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Topological Radon Transforms and Their Applications

By

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§ 1. Introduction

Let X be a real analytic manifold. We say that a $\mathbb{Z}$-valued function $\varphi\colon X \to \mathbb{Z}$ is constructible if there exists a locally finite family $\{X_i\}_{i\in I}$ of compact subanalytic subsets X_i of X such that φ is expressed by

$$\varphi = \sum_{i\in I} c_i \mathbf{1}_{X_i} \qquad (c_i \in \mathbb{Z}).$$

Let us consider the diagram:

$$\begin{tikzcd} & X \times Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ & S \arrow[u, hook] \arrow[dl, "f"] \arrow[dr, "g"'] & \\ X & & Y. \end{tikzcd}$$

Here X and Y are real analytic manifolds, S is a subanalytic subset of $X \times Y$ and f and g are restrictions of natural projections p_1 and p_2 to S respectively.

In [24], [38], [45], several operations, such as direct and inverse images and so on, on constructible functions were introduced. See Section 2.1 for the precise definitions. Therefore, we can associate a constructible function φ on X with a constructible function $\mathcal{R}_S(\varphi)$ on Y, which is called the topological Radon transform of φ, by applying the inverse image by f and the direct image by g to φ. Now let X be a projective space $\mathbb{P}_N$, Y its dual space $\mathbb{P}_N^*$, and S the incidence submanifold of $X \times Y$. Note that $Y = \mathbb{P}_N^*$ is

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naturally identified with the set of hyperplanes (linear subspaces of codimension one) $H \simeq \mathbb{P}_{N-1}$ in $X = \mathbb{P}_N$. In this situation, for a subanalytic subset K of X, the topological Radon transform $\mathcal{R}_S(\mathbf{1}_K)$ of $\mathbf{1}_K$ satisfies

$$\mathcal{R}_S(\mathbf{1}_K)(H) = \chi(K \cap H)$$

for any hyperplane $H \in Y = \mathbb{P}_N^*$. Namely, the values of our topological Radon transform $\mathcal{R}_S(\mathbf{1}_K)$ are the topological Euler characteristics of hyperplane sections of K.

We studied these topological Radon transforms on projective spaces or on Grassmann manifolds by the combinatorial Schubert calculus and the microlocal theory of sheaves developed by Kashiwara-Schapira [24]. In this note, we shall explain topological Radon transforms and survey our following recent results.

(i) Inversion formulas [29]: We generalized Schapira's formula in [37]. An intuitive meaning of one of our results is as follows. For an integer $0 \leq q \leq N-1$ and a subanalytic subset K of $X = \mathbb{P}_N$, we can reconstruct K from the Euler characteristics of the sections of K by q-dimensional linear subspaces $L \simeq \mathbb{P}_q$ in $X = \mathbb{P}_N$ under appropriate conditions.

(ii) Projective duality [30]: We generalized Ernström's result [7] obtained in the complex case to the real case. For a smooth real projective variety $M \subset \mathbb{RP}_N$, we completely described the way of changes of the Euler characteristics of the hyperplane sections of M (the values of $\mathcal{R}_S(\mathbf{1}_M)$) in terms of the singularities of the dual variety $M^* \subset \mathbb{RP}_N^*$ of M. Recall that the dual variety M^* of M is the set of hyperplanes tangent to M:

$$M^* = \overline{\{H \in \mathbb{RP}_N^* \mid \exists x \in M_{\mathrm{reg}} \ \text{ s.t. } \ T_xM \subset T_xH\}} \subset \mathbb{RP}_N^*.$$

(iii) Class formulas [31], [32]: As an application of topological Radon transforms to algebraic geometry, we obtained some class formulas which express the algebraic degrees of dual varieties. Our formulas are reformulations or generalizations of Ernström's one [8], and we can recover and even extend various previous results by Plücker, Teissier [41], Kleiman [27], Holme [17] and so on. We also generalize them to the case of associated varieties studied by Gelfand-Kapranov-Zelevinsky [12] and so on.

§ 2. Preliminaries

§ 2.1. Constructible Functions

Definition 2.1. Let X be a real analytic manifold. We say that a function $\varphi\colon X \to \mathbb{Z}$ is constructible if there exists a locally finite family $\{X_i\}_{i \in I}$ of compact subanalytic subsets X_i of X such that φ is expressed by

$$\varphi = \sum_i c_i \mathbf{1}_{X_i} \qquad (c_i \in \mathbb{Z}).$$

We denote the abelian group of constructible functions on X by $CF(X)$.

We define the operations on constructible functions in the following way.

