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Formal Solutions and True Solutions with Gevery Type Asymptotic Expansion for Some Nonlinear Partial Differential Equations in the Complex Domain

By

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Abstract

Ōuchi [3] showed that for some linear partial differential equations in a complex domain, there exists a true solution $u_S(t,x)$ which is a holomorphic function in a sector S, and has an asymptotic expansion as $t \to 0$ in S. In this paper, we extend these results for nonlinear equations, and give another construction of such a solution.

§ 1. Introduction

Let $\mathbb{C}$ be the complex plane or the set of all complex numbers, t a coordinate of $\mathbb{C}_t$, and $x = (x_1, \dots, x_n)$ coordinates of $\mathbb{C}_x^n = \mathbb{C}_{x_1} \times \cdots \times \mathbb{C}_{x_n}$. Set $\mathbb{N} := \{0, 1, 2, \dots\}$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we set $|\alpha| := \alpha_1 + \cdots + \alpha_n$, and $(\partial/\partial x)^\alpha := (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$. Set $|x| := \max_{1 \le i \le n} \{|x_i|\}$, $D_R := \{x \in \mathbb{C}_x^n;\ |x| < R\}$ and

$$S_\theta(T) := \{t \in \mathbb{C}_t;\ 0 < |t| < T,\ |\arg t| < \theta\}.$$

We denote by $\mathscr{O}(D_R)$ (resp. $\mathscr{O}(S_\theta(T) \times D_R)$) the set of all holomorphic functions defined on D_R (resp. $S_\theta(T) \times D_R$).

In this paper, we consider the following equation:

$$D(u(t,x)) = f(t,x). \tag{1.1}$$

Here $D(u(t,x))$ is a nonlinear partial differential operator with coefficients in holomorphic functions on a neighborhood of the origin for an unknown function $u(t,x)$, and $f(t,x)$ is a given function.

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Problem 1. Suppose that $f(t,x)$ is a holomorphic function. Can we construct a true solution?
Then the answer is "Yes" (Ōuchi [1], [2], Yamazawa [4]).

Problem 2. Suppose that $f(t,x)$ is a function with Gevrey type asymptotic expansion. Can we construct a true solution of the same type?

In this paper we consider Problem 2; we construct a true solution to (1.1), and moreover, we prove that the solution has the same Gevrey type asymptotic expansion as $f(t,x)$.

§ 2. Solvability in $\mathrm{Asy}^0_{\{\gamma\}}$

We define some function spaces which will be used in this paper.

Definition 2.1. Let $\gamma > 0$. Then we define a subspace $\mathrm{Asy}^0_{\{\gamma\}}(S_\theta(T) \times D_R)$ of $\mathscr{O}(S_\theta(T) \times D_R)$ as follows: $f(t,x) \in \mathrm{Asy}^0_{\{\gamma\}}(S_\theta(T) \times D_R)$ if for any $S_0 = S_{\theta_0}(T_0)$ with $0 < \theta_0 < \theta$ and $0 < T_0 < T$ (which we denote by $S_0 \Subset S$), there exist $f_k(x) \in \mathscr{O}(D_R)$ and C and $c_0 > 0$ such that

$$|f(t,x)| \leq C \exp(-c_0 |t|^{-\gamma})$$

holds in S_0.

Definition 2.2. Let $\gamma > 0$. Then we define a subspace $\mathrm{Asy}_{\{\gamma\}}(S_\theta(T) \times D_R)$ of $\mathscr{O}(S_\theta(T) \times D_R)$ as follows: $f(t,x) \in \mathrm{Asy}_{\{\gamma\}}(S_\theta(T) \times D_R)$ if for any $S_0 \Subset S_{\theta_0}(T_0)$, there exist $f_k(x) \in \mathscr{O}(D_R)$ and A_0, $B_0 > 0$ such that for any $N \in \mathbb{N} \setminus \{0\}$

$$\Big| f(t,x) - \sum_{k=0}^{N-1} f_k(x) t^k \Big| \leq A_0 B_0^N |t|^N \Gamma\Big(\frac{N}{\gamma} + 1\Big) \tag{2.1}$$

holds in S_0. If the condition (2.1) is satisfied, then we write

$$f(t,x) \sim_\gamma \widetilde{f}(t,x) = \sum_{k \geq 0} f_k(x) t^k \quad \text{in } S_\theta(T).$$

We call $\widetilde{f}(t,x)$ an Gevrey type asymptotic expansion with index γ for $f(t,x)$.

We consider the following operator $D(u)$:

$$D(u(t,x)) = F(t,x,\{(\partial/\partial t)^j (\partial/\partial x)^\alpha u(t,x)\}_{j+|\alpha| \leq m}). \tag{2.2}$$

We assume that $F(t,x,Z)$ ($Z = \{Z_{j,\alpha}\}_{j+|\alpha| \leq m}$) admits an expansion which is a convergent power series with respect to Z:

$$F(t,x,Z) = \sum_{|q| \geq 1} a_q(t,x) \prod_{j+|\alpha| \leq m} \{Z_{j,\alpha}\}^{q_{j,\alpha}}, \tag{2.3}$$

here $q_{j,\alpha} \in \mathbb{N}$, and we set $q := \{q_{j,\alpha} \in \mathbb{N};\ j + |\alpha| \le m\}$, $|q| := \sum_{j+|\alpha|\le m} q_{j,\alpha}$, and we assume that $a_q(t,x) \in \mathrm{Asy}_{\{\gamma\}}(S_\theta(T) \times D_R)$. Then we consider the following equation (E_0).

$$(E_0) \qquad D(u(t,x)) = f(t,x) \in \mathrm{Asy}^0_{\{\gamma\}}(S_\theta(T) \times D_R).$$

For the equation (E_0) we introduce Newton polygon due to Ōuchi [2], [3]. We write each coefficient $a_q(t,x) \in \mathrm{Asy}_{\{\gamma\}}(S_\theta(T) \times D_R)$ as

$$a_q(t,x) = t^{\sigma_q} b_q(t,x) \qquad (b_q(0,x) \not\equiv 0,\ \sigma_q \in \mathbb{N} \setminus \{0\}).$$

We set

$$\Pi(a,b) := \{(x,y) \in \mathbb{R}^2;\ x \le a \text{ and } y \ge b\}.$$

Moreover, set $l_q := \max\{j + |\alpha|;\ q_{j,\alpha} \in q,\ q_{j,\alpha} \neq 0\}$ and $e_q := \sigma_q - \sum_{q_{j,\alpha}\in q} j q_{j,\alpha}$. Then we define Newton polygon $NP_1(D)$ for the linear part of the operator $D(\cdot)$ by

$$NP_1(D) := CH\Big\{ \bigcup_{|q|=1} \Pi(l_q, e_q);\ b_q(t,x) \not\equiv 0 \Big\},$$

where $CH\{\cdot\}$ is the convex hull of a set.

