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Feynman Path Integrals and Semiclassical Approximation

By

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Abstract

These notes are rough surveys of our papers [24], [12], [14] on the theory of Feynman path integrals by the time slicing approximation. Since the RIMS Kôkyûroku Bessatsu gives us a chance to introduce the ideas which are meaningful but are not suited for publication in ordinary journal, we try to use many figures and to explain the process of our proof.

§ 1. Introduction

In 1948, R. P. Feynman [6] expressed the integral kernel $K(T,x,x_0)$ of the fundamental solution for the Schrödinger equation, using the path integral as follows:

$$K(T,x,x_0) = \int e^{\frac{i}{\hbar}S[\gamma]}\mathcal{D}[\gamma]\,. \tag{1.1}$$

Here $0 < \hbar < 1$ is Planck's parameter, $\gamma\colon [0,T] \to \mathbf{R}^d$ is a path with $\gamma(0) = x_0$ and $\gamma(T) = x$ (see Figure 1), $S[\gamma]$ is the action along the path γ defined by

$$S[\gamma] = \int_0^T \Big(\frac{1}{2}\Big|\frac{d\gamma}{dt}\Big|^2 - V(t,\gamma(t))\Big)dt, \tag{1.2}$$

and the path integral $\displaystyle\int \sim \mathcal{D}[\gamma]$ is a new sum of $e^{\frac{i}{\hbar}S[\gamma]}$ over all the paths γ. Feynman explained his integral (1.1) as a limit of a finite dimensional integral, which is now

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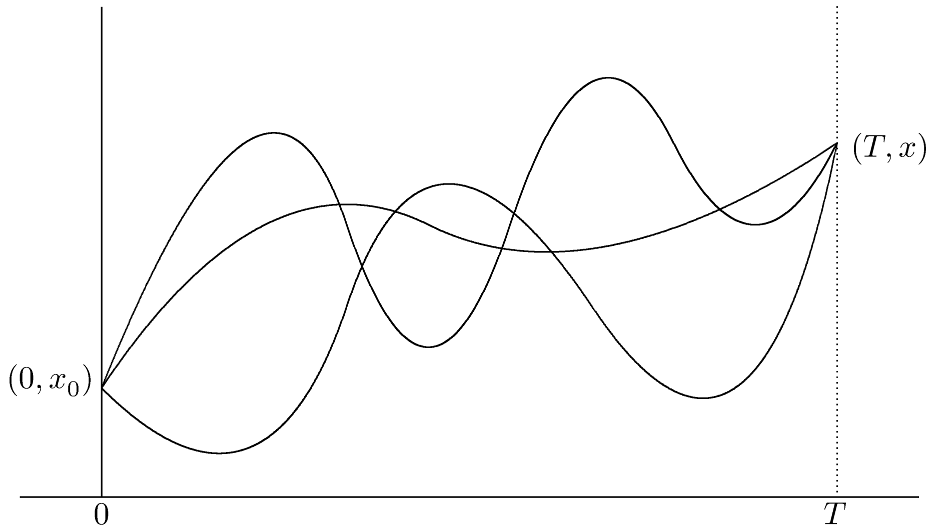

Figure 1.

called the time slicing approximation. Furthermore, Feynman considered path integrals with general functional $F[\gamma]$ as integrand, and suggested a new analysis on a path space with the functional integration $\int F[\gamma]e^{\frac{i}{\hbar}S[\gamma]}\mathcal{D}[\gamma]$ and the functional differentiation $(DF)[\gamma][\eta]$ (cf. L. S. Schulman [29, Chapter 8]). However, in 1960, R. H. Cameron [4] proved that the measure $e^{\frac{i}{\hbar}S[\gamma]}\mathcal{D}[\gamma]$ of Feynman path integrals does not exist in mathematics.

Therefore, using the time slicing approximation, we prove the existence of the Feynman path integrals

$$\int e^{\frac{i}{\hbar}S[\gamma]}F[\gamma]\mathcal{D}[\gamma]\,, \tag{1.3}$$

with the smooth functional derivatives $(DF)[\gamma][\eta]$. More precisely, we give a fairly general class $\mathcal{F}$ of functionals $F[\gamma]$ so that for any $F[\gamma]\in\mathcal{F}$, the time slicing approximation of the Feynman path integral (1.3) converges uniformly on any compact subset of the configuration space $\mathbf{R}^{2d}$ of the endpoints (x, x_0).

There are some mathematical works which proved the time slicing approximation of (1.1) converges uniformly on any compact subset. See D. Fujiwara [7], [9], [10], [11], H. Kitada and H. Kumano-go [21], K. Yajima [32], N. Kumano-go [23], D. Fujiwara and T. Tsuchida [15], and W. Ichinose [17]. However these works treated (1.1), that is the particular case of (1.3) with $F[\gamma]\equiv 1$.

Many people have given mathematically rigorous meanings to Feynman path integral. E. Nelson [27] succeeded in connecting Feynman path integral to Wiener measure by analytic continuation. K. Itô [19], S. Albeverio and Høegh Krohn [1], A. Truman [31], J. Rezende [28], S. Albeverio and S. Mazzucchi [2] defined Feynman path integrals via Fresnel integral transform and applied many problems. G. W. Johnson and M. Lapidus

[20], T. L. Gill and W. W. Zachary [16] developed the Feynman operational calculus.

§ 2. Existence of Feynman Path Integrals

Our assumption of the potential $V(t,x)$ of (1.2) is the following:

Assumption 1 (Potential). $V(t,x)$ is a real-valued function of $(t,x) \in \mathbf{R} \times \mathbf{R}^d$, and for any multi-index α, $\partial_x^\alpha V(t,x)$ is continuous in $\mathbf{R} \times \mathbf{R}^d$. For any multi-index α with $|\alpha| \geq 2$, there exists a positive constant A_α such that $|\partial_x^\alpha V(t,x)| \leq A_\alpha$.

Typical examples of the functionals $F[\gamma]$ in our functional class $\mathcal{F}$ are the following:

Example 1. (1) Let $m \geq 0$ and $B(t,x)$ be a function of $(t,x) \in \mathbf{R} \times \mathbf{R}^d$. For any multi-index α, $\partial_x^\alpha B(t,x)$ is continuous in $\mathbf{R} \times \mathbf{R}^d$ and there exists a positive constant C_α such that $|\partial_x^\alpha B(t,x)| \leq C_\alpha (1+|x|)^m$. Then, the value at time t, $0 \leq t \leq T$,

$$F[\gamma] = B(t, \gamma(t)) \in \mathcal{F}\,. \tag{2.1}$$

In particular, if $F[\gamma] \equiv C$, then $F[\gamma] \in \mathcal{F}$.

(2) Let $0 \leq T' \leq T'' \leq T$. Then, the Riemann(-Stieltjes) integrals,

$$F[\gamma] = \int_{T'}^{T''} B(t, \gamma(t))dt \in \mathcal{F}\,. \tag{2.2}$$

(3) If $|\partial_x^\alpha B(t,x)| \leq C_\alpha$, then

$$F[\gamma] = \exp\Big(\int_{T'}^{T''} B(t, \gamma(t))dt\Big) \in \mathcal{F}\,. \tag{2.3}$$

(4) Let $Z(t,x)$ be a vector-valued function of $(t,x) \in \mathbf{R} \times \mathbf{R}^d$ into $\mathbf{C}^d$. For any multi-index α, $\partial_x^\alpha Z(t,x)$ and $\partial_x^\alpha \partial_t Z(t,x)$ are continuous in $\mathbf{R} \times \mathbf{R}^d$ and there exists a positive constant C_α such that $|\partial_x^\alpha Z(t,x)| + |\partial_x^\alpha \partial_t Z(t,x)| \leq C_\alpha (1+|x|)^m$ and ${}^t(\partial_x Z) = (\partial_x Z)$. Then the curvilinear integral along paths

$$F[\gamma] = \int_{T'}^{T''} Z(t, \gamma(t)) \cdot d\gamma(t) \in \mathcal{F}\,. \tag{2.4}$$

We will state how to define the class $\mathcal{F}$ of functionals $F[\gamma]$ in § 4 and § 5. Because, even if we do not state the definition of $\mathcal{F}$ here, we can produce many functionals $F[\gamma] \in \mathcal{F}$, applying Theorem 1 to Example 1.

Theorem 1 (Smooth Algebra). *For any $F[\gamma]$, $G[\gamma] \in \mathcal{F}$, any broken line path $\zeta\colon [0,T] \to \mathbf{R}^d$ and any real $d \times d$ matrix P, we have the following.*

(1) $F[\gamma] + G[\gamma] \in \mathcal{F}, \quad F[\gamma]G[\gamma] \in \mathcal{F}$.
(2) $F[\gamma + \zeta] \in \mathcal{F}, \quad F[P\gamma] \in \mathcal{F}$.
(3) $(DF)[\gamma][\zeta] \in \mathcal{F}$.