Definition 2.2 ([24], [45]). Let $f\colon Y \to X$ be a morphism of real analytic manifolds.
(i) (The inverse image) For $\varphi \in CF(X)$, we define a function $f^*\varphi \in CF(Y)$ by

$$f^*\varphi(y) := \varphi(f(y)).$$

(ii) (The integral) Let $\varphi = \sum_i c_i \mathbf{1}_{X_i} \in CF(X)$ and assume that $\mathrm{supp}(\varphi)$ is compact. Then we define a topological (Euler) integral $\int_X \varphi \in \mathbb{Z}$ of φ by

$$\int_X \varphi := \sum_i c_i \cdot \chi(X_i),$$

where $\chi(X_i)$ is the topological Euler characteristic of X_i.
(iii) (The direct image) Let $\psi \in CF(Y)$ such that $f|_{\mathrm{supp}(\psi)}\colon \mathrm{supp}(\psi) \to X$ is proper. Then we define a function $\int_f \psi \in CF(X)$ by

$$\Big(\int_f \psi\Big)(x) := \int_Y (\psi \cdot \mathbf{1}_{f^{-1}(x)}).$$

§ 2.2. Topological Radon Transforms

Let X and Y be real analytic manifolds and S a real analytic submanifold of $X \times Y$. Consider the diagram:

(2.1)
$$\begin{tikzcd}
 & X \times Y \arrow[ldd, "p_1"'] \arrow[rdd, "p_2"] & \\
 & S \arrow[u, hook] \arrow[ld, "f"] \arrow[rd, "g"'] & \\
X & & Y,
\end{tikzcd}$$

where p_1 and p_2 are natural projections and f and g are restrictions of p_1 and p_2 to S respectively.

Definition 2.3. Let $\varphi \in CF(X)$. We define the topological Radon transform $\mathcal{R}_S(\varphi) \in CF(Y)$ of φ by

$$\mathcal{R}_S(\varphi) := \int_g f^*\varphi.$$

We denote the projective space of dimension N over a field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) by $\mathbb{P}_N$ and its dual space by $\mathbb{P}_N^*$. Then we have the following natural identifications:

$$\begin{aligned}\mathbb{P}_N &= \{l \mid l \text{ is a line in } \mathbb{K}^{N+1} \text{ through the origin}\},\\ \mathbb{P}_N^* &= \{H' \mid H' \text{ is a hyperplane in } \mathbb{K}^{N+1} \text{ through the origin}\}.\end{aligned}$$

Note that if we projectivize a hyperplane H' in $\mathbb{K}^{N+1}$ we obtain a hyperplane (i.e. a linear subspace of codimension one) $H \simeq \mathbb{P}_{N-1}$ in $\mathbb{P}_N$. Therefore in what follows, we identify the dual projective space $\mathbb{P}_N^*$ with the set

$$\{H \mid H \text{ is a hyperplane in } \mathbb{P}_N\}.$$

Example 2.4. Let $X = \mathbb{P}_N$, $Y = \mathbb{P}_N^*$, $S = \{(l, H) \in X \times Y \mid l \subset H\}$ (the incidence submanifold of $X \times Y$) and M a real analytic submanifold of $X = \mathbb{P}_N$. Then for any hyperplane $H \in Y$ we have

$$\mathcal{R}_S(\mathbf{1}_M)(H) = \chi(M \cap H).$$

§ 3. Inversion Formulas for Topological Radon Transforms

In this section, we introduce our results in [29].

For $0 \leq k \leq N-1$, we denote by $\mathbb{G}_{N,k}$ the Grassmann manifold consisting of k-dimensional linear subspaces $L \simeq \mathbb{P}_k$ in $\mathbb{P}_N$. Namely we set

$$\begin{aligned}\mathbb{G}_{N,k} &= \{L' \mid L' \text{ is a } (k+1)\text{-dimensional linear subspace in } \mathbb{K}^{N+1} \text{ through the origin}\}\\ &= \{L \mid L \text{ is a } k\text{-dimensional linear subspace in } \mathbb{P}_N\}.\end{aligned}$$

Let us consider the diagram (2.1) for $X = \mathbb{G}_{N,p}$, $Y = \mathbb{G}_{N,q}$ ($p < q$) and $S = \{(L_p, L_q) \in \mathbb{G}_{N,p} \times \mathbb{G}_{N,q} \mid L_p \subset L_q\}$ (the incidence submanifold of $X \times Y$).

In this case, unfortunately the formal dual ${}^t\mathcal{R}_S = \displaystyle\int_f g^*$ of $\mathcal{R}_S$ is not a left inverse of our topological Radon transform $\mathcal{R}_S = \displaystyle\int_g f^*$ in general. By modifying the kernel function of the formal dual ${}^t\mathcal{R}_S$ by the Schubert calculus on Grassmann manifolds, we could construct a left inverse of $\mathcal{R}_S$ as follows.