The boundary of Newton polygon $NP_1(D)$ consists of a vertical half line Σ_p^*, a horizontal half line Σ_p^* and segments Σ_i^* $(1 \le i \le p-1)$. Let γ_i^* be the slope of Σ_i^* for $i = 0, \ldots, p$. Then we have $0 = \gamma_p^* < \gamma_{p-1}^* < \cdots < \gamma_0^* = \infty$. Further Newton polygon $NP_1(D)$ has p-point vertices which we denote by (l_i^*, e_i^*) with $l_{p-1}^* < l_{p-2}^* < \cdots < l_0^* = m$.

Next let us define an operator $\mathcal{L}_i$ with respect to Σ_i^* for $i = 1, \cdots, p-1$. We set

$$I_i := \{q \subset \mathbb{N};\ e_i^* - e_q = \gamma_i^*(l_i^* - l_q) \text{ and } |q| = 1\}.$$

Then we set

$$\begin{aligned}\mathcal{L}_i u(t,x) &:= \sum_{q\in I_i} t^{\sigma_q} b_q(t,x) \prod_{j+|\alpha|\le m} \Big\{ \Big(\frac{\partial}{\partial t}\Big)^j \Big(\frac{\partial}{\partial x}\Big)^\alpha u(t,x) \Big\}^{q_{j,\alpha}} \\ &= \sum_{(j,\alpha)\in J_i} t^{\sigma_{j,\alpha}} b_{j,\alpha}(t,x) \Big(\frac{\partial}{\partial t}\Big)^j \Big(\frac{\partial}{\partial x}\Big)^\alpha u(t,x),\end{aligned}$$

where

$$J_i := \{j + |\alpha| \le m;\ e_i^* - (\sigma_{j,\alpha} - j) = \gamma_i^*(l_i^* - j - |\alpha|)\}.$$

Let m_i^* be the order with respect to $\partial/\partial x$ of $\mathcal{L}_i$.

We assume the following conditions for the equation (E_0):

(C_1) $D(u)$ has a linear part of order m,

$(C_2)_i$ the operator $\mathcal{L}_i$ satisfies

(1) if $j+|\alpha|<l_i^*$ then $|\alpha|<m_i^*$ and
(2) $\sum_{\substack{j+|\alpha|=l_i^*\\ |\alpha|=m_i^*}} b_{j,\alpha}(0,0)\widehat{\xi}^\alpha \neq 0$ with $\widehat{\xi}:=(1,0,\cdots,0)$.

If we assume that the equation (E_0) satisfies the condition (C_1), then it is sufficient to define Newton polygon for the only linear part (see Ōuchi [2, Proposition 1.7]).

We have an existence theorem concerning exponential decay solutions.

Theorem 2.3. *Let $\gamma_{s+1}^* \leq \gamma < \gamma_s^*$ and $f(t,x) \in \mathrm{Asy}^0_{\{\gamma\}}(S_\theta(T)\times D_R)$ ($|\theta|<\pi/2\gamma$). Suppose the conditions (C_1) and $(C_2)_i$ ($i=0,\ldots,s$). Then for any $0<r<R$, there exists a solution $u(t,x)\in \mathrm{Asy}^0_{\{\gamma\}}(S_{\theta'}(T)\times D_r)$ ($|\theta'|<\pi/2\gamma_1^*$) to the equation (E_0).*

In the case where $\gamma=\gamma_{s+1}^*$, Theorem 2.3 was obtained in Ōuchi [2] under the condition that $F(t,x,Z)$ is a polynomial in Z, and this condition was removed in Yamazawa [4]. If $\gamma>\gamma_{s+1}^*$, we can prove this theorem as same way as in the case where $\gamma=\gamma_{s+1}^*$.

§ 3. Construction of True Solution

We consider the following equation:

$$(E_1) \qquad D(u(t,x)) = f(t,x),$$

where each $a_q(t,x)$ in (2.3) is holomorphic in a neighborhood of the origin, and $f(t,x)$ belongs to $\mathrm{Asy}_{\{\gamma\}}(S_\theta(T)\times D_R)$.

For the equation (E_1) we want to construct a solution that belongs to the same class as $f(t,x)$. In a linear case we shall recall Ōuchi's result in [3].

Let us consider the following condition $(C_2')_i$ for the operator $\mathcal{L}_i$:

$(C_2')_i$ (1) $\sigma_{j,\alpha}=0$ for any (j,α) with $j+|\alpha|=l_i^*$ and $|\alpha|=m_i^*$.
(2) $\sum_{\substack{j+|\alpha|=l_i^*\\ |\alpha|=m_i^*}} b_{j,\alpha}(0,0)\widehat{\xi}^\alpha \neq 0$.

Then we have the following theorem:

Theorem 3.1 (Ōuchi [3]). *Assume that $D(u)$ satisfies $(C_2')_i$ ($i=1,\ldots,s$). Let $\gamma\geq\gamma_{s+1}^*$. Then for any $S_0 \Subset S$ with $0<\theta_0<\pi/(2\gamma_1^*)$ there exists a solution $u(t,x)\in \mathrm{Asy}_{\{\gamma\}}(S_0\times D_r)$ to (E_1).*

Remark. In [3], Ōuchi constructed a solution to (E_1) by using an integral kernel G and the solution is expressed as follows:

$$\int_c G(t,x;w)f(w)dw,$$

but it is very complicated to construct $G(t,x;w)$.

Next let us consider a nonlinear case. We shall get the same result as in a linear case, and give another construction of a true solution.

Let s be a nonnegative integer with $\gamma^*_{s+1} \le \gamma < \gamma^*_s$, and we set $k^*_s = l^*_s - m^*_s$. We will give a condition for nonlinear terms.

For all nonlinear term, let us assume the following condition:

(A_1) For any (l_q, e_q) with $|q| \ge 2$, there exist $J^-_q > 0$ and $J^+_q \ge 0$ such that

$$l_q = \begin{cases} -\dfrac{e^* - e_q}{\gamma^*_{s+1}} + l^*_s - J^-_q & \text{for} \quad e^* \ge e_q, \\ \dfrac{e_q - e^*}{\gamma^*_s} + l^*_s - J^+_q & \text{for} \quad e_q > e^*. \end{cases}$$

Further we assume

$$\Big[\frac{m_s}{\gamma^*_{s+1}}\Big]_0 < J_q \quad \text{if} \quad e^* > e_q \text{ and } |q| \ge 2,$$

where $[a]_0$ is the decimal part of a number a.

Then we have the following result for a formal solution.

