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Explicit construction of semi-stable models of Lubin-Tate curves with low level

By

Naoki Imai * and Takahiro Tsushima **

Abstract

In this paper, we construct semi-stable models of Lubin-Tate curves with level one or two, and determine their reductions. We also give a description of an action of a subgroup of the product group of the Weil group and the multiplicative group of the central division algebra of invariant 1/2 on the reductions.

Introduction

Let K be a non-archimedean local field with a finite residue field k of characteristic p. Let $\mathcal{O}_K$ be the ring of integers of K and $\mathfrak{p}$ the maximal ideal of $\mathcal{O}_K$. Let $v\colon K \to \mathbb{Z} \cup \{\infty\}$ be the valuation such that $v(\mathfrak{p}\backslash\mathfrak{p}^2) = 1$. We fix an algebraic closure of K, for which we write K^{ac}. Let K^{ur} denote the maximal unramified extension of K in K^{ac}. The completions of K^{ac} and K^{ur} are denoted by $\mathbf{C}$ and $\widehat{K}^{\mathrm{ur}}$ respectively. Let n be a natural number. We put

$$K_1(\mathfrak{p}^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_K) \;\middle|\; c \equiv 0,\ d \equiv 1 \bmod \mathfrak{p}^n \right\}.$$

Let $\mathbf{X}_1(\mathfrak{p}^n)$ be the connected Lubin-Tate curve with level $K_1(\mathfrak{p}^n)$ over $\widehat{K}^{\mathrm{ur}}$. Let W_K be the Weil group of K and D the central division algebra over K of invariant 1/2. Let

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*Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan.
e-mail: `naoki@ms.u-tokyo.ac.jp`
**Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan.
e-mail: `tsushima@ms.u-tokyo.ac.jp`

$\mathrm{Art}_K \colon K^\times \xrightarrow{\sim} W_K^{\mathrm{ab}}$ be the Artin reciprocity map normalized such that a uniformizer is sent to a lift of the geometric Frobenius. We write $v \colon W_K \to \mathbb{Z}$ for the composite

$$W_K \twoheadrightarrow W_K^{\mathrm{ab}} \xrightarrow{\mathrm{Art}_K^{-1}} K^\times \xrightarrow{v} \mathbb{Z}.$$

Let $\mathrm{Nrd}_{D/K} \colon D^\times \to K^\times$ be the reduced norm map. We set

$$(W_K \times D^\times)^0 = \{(\sigma, d) \in W_K \times D^\times \mid v(\mathrm{Nrd}_{D/K}(d)) + v(\sigma) = 0\}.$$

Then, $(W_K \times D^\times)^0$ acts on $\mathbf{X}_1(\mathfrak{p}^n)_{\mathbf{C}} = \mathbf{X}_1(\mathfrak{p}^n) \times_{\hat{K}^{\mathrm{ur}}} \mathbf{C}$ and induces an action on the stable reduction $\overline{\mathbf{X}_1(\mathfrak{p}^n)}$.

In this paper, we determine semi-stable coverings of $\mathbf{X}_1(\mathfrak{p}^n)$ for $n = 1, 2$ and calculate the action of $(W_K \times D^\times)^0$ on the reductions. A notion of semi-stable covering is due to Coleman in [CM]. It is known that we can construct a semi-stable model from a semi-stable covering (cf. [IT1, Theorem 3.5]).

There are a lot of researches on the stable models of modular curves. The calculation of the stable models in the modular curve setting is equivalent to that in the Lubin-Tate setting where $K = \mathbb{Q}_p$.

The stable reduction of the modular curve $X_0(p^2)$ is computed by Edixhoven in [Ed]. He finds an irreducible component, whose affine model is defined by $xy(x-y)^{p-1} = 1$. In the stable reduction of $\mathbf{X}_1(\mathfrak{p}^2)$, we will find a component, whose affine model is defined by $x^q y - xy^q = 1$. As for the calculation of the stable reduction of the modular curve $X_1(p^n)$, it is given in [DR] if $n = 1$. In [CM], Coleman-McMurdy calculate the stable reduction of the modular curve $X_0(p^3)$ under the assumption $p \geq 13$. In [IT2], we compute the stable reduction of $\mathbf{X}_1(\mathfrak{p}^3)$ by a similar method as in [CM]. However, detailed studies of the stable reduction of $\mathbf{X}_1(\mathfrak{p}^n)$ for $n \leq 2$ are not written anywhere. Therefore we write down the stable coverings of them in this paper.