Remark (Functional Derivative). For any broken line paths $\gamma\colon [0,T] \to \mathbf{R}^d$ and $\zeta\colon [0,T] \to \mathbf{R}^d$, we have

$$(DF)[\gamma][\eta] = \frac{d}{d\theta} F[\gamma + \theta\eta]\Big|_{\theta=0}. \tag{2.5}$$

Now we recall the time slicing approximation:

Let $\Delta_{T,0}$ be an arbitrary division of the interval $[0,T]$ into subintervals, i.e.

$$\Delta_{T,0} : T = T_{J+1} > T_J > \cdots > T_1 > T_0 = 0\,. \tag{2.6}$$

Let $t_j = T_j - T_{j-1}$ and $|\Delta_{T,0}| = \max_{1 \le j \le J+1} t_j$. Set $x_{J+1} = x$. Let x_j $(j = 1, 2, \ldots, J)$ be arbitrary points of $\mathbf{R}^d$. Let

$$\gamma_{\Delta_{T,0}} = \gamma_{\Delta_{T,0}}(t, x_{J+1}, x_J, \ldots, x_1, x_0), \tag{2.7}$$

be the broken line path which connects (T_j, x_j) and (T_{j-1}, x_{j-1}) by a line segment for any $j = 1, 2, \ldots, J, J+1$ (see Figure 2).

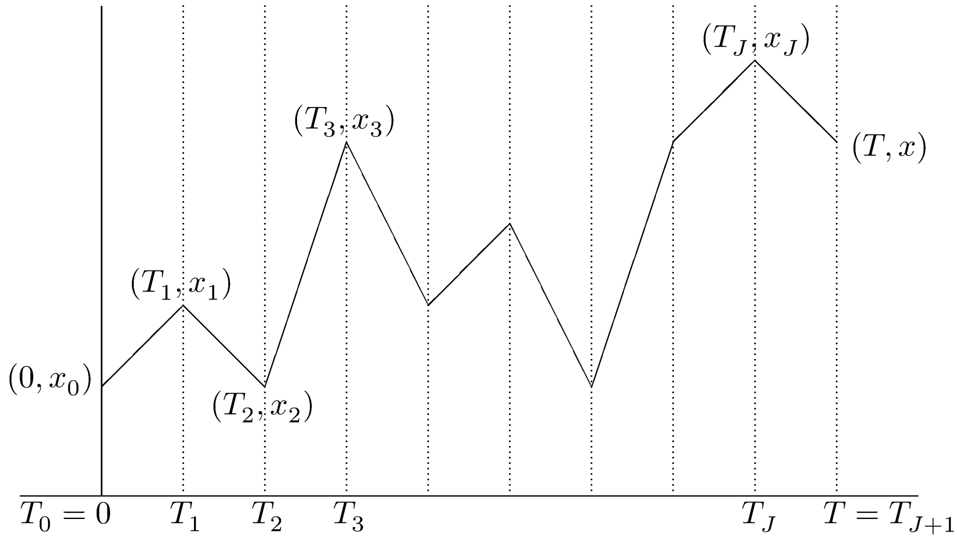

Figure 2.

As Feynman [6] had first defined (1.1) by the time slicing approximation, we define the Feynman path integrals (1.3) by

$$\int e^{\frac{i}{\hbar}S[\gamma]} F[\gamma] \mathcal{D}[\gamma] = \lim_{|\Delta_{T,0}| \to 0} \prod_{j=1}^{J+1} \Big(\frac{1}{2\pi i \hbar t_j}\Big)^{d/2} \int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]} F[\gamma_{\Delta_{T,0}}] \prod_{j=1}^{J} dx_j\,, \tag{2.8}$$

whenever the limit exists.

Theorem 2 (Existence of Feynman Path Integral). *Let T be sufficiently small. Then, for any $F[\gamma] \in \mathcal{F}$, the right-hand side of* (2.8) *converges uniformly on compact sets of the configuration space $\mathbf{R}^{2d}$ of the endpoints (x, x_0), together with all its derivatives in x and x_0, i.e.* (2.8) *is well-defined.*

Remark. There are two hurdles if we try to treat (2.8) mathematically. The first hurdle is that even when $F[\gamma] \equiv 1$, the integrals of the right-hand side of (2.8) do not converge absolutely, i.e.

$$\int_{\mathbf{R}^d} dx_j = \infty.$$

In order to get over the first hurdle, we treat integrals of this type as oscillatory integrals (cf. H. Kumano-go [22]). The second hurdle is that if $|\Delta_{T,0}| \to 0$, the number J of the integrals of the right-hand side of (2.8) tends to ∞, i.e.

$$\infty \times \infty \times \infty \times \infty \times \cdots\cdots\cdots .$$

In order to get over the second hurdle, we go back to Feynman's first paper [6]. Since the functionals $S[\gamma_{\Delta_{T,0}}]$ and $F[\gamma_{\Delta_{T,0}}]$ are functions of $x_{J+1}, x_J, \ldots, x_1, x_0$, i.e.

$$\begin{aligned} S[\gamma_{\Delta_{T,0}}] &= S_{\Delta_{T,0}}(x_{J+1}, x_J, \ldots, x_1, x_0), \\ F[\gamma_{\Delta_{T,0}}] &= F_{\Delta_{T,0}}(x_{J+1}, x_J, \ldots, x_1, x_0), \end{aligned} \tag{2.9}$$

Feynman used the form of function

$$\prod_{j=1}^{J+1} \Big(\frac{1}{2\pi i \hbar t_j}\Big)^{d/2} \int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar} S_{\Delta_{T,0}}(x_{J+1}, x_J, \ldots, x_1, x_0)} \prod_{j=1}^{J} dx_j\,. \tag{2.10}$$

Furthermore, in order to treat the integrals one by one mathematically via the Trotter formula, Nelson [27] used an approximation of $S[\gamma_{\Delta_{T,0}}]$, i.e.

$$\prod_{j=1}^{J+1} \Big(\frac{1}{2\pi i \hbar t_j}\Big)^{d/2} \int_{\mathbf{R}^{dJ}} \exp\Big(\frac{i}{\hbar} \sum_{j=1}^{J+1} \Big(\frac{(x_j - x_{j-1})^2}{2t_j} - t_j V(T_{j-1}, x_{j-1})\Big)\Big) \prod_{j=1}^{J} dx_j\,. \tag{2.11}$$

Note that (2.11) is not equal to (2.10) (cf. Johnson and Lapidus [20, pp. 109–110]).

On the other hand, treating the multi oscillatory integral of (2.8) directly, we keep the first step $S[\gamma_{\Delta_{T,0}}]$, $F[\gamma_{\Delta_{T,0}}]$.

We will prove Theorems 1 and 2 in § 4 and § 5.

Remark. For the formulation by broken line paths via Fresnel integral transform, see A. Truman [31].

Remark. S. Albeverio and S. Mazzucchi [2] generalized Fresnel integrals for the polynomially growth potential. T. L. Gill and W. W. Zachary [16] developed an operator version of the Henstock-Kurzweil integrals for the Feynman time-ordered operator calculus.

§ 3. Properties of Feynman Path Integrals

Assuming Theorems 1 and 2, we state some properties of path integrals.

§ 3.1. Fundamental Theorem of Calculus

Theorem 3 (Fundamental Theorem of Calculus). *Let T be sufficiently small. Let $m \geq 0$ and $0 \leq T' \leq T'' \leq T$. Let $f(t,x)$ be a function of $(t,x) \in \mathbf{R} \times \mathbf{R}^d$. For any multi-index α, $\partial_x^\alpha f(t,x)$, $\partial_x^\alpha \partial_t f(t,x)$ are continuous in $\mathbf{R} \times \mathbf{R}^d$, and there exists a positive constant C_α such that $|\partial_x^\alpha f(t,x)| + |\partial_x^\alpha \partial_t f(t,x)| \leq C_\alpha (1+|x|)^m$. Then we have*

$$\begin{aligned}\int & e^{\frac{i}{\hbar}S[\gamma]}\big(f(T'',\gamma(T'')) - f(T',\gamma(T'))\big)\mathcal{D}[\gamma] \\ &= \int e^{\frac{i}{\hbar}S[\gamma]}\Big(\int_{T'}^{T''} (\partial_x f)(t,\gamma(t)) \cdot d\gamma(t) + \int_{T'}^{T''} (\partial_t f)(t,\gamma(t))dt\Big)\mathcal{D}[\gamma].\end{aligned}$$

Remark. The integral $\displaystyle\int_{T'}^{T''} (\partial_x f)(t,\gamma(t)) \cdot d\gamma(t)$ is our new curvilinear integral along paths on a path space because the usual curvilinear integral can not be defined for all continuous paths γ or the Brownian motion $\mathbf{B}(t)$. In order to explain the difference with known curvilinear integrals on a path space, please forgive very rough sketch. If we can set $\mathbf{B}(T_j) = x_j$, the Itô integral [18] is approximated by initial points, i.e.

$$\int_{T'}^{T''} (\partial_x f)(t,\mathbf{B}(t)) \cdot d\mathbf{B}(t) \approx \sum_j (\partial_x f)(T_{j-1}, x_{j-1}) \cdot (x_j - x_{j-1}).$$

and the Stratonovich integral [30] is approximated by middle points, i.e.

$$\int_{T'}^{T''} (\partial_x f)(t,\mathbf{B}(t)) \circ d\mathbf{B}(t) \approx \sum_j (\partial_x f)\Big(\frac{T_j + T_{j-1}}{2}, \frac{x_j + x_{j-1}}{2}\Big) \cdot (x_j - x_{j-1}).$$