Theorem 3.2. *Suppose the conditions (A_1) and $(C'_2)_s$ for (E_1). Then for $|\theta| < \pi/(2\gamma)$, we can construct a formal power series $\widetilde{u}(t,x) = \sum_{h \le 0} u_{(h)}(t,x)$ that formally satisfeis (E_1). Further $u_{(h)}(t,x)$ satisfies the following estimate*:

$$(3.1) \qquad |u_{(h)}(t,x)| \le \tau^{-m_s}\widetilde{U}_{(h)} B^h \Gamma\Big(-\frac{h}{\gamma^*_s}+1\Big)\Gamma\Big(\frac{k^*_s}{\gamma}+1\Big)|t|^{k^*_s - h} \ \text{in } S_0 \Subset S_\theta(T),$$

and $\sum_{h \le 0} \widetilde{U}_{(h)} t^{-h}$ is a convergent power series in a neighborhood of the origin.

We shall give a sketch of proof of Theorem 3.2 in Section 5.

As for the formal solution $\widetilde{u}(t,x)$ we have the following fact.

Lemma 3.3. *There exists a function $u_{S_0}(t,x) \in \mathrm{Asy}_{\{\gamma^*_s\}}(S_0 \times D_r)$ with $0 < \theta_0 < \pi/(2\gamma^*_s)$ such that for any $S_1 \Subset S_0$*

$$|u_{S_0}(t,x) - \sum_{h=0}^{-N+1} u_{(h)}(t,x)| \le \widetilde{U}'_{(h)}|t|^N \Gamma\Big(\frac{N}{\gamma^*_s}+1\Big) \quad \textit{for} \quad t \in S_1,$$

where $\sum_{n\geq 0}\widetilde{U}'_{(h)}t^N$ *is a convergent series.*

Proof. Put

$$\widehat{u}_{(h)}(t,x;\xi)=\frac{u_{(h)}(t,x)}{t^{k_s-h+\gamma_s^*}}\frac{\xi^{-h/\gamma_s^*}}{\Gamma(-h/\gamma_s^*+1)},\quad \widehat{u}_H(t,x;\xi)=\sum_{h\leq -H-1}\widehat{u}_{(h)}(x,t,\xi),$$
$$\widehat{u}(t,x;\xi)=\sum_{h\leq 0}\widehat{u}_{(h)}(x,t,\xi).$$

It follows from Theorem 3.2 that $\widehat{u}_H(t,x;\xi)$ and $\widehat{u}(t,x;\xi)$ converge on $D_R\times S_0\times\{|\xi|\leq\widehat{\xi}_0\}$ for some $\widehat{\xi}_0>0$. Then there exist $\widehat{\xi}$ with $0<\widehat{\xi}<\widehat{\xi}_0$ and B_0 such that

$$|\widehat{u}_H(t,x;\xi)|\leq AB_0^{H+1}|t|^{-\gamma_s^*}|\xi|^{(H+1)/\gamma_s^*}\quad\text{on } D_R\times S_0\times\{|\xi|\leq\widehat{\xi}\},$$
$$\sum_{h=0}^{-H}|\widehat{u}_{(h)}(t,x;\xi)|\leq AB_0^{H+1}|t|^{-\gamma_s^*}|\xi|^{(H+1)/\gamma_s^*}\quad\text{on } D_R\times S_0\times\{|\xi|\geq\widehat{\xi}\}.$$

Define

$$u_{S_0}(t,x)=t^{k_s^*}\int_0^{\widehat{\xi}}\exp(-\xi t^{-\gamma_0^*})\widehat{u}(t,x;\xi)d\xi.$$

By the estimates above we can see that $u_{S_0}(t,x)$ is the desired function. □

We can construct a true solution as $u(t,x)=u_{S_0}(t,x)+v(t,x)$, where $v(t,x)$ is an unknown function. Set $D^{u_{S_0}}(v):=D(u_{S_0}+v)-D(u_{S_0})$. Then we have the following theorem.

Theorem 3.4. *Suppose that for the differential equation* $D^{u_{S_0}}(v)$ *with respect to* $v(t,x)$*, the conditions* $(C_2)_i$ *are satisfied* $(i=0,\dots,s-1)$*. Then for any* $S_1\Subset S_0$ *with* $0<\theta_1<\pi/(2\gamma_1^*)$*, there exists a solution* $u_{S_1}(t,x)\in\mathrm{Asy}_{\{\gamma\}}(S_1\times D_r)$ *to* (E_1)*.*

Proof. If the condition (C_1) is satisfied for the equation (E_1), then the same condition is also satisfied for the equation $D^{u_{S_0}}(v)$. As the proof of in [2, Lemma 4.4] we can prove that $f(t,x)-D(u_{S_0})\in\mathrm{Asy}^0_{\{\gamma_s^*\}}(S_0\times D_r)$. Therefore we can adapt Theorem 2.3 to an equation $D^{u_{S_0}}(v)=f(t,x)-D(u_{S_0})$, and this equation has a solution $v_{S_1}(x)\in\mathrm{Asy}^0_{\{\gamma_s^*\}}(S_1\times D_r)$. Then $u_{S_0}(t,x)+v_{S_1}(t,x)$ is a solution to the equation (E_1). □

Remark. We can prove Theorem 3.4 if we replace the condition $(C_2)_i$ by $(C_2')_i$.

§ 4. Majorant Function

First let us introduce a majorant function in [2]. Set

$$\theta(t)=\sum_{n\geq 0}\frac{ct^n}{(n+1)^{m+2}}\quad\text{and}\quad\theta^{(k)}(t)=\left(\frac{\partial}{\partial t}\right)^k\theta(t),$$

where $m \in \mathbb{N}$.

Lemma 4.1. *There exists a positive constant c such that for $0 \le k' \le k \le m$*

$$\theta^{(k)}(t)\,\theta^{(k')}(t) \ll \theta^{(k)}(t). \tag{4.1}$$

We fix $c > 0$ so that (4.1) holds.

Lemma 4.2. (1) *There exists a constant $C > 0$ such that the following holds for any k:*

$$\theta^{(k)}(t) \ll \frac{C}{k+1}\theta^{(k+1)}(t).$$

(2) *Let $0 \le k' \le k \le m$. Then*

$$\sum_{i=0}^{n} \frac{n!}{i!(n-i)!}\theta^{(n-i+k+p)}(t)\theta^{(i+k'+p')}(t) \ll \frac{p!p'!}{(p+p')!}\theta^{(n+k+p+p')}(t).$$

Set $\Psi_R(t) = \theta(t/R)$ and $\Psi_R^{(k)}(t) = (\partial/\partial t)^k \Psi_R(t)$ for $0 < R < 1$. By Lemma 4.2 we can prove the following proposition:

Proposition 4.3. (1) *There exists a constant $C > 0$ such that the following holds for any k:*

$$\Psi_R^{(k)}(t) \ll \frac{C}{k+1} R \Psi_R^{(k+1)}(t).$$

(2) *Let $0 \le k' \le k \le m$. Then for $\delta \ge 1$*

$$\sum_{i=0}^{n} \frac{n!}{i!(n-i)!}\Psi_R^{([(n-i)/\delta]+k+p)}(t)\Psi_R^{([i/\delta]+k'+p')}(t) \ll \frac{p!p'!}{(p+p')!}R^{-k'}\Psi_R^{([n/\delta]+k+p+p')}(t),$$

where $[a]$ denotes the integer part of a number a.