The method in this paper is a bit different from that in [IT2]. In [IT2], we use a cohomological argument to show that a constructed covering is semi-stable. On the other hand, we show it by direct calculations in this paper. A merit of direct calculations is that we can determine the widths of open annuli in a constructed semi-stable covering.

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Notation

We fix some notations. We fix a uniformizer ϖ of K. Let $q = |k|$. Let $\mathcal{O}_{\mathbf{C}}$ be the ring of integers of $\mathbf{C}$ and k^{ac} the residue field of $\mathbf{C}$. For an element $a \in \mathcal{O}_{\mathbf{C}}$, we write $\bar{a}$ for the image of a by the reduction map $\mathcal{O}_{\mathbf{C}} \twoheadrightarrow k^{\mathrm{ac}}$. Let $v(\cdot)$ denote the normalized valuation of $\mathbf{C}$ such that $v(\varpi) = 1$. For $a, b \in \mathbf{C}$ and a rational number $\alpha \in \mathbb{Q}_{\geq 0}$, we write $a \equiv b \pmod{\alpha}$ if we have $v(a-b) \geq \alpha$, and $a \equiv b \pmod{\alpha+}$ if we have $v(a-b) > \alpha$. Let $\mathcal{O}_D$ be the ring of integers of D and $\mathfrak{p}_D$ the unique maximal left ideal of $\mathcal{O}_D$. For a smooth curve X over k^{ac}, we denote by X^{c} the smooth compactification of X, and the genus of X means the genus of X^{c}. For an affinoid $\mathbf{X}$, we write $\overline{\mathbf{X}}$ for its canonical reduction. The category of sets is denoted by **Set**.

§ 1. Preliminaries

§ 1.1. The universal deformation

Let Σ denote the formal $\mathcal{O}_K$-module of dimension 1 and height 2 over k^{ac}, which is unique up to isomorphism. Let n be a non-negative integer. We define $K_1(\mathfrak{p}^n)$ as in the introduction. In the following, we define the connected Lubin-Tate space $\mathbf{X}_1(\mathfrak{p}^n)$ with level $K_1(\mathfrak{p}^n)$.

Let $\mathcal{C}$ be the category of Noetherian complete local $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}$-algebras with residue field k^{ac}. For $A \in \mathcal{C}$, a formal $\mathcal{O}_K$-module $\mathcal{F} = \operatorname{Spf} A[[X]]$ over A and an A-valued point P of $\mathcal{F}$, the corresponding element of the maximal ideal of A is denoted by $x(P)$. We consider the functor

$$\mathcal{A}_1(\mathfrak{p}^n)\colon \mathcal{C} \to \mathbf{Set}; A \mapsto [(\mathcal{F}, \iota, P)],$$

where $\mathcal{F}$ is a formal $\mathcal{O}_K$-module over A with an isomorphism $\iota\colon \Sigma \simeq \mathcal{F} \otimes_A k^{\mathrm{ac}}$ and P is a ϖ^n-torsion point of $\mathcal{F}$ such that

$$\prod_{a \in \mathcal{O}_K/\mathfrak{p}^n} \bigl(X - x([a]_{\mathcal{F}}(P))\bigr) \,\Bigg|\, [\varpi^n]_{\mathcal{F}}(X)$$

in $A[[X]]$. This functor is represented by a regular local ring $\mathcal{R}_1(\mathfrak{p}^n)$ by [Dr, Lemma in p.572]. We write $\mathfrak{X}_1(\mathfrak{p}^n)$ for $\operatorname{Spf} \mathcal{R}_1(\mathfrak{p}^n)$. Its generic fiber is denoted by $\mathbf{X}_1(\mathfrak{p}^n)$, which we call the connected Lubin-Tate space with level $K_1(\mathfrak{p}^n)$. The space $\mathbf{X}_1(\mathfrak{p}^n)$ is a rigid analytic curve over $\widehat{K}^{\mathrm{ur}}$. We can define the Lubin-Tate space $\mathrm{LT}_1(\mathfrak{p}^n)$ of height 2 with level n by changing $\mathcal{C}$ to the category of $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}$-algebras where ϖ is nilpotent, and ι to a quasi-isogeny $\Sigma \otimes_{k^{\mathrm{ac}}} A/\mathfrak{p}A \to \mathcal{F} \otimes_A A/\mathfrak{p}A$. We consider $\mathrm{LT}_1(\mathfrak{p}^n)$ as a rigid analytic curve over $\widehat{K}^{\mathrm{ur}}$.

