
\end{aligned} \tag{1.5}$$

Set $\mathscr{D}_N^A := \mathscr{D}_Y|_N$ and $\mathscr{D}_{M\leftarrow N}^A := \mathscr{D}_{X\leftarrow Y}|_N \otimes or_{N/M}$. We can write a section (or a germ) of $\mathscr{D}_{M\leftarrow N}^A$ as $\sum_j a_j(x, \partial_x)\, \partial_t^j \otimes |dt|^{\otimes -1}$, where $|dt|^{\otimes -1}$ is a generator of $or_{N/M}$. Then by (1.5), the isomorphism (1.4) is equivalent to:

$$\boldsymbol{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_{M\leftarrow N}^A \underset{\mathscr{D}_N^A}{\otimes} \mathscr{B}_N) \xrightarrow{\sim} \boldsymbol{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_N(\mathscr{B}_M)).$$

Here, a canonical morphism $\mathscr{D}_{M\leftarrow N}^A \underset{\mathscr{D}_N^A}{\otimes} \mathscr{B}_N \to \Gamma_N(\mathscr{B}_M)$ is induced by

$$\mathscr{D}_{M\leftarrow N}^A \ni \partial_t^j \otimes |dt|^{\otimes -1} \mapsto \partial_t^j \delta(t). \tag{1.6}$$

Next we replace hyperfunctions by distributions: Let $\mathcal{D}b_M$ be the sheaf on M of *Schwartz distributions.* By a structure theorem, we see that (1.6) induces an isomorphism:

$$\mathscr{D}_{M\leftarrow N}^A \underset{\mathscr{D}_N^A}{\otimes} \mathcal{D}b_N \simeq \Gamma_N(\mathcal{D}b_M).$$

Therefore by (1.5), we see that (1.4) induces the following division isomorphism:

$$\boldsymbol{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_N(\mathcal{D}b_M)) \simeq \boldsymbol{R}\mathcal{H}om_{\mathscr{D}_Y}(\boldsymbol{D}i^!\mathscr{M}, \mathcal{D}b_N) \otimes or_{N/M}[-1]\,.$$

§ 2. Statement of Main Results

We shall consider the same division problem for Gevrey ultradistributions. In what follows, we use the symbol $* = (s)$ or $\{s\}$ to indicate the Gevrey growth order for $1 < s < \infty$. Let $\mathcal{D}b_M^*$ be the sheaf on M of *Gevrey ultradistributions* of class $*$ (for the convenience of the reader, we recall the definition of ultradistributions in the next section).

Take $k \in \mathbb{N}_0$, $m \in \mathbb{N}$ with $k \leqslant m$. We shall consider the following holomorphic differential operator of order m defined on a complex neighborhood of Ω:

$$P(z,\tau,\partial_z,\partial_\tau) = \tau^k \partial_\tau^m + \sum_{j=m-k}^{m-1} a_j(z)\,\tau^{j-m+k}\,\partial_\tau^j + \sum_{|\alpha|+j\leqslant m} p_{\alpha,j}(z,\tau)\,\partial_z^\alpha \tau^{\max\{0,j-m+k+1\}}\,\partial_\tau^j,$$

P is said to be of *Fuchsian type with weight* (k,m) due to Baouendi-Goulaouic [1]. We set $\mathscr{M} := \mathscr{D}_X/\mathscr{D}_X P$. Then $\mathscr{M}$ is a Fuchsian $\mathscr{D}_X$-Module along Y in the sense of Laurent-Monteiro Fernandes [11].

In order to state our main theorem, we need the notation due to Tahara [16]. First, assume that the weight of P is (m,m). Then we can write P as

$$P(z,\tau,\partial_z,\partial_\tau) = b(z,\vartheta) + \sum_{|\alpha|+j\leqslant m} \tau^{l(\alpha,j)} q_{\alpha,j}(z,\tau)\,\partial_z^\alpha\,\vartheta^j,$$

where $\vartheta := \tau\partial_\tau$ (or $t\partial_t$ in real cases), $l(\alpha,j) \in \mathbb{N}$, $q_{\alpha,j}(z,0) \neq 0$ if $q_{\alpha,j}(z,\tau) \not\equiv 0$, and $b(z,\vartheta)$ is of the form

$$b(z,\vartheta) = \vartheta^m + \sum_{j=0}^{m-1} b_j(z)\,\vartheta^j. \tag{2.1}$$

We set

$$\mathscr{S}_P := \{(\alpha,j);\, q_{\alpha,j} \not\equiv 0 \text{ and } |\alpha| - l(\alpha,j) \geqslant 1\}.$$

Then Tahara's index is defined by

$$\mathscr{I}_T^0(P) := \begin{cases} \min\limits_{(\alpha,j)\in\mathscr{S}_P} \left\{\dfrac{m-j-l(\alpha,j)}{|\alpha|-l(\alpha,j)}\right\} & (\mathscr{S}_P \neq \emptyset), \\ \infty & (\mathscr{S}_P = \emptyset). \end{cases}$$

Remark 2.1. If $\mathscr{I}_T^0(P) = \infty$, then $q_{\alpha,j} \not\equiv 0$ imply $l(\alpha,j) \geqslant |\alpha|$. Hence P is written as

$$P(z,\tau,\partial_z,\partial_\tau) = b(z,\vartheta) + \sum_{j+|\alpha|\leqslant m} \tau^{l(\alpha,j)-|\alpha|} q_{\alpha,j}(z,\tau)\,(\tau\partial_z)^\alpha\,\vartheta^j.$$

This means that P has regular singularities along Y in the sense of Kashiwara-Oshima [3]. Hence $\mathscr{I}_T^0(P)$ measures the difference of Fuchsian operators from operators with regular singularities.