Feynman also used middle point method (cf. Schulman [29, p. 23, p. 27], K. L. Chung-J. C. Zambrini [5, pp.131–132]). On the other hand, if $\gamma = \gamma_{\Delta_{T,0}}$, our new curvilinear integral is the classical curvilinear integral itself along the broken line path $\gamma_{\Delta_{T,0}}$ (see Figure 3), i.e.

$$\int_{T'}^{T''} (\partial_x f)(t,\gamma_{\Delta_{T,0}}(t)) \cdot d\gamma_{\Delta_{T,0}}(t). \tag{3.1}$$

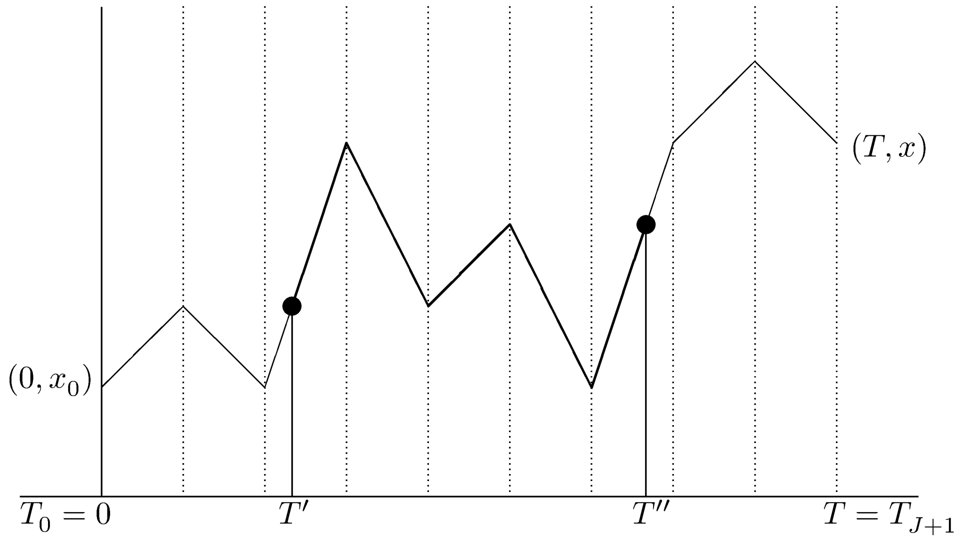

Figure 3.

In other words, the Itô integral and the Stratonovich integral are some limits of the Riemann sums. On the other hand, our new integral is a limit of curvilinear integrals. (3.1) is the key of the proof of Theorem 3.

Proof of Theorem 3. By Example 1 (1) and Theorem 1 (1), we have

$$F_1[\gamma] = f(T'', \gamma(T'')) - f(T', \gamma(T')) \in \mathcal{F}.$$

By Example 1 (4), (2) and Theorem 1 (1), we have

$$F_2[\gamma] = \int_{T'}^{T''} (\partial_x f)(t, \gamma(t)) \cdot d\gamma(t) + \int_{T'}^{T''} (\partial_t f)(t, \gamma(t)) dt \in \mathcal{F}.$$

By the fundamental theorem of calculus, we have $F_1[\gamma_{\Delta_{T,0}}] = F_2[\gamma_{\Delta_{T,0}}]$ for any broken line path $\gamma_{\Delta_{T,0}}$ (see Figure 3). By Theorem 2, we get

$$\int e^{\frac{i}{\hbar}S[\gamma]} F_1[\gamma] \mathcal{D}[\gamma] = \lim_{|\Delta_{T,0}| \to 0} \prod_{j=1}^{J+1} \Big(\frac{1}{2\pi i\hbar t_j}\Big)^{d/2} \int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]} F_1[\gamma_{\Delta_{T,0}}] \prod_{j=1}^{J} dx_j$$

$$= \lim_{|\Delta_{T,0}| \to 0} \prod_{j=1}^{J+1} \Big(\frac{1}{2\pi i\hbar t_j}\Big)^{d/2} \int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]} F_2[\gamma_{\Delta_{T,0}}] \prod_{j=1}^{J} dx_j = \int e^{\frac{i}{\hbar}S[\gamma]} F_2[\gamma] \mathcal{D}[\gamma].$$

□

§ 3.2. Interchange of the Order with Riemann Integrals

Theorem 4 (Interchange of the Order with Riemann Integrals). *Let T be sufficiently small. Let $m \geq 0$ and $0 \leq T' \leq T'' \leq T$. Let $B(t, x)$ be a function of*

$(t,x)\in\mathbf{R}\times\mathbf{R}^d$. For any multi-index α, $\partial_x^\alpha B(t,x)$ is continuous on $\mathbf{R}\times\mathbf{R}^d$, and there exists a positive constant C_α such that $|\partial_x^\alpha B(t,x)|\le C_\alpha(1+|x|)^m$. Then we have

$$\int_{T'}^{T''}\Big(\int e^{\frac{i}{\hbar}S[\gamma]}B(t,\gamma(t))\mathcal{D}[\gamma]\Big)dt=\int e^{\frac{i}{\hbar}S[\gamma]}\Big(\int_{T'}^{T''}B(t,\gamma(t))dt\Big)\mathcal{D}[\gamma]\,.$$

Remark (Perturbative Expansion). We can also interchange the order with some analytic limit. Therefore, if $|\partial_x^\alpha B(t,x)|\le C_\alpha$, we can prove the perturbative expansion

$$\begin{aligned}\int \exp\Big(\frac{i}{\hbar}S[\gamma]+\frac{i}{\hbar}\int_{T'}^{T''}B(\tau,\gamma(\tau))d\tau\Big)\mathcal{D}[\gamma]\\
&=\sum_{n=1}^{\infty}\Big(\frac{i}{\hbar}\Big)^n\int_{T'}^{T''}d\tau_n\int_{T'}^{\tau_n}d\tau_{n-1}\cdots\int_{T'}^{\tau_2}d\tau_1\\
&\quad\times\int e^{\frac{i}{\hbar}S[\gamma]}B(\tau_n,\gamma(\tau_n))B(\tau_{n-1},\gamma(\tau_{n-1}))\cdots B(\tau_1,\gamma(\tau_1))\mathcal{D}[\gamma].\end{aligned}$$

Remark. We explain the key of the proof of Theorem 4 roughly. In order to treat the integrals of (2.8) one by one mathematically via the Trotter formula, many books about Feynman path integrals approximate the position of the particle at time t by the endpoint x_j or x_{j-1}. On the other hand, using the number j so that $T_{j-1}<t\le T_j$, we keep the position of the particle at time t, i.e.

$$\gamma_{\Delta_{T,0}}(t)=\frac{t-T_{j-1}}{T_j-T_{j-1}}x_j+\frac{T_j-t}{T_j-T_{j-1}}x_{j-1}\,,$$

inside the finite dimensional oscillatory integral of (2.8). Furthermore, we treat the multi integral of (2.8) directly. Therefore, we can use the continuity of the broken line path $\gamma_{\Delta_{T,0}}(t)$ with respect to t (see Figure 4).

Proof of Theorem 4. Note that $B(t,\gamma_{\Delta_{T,0}}(t))$ is a continuous function of t on $[T',T'']$, together with all its derivatives in x_j, $j=0,1,\dots,J,J+1$. By Lebesgue's dominated convergence theorem after integrating by parts by x_j, $j=1,2,\dots,J$ (Oscillatory integrals), for any division $\Delta_{T,0}$,

$$\prod_{j=1}^{J+1}\Big(\frac{1}{2\pi i\hbar t_j}\Big)^{d/2}\int_{\mathbf{R}^{dJ}}e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]}B(t,\gamma_{\Delta_{T,0}}(t))\prod_{j=1}^{J}dx_j$$

is also a continuous function of t on $[T',T'']$. By Theorem 2, the convergence of

$$\int e^{\frac{i}{\hbar}S[\gamma]}B(t,\gamma(t))\mathcal{D}[\gamma]=\lim_{|\Delta_{T,0}|\to0}\prod_{j=1}^{J+1}\Big(\frac{1}{2\pi i\hbar t_j}\Big)^{d/2}\int_{\mathbf{R}^{dJ}}e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]}B(t,\gamma_{\Delta_{T,0}}(t))\prod_{j=1}^{J}dx_j$$

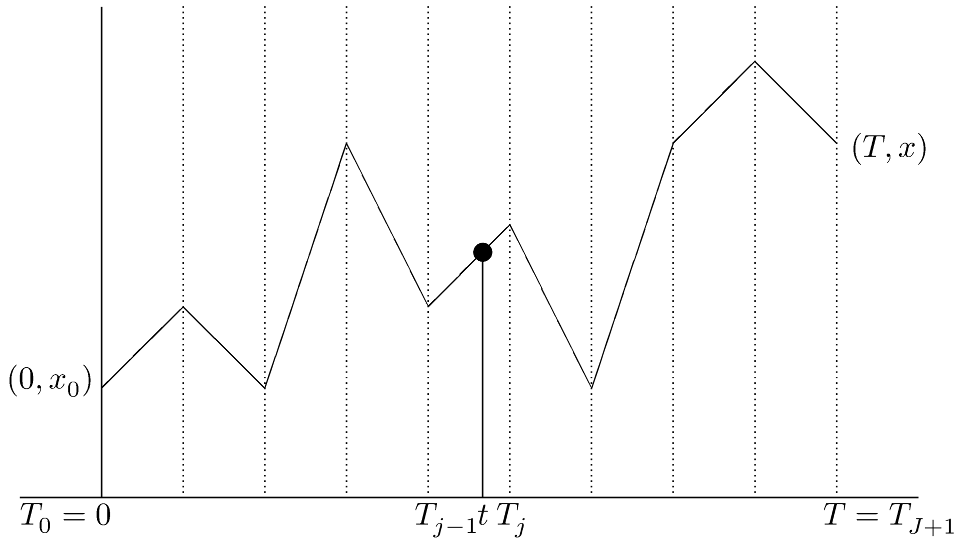

Figure 4.