§ 5. Sketch of Proof of Theorem 3.2

First we study a solvability for an operator with respect to a vertex of Newton polygon.

We set

$$\Delta(s) := \{q;\ l_q = l_s^* \text{ and } e_q = e_s^*\}.$$

Then we define an operator by

$$Su(t,x) = \sum_{q \in \Delta(s)} a_q(t,x) \prod_{j+|\alpha| \le m} \left\{\left(\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u\right\}^{q_{j,\alpha}}.$$

Under the condition (A_1), the operator S is a linear operator. So we can write:

$$S = \sum_{(j,\alpha)\in\Delta(s)} t^{\sigma_{j,\alpha}} b_{j,\alpha}(t,x)\Big(\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha,$$

where

$$\Delta(s) := \{(j,\alpha);\ j+|\alpha| = l_s^* \text{ and } \sigma_{j,\alpha} - j = e_s^*\}.$$

We fix s and set $\Delta^* = \Delta(s)$, $k^* = k_s^*$ and $m^* = m_s^*$ and $(l^*, e^*) = (l_s^*, e_s^*)$ for short.

Let us consider the following equation:

$$(E_S) \qquad Su(t,x) = F(t,x),$$

where $F(t,x)$ satisfies the following estimate as a formal power series in (t,x) for some $F > 0$:

$$F(t,x) \ll F \sum_{k\ge m_h} \frac{1}{k!}\Psi_R^{([(k+k^*)/\delta]+m^*+c_h)}(\chi)\Big(\frac{t}{\zeta}\Big)^k,$$

where $\chi = \tau x_1 + \sum_{i=2}^n x_i$, τ, $\zeta > 0$.

Lemma 5.1. *Under the conditions $(C_2')_s$ and (A_1), the equation (E_S) has a solution $u(t,x)$ satisfying*

$$u(t,x) \ll \tau^{-m^*}\zeta^{-e^*}c(\tau)\,F \sum_{k\ge m_h+k^*} \frac{1}{k!}\Psi_R^{([k/\delta]+c_h)}(\chi)\Big(\frac{t}{\zeta}\Big)^k,$$

where $c(\tau)$ is a positive and bounded function of $\tau > 0$.

We will construct a formal solution $u(t,x) = \sum_{g\ge 0} u^g(t,x)$ to the equation (E_1): We choose the sequence $\{u^g(t,x)\}$ satisfying the following equations:

$$(E^g)\quad \begin{cases} Su^0(t,x) + \displaystyle\sum_{e_q\le e^*} t^{\sigma_q} b_q(t,x) \prod_{j+|\alpha|\le m}\Big\{\Big(\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha u^0(t,x)\Big\}^{q_{j,\alpha}} = f(t,x), \\ Su^g(t,x) + \displaystyle\sum_{\substack{e_q\le e^* \\ |g'|=g}} t^{\sigma_q} b_q(t,x) \prod_{j+|\alpha|\le m}\prod_{i=1}^{q_{j,\alpha}}\Big(\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha u^{g_i}(t,x) \\ \quad = -\displaystyle\sum_{\substack{e_q> e^* \\ |g'|=g-(e_q-e^*)}} t^{\sigma_q} b_q(t,x) \prod_{j+|\alpha|\le m}\prod_{i=1}^{q_{j,\alpha}}\Big(\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha u^{g_i}(t,x) \quad (g\ge 1). \end{cases}$$

Here we set $|g'| := \sum_{j+|\alpha|\le m}\sum_{i=1}^{q_{j,\alpha}} g_i$.

We solve the equation (E^g) inductively: First let us solve the equation (E^0). We construct a solution $u^0(t,x) = \sum_{h\geq 0} u_h^0(t,x)$ as

$$
\begin{aligned}
&Su_0^0(t,x) = f(t,x),\\
&Su_h^0(t,x) := W_h^0(t,x)\\
&\qquad = -\sum_{\substack{e_q<e^*\\ |h'|=h-(e^*-e_q)}} t^{\sigma_q} b_q(t,x) \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j,\alpha}} \Big(\frac{\partial}{\partial t}\Big)^j \Big(\frac{\partial}{\partial x}\Big)^\alpha u_{h_i}^0(t,x)\\
&\qquad\quad - \sum_{\substack{e_q=e^*\\ |h'|=h-1}} t^{\sigma_q} b_q(t,x) \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j,\alpha}} \Big(\frac{\partial}{\partial t}\Big)^j \Big(\frac{\partial}{\partial x}\Big)^\alpha u_{h_i}^0(t,x) \qquad (h\geq 1).
\end{aligned}
$$

Here we set $|h'| := \sum_{j+|\alpha|\leq m} \sum_{i=1}^{q_{j,\alpha}} h_i$.

Since the function $f(t,x)$ belongs to $\mathrm{Asy}_{\{\gamma\}}(S_\theta(T)\times D_R)$ we can assume that $f(t,x)$ satisfies

$$f(t,x) \ll F \sum_{k\geq 0} \frac{1}{k!} \Psi_R^{([(k+k^*)/\delta]+m^*)}(\chi) \Big(\frac{t}{\zeta}\Big)^k,$$

where $\dfrac{1}{\delta} = 1 + \dfrac{1}{\gamma}$, and each $b_q(t,x)$ satisfies for some $B_q > 0$:

$$b_q(t,x) \ll B_q \sum_{k\geq 0} \frac{1}{k!} \Psi_R^{(k)}(\chi) \Big(\frac{t}{\zeta}\Big)^k.$$

Then by Lemma 5.1, we get

$$u_0^0(t,x) \ll \tau^{-m^*} U_0^0 \sum_{k\geq k^*} \frac{1}{k!} \Psi_R^{([k/\delta])}(\chi) \Big(\frac{t}{\zeta}\Big)^k,$$

where $U_0^0 := \zeta^{-e^*} c(\tau) F$. By induction on $h > 0$, we can obtain $u_h^0(t,x)$ by Lemma 5.1. So we will inductively give an estimate for $u_h^0(t,x)$.

Proposition 5.2. *Under the conditions $(C_2')_s$ and (A_1), for any $h \in \mathbb{N}$ there exists $U_h^0 > 0$ such that*

$$u_h^0(t,x) \ll \tau^{-m^*} U_h^0 \sum_{k\geq k^*} \frac{1}{k!} \Psi_R^{([k/\delta])}(\chi) \Big(\frac{t}{\zeta}\Big)^k, \tag{5.1}$$

and that $\sum_{h\geq 0} U_h^0$ converges for a sufficiently large $\tau > 0$ and a sufficiently small $R > 0$.