Definition 2.2. If the weight of P is (k,m), then since $\tau^{m-k}P$ is of weigh (m,m), we set

$$\mathscr{I}_T(P) := \mathscr{I}_T^0(\tau^{m-k}P).$$

Remark 2.3. (1) If the weight of P is (m,m), then $\mathscr{I}_T(P)=\mathscr{I}^0_T(P)$.

(2) We denote by ${}^tP$ the formal adjoint of P. Then it is easy to see that

$$\mathscr{I}_T(P)=\mathscr{I}_T({}^tP). \tag{2.2}$$

We are ready to state our main theorems:

Theorem 2.4. *Assume the condition*

$$1<s\leqslant \mathscr{I}_T(P). \tag{2.3}$$

Then there exists the following division isomorphism for $*=\{s\}$ *or* (s):

$$\boldsymbol{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M},\Gamma_N(\mathcal{D}b^*_M))\simeq \boldsymbol{R}\mathcal{H}om_{\mathscr{D}_Y}(\boldsymbol{D}i^!\mathscr{M},\mathcal{D}b^*_N)\otimes or_{N/M}[-1]\,. \tag{2.4}$$

Note that by (1.5), the isomorphism (2.4) is equivalent to the isomorphism:

$$\boldsymbol{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{D}^A_{M\leftarrow N}\underset{\mathscr{D}^A_N}{\otimes}\mathcal{D}b^*_N)\xrightarrow{\sim}\boldsymbol{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M},\Gamma_N(\mathcal{D}b^*_M)).$$

Remark 2.5. (1) If Y is non-characteristic for $Q\in\mathscr{D}_X$ of order m, then Q is Fuchsian with weight $(0,m)$, and $\mathscr{I}_T(Q)=\mathscr{I}_T(t^mQ)=\infty$. In particular, Q satisfies (2.3) for any $*=\{s\}$ or (s) with $1<s<\infty$. Thus Theorem 2.4 is a special case of Komatsu [9].

(2) Since $s>1$, if $\mathscr{I}_T(P)<\infty$ then the assumption (2.3) can be written as

$$\max\Big\{1,\max_{\substack{|\alpha|+j\leqslant m\\ 1\leqslant|\alpha|}}\Big\{\frac{m-l(\alpha,j)-j}{m-|\alpha|-j}\Big\}\Big\}\leqslant\frac{s}{s-1}.$$

Therefore, we can regard (2.3) as a counterpart of an irregularity condition for ordinary differential equation.

As is mentioned in § 1, we can prove the division isomorphism without assumption (2.3) in the category of hyperfunctions or of distributions. On the contrary, we state:

Theorem 2.6. *For any* $\sigma_1\in\mathbb{Q}$ *with* $\sigma_1\geqslant 1$, *there exists an operator* $P\in\mathscr{D}_X\big|_M$ *such that* $\mathscr{I}_T(P)=\sigma_1$ *and if*

$$\mathscr{I}_T(P)<s<\infty,$$

then the isomorphism (2.4) *does not hold.*

§ 3. Gevrey Ultradistributions

We refer to [6], [7] (and [8] in Japanese) for exposition of ultradistributions from the viewpoint of algebraic analysis. We inherit the notation from the preceding section. We denote by C^∞_M the sheaf on M of (complex-valued) functions of class C^∞. Let $U\subset M$ be an open set.

Definition 3.1. Let $K \Subset U$ be a compact set with sufficiently smooth boundary. We set

$$C_M^\infty(K) := \varinjlim_{K \subset V} C_M^\infty(V)\big|_K,$$

where V ranges through the family of open neighborhoods of K in U.

For $u(x,t) \in C_M^\infty(K)$ and $h > 0$, we set

$$\mathfrak{p}_{h,K}^{\{s\}}(u) := \sup_{\substack{(x,t)\in K \\ (\alpha,\nu)\in \mathbb{N}_0^{n+1}}} \frac{|\partial_x^\alpha \partial_t^\nu u(x,t)|}{h^{|\alpha|+\nu}(|\alpha|+\nu)!^s},$$

$$\mathscr{G}_M^{\{s\},h}(K) := \{u(x,t) \in C_M^\infty(K);\, \mathfrak{p}_{h,K}^{\{s\}}(u) < \infty\}.$$

$\mathscr{G}_M^{\{s\},h}(K)$ is a Banach space under the norm $\mathfrak{p}_{h,K}^{\{s\}}(\,\cdot\,)$. Further, we set

$$\mathscr{G}_K^{\{s\},h} := \{u(x,t) \in \mathscr{G}_M^{\{s\},h}(K)\,;\, \operatorname{supp} u \subset K\}.$$

$\mathscr{G}_K^{\{s\},h}$ is also a Banach space as a closed subspace of $\mathscr{G}_M^{\{s\},h}(K)$. Under this notation, we set

(3.1)
$$\mathscr{G}_M^{(s)}(U) := \varprojlim_{K\Subset U} \varprojlim_{h\to 0} \mathscr{G}_M^{\{s\},h}(K), \qquad \mathit{\Gamma}_{\mathrm{c}}(\mathscr{G}_M^{(s)})(U) := \varinjlim_{K\Subset U} \varprojlim_{h\to 0} \mathscr{G}_K^{\{s\},h},$$
$$\mathscr{G}_M^{\{s\}}(U) := \varprojlim_{K\Subset U} \varinjlim_{h\to 0} \mathscr{G}_M^{\{s\},h}(K), \qquad \mathit{\Gamma}_{\mathrm{c}}(\mathscr{G}_M^{\{s\}})(U) := \varinjlim_{K\Subset U} \varinjlim_{h\to 0} \mathscr{G}_K^{\{s\},h}.$$

$\mathit{\Gamma}_{\mathrm{c}}(\mathscr{G}_M^*)(U)$ $(* = (s), \{s\})$ is nothing but the subspace of $\mathscr{G}_M^*(U)$ consisting of compactly supported elements.