is uniform with respect to t on $[T', T'']$. Therefore, the limit function

$$\int e^{\frac{i}{\hbar}S[\gamma]}B(t,\gamma(t))\mathcal{D}[\gamma]$$

is also a continuous function of t on $[T', T'']$ and Riemann integrable. Furthermore, by the uniform convergence, we can interchange the order of $\int_{T'}^{T''} \cdots dt$ and $\lim_{|\Delta_{T,0}|\to 0}$.

$$\begin{aligned}
&\int_{T'}^{T''}\Big(\int e^{\frac{i}{\hbar}S[\gamma]}B(t,\gamma(t))\mathcal{D}[\gamma]\Big)dt \\
&\quad \equiv \int_{T'}^{T''} \lim_{|\Delta_{T,0}|\to 0} \prod_{j=1}^{J+1}\Big(\frac{1}{2\pi i\hbar t_j}\Big)^{d/2} \int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]} B(t,\gamma_{\Delta_{T,0}}(t)) \prod_{j=1}^{J} dx_j dt \\
&\quad = \lim_{|\Delta_{T,0}|\to 0} \int_{T'}^{T''} \prod_{j=1}^{J+1}\Big(\frac{1}{2\pi i\hbar t_j}\Big)^{d/2} \int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]} B(t,\gamma_{\Delta_{T,0}}(t)) \prod_{j=1}^{J} dx_j dt.
\end{aligned}$$

By Fubini's theorem after integrating by parts by x_j, $j = 1, 2, \ldots, J$ (oscillatory integrals), we have

$$\begin{aligned}
&= \lim_{|\Delta_{T,0}|\to 0} \prod_{j=1}^{J+1}\Big(\frac{1}{2\pi i\hbar t_j}\Big)^{d/2} \int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}]} \int_{T'}^{T''} B(t,\gamma_{\Delta_{T,0}}(t))dt \prod_{j=1}^{J} dx_j \\
&= \int e^{\frac{i}{\hbar}S[\gamma]}\Big(\int_{T'}^{T''} B(t,\gamma(t))dt\Big)\mathcal{D}[\gamma].
\end{aligned}$$

□

§ 3.3. Translation

Theorem 5 (Translation). *Let T be sufficiently small. For any $F[\gamma] \in \mathcal{F}$ and any broken line path $\eta\colon [0,T] \to \mathbf{R}^d$, we have*

$$\int_{\gamma(0)=x_0,\gamma(T)=x} e^{\frac{i}{\hbar}S[\gamma+\eta]}F[\gamma+\eta]\mathcal{D}[\gamma] = \int_{\gamma(0)=x_0+\eta(0),\gamma(T)=x+\eta(T)} e^{\frac{i}{\hbar}S[\gamma]}F[\gamma]\mathcal{D}[\gamma]\,.$$

Remark (Orthogonal Transformation). Let T be sufficiently small. Then, for any $F[\gamma] \in \mathcal{F}$ and any $d \times d$ orthogonal matrix Q, we have

$$\int_{\gamma(0)=x_0,\gamma(T)=x} e^{\frac{i}{\hbar}S[Q\gamma]}F[Q\gamma]\mathcal{D}[\gamma] = \int_{\gamma(0)=Qx_0,\gamma(T)=Qx} e^{\frac{i}{\hbar}S[\gamma]}F[\gamma]\mathcal{D}[\gamma].$$

Proof of Theorem 5. By Theorem 2,

$$\int_{\gamma(0)=x_0,\gamma(T)=x} e^{\frac{i}{\hbar}S[\gamma+\eta]}F[\gamma+\eta]\mathcal{D}[\gamma]$$
$$= \lim_{|\Delta_{T,0}|\to 0}\prod_{j=1}^{J+1}\Big(\frac{1}{2\pi i\hbar t_j}\Big)^{d/2}\int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S[\gamma_{\Delta_{T,0}}+\eta]}F[\gamma_{\Delta_{T,0}}+\eta]\prod_{j=1}^{J}dx_j$$

exists. Choose $\Delta_{T,0}$ which contains all times when the broken line path η breaks (see Figure 5). Set $\eta(T_j) = y_j$, $j = 0, 1, \ldots, J, J+1$.

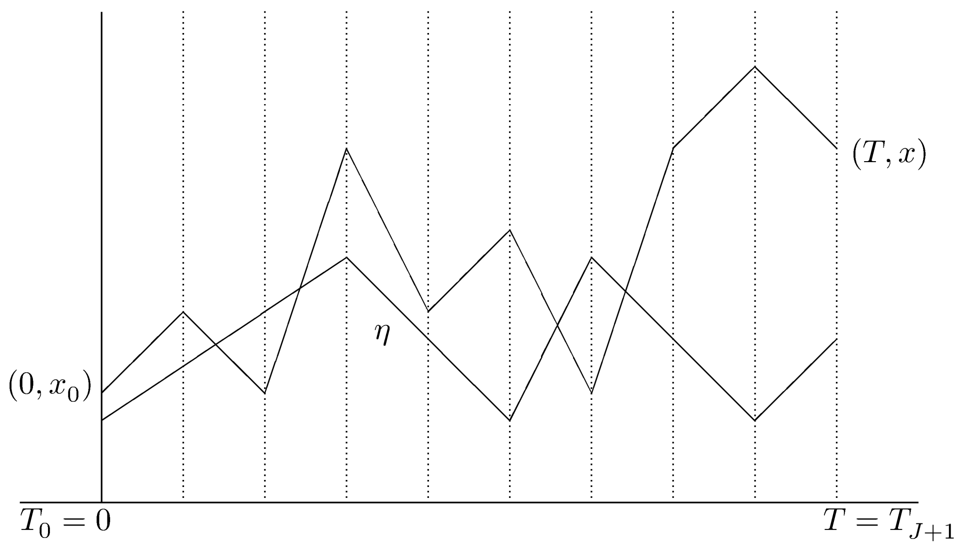

Figure 5.

Since $\gamma_{\Delta_{T,0}}+\eta$ is the broken line path which connects (T_j, x_j+y_j) and $(T_{j-1}, x_{j-1}+y_{j-1})$

by a line segment for $j = 1, 2, \dots, J, J+1$, we can write

$$= \lim_{|\Delta_{T,0}|\to 0} \prod_{j=1}^{J+1} \Big(\frac{1}{2\pi i\hbar t_j}\Big)^{d/2} \int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S_{\Delta_{T,0}}(x_{J+1}+y_{J+1},x_J+y_J,\dots,x_1+y_1,x_0+y_0)}$$
$$\times F_{\Delta_{T,0}}(x_{J+1}+y_{J+1}, x_J+y_J, \dots, x_1+y_1, x_0+y_0) \prod_{j=1}^{J} dx_j\,.$$

By the change of variables: $x_j + y_j \to x_j$, $j = 1, 2, \dots, J$, we have

$$= \lim_{|\Delta_{T,0}|\to 0} \prod_{j=1}^{J+1} \Big(\frac{1}{2\pi i\hbar t_j}\Big)^{d/2} \int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar}S_{\Delta_{T,0}}(x_{J+1}+y_{J+1},x_J,\dots,x_1,x_0+y_0)}$$
$$\times F_{\Delta_{T,0}}(x_{J+1}+y_{J+1}, x_J, \dots, x_1, x_0+y_0) \prod_{j=1}^{J} dx_j$$
$$= \int_{\gamma(0)=x_0+\eta(0),\gamma(T)=x+\eta(T)} e^{\frac{i}{\hbar}S[\gamma]} F[\gamma]\mathcal{D}[\gamma].$$

□

§ 3.4. Functional Derivative

Theorem 6 (Integration by Parts). *Let T be sufficiently small. Then, for any $F[\gamma] \in \mathcal{F}$ and any broken line path η: $[0,T] \to \mathbf{R}^d$ with $\eta(0) = \eta(T) = 0$, we have*

$$\int e^{\frac{i}{\hbar}S[\gamma]}(DF)[\gamma][\eta]\mathcal{D}[\gamma] = -\frac{i}{\hbar}\int e^{\frac{i}{\hbar}S[\gamma]}(DS)[\gamma][\eta]F[\gamma]\mathcal{D}[\gamma].$$

Remark (Functional Derivative). Let $\Delta_{T,0}$ contain all times when the broken line path γ or the broken line path η breaks (see Figure 6). Set $\gamma(T_j) = x_j$ and $\eta(T_j) = y_j$, $j = 0, 1, \dots, J, J+1$.