The ring $\mathcal{R}_1(1)$ is isomorphic to the ring of formal power series $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}[[u]]$. We simply write $\mathcal{B}(1)$ for $\operatorname{Spf} \mathcal{O}_{\widehat{K}^{\mathrm{ur}}}[[u]]$. Let $B(1)$ denote an open unit ball such that $B(1)(\mathbf{C}) = \{u \in \mathbf{C} \mid v(u) > 0\}$. The generic fiber of $\mathcal{B}(1)$ is equal to $B(1)$. Then, the space $\mathbf{X}_1(1)$ is identified with $B(1)$. Let $\mathcal{F}^{\mathrm{univ}}$ denote the universal formal $\mathcal{O}_K$-module over $\mathfrak{X}_1(1)$.

In this subsection, we choose a parametrization of $\mathfrak{X}_1(1) \simeq \mathcal{B}(1)$ such that the universal formal $\mathcal{O}_K$-module has a simple form. Let $\mathcal{F}$ be a formal $\mathcal{O}_K$-module of dimension 1 over a flat $\mathcal{O}_K$-algebra R. For an invariant differential $\omega \neq 0$ on $\mathcal{F}$, a logarithm of $\mathcal{F}$ means a unique isomorphism $F\colon \mathcal{F} \xrightarrow{\sim} \mathbb{G}_a$ over $R \otimes K$ with $dF = \omega$

(cf. [GH, 3]). In the sequel, we always take an invariant differential ω on $\mathcal{F}$ so that a logarithm F has the following form;

$$F(X) = X + \sum_{i \geq 1} f_i X^{q^i} \text{ with } f_i \in R \otimes K.$$

Let $F(X) = \sum_{i \geq 0} f_i X^{q^i} \in K[[u, X]]$ be the universal logarithm over $\mathcal{O}_K[[u]]$. By [GH, (5.5), (12.3), Proposition 12.10], the coefficients $\{f_i\}_{i \geq 0}$ satisfy $f_0 = 1$ and $\varpi f_i = \sum_{0 \leq j \leq i-1} f_j v_{i-j}^{q^j}$ for $i \geq 1$, where $v_1 = u$, $v_2 = 1$ and $v_i = 0$ for $i \geq 3$. Hence, we have the following;

$$f_0 = 1,\ f_1 = \frac{u}{\varpi},\ f_2 = \frac{1}{\varpi}\left(1 + \frac{u^{q+1}}{\varpi}\right),\ f_3 = \frac{1}{\varpi^2}\left(u + u^{q^2} + \frac{u^{q^2+q+1}}{\varpi}\right), \cdots . \tag{1.1}$$

By [GH, Proposition 5.7] or [Ha, 21.5], if we set

$$\mathcal{F}^{\mathrm{univ}}(X, Y) = F^{-1}(F(X) + F(Y)), \quad [a]_{\mathcal{F}^{\mathrm{univ}}}(X) = F^{-1}(aF(X)) \tag{1.2}$$

for $a \in \mathcal{O}_K$, it is known that these power series have coefficients in $\mathcal{O}_K[[u]]$ and define the universal formal $\mathcal{O}_K$-module $\mathcal{F}^{\mathrm{univ}}$ over $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}[[u]]$ of dimension 1 and height 2 with logarithm $F(X)$. We have the following approximation formula for $[\varpi]_{\mathcal{F}^{\mathrm{univ}}}(X)$.