The assignment $U \mapsto \mathscr{G}_M^*(U)$ defines the sheaf $\mathscr{G}_M^*$ of *Gevrey ultradifferentiable functions* of growth order $*$ on M. Further $\mathit{\Gamma}_{\mathrm{c}}(\mathscr{G}_M^*)(U) = \mathit{\Gamma}_{\mathrm{c}}(U;\mathscr{G}_M^*)$.

Using the expressions (3.1), we can endow $\mathit{\Gamma}(U;\mathscr{G}_M^*)$ and $\mathit{\Gamma}_{\mathrm{c}}(U;\mathscr{G}_M^*)$ with natural locally convex topologies respectively, and consequently $\mathit{\Gamma}(U;\mathscr{G}_M^{(s)})$ is an (FS) space, $\mathit{\Gamma}(U;\mathscr{G}_M^{\{s\}})$ a (DLFS) space, $\mathit{\Gamma}_{\mathrm{c}}(U;\mathscr{G}_M^{(s)})$ an (LFS) space, and $\mathit{\Gamma}_{\mathrm{c}}(U;\mathscr{G}_M^{\{s\}})$ a (DFS) space. These all spaces are reflexive.

Definition 3.2. We take $* = (s)$ or $\{s\}$ with $1 < s < \infty$. Let $\mathscr{V}_M^*$ be the sheaf on M of volume elements with coefficients in $\mathscr{G}_M^*$; that is,

$$\mathscr{V}_M^* := \bigwedge^{n+1} \mathscr{G}_M^* \otimes or_M\,.$$

Remark 3.3. Since we fix the coordinates, we have a global isomorphism

(3.2)
$$\begin{array}{ccc} \mathscr{G}_M^* & \xrightarrow{\ \sim\ } & \mathscr{V}_M^* \\ \Cup & & \Cup \\ u(x,t) & \longmapsto & u(x,t)\,dx\,dt, \end{array}$$

where $dx\,dt$ denotes the standard Lebesgue mesure on $M \simeq \mathbb{R}^{n+1}$. We endow $\mathscr{V}_M^*$ with a locally convex topology under which (3.2) is a topological isomorphism.

Definition 3.4. For $* = (s)$ or $\{s\}$ with $1 < s < \infty$, we set:

$$\mathcal{D}b_M^*(U) := \varGamma_{\mathrm{c}}(U; \mathscr{V}_M^*)'.$$

Here the prime means the strong dual of a topological vector space, and the subscript c means the sections with compact support.

We can prove that the assignment $U \mapsto \mathcal{D}b_M^*(U)$ defines a sheaf $\mathcal{D}b_M^*$ on M. $\mathcal{D}b_M^{(s)}$ (resp. $\mathcal{D}b_M^{\{s\}}$) is called the sheaf on M of *Gevrey ultradistributions of Beurling-Björck type* (resp. *of Roumieu type*).

By Remark 3.3, we identify $\varGamma(U; \mathcal{D}b_M^*)$ with $\varGamma_{\mathrm{c}}(U; \mathscr{G}_M^*)'$ as usual, and we can show

$$\varGamma(U; \mathscr{G}_M^*)' = \varGamma_{\mathrm{c}}(U; \mathcal{D}b_M^*) \subset \varGamma(U; \mathcal{D}b_M^*) = \varGamma_{\mathrm{c}}(U; \mathscr{G}_M^*)'.$$

Hence $\varGamma(U; \mathcal{D}b_M^{(s)})$ is a (DLFS) space, $\varGamma(U; \mathcal{D}b_M^{\{s\}})$ is an (FS) space, $\varGamma_{\mathrm{c}}(U; \mathcal{D}b_M^{(s)})$ is a (DFS) space, and $\varGamma_{\mathrm{c}}(U; \mathcal{D}b_M^{\{s\}})$ is a (LFS) space. If $1 < s < t$, then as subsheaves

$$\mathcal{D}b_M \subset \mathcal{D}b_M^{\{t\}} \subset \mathcal{D}b_M^{(t)} \subset \mathcal{D}b_M^{\{s\}} \subset \mathcal{D}b_M^{(s)} \subset \mathscr{B}_M\,.$$

We prove Theorems 2.4 and 2.6 by the duality method. To this end, we introduce several function spaces: Let $U_0 \subset N$ be an open subset, and $K \Subset U_0$ a compact set with sufficiently smooth boundary. We define $C_N^\infty(K)$ as in Definition 3.1.