Then, for any $\theta \in \mathbf{R}$, $\gamma + \theta\eta$ is the broken line path which connects $(T_j, x_j + \theta y_j)$ and $(T_{j-1}, x_{j-1} + \theta y_{j-1})$ by a line segment for $j = 1, 2, \dots, J, J+1$. Hence we have

$$F[\gamma+\theta\eta] = F_{\Delta_{T,0}}(x_{J+1}+\theta y_{J+1}, x_J+\theta y_J, \dots, x_1+\theta y_1, x_0+\theta y_0).$$

Therefore, we can write $(DF)[\gamma][\eta]$ as a finite sum as follows:

$$(DF)[\gamma][\eta] = \frac{d}{d\theta}F[\gamma+\theta\eta]\Big|_{\theta=0} = \sum_{j=0}^{J+1}(\partial_{x_j}F_{\Delta_{T,0}})(x_{J+1}, x_J, \dots, x_1, x_0)\cdot y_j\,.$$

Note that we 'restrict' the direction of functional derivatives to broken line paths (cf. Malliavin's derivatives [25]).

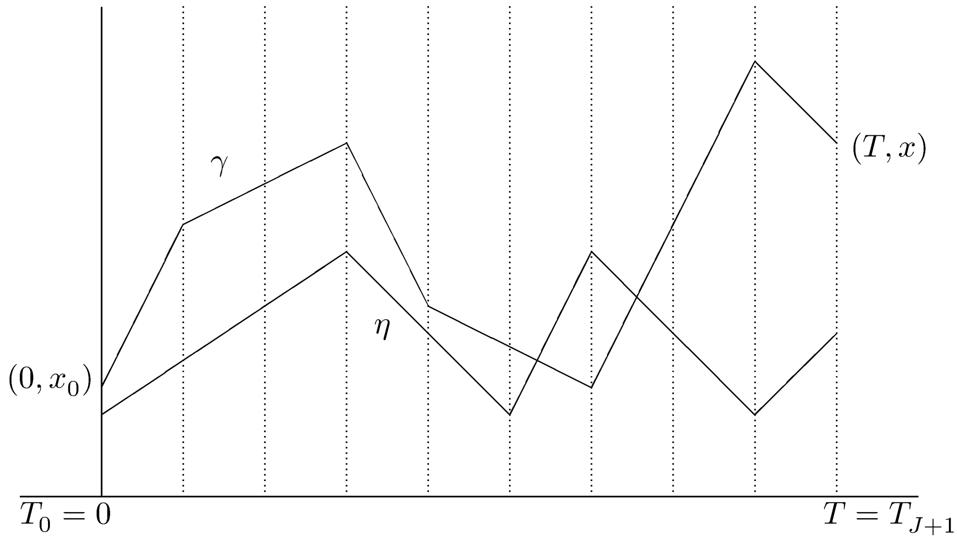

Figure 6.

Remark (Taylor's Expansion Formula). Let T be sufficiently small. For any $F[\gamma] \in \mathcal{F}$ and any broken line path $\eta\colon [0,T] \to \mathbf{R}^d$, we have

$$\int e^{\frac{i}{\hbar}S[\gamma]} F[\gamma+\eta]\mathcal{D}[\gamma] = \sum_{l=0}^{L} \frac{1}{l!} \int e^{\frac{i}{\hbar}S[\gamma]} (D^l F)[\gamma][\eta]\cdots[\eta]\mathcal{D}[\gamma] + \int_0^1 \frac{(1-\theta)^L}{L!} \int e^{\frac{i}{\hbar}S[\gamma]} (D^{L+1}F)[\gamma+\theta\eta][\eta]\cdots[\eta]\mathcal{D}[\gamma]d\theta.$$

§ 3.5. Semiclassical Approximation as $\hbar \to 0$

Theorem 7 (Semiclassical Approximation as $\hbar \to 0$). *Let T be sufficiently small. Let $F[\gamma] \in \mathcal{F}$ and the domain of $F[\gamma]$ be continuously extended to $C([0,T] \to \mathbf{R}^d)$ with respect to the norm $\|\gamma\| = \max_{0 \le t \le T} |\gamma(t)|$. Then we can write*

$$\int e^{\frac{i}{\hbar}S[\gamma]} F[\gamma]\mathcal{D}[\gamma] = \Big(\frac{1}{2\pi i\hbar T}\Big)^{d/2} e^{\frac{i}{\hbar}S[\gamma^{cl}]} \big(D(T,x,x_0)^{-1/2}F[\gamma^{cl}] + \hbar\Upsilon(\hbar,T,x,x_0)\big).$$

Here γ^{cl} be the classical path with $\gamma^{cl}(0) = x_0$ and $\gamma^{cl}(T) = x$, $D(T,x,x_0)$ be the Morette-Van Vleck determinant [26] *and for any multi-indices α, β, there exists a positive constant $C_{\alpha,\beta}$ such that*

$$|\partial_x^\alpha \partial_{x_0}^\beta \Upsilon(\hbar,T,x,x_0)| \le C_{\alpha,\beta}(1+|x|+|x_0|)^m.$$

Remark. When $F[\gamma] \equiv 1$, using piecewise classical paths (see Figure 7) instead of broken line paths in (2.7), Fujiwara [8], [9], [10], [11] proved Theorem 7.

Remark. For the semiclassical approximation as $\hbar \to 0$ of the path integral via Fresnel integral transform, see J. Rezende [28].

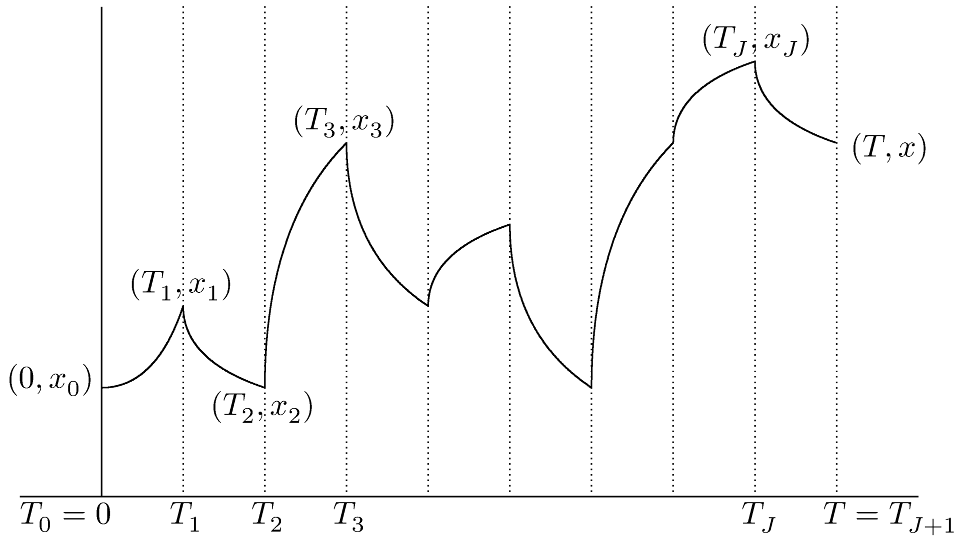

Figure 7.

§ 4. Proof of Existence of Feynman Path Integrals

§ 4.1. Starting Point is Fujiwara's Result when $F[\gamma] \equiv 1$

Using piecewise classical paths, Fujiwara [8], [9], [10], [11] proved the semiclassical approximation when $F[\gamma] \equiv 1$ as $\hbar \to 0$ as follows:

First, Fujiwara divided the time slicing approximation of (1.1) via piecewise classical paths into the phase function $S^{\dagger}_{\Delta_{T,0}}(x, x_0)$, the main term $D_{\Delta_{T,0}}(x, x_0)^{-1/2}$ and the remainder term $\Upsilon_{\Delta_{T,0}}(\hbar, x, x_0)$, i.e.

$$\prod_{j=1}^{J+1} \Big(\frac{1}{2\pi i\hbar t_j}\Big)^{d/2} \int_{\mathbf{R}^{dJ}} e^{\frac{i}{\hbar} S_{\Delta_{T,0}}(x_{J+1}, x_J, \ldots, x_1, x_0)} \prod_{j=1}^{J} dx_j$$
$$= \Big(\frac{1}{2\pi i\hbar T}\Big)^{d/2} e^{\frac{i}{\hbar} S^{\dagger}_{\Delta_{T,0}}(x, x_0)} \big(D_{\Delta_{T,0}}(x, x_0)^{-1/2} + \hbar \Upsilon_{\Delta_{T,0}}(\hbar, x, x_0)\big).$$

Remark. Piecewise classical paths are sharper as an approximation than broken line paths. Especially, the phase function $S^{\dagger}_{\Delta_{T,0}}(x, x_0)$ is the action defined by the classical path γ^{cl}, i.e. $S^{\dagger}_{\Delta_{T,0}}(x, x_0) = S[\gamma^{cl}]$.

Remark. The main term $D_{\Delta_{T,0}}(x, x_0)^{-1/2}$ is defined by the Hessian of the phase function $S_{\Delta_{T,0}}(x_{J+1}, x_J, \ldots, x_1, x_0)$.