Theorem 3.1 ([29]). *Assume that one of the following conditions are satisfied.*

(i) $\mathbb{K} = \mathbb{C}$ *and* $p + q \leq n + 1$,
(ii) $\mathbb{K} = \mathbb{R}$, $p + q \leq n + 1$ *and* $q - p$ *is even.*

Then there exist a group homomorphism $\widehat{\mathcal{R}}\colon CF(Y) \to CF(X)$ *and a constant* $C_{p,q} \neq 0$ *which depends only on* p *and* q *such that*

$$\widehat{\mathcal{R}} \circ \mathcal{R}_S(\varphi) = C_{p,q} \cdot \varphi \quad \textit{for any } \varphi \in CF(X).$$

Note that our construction of the left inverse $C_{p,q}^{-1} \cdot \widehat{\mathcal{R}}$ of $\mathcal{R}_S$ in [29] is quite explicit. We thus can completely reconstruct the original function $\varphi \in CF(X)$ from its topological Radon transform $\mathcal{R}_S(\varphi)$. In particular, when $\mathbb{K} = \mathbb{R}$, $p = 0$ and q is even, Theorem 3.1 implies that for any subanalytic set K of $X = \mathbb{RP}_N$ we can reconstruct K from the topological Euler characteristics of its sections by q-dimensional linear subspaces $L_q \simeq \mathbb{RP}_q$ in $X = \mathbb{RP}_N$.

Remark. The meaning of our integrations is not the (usual) analytic one but the topological one based on Euler characteristics. Nevertheless, our results above are very similar to the ones obtained in the case of analytic Radon transforms. For example, by using invariant differential operators, Kakehi [19] obtained an inversion formula for analytic Radon transforms of C^∞-functions on $\mathbb{G}_{N,p}$ under the same condition that $\mathbb{K} = \mathbb{R}$ and $q - p$ is even. Namely, in spite of the difference of the definitions of integrations, the sufficient conditions under which we obtain an inversion formula coincide with each other. It would be an interesting problem to investigate the reason why we need the same condition. Note that very recently in [14], Grinberg and Rubin constructed an inversion formula for analytic Radon transforms of C^∞-functions on $\mathbb{G}_{N,p}$ for $\mathbb{K} = \mathbb{R}$ and any p, q by using the Gårding-Gindikin fractional integrals.

In some special cases, we can prove also that the left inverse $C_{p,q}^{-1} \cdot \widehat{\mathcal{R}}$ in Theorem 3.1 is actually the inverse of $\mathcal{R}_S$ as follows.

Theorem 3.2 ([29]). *Assume that one of the following conditions are satisfied.*

(i) $\mathbb{K} = \mathbb{C}$ *and* $p + q = n + 1$,
(ii) $\mathbb{K} = \mathbb{R}$, $p + q = n + 1$ *and* $q - p$ *is even.*

Then the topological Radon transform $\mathcal{R}_S$ *induces a non-trivial group isomorphism between* $CF(\mathbb{G}_{N,p})$ *and* $CF(\mathbb{G}_{N,q})$.

In the special case where $\mathbb{K} = \mathbb{R}$, $p = 0$, $q = N - 1$ (i.e. when $X = \mathbb{RP}_N$, $Y = \mathbb{RP}_N^*$) and N is odd, Schapira [37] already proved that

$${}^t\mathcal{R}_S \circ \mathcal{R}_S(\varphi) = \varphi \quad \text{for any} \quad \varphi \in CF(X).$$

Hence our result is a generalization of this result to Grassmann cases.

§ 4. Helgason-Type Support Theorem

In [30], we obtained an analogue of Helgason's support theorem [16] for topological Radon transforms.

Theorem 4.1 ([30]). *Let $K \subset \mathbb{R}^N$ be a compact convex subset and $\varphi \in CF(\mathbb{R}^N)$ a constructible function on $\mathbb{R}^N$ such that $\{x \in \mathbb{R}^N \mid \varphi(x) \neq 0\}$ is a compact set in $\mathbb{R}^N$. Assume that for any hyperplane $H \subset \mathbb{R}^N$ such that $K \cap H = \emptyset$, we have $\int_H \varphi = 0$. Then* $\mathrm{supp}(\varphi) = \{x \in \mathbb{R}^N \mid \varphi(x) \neq 0\}$ *is contained in K.*

Our proof of this theorem is similar to that of the microlocal proof of Helgason's support theorem given by Boman-Quinto [3].