Lemma 1.1. *Let the notation be as above. Then, we have*

$$[\varpi]_{\mathcal{F}^{\mathrm{univ}}}(X) \equiv \varpi X + uX^q + X^{q^2} \bmod (u\varpi X^q, uX^{q^2}, \varpi X^{q^2}, X^{q^2+1}).$$

Proof. This follows from a direct computation using the relation $F([\varpi]_{\mathcal{F}^{\mathrm{univ}}}(X)) = \varpi F(X)$ and (1.1). □

In the sequel, $\mathcal{F}^{\mathrm{univ}}$ means the universal formal $\mathcal{O}_K$-module with the identification $\mathfrak{X}_1(1) \simeq \mathcal{B}(1)$ given by (1.2), and we simply write $[a]_{\mathrm{u}}$ for $[a]_{\mathcal{F}^{\mathrm{univ}}}$. The reduction of (1.2) gives a simple model of Σ such that

$$X +_\Sigma Y = X + Y, \quad [\zeta]_\Sigma(X) = \bar{\zeta} X \text{ for } \zeta \in \mu_{q-1}(\mathcal{O}_K), \quad [\varpi]_\Sigma(X) = X^{q^2}. \tag{1.3}$$

We put

$$\mathfrak{A}_n = \mathcal{O}_{\widehat{K}^{\mathrm{ur}}}[[u, X_n]] / \big([\varpi^n]_{\mathrm{u}}(X_n)/[\varpi^{n-1}]_{\mathrm{u}}(X_n)\big).$$

Then there is a natural identification

$$\mathfrak{X}_1(\mathfrak{p}^n) \simeq \operatorname{Spf} \mathfrak{A}_n \tag{1.4}$$

that is compatible with the identification $\mathfrak{X}_1(1) \simeq \mathcal{B}(1)$. Note that, under (1.4), we have $x(P^{\mathrm{univ}}) = X_n$ for the level structure P^{univ} of the universal family over $\mathfrak{X}_1(\mathfrak{p}^n)$. The

Lubin-Tate space $\mathbf{X}_1(\mathfrak{p}^n)$ is identified with the generic fiber of the right hand side of (1.4). We write $\mathfrak{X}(1)$ for $\mathfrak{X}_1(1)$.

§ 1.2. Group action on Lubin-Tate curve

In this subsection, we recall a left action of $(W_K \times D^\times)^0$ on the space $\mathbf{X}_1(\mathfrak{p}^n)_{\mathbf{C}}$.

First we recall an action of $\mathcal{O}_D^\times$ on $\mathbf{X}_1(\mathfrak{p}^n)$. Let K_2 be the unramified quadratic extension of K. Let k_2 be the residue field of K_2, and $\sigma \in \mathrm{Gal}(K_2/K)$ be the non-trivial element. The ring of integers $\mathcal{O}_D$ in D has the following description; $\mathcal{O}_D = \mathcal{O}_{K_2} \oplus \varphi \mathcal{O}_{K_2}$ with $\varphi^2 = \varpi$ and $a\varphi = \varphi a^\sigma$ for $a \in \mathcal{O}_{K_2}$. The maximal ideal $\mathfrak{p}_D$ is generated by φ. We define an action of $\mathcal{O}_D$ on Σ by $\zeta(X) = \bar{\zeta} X$ for $\zeta \in \mu_{q^2-1}(\mathcal{O}_{K_2})$ and $\varphi(X) = X^q$. Then this gives an isomorphism $\mathcal{O}_D \simeq \mathrm{End}(\Sigma)$ by [GH, Proposition 13.10]. Note that the action of the subring $\mathcal{O}_K \subset \mathcal{O}_D$ on Σ coincides with the $\mathcal{O}_K$-multiplication (1.3) on Σ.

Let $d = d_1 + \varphi d_2 \in \mathcal{O}_D^\times$, where $d_1 \in \mathcal{O}_{K_2}^\times$ and $d_2 \in \mathcal{O}_{K_2}$. By the definition of the action of $\mathcal{O}_D$ on Σ, we have

$$d(X) \equiv \bar{d}_1 X + (\bar{d}_2 X)^q \bmod (X^{q^2}). \tag{1.5}$$

We take a lifting $\tilde{d}(X) \in \mathcal{O}_{K_2}[[X]]$ of $d(X) \in k_2[[X]]$. Let $\mathcal{F}_{\tilde{d}}$ be the formal $\mathcal{O}_K$-module defined by

$$\mathcal{F}_{\tilde{d}}(X,Y) = \tilde{d}\big(\mathcal{F}^{\mathrm{univ}}(\tilde{d}^{-1}(X), \tilde{d}^{-1}(Y))\big), \quad [a]_{\mathcal{F}_{\tilde{d}}}(X) = \tilde{d}\big([a]_{\mathrm{u}}(\tilde{d}^{-1}(X))\big)$$

for $a \in \mathcal{O}_K$. Then, we have an isomorphism

$$\tilde{d} \colon \mathcal{F}^{\mathrm{univ}} \xrightarrow{\sim} \mathcal{F}_{\tilde{d}};\ (u, X) \mapsto (u, \tilde{d}(X)).$$