For any division $\Delta_{T,0}$, define the coarser division $(\Delta_{T,T_{N+1}}, \Delta_{T_{n-1},0})$ by

$$(4.1) \qquad (\Delta_{T,T_{N+1}}, \Delta_{T_{n-1},0}) : T = T_{J+1} > \cdots > T_{N+1} > T_{n-1} > \cdots > T_0 = 0.$$

Fujiwara obtained the following estimates

$$
\begin{aligned}
|D_{\Delta_{T,0}}(x,x_0) - D_{(\Delta_{T,T_{N+1}},\Delta_{T_{n-1},0})}(x,x_0)| &\le C(T_{N+1}-T_{n-1})^2,\\
|\Upsilon_{\Delta_{T,0}}(\hbar,x,x_0) - \Upsilon_{(\Delta_{T,T_{N+1}},\Delta_{T_{n-1},0})}(\hbar,x,x_0)| &\le \frac{C}{\hbar}(T_{N+1}-T_{n-1})^2,\\
|\Upsilon_{\Delta_{T,0}}(\hbar,x,x_0)| &\le C,
\end{aligned}
$$

and showed all convergences as $|\Delta_{T,0}| \to 0$, i.e.

$$
\begin{aligned}
|D_{\Delta_{T,0}}(x,x_0) - D(T,x,x_0)| &\le C|\Delta_{T,0}|T,\\
|\Upsilon_{\Delta_{T,0}}(\hbar,x,x_0) - \Upsilon(\hbar,T,x,x_0)| &\le \frac{C}{\hbar}|\Delta_{T,0}|T,\\
|\Upsilon(\hbar,T,x,x_0)| &\le C.
\end{aligned}
\tag{4.2}
$$

These imply Theorem 7 when $F[\gamma] \equiv 1$.

§ 4.2. Question

As $\hbar \to 0$, the right-hand side of (4.2) $\to \infty$. Instead of (4.2), why did not he write

$$
|\Upsilon_{\Delta_{T,0}}(\hbar,x,x_0) - \Upsilon(\hbar,T,x,x_0)| \le C|\Delta_{T,0}|T \ ?
$$

It is impossible because he defined the remainder term by the main term as follows:

$$
\boxed{\text{Remainder}} = \frac{1}{\hbar}\Big(\boxed{\text{Total}} - \boxed{\text{Main}}\Big). \tag{4.3}
$$

Our first problem was the following: When $F[\gamma] \equiv 1$, using broken line paths which are rougher as an approximation than piecewise classical paths, can we get $\hbar$ times shaper estimate:

$$
|\Upsilon_{\Delta_{T,0}}(\hbar,x,x_0) - \Upsilon(\hbar,T,x,x_0)| \le C|\Delta_{T,0}|T(1+|x|+|x_0|) \ ? \tag{4.4}
$$

Here the term $(1+|x|+|x_0|)$ appears because broken line paths are rougher as an approximation than piecewise classical paths.

§ 4.3. Change the Definition

Instead of the definition (4.3), we try to define $\Upsilon_{\Delta_{T,0}}(\hbar,x,x_0)$, using $\Upsilon_\epsilon(\hbar,x,x_0)$ which connects $\Delta_{T,0}$ and $(\Delta_{T,T_{N+1}},\Delta_{T_{n-1},0})$ with a parameter $0 \le \epsilon \le 1$. More precisely, we try to find S^*_ϵ, D_ϵ, Υ_ϵ so that $S^*_1 = S^\dagger_{\Delta_{T,0}}$, $D_1 = D_{\Delta_{T,0}}$, $\Upsilon_1 = \Upsilon_{\Delta_{T,0}}$ and that

$S_0^* = S^\dagger_{(\Delta_{T,T_{N+1}},\Delta_{T_{n-1},0})}$, $D_0 = D_{(\Delta_{T,T_{N+1}},\Delta_{T_{n-1},0})}$, $\Upsilon_0 = \Upsilon_{(\Delta_{T,T_{N+1}},\Delta_{T_{n-1},0})}$ as follows:

$$
\begin{array}{ll}
\epsilon = 1: & \left(\frac{1}{2\pi i\hbar T}\right)^{d/2} e^{\frac{i}{\hbar}S^\dagger_{\Delta_{T,0}}}(D^{-1/2}_{\Delta_{T,0}} + \hbar\Upsilon_{\Delta_{T,0}}) \\
\Uparrow & \Uparrow \\
\epsilon & \left(\frac{1}{2\pi i\hbar T}\right)^{d/2} e^{\frac{i}{\hbar}S^*_\epsilon}(D^{-1/2}_\epsilon + \hbar\Upsilon_\epsilon) \\
\Downarrow & \Downarrow \\
\epsilon = 0: & \left(\frac{1}{2\pi i\hbar T}\right)^{d/2} e^{\frac{i}{\hbar}S^\dagger_{(\Delta_{T,T_{N+1}},\Delta_{T_{n-1},0})}}(D^{-1/2}_{(\Delta_{T,T_{N+1}},\Delta_{T_{n-1},0})} + \hbar\Upsilon_{(\Delta_{T,T_{N+1}},\Delta_{T_{n-1},0})}).
\end{array}
$$

Note that

$$\Upsilon_{\Delta_{T,0}} - \Upsilon_{(\Delta_{T,T_{N+1}},\Delta_{T_{n-1},0})} = \Upsilon_1 - \Upsilon_0 = \int_0^1 (\partial_\epsilon \Upsilon_\epsilon)d\epsilon.$$

If we can define the remainder term $\Upsilon_\epsilon(\hbar, x, x_0)$ such that

$$|\partial_\epsilon \Upsilon_\epsilon| \leq C(T_{N+1} - T_{n-1})^2(1 + |x| + |x_0|),$$

then we will be able to get the estimate (4.4) independent of $\hbar$.

§ 4.4. However, How Can We Define the Remainder Term Υ_ϵ?

The remainder term Υ_ϵ is defined by the main term $D_\epsilon^{-1/2}$. We can not change this fact. Thus, in order to define the remainder term, we must define the main term. Furthermore, the main term $D_\epsilon^{-1/2}$ is defined by the phase function S_ϵ. Therefore, in order to define the main term, we must define the phase function. Furthermore, the phase function S_ϵ is defined by the path.

§ 4.5. All are Defined by Paths

We have only to prove all convergences from the beginning in the following order:

$$\boxed{\text{Path}} \Longrightarrow \boxed{\text{Phase}} \Longrightarrow \boxed{\text{Main}} \Longrightarrow \boxed{\text{Remainder}}$$

We repeat similar discussions about convergence four times. At the first step, since we consider only the broken line paths $\gamma_{\Delta_{T,0}}$, the assumption that $F[\gamma] \equiv 1$ is not necessary. In other words, we have no assumption about $F[\gamma]$ at the first step.

To make Cauchy sequences, we compare the functions for the division $\Delta_{T,0}$ and the functions for the division $(\Delta_{T,T_{N+1}}, \Delta_{T_{n-1},0})$ four times. At the first step, we compare the broken path $\gamma_{\Delta_{T,0}}$ of Figure 2 and the broken line path $\gamma_{(\Delta_{T,T_{N+1}},\Delta_{T_{n-1},0})}$ of Figure 8.

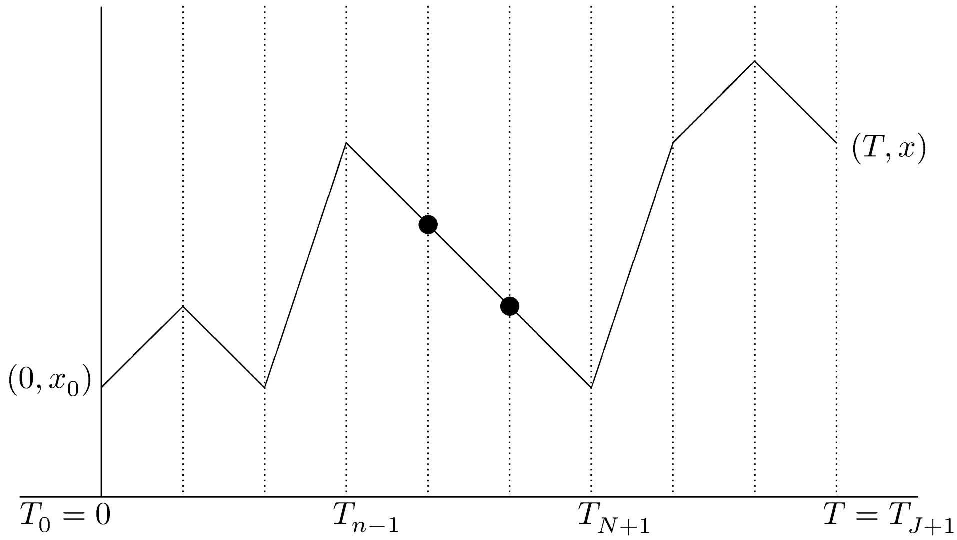

Figure 8.

For simplicity, for $1 \leq l \leq L \leq J+1$, we write

$$x_{L,l} = (x_L, x_{L-1}, \dots, x_l)\,. \tag{4.5}$$

Then the key lemma is the following:

Lemma 4.1. *For any $1 \leq n \leq N \leq J$, define $x^{\triangleleft}_{N,n} = x^{\triangleleft}_{N,n}(x_{N+1}, x_{n-1})$ by*

$$x^{\triangleleft}_j = \frac{T_j - T_{n-1}}{T_{N+1} - T_{n-1}} x_{N+1} + \frac{T_{N+1} - T_j}{T_{N+1} - T_{n-1}} x_{n-1}\,, \quad j = n, n+1, \dots, N\,.$$

Set $x_{N,n} = x^{\triangleleft}_{N,n}$. Then the broken line path $\gamma_{\Delta_{T,0}}$ becomes the broken line path

$$\gamma_{(\Delta_{T,T_{N+1}}, \Delta_{T_{n-1},0})} \quad (\textit{see Figure } 8).$$

We use this lemma four times.