Remark. This theorem does not hold for all constructible functions on $\mathbb{R}^N$. Even for constructible functions having closed supports, there are some counterexamples. Let us give a very simple example. In $\mathbb{R}^2$, consider the following disjoint subsets.

$$\begin{aligned} K &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 2\}, \\ A_1 &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 > 2,\ x_1 = 1\}, \\ A_2 &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 > 2,\ x_1 = -1\}. \end{aligned}$$

We define a constructible function $\varphi \colon \mathbb{R}^2 \to \mathbb{Z}$ on $\mathbb{R}^2$ by

$$\varphi(x) = \begin{cases} 1 & x \in K \cup A_1, \\ -1 & x \in A_2, \\ 0 & \text{otherwise}. \end{cases}$$

Then $\mathrm{supp}(\varphi) = \{x \in \mathbb{R}^2 \mid \varphi(x) \neq 0\}$ is a non-compact closed subset of $\mathbb{R}^2$ and for any line l in $\mathbb{R}^2$ such that $K \cap l = \emptyset$ we have $\int_l \varphi = 0$. However $\mathrm{supp}(\varphi)$ is not contained in K in this case.

§ 5. Application (1): Projective Duality

In this section, we consider the case where X is a projective space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and Y is its dual space.

§ 5.1. Dual Varieties

Definition 5.1. Let V be a projective variety in $X = \mathbb{CP}_N$. We define the dual variety V^* of V by

$$V^* := \overline{\{H \in \mathbb{CP}_N^* \mid \exists x \in V_{\text{reg}} \cap H \text{ s.t. } T_x V \subset T_x H\}} \quad (\subset \mathbb{CP}_N^*).$$

When V is smooth, V^* is the set of hyperplanes tangent to V. Note that even if V is smooth V^* may be very singular in general.

Many mathematicians were interested in the mysterious relations between projective varieties and their duals. Above all, they observed that the tangency of a hyperplane $H \in V^*$ with V is related to the singularity of the dual V^* at H. For example, consider the case of a plane curve $C \subset \mathbb{CP}_2$. Then a tangent line l at an inflection point of C corresponds to a cusp of the dual curve C^*, and a bitangent (double tangent) line l of C corresponds to an ordinary double point of C^*. The most general results for complex plane curves were found in the 19th century by Klein, Plücker and Clebsch and so on (see for example, [42, Theorem 1.6] and [46, Chapter 7] and so on).

In the last two decades, this beautiful correspondence was extended to higher-dimensional complex projective varieties from the viewpoint of the geometry of hyperplane sections. In particular, after some important contributions by Viro [45] and Dimca [6] and so on, Ernström proved the following remarkable result in 1994.

Theorem 5.2 ([7]). *Let $V \subset \mathbb{CP}_N$ be a smooth projective variety over $\mathbb{C}$. Take a generic hyperplane H in $\mathbb{CP}_N$ such that $H \notin V^*$. Then for any hyperplane $L \in V^*$, we have*

$$\chi(V \cap L) - \chi(V \cap H) = (-1)^{N-1+\dim V - \dim V^*} \mathrm{Eu}_{V^*}(L),$$

where $\mathrm{Eu}_{V^*} \colon V^* \to \mathbb{Z}$ is the Euler obstruction of V^* (introduced by Kashiwara [20] *and MacPherson* [28] *independently).*

Recall that the Euler obstruction Eu_{V^*} of V^* is a $\mathbb{Z}$-valued function on V^* which measures the singularity of V^* at each point of V^*. For example, Eu_{V^*} takes the value 1 on the regular part of V^*. Moreover, if we take a complex Whitney stratification $\bigsqcup_{\alpha \in A} V^*_\alpha$ of V^* consisting of connected strata, then Eu_{V^*} is constant on each stratum V^*_α. The values of Eu_{V^*} on a stratum V^*_α is determined by those on V^*_β's satisfying the condition $V^*_\alpha \subset \overline{V^*_\beta}$ (for more detail, see for example [21]).

Hence Ernström's result above says that the jumping number of the topological Euler characteristics of hyperplane sections of V at L is expressed by $\mathrm{Eu}_{V^*}(L)$, that is, the singularity of the dual variety V^* at L.

In [30], we gave a more transparent proof to Theorem 5.2 by using the microlocal theory of sheaves developed by Kashiwara-Schapira [24]. By our methods, we can also generalize Theorem 5.2 to the real case. Namely, for any smooth real projective variety $M \subset \mathbb{RP}_N$ we completely described the way of changes of the Euler characteristics of the hyperplane sections of M in terms of the singularities of the dual $M^* \subset \mathbb{RP}_N^*$ of M. In the real setting, we need also the number of positive principal curvatures of the dual variety M^* in order to state our results. For more precise statements of our results and examples, see [30], [33].