Proof. We set $\gamma_0 = \gamma^*_{s+1}$ for short. We already obtain the estimate (5.1) for $h = 0$. Let us assume the following estimate for $h_i < h$:

$$\Big(\frac{\partial}{\partial t}\Big)^j \Big(\frac{\partial}{\partial x}\Big)^\alpha u^0_{h_i}(t,x) \ll \tau^{\alpha_1 - m^*} \zeta^{-j} U^0_{h_i} \sum_{k \geq (k^*-j)_+} \frac{1}{k!} \Psi_R^{([(k+j)/\delta]+|\alpha|)}(\chi) \Big(\frac{t}{\zeta}\Big)^k,$$

where $(a)_+ = a$ if $a \geq 0$ and $(a)_+ = 0$ if $a < 0$ for $a \in \mathbb{R}$. By Proposition 4.3, we have

$$t^{\sigma_q} \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j,\alpha}} \Big(\frac{\partial}{\partial t}\Big)^j \Big(\frac{\partial}{\partial x}\Big)^\alpha u^0_{h_i}(t,x)$$
$$\ll C^{\sigma_q} \zeta^{e_q} \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j,\alpha}} (\tau^{\alpha_1 - m^*} U^0_{h_i}) \sum_{k\geq 0} \frac{1}{k!} \Psi_R^{([k/\delta - e_q/\gamma + l_q])}(\chi) \Big(\frac{t}{\zeta}\Big)^k.$$

It follows $[[k^*/\gamma]_0 + 1 - J_q^-] \leq 0$ from the assumption (A_1). If $e_q < e^*$ and $|h'| = h - (e^* - e_q)$, then we have

$$\begin{aligned}
\Big[\frac{-e_q}{\gamma} + l_q\Big] &= \Big[\frac{-e_q}{\gamma} - \frac{e^* - e_q}{\gamma_0} + k^* + m^* - J_q^-\Big] \\
&= \Big[\frac{k^*}{\delta} + \frac{e^* - e_q}{\gamma} - \frac{e^* - e_q}{\gamma_0} + m^* - J_q^-\Big] \\
&\leq \Big[\frac{k^*}{\delta} + m^* - J_q^-\Big] = k^* + \Big[\frac{k^*}{\gamma}\Big] + m^* - 1 + \Big[\Big[\frac{k^*}{\gamma}\Big]_0 + 1 - J_q^-\Big] \\
&\leq \Big[\frac{k^*}{\delta}\Big] + m^* - 1,
\end{aligned} \tag{5.2}$$

and if $e_q = e^*$, by the same manner we have

$$\Big[\frac{-e_q}{\gamma} + l_q\Big] = \Big[\frac{k^*}{\gamma} + k^* + m^* - 1\Big] = \Big[\frac{k^*}{\delta}\Big] + m^* - 1. \tag{5.3}$$

Thus by Proposition 4.3, we have

$$\begin{aligned}
&W^0_h(t,x) \\
&\ll \sum_{\substack{e_q < e^* \\ |h'| = h - (e^* - e_q)}} C^{\sigma_q} \zeta^{e_q} R B_q \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j,\alpha}} (\tau^{\alpha_1 - m^*} U^0_{h_i}) \sum_{k\geq 0} \frac{1}{k!} \Psi_R^{([(k+k^*)/\delta]+m^*)}(\chi) \Big(\frac{t}{\zeta}\Big)^k \\
&\quad + \sum_{\substack{e_q = e^* \\ |h'| = h-1}} C^{\sigma_q} \zeta^{e_q} R B_q \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j,\alpha}} (\tau^{\alpha_1 - m^*} U^0_{h_i}) \sum_{k\geq 0} \frac{1}{k!} \Psi_R^{([(k+k^*)/\delta]+m^*)}(\chi) \Big(\frac{t}{\zeta}\Big)^k.
\end{aligned}$$

Thus by Lemma 5.1, we get the estimate (5.1) for any $h \geq 0$, where

$$
\begin{aligned}
U_h^0 :=\ & c(\tau) \sum_{\substack{e_q < e^* \\ |h'| = h-(e^*-e_q)}} C^{\sigma_q} \zeta^{e_q - e^*} R B_q \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \tau^{\alpha_1 - m^*} U_{h_i}^0 \\
& + c(\tau) \sum_{e_q = e^*, |h'| = h-1} C^{\sigma_q} R B_q \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \tau^{\alpha_1 - m^*} U_{h_i}^0 .
\end{aligned} \tag{5.4}
$$

Next let us give an estimate of the coefficient U_h^0. Let us consider the following functional equation for $Z(t)$:

$$
\begin{aligned}
Z(t) = c(\tau) & \sum_{e_q < e^*} C^{\sigma_q} \zeta^{e_q - e^*} t^{e^* - e_q} B_q \prod_{j+|\alpha| \leq m} \{\tau^{\alpha_1 - m^*} Z(t)\}^{q_{j,\alpha}} \\
& + c(\tau) \sum_{e_q = e^*} C^{\sigma_q} t \prod_{j+|\alpha| \leq m} \{\tau^{\alpha_1 - m^*} Z(t)\}^{q_{j,\alpha}} + \zeta^{-e^*} c(\tau) F.
\end{aligned} \tag{5.5}
$$

By the implicit function theorem, we have the holomorphic solution near $t = 0$. Moreover substituting $Z(t) = \sum_{h \geq 0} Z_h^0 t^h$ into (5.5), for any $T > 0$, we can take a sufficiently large $\tau > 0$ and a sufficiently small $R > 0$ such that $U_h^0 \leq Z_h^0 T^h$. Hence we have the desired result. $\square$

We construct a true solution to (E^0). Set $\widetilde{u}^0(t,x) := \sum_{h \geq 0} u_h^0(t,x)$. Then we have:

$$
\widetilde{u}^0(t,x) \ll \tau^{-m^*} \sum_{h \geq 0} U_h^0 \sum_{k \geq k^*} \frac{1}{k!} \Psi_R^{([k/\delta])}(\chi) \Big(\frac{t}{\zeta}\Big)^k .
$$

If we set $\widetilde{U}_0^0 := \sum_{h \geq 0} U_h^0$, then by Proposition 5.2, we get

$$
\widetilde{u}^0(t,x) \ll \tau^{-m^*} \widetilde{U}_0^0 \sum_{k \geq k^*} \frac{1}{k!} \Psi_R^{([k/\delta])}(\chi) \Big(\frac{t}{\zeta}\Big)^k .
$$

For $|\theta| < \pi/(2\gamma)$, as in Lemma 3.3 we can prove the existence of a holomorphic function $u_S^0(t,x)$ such that

$$
u_S^0(t,x) \sim_\gamma \widetilde{u}^0(t,x) \quad \text{in} \quad S_\theta(T).
$$

So we construct a true solution $u^0(t,x)$ to (E^0) such that $u^0(t,x) = u_S^0(t,x) + v^0(t,x)$, where $v^0(t,x)$ is an unknown function; we can adapt Theorem 2.3 to the equation for $v^0(t,x)$, and we can solve (E^0) with $v^0(t,x) \in \mathrm{Asy}^0_{\{\gamma\}}(S_\theta(T) \times D_R)$.