By [GH, Proposition 14.7], the formal $\mathcal{O}_K$-module $\mathcal{F}_{\tilde{d}}$ with

$$\Sigma \xrightarrow{d^{-1}} \Sigma \xrightarrow{\iota} \mathcal{F}^{\mathrm{univ}} \otimes k^{\mathrm{ac}} \xrightarrow{\tilde{d} \otimes k^{\mathrm{ac}}} \mathcal{F}_{\tilde{d}} \otimes k^{\mathrm{ac}}$$

gives an isomorphism

$$d \colon \mathfrak{X}(1) \to \mathfrak{X}(1), \tag{1.6}$$

which is independent of a choice of a lifting $\tilde{d}$, such that there is the unique isomorphism

$$j \colon d^* \mathcal{F}^{\mathrm{univ}} \xrightarrow{\sim} \mathcal{F}_{\tilde{d}};\ (u, X) \mapsto (u, j(X))$$

satisfying $j(X) \equiv X \bmod (\varpi, u)$, where $d^* \mathcal{F}^{\mathrm{univ}}$ denotes the pull-back of $\mathcal{F}^{\mathrm{univ}}$ over $\mathfrak{X}(1)$ by the map (1.6). Hence, we have

$$[\varpi]_{d^* \mathcal{F}^{\mathrm{univ}}}(j^{-1}(X)) = j^{-1}([\varpi]_{\mathcal{F}_{\tilde{d}}}(X)). \tag{1.7}$$

On the other hand, we have an isomorphism

$$d^*\mathcal{F}^{\mathrm{univ}} \xrightarrow{\sim} \mathcal{F}^{\mathrm{univ}};\ (u, X') \mapsto (d(u), X').$$

Furthermore, under the identification (1.4), we consider an isomorphism

$$\psi_d \colon \mathfrak{X}_1(\mathfrak{p}^n) \longrightarrow \mathfrak{X}_1(\mathfrak{p}^n);\ (u, X_n) \mapsto \bigl(d(u), j^{-1}(\tilde{d}(X_n))\bigr), \tag{1.8}$$

which depends only on d as in [GH, Proposition 14.7]. We put $d^*(X) = j^{-1}(\tilde{d}(X))$. We define a left action of d on $\mathfrak{X}_1(\mathfrak{p}^n)$ by $[(\mathcal{F}, \iota, P)] \mapsto [(\mathcal{F}, \iota \circ d^{-1}, P)]$. Then this action coincides with ψ_d by the definition.

By (1.5), we have

$$\tilde{d}^{-1}(X) \equiv d_1^{-1}X - d_1^{-(q+1)} d_2^q X^q \bmod (\varpi, X^{q^2}) \tag{1.9}$$

in $\mathcal{O}_{K_2}[[X]]$.

Next, we recall a left action of $W'_K = \{(\sigma, \varphi^{-r_\sigma}) \in W_K \times D^\times\}$ on the space $\mathbf{X}_1(\mathfrak{p}^n)_{\mathbf{C}}$. The formal scheme $\mathfrak{A}_n$ is the base change to $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}$ of the formal scheme

$$\mathfrak{A}'_n = \mathcal{O}_K[[u, X_n]]/\bigl([\varpi^n]_{\mathrm{u}}(X_n)/[\varpi^{n-1}]_{\mathrm{u}}(X_n)\bigr).$$

Therefore an element of $\sigma \in W_K$ induces $(\sigma^{-1})^* : \mathbf{X}_1(\mathfrak{p}^n)_{\mathbf{C}} \to \mathbf{X}_1(\mathfrak{p}^n)_{\mathbf{C}}$. We define an action of $(\sigma, \varphi^{-r_\sigma}) \in W'_K$ by $(\sigma^{-1})^*$.

Remark 1.2. Usually, we define an action of $\sigma \in W_K$ on $\mathrm{LT}_1(\mathfrak{p}^n)_{\mathbf{C}}$ as the composite of $(\sigma^{-1})^*$ and φ^{r_σ}. Hence, the above definition is compatible with the usual one.