Definition 3.5. Let $C_N^\infty(K)[[t]]$ be the space of formal power series of t with coefficients in $C_N^\infty(K)$. Set

$$C_K^\infty[[t]] := \{\sum_{\nu=0}^{\infty} u_\nu(x)\,\frac{t^\nu}{\nu!} \in C_N^\infty(K)[[t]]\,;\, \operatorname{supp} u_\nu \subset K\}.$$

For $u(x,\langle t\rangle) = \sum\limits_{\nu=0}^{\infty} u_\nu(x)\dfrac{t^\nu}{\nu!} \in C_N^\infty(K)[[t]]$ and $h > 0$, we set

$$\widehat{\mathfrak{p}}_{h,K}^{\{s\}}(u) := \sup_{\substack{x \in K \\ (\alpha,\nu) \in \mathbb{N}_0^{n+1}}} \frac{|\partial_x{}^\alpha u_\nu(x)|}{h^{|\alpha|+\nu}(|\alpha|+\nu)!^s}\,.$$

Then we set

$$\widehat{\mathscr{G}}_M^{\{s\},h}(K) := \{u(x,\langle t\rangle) \in C_N^\infty(K)[[t]]\,;\, \widehat{\mathfrak{p}}_{h,K}^{\{s\}}(u) < \infty\}.$$

$\widehat{\mathscr{G}}_M^{\{s\},h}(K)$ is a Banach space under the norm $\widehat{\mathfrak{p}}_{h,K}^{\{s\}}(\,\cdot\,)$. Further, we define

$$\widehat{\mathscr{G}}_K^{\{s\},h} := \{u(x,\langle t\rangle) \in C_K^\infty[[t]]\,;\, \widehat{\mathfrak{p}}_{h,K}^{\{s\}}(u) < \infty\} \subset \widehat{\mathscr{G}}_M^{\{s\},h}(K).$$

$\widehat{\mathscr{G}}_K^{\{s\},h}$ is also a Banach space as a closed subspace of $\widehat{\mathscr{G}}_M^{\{s\},h}(K)$. Under this notation, we set

$$\begin{aligned} &\widehat{\mathscr{G}}_{M|N}^{(s)}(U_0) := \varprojlim_{K\Subset U_0} \varprojlim_{h\to 0} \widehat{\mathscr{G}}_M^{\{s\},h}(K), \qquad \Gamma_{\mathrm{c}}(\widehat{\mathscr{G}}_{M|N}^{(s)})(U_0) := \varinjlim_{K\Subset U_0} \varprojlim_{h\to 0} \widehat{\mathscr{G}}_K^{\{s\},h}, \\ &\widehat{\mathscr{G}}_{M|N}^{\{s\}}(U_0) := \varprojlim_{K\Subset U_0} \varinjlim_{h\to 0} \widehat{\mathscr{G}}_M^{\{s\},h}(K), \qquad \Gamma_{\mathrm{c}}(\widehat{\mathscr{G}}_{M|N}^{\{s\}})(U_0) := \varinjlim_{K\Subset U_0} \varinjlim_{h\to 0} \widehat{\mathscr{G}}_K^{\{s\},h}. \end{aligned} \tag{3.3}$$

The support of $u(x,\langle t\rangle) = \sum\limits_{\nu=0}^{\infty} u_\nu(x)\dfrac{t^\nu}{\nu!} \in \widehat{\mathscr{G}}_{M|N}^*(U_0)$ is defined by

$$\operatorname{supp} u := \mathrm{Cl}[\bigcup_{\nu=0}^{\infty} \operatorname{supp} u_\nu].$$

Here Cl means the closure. $\Gamma_{\mathrm{c}}(\widehat{\mathscr{G}}_{M|N}^*)(U_0)$ is nothing but the subspace of $\widehat{\mathscr{G}}_{M|N}^*(U_0)$ consisting of compactly supported elements.

Note that the assignment $U_0 \mapsto \widehat{\mathscr{G}}_{M|N}^*(U_0)$ defines the sheaf $\widehat{\mathscr{G}}_{M|N}^*$ on N. Further we see that $\Gamma_{\mathrm{c}}(U_0;\widehat{\mathscr{G}}_{M|N}^*) = \Gamma_{\mathrm{c}}(\widehat{\mathscr{G}}_{M|N}^*)(U_0)$.

Using the expressions (3.3), we can endow $\Gamma(U_0;\widehat{\mathscr{G}}_{M|N}^*)$ and $\Gamma_{\mathrm{c}}(U_0;\widehat{\mathscr{G}}_{M|N}^*)$ with natural locally convex topologies respectively, and consequently $\Gamma(U_0;\widehat{\mathscr{G}}_{M|N}^{(s)})$ is an (FS) space, $\Gamma(U_0;\widehat{\mathscr{G}}_{M|N}^{\{s\}})$ a (DLFS) space, $\Gamma_{\mathrm{c}}(U_0;\widehat{\mathscr{G}}_{M|N}^{(s)})$ an (LFS) space, and $\Gamma_{\mathrm{c}}(U_0;\widehat{\mathscr{G}}_{M|N}^{\{s\}})$ a (DFS) space. These all spaces are reflexive. Further,

Theorem 3.6 ([7, Theorem 4.4]). *For any open subset $U_0 \subset N$,*

$$\Gamma_{\mathrm{c}}(U_0;\widehat{\mathscr{G}}_{M|N}^*)' = \Gamma(U_0;\Gamma_N(\mathcal{D}b_M^*)) \supset \Gamma(U_0;\widehat{\mathscr{G}}_{M|N}^*)' = \Gamma_{\mathrm{c}}(U_0;\Gamma_N(\mathcal{D}b_M^*)).$$

Therefore we see that $\Gamma(U_0;\Gamma_N(\mathcal{D}b_M^{(s)}))$ is a (DLFS) space, $\Gamma_{\mathrm{c}}(U_0;\Gamma_N(\mathcal{D}b_M^{\{s\}}))$ an (FS) space, $\Gamma_{\mathrm{c}}(U_0;\Gamma_N(\mathcal{D}b_M^{(s)}))$ a (DFS) space, and $\Gamma_{\mathrm{c}}(U_0;\Gamma_N(\mathcal{D}b_M^{\{s\}}))$ a (LFS) space.