We can solve each equation (E^g) for $g \geq 0$ by repeating the procedure of constructing a true solution $u^0(t,x)$:

First we construct a formal solution $u^g(t,x) = \sum_{h\geq -g} u^g_h(t,x)$ for $g \geq 0$ as follows:

$$Su^g_{-g}(t,x) = -\sum_{e_q > e^*} \sum_{\substack{|g'|=g-(e_q-e^*)\\ |h'|=-g+(e_q-e^*)}} a_q(t,x) \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j,\alpha}} \Big(\frac{\partial}{\partial t}\Big)^j \Big(\frac{\partial}{\partial x}\Big)^\alpha u^{g_i}_{h_i}(t,x),$$

$$\begin{aligned} Su^g_h(t,x) = &-\sum_{e_q > e^*} \sum_{\substack{|g'|=g-(e_q-e^*)\\ |h'|=h+(e_q-e^*)}} a_q(t,x) \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j,\alpha}} \Big(\frac{\partial}{\partial t}\Big)^j \Big(\frac{\partial}{\partial x}\Big)^\alpha u^{g_i}_{h_i}(t,x) \\ &-\sum_{e_q < e^*} \sum_{\substack{|g'|=g\\ |h'|=h-(e^*-e_q)}} a_q(t,x) \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j,\alpha}} \Big(\frac{\partial}{\partial t}\Big)^j \Big(\frac{\partial}{\partial x}\Big)^\alpha u^{g_i}_{h_i}(t,x) \\ &-\sum_{e_q = e^*} \sum_{\substack{|g'|=g\\ |h'|=h-1}} a_q(t,x) \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j,\alpha}} \Big(\frac{\partial}{\partial t}\Big)^j \Big(\frac{\partial}{\partial x}\Big)^\alpha u^{g_i}_{h_i}(t,x). \end{aligned}$$

Proposition 5.3. *Under the conditions $(C_2')_s$ and (A_1), for any $g \in \mathbb{N}$ and $h \geq g$, there exists a $U^g_h > 0$ such that*

$$u^g_h(t,x) \ll \tau^{-m^*} U^g_h \sum_{k\geq k^*-(h)_-} \frac{1}{k!} \Psi_R^{([k/\delta+(h)_-/\gamma-(h)_-/\gamma^*])}(\chi)\Big(\frac{t}{\zeta}\Big)^k, \tag{5.6}$$

and that $\sum_{g\geq 0}\sum_{h\geq -g} U^g_h$ converges for any sufficiently large $\tau > 0$ and sufficiently small ζ, $R > 0$. Here $\gamma^* = \gamma^*_s$, and $(a)_- := 0$ if $a \geq 0$ and $(a)_- := a$ if $a < 0$ for $a \in \mathbb{R}$.

Proof. In the case where $g = 0$, we showed the estimate (5.6) by Lemma 5.2. Let us prove that $u^g_h(t,x)$ satisfies (5.6) by induction on $g > 0$. We assume for $g_i < g$

$$\begin{aligned} &\Big(\frac{\partial}{\partial t}\Big)^j \Big(\frac{\partial}{\partial x}\Big)^\alpha u^{g_i}_{h_i}(t,x) \\ &\ll \tau^{\alpha_1-m^*}\zeta^{-j} U^{g_i}_{h_i} \sum_{k\geq (k^*-(h_i)_--j)_+} \frac{1}{k!}\Psi_R^{([k+(k+j+(h_i)_-)/\gamma-(h_i)_-/\gamma^*+j+|\alpha|])}(\chi)\Big(\frac{t}{\zeta}\Big)^k. \end{aligned}$$

Set $|(h')_-| = \sum_{j+|\alpha|\leq m} \sum_{i=1}^{q_{j,\alpha}} (h_i)_-$. By Proposition 4.3, we have

$$\begin{aligned} &t^{\sigma_q} \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j,\alpha}} \Big(\frac{\partial}{\partial t}\Big)^j \Big(\frac{\partial}{\partial x}\Big)^\alpha u^{g_i}_{h_i}(t,x) \\ &\ll C^{\sigma_q}\zeta^{e_q} \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j,\alpha}} \tau^{\alpha_1-m^*} U^{g_i}_{h_i} \sum_{k\geq k_h} \frac{1}{k!}\Psi_R^{([k+(k+|(h')_-|-e_q)/\gamma-|(h')_-|/\gamma^*+l_q])}(\chi)\Big(\frac{t}{\zeta}\Big)^k. \end{aligned}$$

First let us estimate $u^g_{-g}(t,x)$. Under the conditions that $e_q > e^*$, $|g'| = g-(e_q-e^*)$ and $|h'| = -g+(e_q-e^*)$, we have

$$k_h \geq k^*|q| - |(h')_-| - \sum_{j+|\alpha|\leq m} jq_{j,\alpha} + \sigma_q \geq k^*|q| - |h'| + e_q \geq g, \tag{5.7}$$

and

$$\begin{aligned}\Big[\frac{|(h')_-|-e_q}{\gamma} - \frac{|(h')_-|}{\gamma^*} + l_q\Big] &\leq \Big[\frac{|h'|-e_q}{\gamma} - \frac{|h'|}{\gamma^*} + l_q\Big] \\ &= \Big[\frac{k^*}{\delta} + \frac{-g}{\gamma} - \frac{-g}{\gamma^*} + m^* - J_q\Big] \leq \Big[\frac{k^*}{\delta} + \frac{-g}{\gamma} - \frac{-g}{\gamma^*}\Big] + m^*.\end{aligned} \tag{5.8}$$

Thus by Proposition 4.3 we have

$$\begin{aligned}&t^{\sigma_q} \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j,\alpha}} \Big(\frac{\partial}{\partial t}\Big)^j \Big(\frac{\partial}{\partial x}\Big)^\alpha u^{g_i}_{h_i}(t,x) \\ &\quad \ll C^{\sigma_q}\zeta^{e_q} \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j,\alpha}} \tau^{\alpha_1-m^*} U^{g_i}_{h_i} \sum_{k\geq g} \frac{1}{k!} \Psi_R^{([(k+k^*)/\delta+(-g)/\gamma-(-g)/\gamma^*]+m^*)}(\chi)\Big(\frac{t}{\zeta}\Big)^k.\end{aligned}$$