§ 4.6. Compare Two Integrals by Two Paths

At the last step, the multiple integral

$$\int \cdots \int \cdots \int \cdots \int \quad \sim \quad \prod_{j=1}^{J} dx_j$$

implies the paths of Figure 2 and the multiple integral

$$\int \cdots \int \times \int \cdots \int \sim \prod_{j=N+1}^{J+1} dx_j \prod_{j=1}^{n-1} dx_j$$

implies the paths of Figure 8. The two multiple integrals are different in the number of variables. Moreover the two integrands are different in the number of variables. However, by the following lemma, we can compare the two integrands. Therefore we can compare the two multiple integrals.

Lemma 4.2. *Let $x^{\triangleleft}_{N,n}$ be the same as in Lemma 4.1. Then for any functional $F[\gamma]$ whose domain contains all of broken line paths, we have*

$$F_{\Delta_{T,0}}(x_{J+1,N+1}, x^{\triangleleft}_{N,n}, x_{n-1,0}) = F_{(\Delta_{T,T_{N+1}},\Delta_{T_{n-1},0})}(x_{J+1,N+1}, x_{n-1,0}).$$

Proof. By Lemma 4.1, if $x_{N,n} = x^{\triangleleft}_{N,n}$, we have

$$\begin{aligned}F_{\Delta_{T,0}}(x_{J+1,N+1}, x^{\triangleleft}_{N,n}, x_{n-1,0}) &= F[\gamma_{\Delta_{T,0}}]\\ &= F[\gamma_{(\Delta_{T,T_{N+1}},\Delta_{T_{n-1},0})}] = F_{(\Delta_{T,T_{N+1}},\Delta_{T_{n-1},0})}(x_{J+1,N+1}, x_{n-1,0}).\end{aligned}$$

□

§ 4.7. Now We Prove Theorems 1 and 2

We hope to prove Theorem 2, i.e.

$$F[\gamma] \in \mathcal{F} \implies \text{the time slicing approximation (2.8) converges.} \tag{4.6}$$

In order to prove this, we add many assumptions so that (2.8) converges. Because we have no assumption about $F[\gamma]$ at the first step, we need assumptions. To add many assumption is valid if we have at least one example. In our case, D. Fujiwara proved the case where $F[\gamma] \equiv 1$, using piecewise classical paths. Therefore, using broken line paths, we will probably have at least one example $F[\gamma] \equiv 1$.

If possible, we hope two or three examples. For example, we hope to prove

$$\int_{T'}^{T''} B(t,\gamma(t))dt \in \mathcal{F}, \quad B(t,\gamma(t)) \in \mathcal{F}. \tag{4.7}$$

In order to prove this, we consider only the convergence of (2.8). We must not consider other things. Then $\mathcal{F}$ will become large as a set. If we are lucky, $\mathcal{F}$ may contain other examples.

Furthermore, if possible, we hope to make many examples. We hope to prove

$$F[\gamma],\ G[\gamma] \in \mathcal{F} \Longrightarrow F[\gamma] + G[\gamma],\ F[\gamma]G[\gamma] \in \mathcal{F}\,. \tag{4.8}$$

In order to prove this, All assumptions to add must be closed under $+$ and $\times$.

We found the assumptions satisfying (4.6), (4.8), (4.7). □

Remark. Because of the process of our proof, the conclusions of Theorems 2, 1 (1) are valid for any functional satisfying this assumption. Furthermore, by accident, the conclusions of Theorems 3, 4, 7 are valid for any functional satisfying this assumption.

§ 5. Assumption of the Class $\mathcal{F}$

§ 5.1. Assumption with Non-Decoration

Broken line paths are rougher as an approximation than piecewise classical paths. Furthermore, we added many assumptions so that (2.8) converges. Therefore, the first assumption with no decoration has many critical points and consists of four inequalities as follows. For simplicity, we define the critical point $x^\dagger_{L,l} = x^\dagger_{L,l}(x_{L+1}, x_{l-1})$ by

$$(\partial_{x_{L,l}} S_{\Delta_{T,0}})(x_{J+1,L+1}, x^\dagger_{L,l}, x_{l-1,0}) = 0.$$

Assumption with no Decoration. Let $m \geq 0$. For any non-negative integer M, there exists a positive constant C_M such that for any $\Delta_{T,0}$ of (2.6), any sequence of integers

$$0 = j_0 < j_1 - 1 < j_1 < j_2 - 1 < j_2 < \cdots < j_K \leq J+1,$$

$j_{K+1} - 1 = J + 1$ and any $|\alpha_{j_{k+1}}|$, $|\alpha_{j_k}| \leq M$,

$$\begin{aligned}
&\Big|\Big(\prod_{k=0}^{K} \partial_{x_{j_{k+1}-1}}^{\alpha_{j_{k+1}-1}} \partial_{x_{j_k}}^{\alpha_{j_k}}\Big) F_{\Delta_{T,0}}(x_{J+1}, x^\dagger_{J,j_K+1}, x_{j_K},\\
&\qquad \ldots, x_{j_{s+1}-1}, x^\dagger_{j_{s+1}-2,j_s+1}, x_{j_s}, x_{j_s-1}, x^\dagger_{j_s-2,j_{s-1}+1}, x_{j_{s-1}},\\
(1)\qquad &\qquad x_{j_{s-1}-1}, x^\dagger_{j_{s-1}-2,j_{s-2}+1}, x_{j_{s-2}}, \ldots, x_{j_1-1}, x^\dagger_{j_1-2,1}, x_0)\Big|\\
&\leq (C_M)^{K+1}\Big(1 + |x_{J+1}| + |x_{j_K}| + \cdots + |x_{j_{s+1}-1}| + |x_{j_s}| + |x_{j_s-1}| + |x_{j_{s-1}}|\\
&\qquad + |x_{j_{s-1}-1}| + |x_{j_{s-2}}| + \cdots + |x_{j_1-1}| + |x_0|\Big)^m.
\end{aligned}$$

(2), (3), (4) are omitted (for the details, see [24], [12]).

§ 5.2. Hide Critical Points inside Classical Path

If we use piecewise classical paths instead of broken line paths, the assumption corresponding to Assumption with no decoration does not need any critical points.

Proof. If we push the critical points into a piecewise classical path, the piecewise classical path changes to a single classical path (see Figure 9). We can hide all critical points of Assumption with no decoration inside single classical paths.

$$(5.1)\qquad F_{\Delta_{T,0}}(\ldots, x_{N+1}, x^\dagger_{N,n}, x_{n-1}, \ldots) = F_{(\Delta_{T,T_{N+1}}, \Delta_{T_{n-1},0})}(\ldots, x_{N+1}, x_{n-1}, \ldots).$$

□

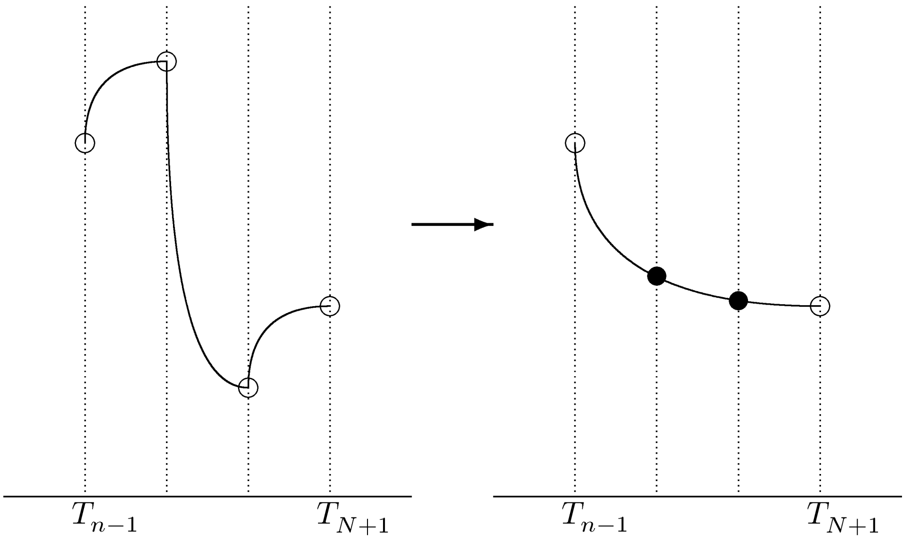

Figure 9.

Assumption for Piecewise Classical Paths. Let $m \geq 0$ and $u_j \geq 0$ with $\sum_{j=1}^{J+1} u_j \leq U < \infty$. For any non-negative integer M, there exists a positive constant C_M such that for any division $\Delta_{T,0}$, any $|\alpha_j| \leq M$, $j = 0, 1, \dots, J+1$ and any $1 \leq k \leq J$,

$$|(\prod_{j=0}^{J+1} \partial_{x_j}^{\alpha_j}) F_{\Delta_{T,0}}(x_{J+1}, x_J, \dots, x_1, x_0)| \leq (C_M)^{J+1}(1 + \sum_{j=0}^{J+1} |x_j|)^m, \tag{5.2}$$

$$|(\prod_{j=0}^{J+1} \partial_{x_j}^{\alpha_j}) \partial_{x_k} F_{\Delta_{T,0}}(x_{J+1}, x_J, \dots, x_1, x_0)| \leq (C_M)^{J+1}(u_{k+1} + u_k)(1 + \sum_{j=0}^{J+1} |x_j|)^m. \tag{5.3}$$

Remark. The conclusions of Theorems 1 (1), 2, 3, 4 and 7 are valid for functionals satisfying this assumption.