For any $\nu_1 \in \mathbb{N}_0$, we have the following splitting exact sequences as $\mathbb{C}$-vector spaces:

$$\begin{array}{ccccccccc} 0 \to & t^{\nu_1}\Gamma_{\mathrm{c}}(U_0;\widehat{\mathscr{G}}_{M|N}^*) & \to & \Gamma_{\mathrm{c}}(U_0;\widehat{\mathscr{G}}_{M|N}^*) & \to & \Gamma_{\mathrm{c}}(U_0;\widehat{\mathscr{G}}_{M|N}^*)/t^{\nu_1}\Gamma_{\mathrm{c}}(U_0;\widehat{\mathscr{G}}_{M|N}^*) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & t^{\nu_1}\Gamma(U_0;\widehat{\mathscr{G}}_{M|N}^*) & \longrightarrow & \Gamma(U_0;\widehat{\mathscr{G}}_{M|N}^*) & \longrightarrow & \Gamma(U_0;\widehat{\mathscr{G}}_{M|N}^*)/t^{\nu_1}\Gamma(U_0;\widehat{\mathscr{G}}_{M|N}^*) & \longrightarrow 0 \end{array} \tag{3.4}$$

$t^{\nu_1}\Gamma(U_0;\widehat{\mathscr{G}}_{M|N}^*)$ (resp. $t^{\nu_1}\Gamma_{\mathrm{c}}(U_0;\widehat{\mathscr{G}}_{M|N}^*)$) is a closed subspace of $\Gamma(U_0;\widehat{\mathscr{G}}_{M|N}^*)$ (resp. $\Gamma_{\mathrm{c}}(U_0;\widehat{\mathscr{G}}_{M|N}^*)$), and moreover inherits the same type of locally convex topology. Then, under the quotient topologies with respect to (3.4), as locally convex spaces we have:

$$\Gamma(U_0;\widehat{\mathscr{G}}_{M|N}^*)/t^{\nu_1}\Gamma(U_0;\widehat{\mathscr{G}}_{M|N}^*) \simeq \{\sum_{\nu=0}^{\nu_1-1} u_\nu(x)\frac{t^\nu}{\nu!} \in \Gamma(U_0;\widehat{\mathscr{G}}_{M|N}^*)\} \simeq \Gamma(U_0;\mathscr{G}_N^*)^{\nu_1},$$

$$\Gamma_{\mathrm{c}}(U_0;\widehat{\mathscr{G}}^*_{M|N})/t^{\nu_1}\Gamma_{\mathrm{c}}(U_0;\widehat{\mathscr{G}}^*_{M|N}) \simeq \{\sum_{\nu=0}^{\nu_1-1} u_\nu(x)\frac{t^\nu}{\nu!} \in \Gamma_{\mathrm{c}}(U_0;\widehat{\mathscr{G}}^*_{M|N})\} \simeq \Gamma_{\mathrm{c}}(U_0;\mathscr{G}^*_N)^{\nu_1}.$$

Hence, we have the following splitting *topologically* exact sequences:

$$\begin{array}{ccccccccc} 0 \to & t^{\nu_1}\Gamma_{\mathrm{c}}(U_0;\widehat{\mathscr{G}}^*_{M|N}) & \to & \Gamma_{\mathrm{c}}(U_0;\widehat{\mathscr{G}}^*_{M|N}) & \to & \Gamma_{\mathrm{c}}(U_0;\mathscr{G}^*_N)^{\nu_1} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & t^{\nu_1}\Gamma(U_0;\widehat{\mathscr{G}}^*_{M|N}) & \longrightarrow & \Gamma(U_0;\widehat{\mathscr{G}}^*_{M|N}) & \longrightarrow & \Gamma(U_0;\mathscr{G}^*_N)^{\nu_1} & \longrightarrow 0 \end{array} \tag{3.5}$$

§ 4. Sketch of Proof of Main Theorems

If the weight of P is (k,m) with $k \neq m$, then we set

$$\mathscr{L} := \mathscr{D}_X/\mathscr{D}_X P, \quad \mathscr{M} := \mathscr{D}_X/\mathscr{D}_X \tau^{m-k}P, \quad \mathscr{N} := \mathscr{D}_X/\mathscr{D}_X \tau^{m-k}.$$

By a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{D}_X & \xrightarrow{\cdot\tau^{m-k}P} & \mathscr{D}_X & \longrightarrow & \mathscr{M} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \cdot\tau^{m-k}} & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{D}_X & \xrightarrow{\cdot P} & \mathscr{D}_X & \longrightarrow & \mathscr{L} & \longrightarrow & 0 \end{array}$$

and Snake Lemma, we have the following exact sequence:

$$0 \to \mathscr{N} \to \mathscr{M} \to \mathscr{L} \to 0.$$

We can prove

$$\begin{aligned} \boldsymbol{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{N},\Gamma_N(\mathcal{D}b^*_M)) &\simeq \boldsymbol{R}\mathcal{H}om_{\mathscr{D}_Y}(\boldsymbol{D}i^!\mathscr{N},\mathcal{D}b^*_N)\otimes or_{N/M}[-1] \\ &\simeq (\mathcal{D}b^*_N \otimes or_{N/M})^{m-k}. \end{aligned}$$