Therefore by Lemma 5.1, the estimate (5.6) holds for $u^g_{-g}(t,x)$, where

$$U^g_{-g} := c(\tau) \sum_{e_q>e^*} \sum_{\substack{|g'|=g-(e_q-e^*)\\ |h'|=-g+(e_q-e^*)}} C^{\sigma_q}\zeta^{e_q-e^*} B_q \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j,\alpha}} \tau^{\alpha_1-m^*} U^{g_i}_{h_i}.$$

Let us assume that a function $u^g_{h_i}(t,x)$ satisfies the estimate (5.6) for $h_i < h$. Here we give an estimate for the following three cases:

(i) $e_q > e^*$, $|g'| = g-(e_q-e^*)$ and $|h'| = h+(e_q-e^*)$;

(ii) $e_q < e^*$, $|g'| = g$ and $|h'| = h-(e^*-e_q)$;

(iii) $e_q = e^*$, $|g'| = g$ and $|h'| = h-1$.

(A) Assume $h \leq 0$. Then as for (5.7), we have $k_h \geq -h$ for the all cases. First consider the case (i). Then as for the estimate (5.8) we have

$$\Big[\frac{|(h')_-|-e_q}{\gamma} - \frac{|(h')_-|}{\gamma^*} + l_q\Big] \leq \Big[\frac{k^*}{\delta} + \frac{h}{\gamma} - \frac{h}{\gamma^*}\Big] + m^*.$$

Therefore by Proposition 4.3, we have

$$\begin{aligned}&t^{\sigma_q} \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j,\alpha}} \Big(\frac{\partial}{\partial t}\Big)^j \Big(\frac{\partial}{\partial x}\Big)^\alpha u^{g_i}_{h_i}(t,x) \\ &\quad \ll C^{\sigma_q}\zeta^{e_q} \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j,\alpha}} \tau^{\alpha_1-m^*} U^{g_i}_{h_i} \sum_{k\geq -h} \frac{1}{k!} \Psi_R^{([(k+k^*)/\delta+h/\gamma-h)/\gamma^*]+m^*)}(\chi)\Big(\frac{t}{\zeta}\Big)^k.\end{aligned}$$

Next consider the cases (ii) and (iii). Then as for the estimates (5.2) and (5.3) we have

$$\Big[\frac{|(h')_-|-e_q}{\gamma}-\frac{|(h')_-|}{\gamma^*}+l_q\Big]\le\Big[\frac{k^*}{\delta}+\frac{h}{\gamma}-\frac{h}{\gamma^*}\Big]+m^*-1,$$

Therefore by the same proposition, we have

$$\begin{aligned}(5.9)\quad & t^{\sigma_q}\prod_{j+|\alpha|\le m}\prod_{i=1}^{q_{j,\alpha}}\Big(\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha u_{h_i}^{g_i}(t,x)\\ &\ll C^{\sigma_q}\zeta^{e_q}R\prod_{j+|\alpha|\le m}\prod_{i=1}^{q_{j,\alpha}}\tau^{\alpha_1-m^*}U_{h_i}^{g_i}\\ &\quad\times\sum_{k\ge-(h)_-}\frac{1}{k!}\Psi_R^{([(k+k^*)/\delta+(h)_-/\gamma-(h)_-/\gamma^*]+m^*)}(\chi)\Big(\frac{t}{\zeta}\Big)^k.\end{aligned}$$

(B) If $h>0$, then we have $k_h\ge 0$ for the all cases.

In the case (i), we have

$$\Big[\frac{|(h')_-|-e_q}{\gamma}-\frac{|(h')_-|}{\gamma^*}+l_q\Big]\le\Big[\frac{-e_q}{\gamma}+l_q\Big]\le\Big[\frac{k^*}{\delta}+m^*-J_q\Big]\le\Big[\frac{k^*}{\delta}\Big]+m^*.$$

Therefore we have

$$\begin{aligned}& t^{\sigma_q}\prod_{j+|\alpha|\le m}\prod_{i=1}^{q_{j,\alpha}}\Big(\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha u_{h_i}^{g_i}(t,x)\\ &\ll C^{\sigma_q}\zeta^{e_q}\prod_{j+|\alpha|\le m}\prod_{i=1}^{q_{j,\alpha}}\tau^{\alpha_1-m^*}U_{h_i}^{g_i}\sum_{k\ge 0}\frac{1}{k!}\Psi_R^{([(k+k^*)/\delta]+m^*)}(\chi)\Big(\frac{t}{\zeta}\Big)^k.\end{aligned}$$

In the cases (ii) and (iii), as for the estimates (5.2) and (5.3) we have

$$\Big[\frac{|(h')_-|-e_q}{\gamma}-\frac{|(h')_-|}{\gamma^*}+l_q\Big]\le\Big[\frac{-e_q}{\gamma}+l_q\Big]\le\Big[\frac{k^*}{\delta}\Big]+m^*-1.$$

Therefore we have the estimate (5.9). Hence by Lemma 5.1, we have the estimate (5.6), where

$$\begin{aligned}U_h^g&=c(\tau)\sum_{e_q>e^*}\sum_{\substack{|g'|=g-(e_q-e^*)\\|h'|=h+(e_q-e^*)}}C^{\sigma_q}\zeta^{e_q-e^*}B_q\prod_{j+|\alpha|\le m}\prod_{i=1}^{q_{j,\alpha}}\tau^{\alpha_1-m^*}U_{h_i}^{g_i}\\ &\quad+c(\tau)\sum_{e_q<e^*}\sum_{\substack{|g'|=g\\|h'|=h-(e^*-e_q)}}C^{\sigma_q}\zeta^{e_q-e^*}RB_q\prod_{j+|\alpha|\le m}\prod_{i=1}^{q_{j,\alpha}}\tau^{\alpha_1-m^*}U_{h_i}^{g_i}\\ &\quad+c(\tau)\sum_{e_q=e^*}\sum_{\substack{|g'|=g\\|h'|=h-1}}C^{\sigma_q}RB_q\prod_{j+|\alpha|\le m}\prod_{i=1}^{q_{j,\alpha}}\tau^{\alpha_1-m^*}U_{h_i}^{g_i}.\end{aligned}$$