§ 5.3. Define Assumption by Path

Definition 5.1 (Functional Derivatives)**.** For any division $\Delta_{T,0}$, assume that $F_{\Delta_{T,0}}(x_{J+1}, x_J, \dots, x_1, x_0) \in C^\infty(\mathbf{R}^{d(J+2)})$. Let $\gamma\colon [0, T] \to \mathbf{R}^d$ and $\eta_l\colon [0, T] \to \mathbf{R}^d$, $l = 1, 2, \dots, L$ be any broken line paths. We define the functional derivative $(D^L F)[\gamma] \prod_{l=1}^{L} [\eta_l]$ by

$$(D^L F)[\gamma] \prod_{l=1}^{L} [\eta_l] = \Big(\prod_{l=1}^{L} \frac{\partial}{\partial \theta_l}\Big) F[\gamma + \sum_{l=1}^{L} \theta_l \eta_l]\Big|_{\theta_1 = \theta_2 = \cdots = \theta_L = 0}.$$

Then our assumption for the functionals $F[\gamma] \in \mathcal{F}$ is the following:

Assumption (The class $\mathcal{F}$ of Functionals $F[\gamma]$). Let $m \geq 0$ and $\rho(t)$ be a function of bounded variation on $[0,T]$. For any non-negative integer M, there exists a positive constant C_M such that

$$\Big|(D^{\sum_{j=0}^{J+1} L_j} F)[\gamma] \prod_{j=0}^{J+1} \prod_{l_j=1}^{L_j} [\eta_{j,l_j}]\Big| \leq (C_M)^{J+2}(1+\|\gamma\|)^m \prod_{j=0}^{J+1} \prod_{l_j=1}^{L_j} \|\eta_{j,l_j}\|, \tag{5.4}$$

$$\Big|(D^{1+\sum_{j=0}^{J+1} L_j} F)[\gamma][\eta] \prod_{j=0}^{J+1} \prod_{l_j=1}^{L_j} [\eta_{j,l_j}]\Big| \tag{5.5}$$

$$\leq (C_M)^{J+2}(1+\|\gamma\|)^m \int_0^T |\eta(t)| d|\rho|(t) \prod_{j=0}^{J+1} \prod_{l_j=1}^{L_j} \|\eta_{j,l_j}\|,$$

for any $\Delta_{T,0}$, any $L_j = 0, 1, \ldots, M$, any broken line path γ: $[0,T] \to \mathbf{R}^d$, any broken line path η: $[0,T] \to \mathbf{R}^d$, and any broken line paths η_{j,l_j} : $[0,T] \to \mathbf{R}^d$, $l_j = 1, 2, \ldots, L_j$ whose supports are contained in $[T_{j-1}, T_{j+1}]$ (see Figure 10). Here $0 = T_{-1} = T_0$, $T_{J+1} = T_{J+2} = T$, $\|\gamma\| = \max_{0 \leq t \leq T} |\gamma(t)|$ and $|\rho|(t)$ is the total variation of $\rho(t)$.

Remark. The conclusions of Theorems 1 through 7 are valid for functionals satisfying this assumption.

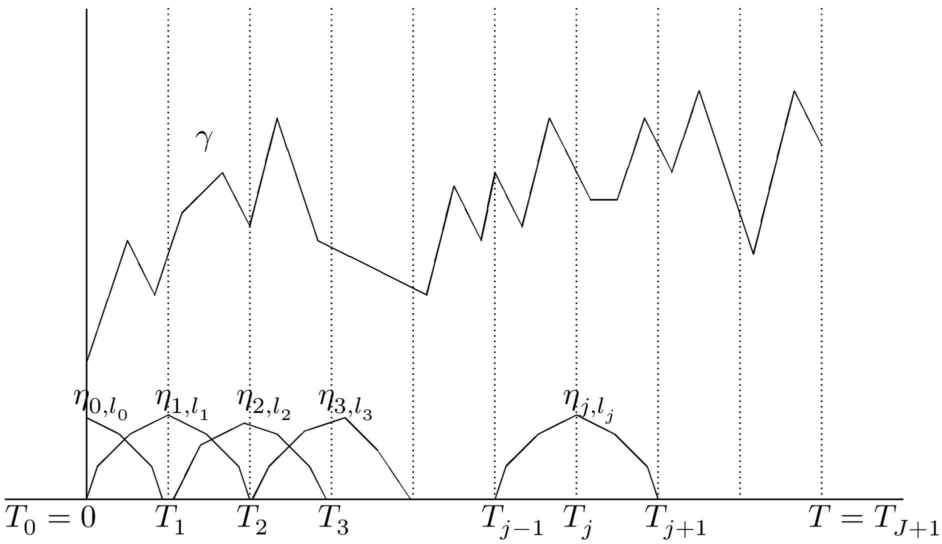

Figure 10.

§ 6. The second Term of Semiclassical Approximation as $\hbar \to 0$

As a recent development of our approach, we state the second term of semiclassical approximation that Fujiwara gave in [13], [14].

Lemma 6.1. *Let $0 < T_1 < T_2 < T$. Let $\gamma_{T,T_2,T_1,0}$ be the piecewise classical path which connects $(0, x_0)$, (T_1, x_1), (T_2, x_2) and (T, x) by classical paths (see Figure 11). Then*

$$
\begin{aligned}
q(T_1) &\equiv q(T_1, x, x_0) \\
&= \lim_{T_2 \downarrow T_1} \triangle_{x_1} \Big(D(T_1, x_1, x_0)^{-1/2} F[\gamma_{T,T_2,T_1,0}] \Big) \Big|_{x_2=\gamma^{cl}(T_2),\ x_1=\gamma^{cl}(T_1)}
\end{aligned} \tag{6.1}
$$

converges on any compact set of $(x, x_0) \in \mathbf{R}^{2d}$.

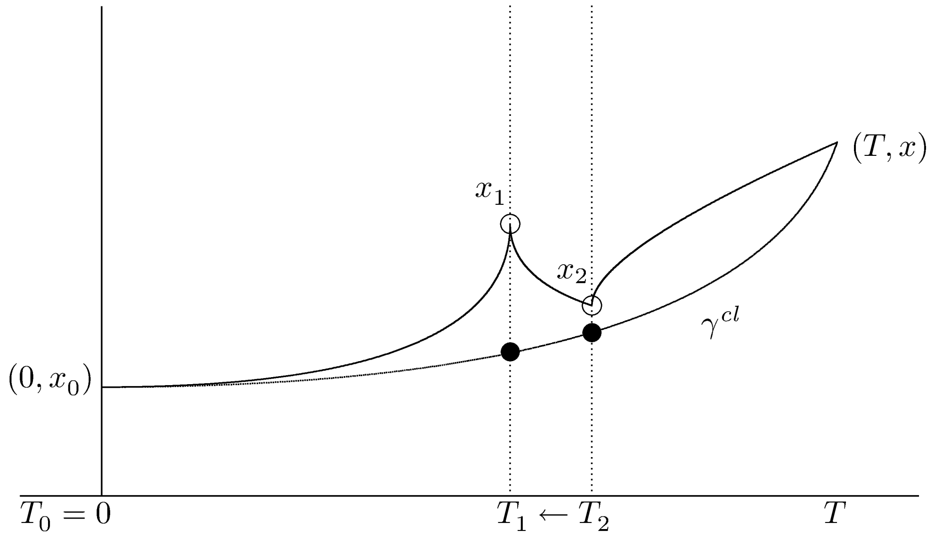

Figure 11.

Theorem 8 (The Second Term of Semiclassical Approximation as $\hbar \to 0$). *Assume that $q(t)$ in Lemma 6.1 is continuous on $[0, T]$. Then we have*

(6.2)
$$
\begin{aligned}
\int e^{\frac{i}{\hbar}S[\gamma]} F[\gamma] \mathcal{D}[\gamma] &= \Big(\frac{1}{2\pi i \hbar T}\Big)^{d/2} e^{\frac{i}{\hbar}S[\gamma^{cl}]} D(T, x, x_0)^{-1/2} \\
&\quad \times \Big(F[\gamma^{cl}] + \frac{i\hbar}{2} \int_0^T D(t, \gamma^{cl}(t), x_0)^{1/2} q(t) dt + \hbar^2 \Upsilon'(T, \hbar, x, x_0) \Big).
\end{aligned}
$$

Here for any multi-indices α, β, there exists a positive constant $C_{\alpha,\beta}$ such that

$$
|\partial_x^\alpha \partial_{x_0}^\beta \Upsilon'(T, \hbar, x, x_0)| \le C_{\alpha,\beta}(1 + |x| + |x_0|)^m.
$$

Remark. If $F[\gamma] \equiv 1$, this second term

$$
\frac{i\hbar}{2} \int_0^T D(t, \gamma^{cl}(t), x_0)^{1/2} \Big(\triangle_y D(t, y, x_0)^{-1/2} \Big) \Big|_{y=\gamma^{cl}(t)} dt
$$

satisfies the Birkhoff equation [3] for the second term of the solution of Schödinger equation.

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