§ 5.2. Outline of Our Proof

In this subsection, we give an outline of our proof of Ernström's result (Theorem 5.2) and its generalization in [30].

First, note that in terms of topological Radon transforms as in Example 2.4, Theorem 5.2 is rewritten as

$$\mathcal{R}_S(\mathbf{1}_V) = d \cdot \mathbf{1}_Y + (-1)^{N-1+\dim V - \dim V^*} \mathrm{Eu}_{V^*},$$

where we set $d = \chi(V \cap H)$ by taking a generic hyperplane H in $X = \mathbb{CP}_N$ such that $H \notin V^*$. Hence it suffices to study the topological Radon transform $\mathcal{R}_S(\mathbf{1}_V)$.

Let $\mathscr{L}_X$ be the sheaf of conic ($\mathbb{R}_{>0}$-invariant) subanalytic Lagrangian cycles in the cotangent bundle T^*X to X. Its global section $H^0(T^*X; \mathscr{L}_X)$ is the abelian group of conic subanalytic Lagrangian cycles in T^*X. In 1985, Kashiwara [22] constructed a group homomorphism $\mathrm{CC}\colon CF(X) \to H^0(T^*X; \mathscr{L}_X)$, called the characteristic cycle homomorphism:

Theorem 5.3 ([22]). *There exists a group isomorphism*

$$\mathrm{CC}\colon CF(X) \to H^0(T^*X; \mathscr{L}_X).$$

In particular, the characteristic function $\mathbf{1}_V \in CF(X)$ of V is sent to the cycle $[T_V^*X]$ generated by its conormal bundle T_V^*X.

By the isomorphism in Theorem 5.3, instead of the topological Radon transform $\mathcal{R}_S\colon CF(X) \to CF(Y)$ itself, we studied the corresponding operation (microlocal Radon transform) Ψ for Lagrangian cycles and found that it induces an isomorphism

$$\Psi\colon H^0(\dot{T}^*X; \mathscr{L}_X) \xrightarrow{\sim} H^0(\dot{T}^*Y; \mathscr{L}_Y),$$

where we set $\dot{T}^*X = T^*X \setminus T_X^*X$ and $\dot{T}^*Y = T^*Y \setminus T_Y^*Y$ (the zero-sections are removed). Moreover Ψ is (up to some sign $\varepsilon = \pm 1$) equal to the isomorphism of Lagrangian cycles induced by the canonical diffeomorphism $\Phi\colon \dot{T}^*X \xrightarrow{\sim} \dot{T}^*Y$ which coincides with the classical Legendre transform in the standard affine charts of $X = \mathbb{CP}_N$ and $Y = \mathbb{CP}_N^*$. Since the characteristic cycle $\mathrm{CC}(\mathbf{1}_V)$ of $\mathbf{1}_V \in CF(X)$ is the conormal cycle $[\dot{T}_V^*X]$ in $\dot{T}^*X$, the characteristic cycle $\mathrm{CC}(\mathcal{R}_S(\mathbf{1}_V))$ of the topological Radon transform $\mathcal{R}_S(\mathbf{1}_V)$

is $\varepsilon[\Phi(\dot{T}^*_V X)]$ in $\dot{T}^*Y$. Let $\dot{\pi}_Y \colon \dot{T}^*Y \to Y$ be the projection. Set $W = (\dot{\pi}_Y \circ \Phi)(\dot{T}^*_V X) \subset Y$. Then we can easily prove that W coincides with the dual variety V^* of V (in the classical literature we call it a caustic or a Legendre singularity). To prove Theorem 5.2 and the corresponding result in [30], we have only to determine the sign $\varepsilon = \pm 1$. We determined it by employing the theory of pure sheaves in [24]. In the real case, we also had to express the Maslov indices of the Lagrangian submanifolds $\dot{T}^*_V X$ and $\dot{T}^*_{W_{\mathrm{reg}}} Y$ by the principal curvatures of V and W_{reg} respectively with the help of results by Fischer-Piontkowski [9].

§ 6. Application (2): Class Formulas

§ 6.1. General Formulas

In the 19th century, Plücker and Clebsch discovered the following remarkable result for plane curves over $\mathbb{C}$.

Theorem 6.1. *Let C be a plane curve of degree d with δ ordinary double points and κ cusps. Then for the dual curve C^* of C, we have*

$$\deg C^* = d(d-1) - 2\delta - 3\kappa.$$

Namely, if C is a smooth plane curve of degree d, we have $\deg C^* = d(d-1)$. But if C has singular points, the above formula for $\deg C^*$ contains some correction terms coming from the singularities of C. In the 20th century, the higher-dimensional analogue of this theorem was studied by many mathematicians. Let V be a projective variety of $\mathbb{CP}_N$. Since the degree of the dual variety V^* is called the class of V, the formulas for the degrees of dual varieties are called class formulas (or Plücker formulas).