Next let us give an estimate of the coefficient U_h^g. Let us consider the following functional equation:

$$
\begin{aligned}
Z(t,s) &= c(\tau)\sum_{e_q>e^*} C^{\sigma_q} s^{e_q-e^*} B_q \prod_{j+|\alpha|\le m} \{\tau^{\alpha_1-m^*} Z(t,s)\}^{q_{j,\alpha}} \\
&\quad + c(\tau)\sum_{e_q<e^*} C^{\sigma_q} \zeta^{e_q-e^*} t^{e^*-e_q} B_q \prod_{j+|\alpha|\le m} \{\tau^{\alpha_1-m^*} Z(t,s)\}^{q_{j,\alpha}} \\
&\quad + c(\tau)\sum_{e_q=e^*} C^{\sigma_q} t B_q \prod_{j+|\alpha|\le m} \{\tau^{\alpha_1-m^*} Z(t,s)\}^{q_{j,\alpha}} + \zeta^{-e^*} c(\tau) F.
\end{aligned}
\tag{5.10}
$$

As in the case of (5.5), we have the holomorphic solution near $(t,s)=(0,0)$. Moreover substituting $Z(t,s) = \sum_{g\ge 0}\sum_{h\ge -g} Z_h^g t^h (ts)^g$ into (5.10), for any $T>0$ and $S>0$ we can take a sufficiently large $\tau>0$ and sufficiently small $R>0$ and $\zeta>0$ such that $U_h^g \le Z_h^g T^h (TS)^g$. Hence we have the desired result. □

Next for $g>0$ we construct a true solution to (E^g).

Set $\widetilde{u}^g(t,x) := \sum_{h\ge -g} u_h^g(t,x) = \sum_{h=-g}^{-1} u_h^g(t,x) + \widetilde{u}_0^g(t,x)$. Then we have

$$
\widetilde{u}_0^g(t,x) \ll \sum_{h\ge 0} \tau^{-m^*} U_h^g \sum_{k\ge k^*} \frac{1}{k!} \Psi_R^{([k/\delta])}(\chi) \Big(\frac{t}{\zeta}\Big)^k.
$$

By Proposition 5.3, $\sum_{h\ge 0} U_h^g$ converges. Further, if we set $\widetilde{U}_0^g := \sum_{h\ge 0} U_h^g$, then $\sum_{g\ge 0} \widetilde{U}_0^g$ also converges by Proposition 5.3. For $h<0$ set $\widetilde{u}_h^g(t,x) := u_h^g(t,x)$ and $\widetilde{U}_h^g := U_h^g$. Then we have

$$
\widetilde{u}^g(t,x) = \sum_{h=-g}^{0} \widetilde{u}_h^g(t,x) \ll \sum_{h=-g}^{0} \tau^{-m^*} \widetilde{U}_h^g \sum_{k\ge k^*-h} \frac{1}{k!} \Psi_R^{([k/\delta+h/\gamma-h/\gamma^*])}(\chi) \Big(\frac{t}{\zeta}\Big)^k.
$$

Therefore for each $g>0$, we can construct a true solution of (E^g) inductively as in the case $g=0$. Hence we have the following result for $g\ge 0$:

Proposition 5.4. *Under the assumptions $(C_2')_s$ and (A_1), there exists a true solution $u^g(t,x)$ to (E^g) such that for $|\theta| \le \pi/(2\gamma)$*

$$
u^g(t,x) \sim_\gamma \widetilde{u}^g(t,x) = \sum_{-g\le h\le 0} \widetilde{u}_h^g(t,x) = \sum_{-g\le h\le 0}\sum_{k\ge k^*-h} \widetilde{u}_{h,k}^g(x) t^k \quad \text{in} \quad S_\theta(T).
$$

Here $\widetilde{u}_{h,k}^g(x) \ll \tau^{-m^*} \widetilde{U}_h^g \frac{1}{k!\zeta^k} \Psi_R^{([k/\delta+h/\gamma-h/\gamma^*])}(\chi)$ and $\sum_{g\ge 0}\sum_{-g\le h\le 0} \widetilde{U}_h^g$ converges for a sufficiently large $\tau>0$ and sufficiently small $R>0$ and $\zeta>0$.

Proof of Theorem 3.2. Let us construct $u_{(h)}(t,x)$. We define

$$\widehat{u}_h^g(x,\xi) := \sum_{k\ge k^*-h} \frac{\widetilde{u}_{h,k}^g(x)}{\Gamma(k/\gamma+1)}\xi^{k/\gamma},$$

$$u_{h,S'}^g(t,x) := t^{-\gamma}\int_0^{\widehat{\xi}} \exp(-\xi t^{-\gamma})\,\widehat{u}_h^g(x,\xi)d\xi.$$

First let us estimate $\widehat{u}_h^g(x,\xi)$. By Proposition 5.4, there exist positive constants A and B such that

$$|\widehat{u}_h^g(x,\xi)| \le \tau^{-m^*}\widetilde{U}_h^g A^g B^h \frac{\Gamma(-h/\gamma^*+1)}{\Gamma(-h/\gamma+1)}|\xi|^{(k^*-h)/\gamma} \quad \text{for} \quad g\ge -h\ge 0.$$

Therefore we have

$$|u_{h,S'}^g(t,x)| \le \tau^{-m^*}\widetilde{U}_h^g A^g B^h \Gamma\Big(-\frac{h}{\gamma^*}+1\Big)\Gamma\Big(\frac{k^*}{\gamma}+1\Big)|t|^{k^*-h} \quad \text{for} \quad g\ge -h\ge 0.$$

Setting $u_{(h)}(t,x) := \sum_{g\ge -h} u_{h,S'}^g(t,x)$, we have

$$|u_{(h)}(t,x)| \le \tau^{-m^*}\sum_{g\ge -h}\widetilde{U}_h^g A^g B^h \Gamma\Big(-\frac{h}{\gamma^*}+1\Big)\Gamma\Big(\frac{k^*}{\gamma}+1\Big)|t|^{k^*-h} \quad \text{for} \quad h\le 0.$$

For a sufficiently large $\tau>0$ and sufficiently small $R>0$ and $\zeta>0$, we can show that $\sum_{g\ge -h}\widetilde{U}_h^g A^g$ converges. Set $\widetilde{U}_{(h)} := \sum_{g\ge -h}\widetilde{U}_h^g A^g$. Then $\sum_{h\le 0}\widetilde{U}_{(h)}t^{-h}$ also converges. This completes a proof of Theorem 3.2. □

References

[1] Ōuchi, S., Formal solutions with Gevrey type estimates of nonlinear partial differential equations, *J. Math. Sci. Univ. Tokyo* **1** (1994), 205–237.

[2] ———, Genuine solutions and formal solutions with Gevrey type estimates of nonlinear partial differential equations, *J. Math. Sci. Univ Tokyo* **2** (1995), 375–417.

[3] ———, Existence of solutions with Asymptotic expansion of linear partial differential equations in the complex domain, *Publ. Res. Inst. Math. Sci.* **40** (2004), 239–294.

[4] Yamazawa, H., On formal solution and genuine solution for some partial differential equations, *Microlocal Analysis and Related Topics* (H. Tahara, ed.), RIMS Kôkyûroku **1431**, 2005, pp. 176–186.