Thererfore, we have the following morphism of distinguished triangles:

$$\begin{array}{ccc} \boldsymbol{R}\mathcal{H}om_{\mathscr{D}_Y}(\boldsymbol{D}i^!\mathscr{L},\mathcal{D}b^*_N)\otimes or_{N/M}[-1] & \xrightarrow{\alpha} & \boldsymbol{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{L},\Gamma_N(\mathcal{D}b^*_M)) \\ \downarrow & & \downarrow \\ \boldsymbol{R}\mathcal{H}om_{\mathscr{D}_Y}(\boldsymbol{D}i^!\mathscr{M},\mathcal{D}b^*_N)\otimes or_{N/M}[-1] & \xrightarrow{\beta} & \boldsymbol{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M},\Gamma_N(\mathcal{D}b^*_M)) \\ \downarrow & & \downarrow \\ \boldsymbol{R}\mathcal{H}om_{\mathscr{D}_Y}(\boldsymbol{D}i^!\mathscr{N},\mathcal{D}b^*_N)\otimes or_{N/M}[-1] & \xrightarrow{\sim} & \boldsymbol{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{N},\Gamma_N(\mathcal{D}b^*_M)) \\ \downarrow{\scriptstyle +1} & & \downarrow{\scriptstyle +1} \end{array}$$

Therefore, if β is an isomorphism, then so is α. Hence the proof of Theorem 2.6 is reduced to the case where the weight is (m,m). For the same reasoning, the proof of

Theorem 4.2 is also reduced to the case where the weight is (m,m). Thus from now on, we assume that the weight of P is (m,m).

Take any open subset $\Omega'_0 \Subset \Omega_0$. Then each $b_j(x)$ in (2.1) is continuous on $\operatorname{Cl}\Omega'_0$. In particular, there is a $\nu_0 \in \mathbb{N}_0$ such that

$$b(x,\nu) \neq 0 \quad \text{for } x \in \operatorname{Cl}\Omega'_0 \text{ and } \nu \geqslant \nu_0.$$

We can prove the following:

Theorem 4.1. *Under the condition* (2.3), *for any* $\nu_1 \geqslant \nu_0$ *the operator* P *induces the following topological isomorphisms*:

$$P\colon t^{\nu_1}\Gamma(\Omega'_0;\widehat{\mathscr{G}}^*_{M|N}) \xrightarrow{\sim} t^{\nu_1}\Gamma(\Omega'_0;\widehat{\mathscr{G}}^*_{M|N}),$$
$$P\colon t^{\nu_1}\Gamma_{\mathrm{c}}(\Omega'_0;\widehat{\mathscr{G}}^*_{M|N}) \xrightarrow{\sim} t^{\nu_1}\Gamma_{\mathrm{c}}(\Omega'_0;\widehat{\mathscr{G}}^*_{M|N}).$$

Further, $\operatorname{supp} u = \operatorname{supp} Pu$ *holds for any* $u \in t^{\nu_1}\Gamma(\Omega'_0;\widehat{\mathscr{G}}^*_{M|N})$.

The proof of this theorem is essentially same as in [16, Theorem 1].

Sketch of Proof of Theorem 2.4. For $P\colon \mathscr{F} \to \mathscr{F}$, we set for short

$$\operatorname{Ker}_P \mathscr{F} := \operatorname{Ker}(P\colon \mathscr{F} \to \mathscr{F}), \quad \operatorname{Coker}_P \mathscr{F} := \operatorname{Coker}(P\colon \mathscr{F} \to \mathscr{F}).$$

We define a filtration $\{\mathscr{D}^{A,(\nu)}_{M\leftarrow N}\}_{\nu=0}^{\infty}$ on $\mathscr{D}^{A}_{M\leftarrow N}$ by

$$\mathscr{D}^{A,(\nu)}_{M\leftarrow N} := \{\sum_{j=0}^{\nu} a_j(x,\partial_x)\,\partial_t{}^j \otimes |dt|^{\otimes -1}\}.$$

Then, by (1.6)

$$\mathscr{D}^{A,(\nu)}_{M\leftarrow N} \underset{\mathscr{D}^A_N}{\otimes} \mathcal{D}b^*_N \subset \mathscr{D}^{A}_{M\leftarrow N} \underset{\mathscr{D}^A_N}{\otimes} \mathcal{D}b^*_N = \bigcup_{\nu\in\mathbb{N}_0} \mathscr{D}^{A,(\nu)}_{M\leftarrow N} \underset{\mathscr{D}^A_N}{\otimes} \mathcal{D}b^*_N \subset \Gamma_N(\mathcal{D}b^*_N).$$

Take any open subset $\Omega' \Subset \Omega$ and set $\Omega'_0 := \Omega' \cap N \Subset \Omega_0$.

Note that for any $\nu \in \mathbb{N}_0$, the operator P induces

$$P\colon \Gamma(\Omega'_0;\mathscr{D}^{A,(\nu)}_{M\leftarrow N} \underset{\mathscr{D}^A_N}{\otimes} \mathcal{D}b^*_N) \to \Gamma(\Omega'_0;\mathscr{D}^{A,(\nu)}_{M\leftarrow N} \underset{\mathscr{D}^A_N}{\otimes} \mathcal{D}b^*_N).$$

Taking duals of splitting exact sequences in (3.5) and using [7, Theorems 3.1 and 4.4], we have

$$0 \leftarrow (t^{\nu}\Gamma_{\mathrm{c}}(\Omega'_0;\widehat{\mathscr{G}}^*_{M|N}))' \leftarrow \Gamma(\Omega'_0;\Gamma_N(\mathcal{D}b^*_M)) \leftarrow \Gamma(\Omega'_0;\mathcal{D}b^*_N)^{\oplus\nu} \leftarrow 0.$$