Our starting point is the following very general topological class formula obtained by Ernström [8].

Theorem 6.2 ([8]). *Let V be a projective variety in $X = \mathbb{CP}_N$ and set $r = \operatorname{codim} V^* = N - \dim V^*$. Then we have*

$$\deg V^* = (-1)^{\dim V + r + 1} \left\{ r \int_X \mathrm{Eu}_V - (r+1) \int_{L_{N-1}} \mathrm{Eu}_V + \int_{L_{N-r-1}} \mathrm{Eu}_V \right\}, \tag{6.1}$$

where $L_i \simeq \mathbb{CP}_i$ is a generic linear subspace of dimension i in $\mathbb{CP}_N$.

In [31], we found a new proof of this theorem which does not use Segre classes nor polar varieties. We used instead Theorem 5.2 ([7]) and some elementary lemmas on constructible functions.

§ 6.2. Various Class Formulas

Theorem 6.2 is quite general, but in practice it is not so easy to calculate the degrees of dual varieties by it. We want to obtain more effective class formulas like the original Plücker formula (Theorem 6.1). For this purpose, we recall the notions of Chern-Mather classes and Chern-Schwartz-MacPherson classes.

Let $\mathcal{V}ar_c$ (resp. $\mathcal{A}b$) be the category of complete algebraic varieties over $\mathbb{C}$ (resp. abelian category of abelian groups). For $V \in \mathcal{V}ar_c$, we denote by $CF(V)$ the abelian group of $\mathbb{Z}$-valued functions which are constructible with respect to a complex stratification of V. Then by the direct images of constructible functions, we obtain a covariant functor

$$CF\colon \mathcal{V}ar_c \to \mathcal{A}b.$$

On the other hand, we have also a natural covariant functor

$$CH_* = \bigoplus_{i\in\mathbb{Z}} CH_i\colon \mathcal{V}ar_c \to \mathcal{A}b$$

which assigns to a complete variety $V \in \mathcal{V}ar_c$ its (total) Chow group $CH_*(V) = \bigoplus_{i\in\mathbb{Z}} CH_i(V) \in \mathcal{A}b$ (for the detail, see [10] and so on). In the 1960's, Deligne and Grothendieck predicted the existence of a natural transformation $CF \to CH_*$ satisfying some natural properties. This conjecture was solved by MacPherson [28] as follows.

Theorem 6.3 ([28]). *There exists a natural transformation $c_*\colon CF \to CH_*$ such that for any smooth complete variety $V \in \mathcal{V}ar_c$ we have $c_*(\mathbf{1}_V) = c^*(TV) \cap [V]$. Here $c^*(TV)$ is the total Chern class of the tangent bundle TV of V (defined in the Chow cohomology class $CH^*(V)$ of V) and $[V] \in CH_{\dim V}(V)$ is the fundamental class of V (see [10] and so on for the definition).*

For the proof of this fundamental theorem, see [25], [28], [36] and so on.

Definition 6.4. Let V be a complete variety over $\mathbb{C}$.
(i) We set

$$c_*^{\mathrm{CM}}(V) := c_*(\mathrm{Eu}_V) \in CH_*(V)$$

and call it the Chern-Mather class of V.
(ii) We set

$$c_*(V) := c_*(\mathbf{1}_V) \in CH_*(V)$$

and call it the Chern-Schwartz-MacPherson class of V.

Now we can rewrite Ernström's topological class formula (6.1) in terms of Chern-Mather classes as follows. From now on, for a projective variety V in $X = \mathbb{CP}_N$ we set $r = \operatorname{codim} V^* = N - \dim V^*$.

Theorem 6.5 ([32]). *Let V be a projective variety in $X = \mathbb{CP}_N$. Then*

$$\deg V^* = (-1)^{\dim V + r + 1} \sum_{j=0}^{r-1} \binom{r+1}{j} (r-j) \int_V \frac{h^j}{(1+h)^{r+1}} c_*^{\mathrm{CM}}(V),$$

where h is the hyperplane class.

By our new proof of Theorem 6.2 in [31] and Theorem 6.5 above, we can also determine the dimension of the dual variety V^* as follows. For each integer $i \geq 1$, take a generic hyperplane $H = L_{N-1} \subset X = \mathbb{CP}_N$ (resp. a generic linear subspace $L_{N-i-1} \subset X = \mathbb{CP}_N$ of dimension $N - i - 1$) and set

$$\delta_i := (-1)^{\dim V + i + 1} \Big\{ i \int_X \mathrm{Eu}_V - (i+1) \int_H \mathrm{Eu}_V + \int_{L_{N-i-1}} \mathrm{Eu}_V \Big\}.$$

Then the codimension of V^* is the integer $k \geq 1$ such that $\delta_1 = \delta_2 = \cdots = \delta_{k-1} = 0$ and $\delta_k \neq 0$. By (the proof of) Theorem 6.5, we can rewrite δ_i's in terms of Chern-Mather classes of V as

$$\delta_i = (-1)^{\dim V + i + 1} \sum_{j=0}^{i-1} \binom{i+1}{j} (i-j) \int_V \frac{h^j}{(1+h)^{i+1}} c_*^{\mathrm{CM}}(V).$$

This last formula is very useful for the exact computation of δ_i's. We thus obtained an effective method to determine the dimensions of the duals of singular varieties, similar to Holme's theorem [17, Theorem 3.4 (i)] (see also [2], [11] and [42, Chapter 6] and so on for related important results for smooth varieties). For the detail, see [31, Remark 3.3].

Now let us apply a localization theorem proved by Parusinski-Pracgacz [35] (see also [39] and [47] and so on) to Theorem 6.5. Then we obtain the following class formulas.

Theorem 6.6 ([32]). *Let V be a hypersurface of the projective space $X = \mathbb{CP}_N$ of degree d. For the decomposition $V_{\mathrm{sing}} = \bigsqcup_{\alpha \in A} V_\alpha$ of V_{sing} into connected components, assume the following conditions.*

(i) *V_α is a smooth complete intersection subvariety of codimension l_α in $X = \mathbb{CP}_N$ with multi-degree $\vec{d}_\alpha = (d_{\alpha,1}, \cdots, d_{\alpha,l_\alpha})$,*
(ii) *$V = V_{\mathrm{reg}} \sqcup \bigsqcup_{\alpha \in A} V_\alpha$ is a Whitney stratification of V.*

Then we have

$$\deg V^* = d(d-1)^{N-r} \frac{d}{dx} \Big\{ \frac{(1-x)^{r+1} - 1}{x} \Big\} \Big|_{x=d} - \sum_{\alpha \in A} R_\alpha,$$

where the correction term R_α *is defined by*

$$R_\alpha = \mu_\alpha \Bigg(\sum_{k=\max\{0, M_\alpha - 1\}}^{N-l_\alpha} \widetilde{D}(l_\alpha, k)\pi'_k(\vec{d}_\alpha) \Bigg) + \mu'_\alpha \Bigg(\sum_{k=\max\{0, M_\alpha\}}^{N-l_\alpha} D(l_\alpha, k)\pi_k(\vec{d}_\alpha) \Bigg).$$

Here μ_α *(resp.* μ'_α*) is a topological invariant of* V *called the Milnor number of* V *along* V_α *(resp. the slice Milnor number (for the precise definition, see* [32]*) of* V *along* V_α*). We set also* $M(l_\alpha) = M_\alpha := N - r - l_\alpha + 1$ *and*

$$\pi_k(\vec{d}_\alpha) := d_{\alpha,1} \cdots d_{\alpha,l_\alpha} \cdot \sum_{\substack{|\vec{m}|=k \\ \vec{m} \in \mathbb{Z}_{\geq 0}^{l_\alpha}}} \prod_{j=1}^{l_\alpha} (d_{\alpha,j} - 1)^{m_j},$$

$$\pi'_k(\vec{d}_\alpha) := d_{\alpha,1} \cdots d_{\alpha,l_\alpha} \cdot \sum_{\substack{|\vec{m}|=k \\ \vec{m} \in \mathbb{Z}_{\geq 0}^{l_\alpha+1}}} (d-1)^{m_0} \prod_{j=1}^{l_\alpha} (d_{\alpha,j} - 1)^{m_j},$$

$$D(l_\alpha, k) := \sum_{j=k-M(l_\alpha)}^{r-1} (-1)^j (j+1) \binom{r+1}{j+2} \binom{j}{j-k+M(l_\alpha)},$$

$$\widetilde{D}(l_\alpha, k) := \sum_{j=k-M(l_\alpha)}^{r-1} (-1)^j (j+1) \binom{r+1}{j+2} \binom{j+1}{j-k+M(l_\alpha)}.$$

By Theorem 6.6, we obtain Plücker-Teissier-Kleiman type formulas in various special cases as follows.