Here we remark

$$\Gamma(\Omega'_0;\mathscr{D}^{(\nu-1)}_{X\leftarrow Y} \underset{\mathscr{D}_Y}{\otimes} \mathcal{D}b^*_N) \simeq \Gamma(\Omega'_0;\mathcal{D}b^*_N)^{\oplus\nu}$$

regarded as subspaces of $\Gamma(\Omega_0';\Gamma_N(\mathcal{D}b_M^*))$. Thus we have

$$(t^\nu \Gamma_{\mathrm{c}}(\Omega_0';\widehat{\mathscr{G}}_{M|N}^*))' \simeq \frac{\Gamma(\Omega_0';\Gamma_N(\mathcal{D}b_M^*))}{\Gamma(\Omega_0';\mathscr{D}_{M\leftarrow N}^{A,(\nu-1)} \underset{\mathscr{D}_N^A}{\otimes} \mathcal{D}b_N^*)}.$$

By (2.2), if P satisfies (2.3), then so does the formal adjoint ${}^tP$. Hence by Theorem 4.1, there is a $\nu_0 \in \mathbb{N}_0$ such that ${}^tP$ induce a topological isomorphism ${}^tP\colon t^\nu\Gamma_{\mathrm{c}}(\Omega_0';\widehat{\mathscr{G}}_{M|N}^*) \simeq t^\nu\Gamma_{\mathrm{c}}(\Omega_0';\widehat{\mathscr{G}}_{M|N}^*)$ for any $\nu \geqslant \nu_0$. Therefore we have an isomorphism

$$[P]\colon \frac{\Gamma(\Omega_0';\Gamma_N(\mathcal{D}b_M^*))}{\Gamma(\Omega_0';\mathscr{D}_{M\leftarrow N}^{A,(\nu-1)} \underset{\mathscr{D}_N^A}{\otimes} \mathcal{D}b_N^*)} \simeq \frac{\Gamma(\Omega_0';\Gamma_N(\mathcal{D}b_M^*))}{\Gamma(\Omega_0';\mathscr{D}_{M\leftarrow N}^{A,(\nu-1)} \underset{\mathscr{D}_N^A}{\otimes} \mathcal{D}b_N^*)}.$$

Hence we have

$$\begin{array}{ccccccccc}
0 \to & \Gamma(\Omega_0';\mathscr{D}_{M\leftarrow N}^{A,(\nu-1)} \underset{\mathscr{D}_N^A}{\otimes} \mathcal{D}b_N^*) & \to & \Gamma(\Omega_0';\Gamma_N(\mathcal{D}b_M^*)) & \to & \dfrac{\Gamma(\Omega_0';\Gamma_N(\mathcal{D}b_M^*))}{\Gamma(\Omega_0';\mathscr{D}_{M\leftarrow N}^{A,(\nu-1)} \underset{\mathscr{D}_N^A}{\otimes} \mathcal{D}b_N^*)} & \to 0 \\
& \downarrow P & & \downarrow P & & \wr\downarrow [P] & \\
0 \to & \Gamma(\Omega_0';\mathscr{D}_{M\leftarrow N}^{A,(\nu-1)} \underset{\mathscr{D}_N^A}{\otimes} \mathcal{D}b_N^*) & \to & \Gamma(\Omega_0';\Gamma_N(\mathcal{D}b_M^*)) & \to & \dfrac{\Gamma(\Omega_0';\Gamma_N(\mathcal{D}b_M^*))}{\Gamma(\Omega_0';\mathscr{D}_{M\leftarrow N}^{A,(\nu-1)} \underset{\mathscr{D}_N^A}{\otimes} \mathcal{D}b_N^*)} & \to 0
\end{array}$$

By Snake Lemma, we have

$$\mathrm{Ker}_P\,\Gamma(\Omega_0';\mathscr{D}_{M\leftarrow N}^{A,(\nu-1)} \underset{\mathscr{D}_N^A}{\otimes} \mathcal{D}b_N^*) \simeq \mathrm{Ker}_P\,\Gamma(\Omega_0';\Gamma_N(\mathcal{D}b_M^*)),$$

$$\mathrm{Coker}_P\,\Gamma(\Omega_0';\mathscr{D}_{M\leftarrow N}^{A,(\nu-1)} \underset{\mathscr{D}_N^A}{\otimes} \mathcal{D}b_N^*) \simeq \mathrm{Coker}_P\,\Gamma(\Omega_0';\Gamma_N(\mathcal{D}b_M^*)).$$

Taking inductive limits, we obtain the theorem. □

By using Theorem 4.1, we also obtain:

Theorem 4.2 (Cauchy-Kovalevskaja type Theorem). *There exists an isomorphism under the condition* (2.3):

$$\boldsymbol{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M},\widehat{\mathscr{G}}_{M|N}^*)\big|_N \simeq \boldsymbol{R}\mathcal{H}om_{\mathscr{D}_Y}(\boldsymbol{D}i^*\mathscr{M},\mathscr{G}_N^*).$$

Sketch of Proof of Theorem 2.6. We use the argument of the proof in [16, Proposition 1]. Take $\sigma_1 \in \mathbb{Q}$ with $\sigma_1 \geqslant 1$ arbitrary. Recall the following operator considered in [16, Proposition 1]:

$$L := \vartheta^m + \sum_{j=0}^{m-1} b_j\vartheta^j - \sum_{|\alpha|+j\leqslant m} t^{l(\alpha,j)}c_{\alpha,j}\,\partial_x^\alpha\,\vartheta^j.$$

We assume that $l(\alpha,j) \in \mathbb{N}$ and further that

(i) $b_j \in \mathbb{R}$ and $b(\nu) \neq 0$ for any $\nu \in \mathbb{Z}$, and $b(\nu) > 0$ for any $\nu \in \mathbb{N}_0$;
(ii) $c_{\alpha,j} \geqslant 0$.