Corollary 6.7 ([32])**.** (i) *In the situation of Theorem* 6.6, *assume moreover that* V^* *is a hypersurface. Then we have*

$$\deg V^* = d(d-1)^{N-1} - \sum_{\alpha \in A} \{\mu_\alpha(\pi'_{N-l_\alpha}(\vec{d}_\alpha) + \pi'_{N-1-l_\alpha}(\vec{d}_\alpha)) + \mu'_\alpha \pi_{N-l_\alpha}(\vec{d}_\alpha)\},$$

where μ_α *(resp.* μ'_α*) is the Milnor number of* V *along* V_α *(resp. the slice Milnor number of* V *along* V_α*).*
(ii) *Assume that* V *is a hypersurface of degree* d *in* $X = \mathbb{CP}_N$ *with only isolated singular points* $V_{\mathrm{sing}} = \{p_1, \cdots, p_q\}$*. Then*

$$\deg V^* = d(d-1)^{N-r} \frac{d}{dx} \Big\{ \frac{(1-x)^{r+1} - 1}{x} \Big\} \Big|_{x=d} + (-1)^r r \sum_{i=1}^{q} (\mu_i + \mu'_i),$$

where $r = \operatorname{codim} V^*$ *and* μ_i *(resp.* μ'_i*) is the Milnor number of* V *along* p_i *(resp. the slice Milnor number of* V *along* p_i*).*

As a special case of Corollary 6.7 (ii), consider the case where V is a hypersurface of degree d in $X = \mathbb{CP}_N$ with only isolated singular points $V_{\text{sing}} = \{p_1, \cdots, p_q\}$ such that V^* is a hypersurface. Then we reobtain Teissier's theorem in [41] (see also [26], [27])

$$\deg V^* = d(d-1)^{N-1} - \sum_{i=1}^{q} (\mu_i + \mu_i').$$

Remark. Even when V is a higher-codimensional complete intersection subvariety of $\mathbb{CP}_N$, we can also obtain formulas for the degree of V^*. See [32] for the detail.

§ 6.3. Grassmann Cases

We can generalize the notion of dual varieties to the Grassmann cases as follows.

Definition 6.8. Let $V \subset \mathbb{CP}_N$ be a projective variety. We define the k-dual variety $V^{\langle k \rangle}$ of V by

$$V^{\langle k \rangle} := \overline{\{L \in \mathbb{G}_{N,k} \mid \exists x \in V_{\text{reg}} \cap L \text{ s.t. } V \not\pitchfork L \text{ at } x\}} \quad (\subset \mathbb{G}_{N,k}).$$

If $k = N-1$ the k-dual $V^{\langle k \rangle} \subset \mathbb{G}_{N,k} \simeq \mathbb{CP}_N^*$ is nothing but the classical dual variety of V. In [12], Gelfand-Kapranov-Zelevinsky called $V^{\langle k \rangle}$ the associated variety of V and showed that $V^{\langle N - \dim V - 1 \rangle}$ is a hypersurface.

Definition 6.9 ([12]). Assume that $V^{\langle k \rangle}$ is a hypersurface in $\mathbb{G}_{N,k}$. Consider the Plücker embedding:

$$V^{\langle k \rangle} \subset \mathbb{G}_{N,k} \subset \mathbb{CP}_{\binom{N+1}{k+1}-1}.$$

We call the degree of the defining polynomial of $V^{\langle k \rangle}$ in $\mathbb{CP}_{\binom{N+1}{k+1}-1}$ the degree of $V^{\langle k \rangle}$ and denote it by $\deg V^{\langle k \rangle}$.

Theorem 6.10 ([31]). *Let $V \subset X = \mathbb{CP}_N$ be a projective variety of dimension n. Assume that $V^{\langle k \rangle}$ is a hypersurface of $\mathbb{G}_{N,k}$. Then we have*
(i) *For generic linear subspaces $L_1 \simeq \mathbb{CP}_{k-1}$, $L_2 \simeq \mathbb{CP}_k$ and $L_3 \simeq \mathbb{CP}_{k+1}$ of $\mathbb{CP}_N$,*

$$\deg V^{\langle k \rangle} = (-1)^{(N-k)+n+1} \left\{ \int_{L_1} \mathrm{Eu}_V - 2 \int_{L_2} \mathrm{Eu}_V + \int_{L_3} \mathrm{Eu}_V \right\}.$$

(ii) *For a generic $(k+1)$-dimensional linear subspace $L \simeq \mathbb{CP}_{k+1}$ of $\mathbb{CP}_N$, we have*

$$\deg V^{\langle k \rangle} = \deg (V \cap L)^*.$$

Here $(V \cap L)^* \subset \mathbb{CP}_{k+1}^* \simeq L^*$ is the usual dual variety of $(V \cap L) \subset \mathbb{CP}_{k+1} \simeq L$.

Remark. By Theorems 6.6 and 6.10, we can also obtain more explicit class formulas for k-dual varieties.

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