Moreover we can choose L as $\mathscr{I}_T(L) = \sigma_1$. Set $b(\nu) := \nu^m + \sum_{j=0}^{m-1} b_j \nu^j$.

Now, set $P := {}^tL$ and $\mathscr{M} := \mathscr{D}_X/\mathscr{D}_X P$. Take s as

$$\mathscr{I}_T(P) = \mathscr{I}_T({}^tP) < s < \infty.$$

By (i) and [12], we have $\boldsymbol{D}i^!\mathscr{M} = 0$. Hence if the isomorphism (2.4) holds for P, then

$$\boldsymbol{R}\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \varGamma_N(\mathcal{D}b^*_M)) = 0;$$

that is, we obtain a sheaf isomorphism:

$$P\colon \varGamma_N(\mathcal{D}b^*_M) \xrightarrow{\sim} \varGamma_N(\mathcal{D}b^*_M). \tag{4.1}$$

Hence, we obtain an isomorphism

$$P\colon \varGamma_N(\Omega; \mathcal{D}b^*_M) \xrightarrow{\sim} \varGamma_N(\Omega; \mathcal{D}b^*_M). \tag{4.2}$$

(1) First assume that $* = \{s\}$. Since $\varGamma_N(\Omega; \mathcal{D}b^{\{s\}}_M)$ is an (FS) space, we can apply Banach's open mapping theorem to prove that (4.2) is a topological isomorphism. Taking the dual, we have a topological isomorphism:

$$L\colon \varGamma_{\mathrm{c}}(\Omega_0; \widehat{\mathscr{G}}^{\{s\}}_{M|N}) \xrightarrow{\sim} \varGamma_{\mathrm{c}}(\Omega_0; \widehat{\mathscr{G}}^{\{s\}}_{M|N}).$$

For the operator L, we take an index $(\alpha, j) \in \mathscr{S}_L$ such that

$$\frac{m - j - l(\alpha, j)}{|\alpha| - l(\alpha, j)} < s.$$

We consider the following differential equation:

$$b(\vartheta)w - C_{\alpha,j} t^{l(\alpha,j)} \partial_x^\alpha \vartheta^j w = \varphi(x). \tag{4.3}$$

Then for any smooth function $\varphi(x)$, the equation (4.3) has a unique formal solution

$$w(x, \langle t \rangle) = \frac{\varphi(x)}{b(0)} + \frac{1}{b(0)} \sum_{p=1}^{\infty} \prod_{q=1}^{p-1} \frac{(ql(\alpha, j))^j}{b(ql(\alpha, j))} \frac{C_{\alpha,j}^{\,p}}{b(pl(\alpha, j))} \partial_x^{p\alpha} \varphi(x)\, t^{pl(\alpha,j)}.$$

Further, if $u(x, \langle t \rangle)$ is a solution to $Lu = \varphi(x)$, then

$$0 \ll w(0, \langle t \rangle) \ll u(0, \langle t \rangle).$$

However we can prove that there is a $\varphi(x) \in \Gamma_{\rm c}(\Omega_0; \mathscr{G}_N^{\{s\}})$ such that the solution $w(x, \langle t\rangle) \in C_0^\infty(\Omega_0)[[t]]$ to (4.3) satisfies

$$\limsup_{\nu\to\infty}\Big(\frac{w_\nu(0)}{\nu!^\sigma}\Big)^{1/\nu} = \infty.$$

Hence

$$\varphi(x) \notin \operatorname{Image}(\Gamma_{\rm c}(\Omega_0; \widehat{\mathscr{G}}^{\{s\}}_{M|N}) \xrightarrow{L} \Gamma_{\rm c}(\Omega_0; \widehat{\mathscr{G}}^{\{s\}}_{M|N})).$$

(2) If $* = (s)$, then $\operatorname{supp} Pu = \operatorname{supp} u$ holds for any $u \in \Gamma_N(\Omega; \mathcal{D}b_M^{(s)})$ by (4.1). Hence we obtain an isomorphism

$$P\colon \Gamma_{\rm c}(\Omega_0; \Gamma_N(\mathcal{D}b_M^{(s)})) \simeq \Gamma_{\rm c}(\Omega_0; \Gamma_N(\mathcal{D}b_M^{(s)})). \tag{4.4}$$

Since $\Gamma_{\rm c}(\Omega_0; \Gamma_N(\mathcal{D}b_M^{(s)}))$ is a (DFS) space, we can apply Pták's open mapping theorem (see [13]) to prove that (4.4) is a topological isomorphism. Taking duals, we obtain an isomorphism

$$L\colon \Gamma(\Omega_0; \widehat{\mathscr{G}}^{(s)}_{M|N}) \simeq \Gamma(\Omega_0; \widehat{\mathscr{G}}^{(s)}_{M|N}).$$

As in (1), we can prove that there is a $\varphi(x) \in \Gamma(\Omega_0; \mathscr{G}_N^{(s)})$ such that

$$\varphi(x) \notin \operatorname{Image}(\Gamma(\Omega_0; \widehat{\mathscr{G}}^{(s)}_{M|N}) \xrightarrow{L} \Gamma(\Omega_0; \widehat{\mathscr{G}}^{(s)}_{M|N})).$$

Therefore we reach a contradiction. □

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