In the sequel, we simply write $\mathbf{X}_1(\mathfrak{p}^n)$ for $\mathbf{X}_1(\mathfrak{p}^n)_{\mathbf{C}}$, and sometimes consider the action of W'_K as the action of W_K by the identification $W_K \cong W'_K; \sigma \mapsto (\sigma, \varphi^{-r_\sigma})$.

§ 2. Semi-stable covering of $\mathbf{X}_1(\mathfrak{p})$

In this section, we construct a semi-stable covering of $\mathbf{X}_1(\mathfrak{p})$. In the subsection 2.1, we prove that $\mathbf{X}_1(\mathfrak{p})$ is an open annulus and, as a result, construct its semi-stable covering. In the subsection 2.2, we calculate the action of $(W_K \times D^\times)^0$ on the semi-stable reduction of $\mathbf{X}_1(\mathfrak{p})$.

§ 2.1. Analysis of the space $\mathbf{X}_1(\mathfrak{p})$

In this subsection, we prove that the space $\mathbf{X}_1(\mathfrak{p})$ is isomorphic to an open annulus of width $1/(q-1)$.

Lemma 2.1. *Let L be a complete subfield of $\mathbf{C}$ with a discrete valuation. Let $\mathbf{W}$ be a smooth rigid analytic space over L of dimension 1 and A an open annulus over L. Let $f\colon \mathbf{W} \to A$ be an L-morphism. Suppose that there is an admissible affinoid covering $\{U_i\}_{i\in I}$ of A such that $f^{-1}(U_i)$ is an affinoid for any $i \in I$. For any circle C in A, let L_C be a finite extension of L over which C reduces to $\mathbb{G}_m$. Suppose that, for any circle C in A, the inverse image $f^{-1}(C)$ is an affinoid and reduces to $\mathbb{G}_m$ over the field L_C, and furthermore f induces an isomorphism $\overline{f^{-1}(C)} \xrightarrow{\sim} \overline{C}$. Then, f is an isomorphism.*

Proof. In this proof, we regard rigid analytic spaces as adic spaces. We note that the quasi-compact admissible open subsets of a quasi-separated rigid space over a complete non-archimedean field correspond to the quasi-compact open subsets of the corresponding adic space (cf. [Hu, p. 43]).

Let C be any circle in A. Let A_C and $A_{f^{-1}(C)}$ be the affinoid algebras corresponding to C and $f^{-1}(C)$. We write A_C° and $A_{f^{-1}(C)}^\circ$ for the subalgebras consisting of the power bounded elements of A_C and $A_{f^{-1}(C)}$. Let $\mathfrak{p}_C$ be the maximal ideal of $\mathcal{O}_{L_C}$. By the assumption that $\overline{f^{-1}(C)}\to\overline{C}$ is an isomorphism, we can see that $A_C^\circ \to A_{f^{-1}(C)}^\circ$ is an isomorphism, because A_C° and $A_{f^{-1}(C)}^\circ$ are $\mathfrak{p}_C$-adic complete separated and torsion free over $\mathcal{O}_{L_C}$. Therefore, $f^{-1}(C) \to C$ is an isomorphism. Hence, we see that f becomes an isomorphism over an open neighborhood of any classical point of A. The morphism f is etale by [Hu, Proposition 1.7.11].

We take $i \in I$ and $x \in U_i$ arbitrarily. By [Hu, Lemma 2.2.8], there is a connected affinoid neighborhood U of x in U_i such that $f^{-1}(U) \to U$ factors through $f^{-1}(U) \to Z \to U$ where the first morphism is an open immersion and the second morphism is finite etale. Then it suffices to show that $f^{-1}(U) \to U$ is an isomorphism. We take a finite etale connected covering V of U such that $Z \times_U V$ decomposes into a finite disjoint union $\coprod_{j\in J} V_j$ where V_j is a copy of V for any $j \in J$. We note that $f^{-1}(U) \times_U V \to V$ becomes an isomorphism over an open neighborhood of any classical point of V. In particular, it induces a bijection between the classical points. Since the image of $f^{-1}(U) \times_U V \to V$ is a quasi-compact open subset of V and contains all classical points of V, the morphism $f^{-1}(U) \times_U V \to V$ is surjective.

We show that the image of the open immersion $f^{-1}(U) \times_U V \to Z \times_U V \cong \coprod_{j\in J} V_j$ is contained in V_j for some $j \in J$. Otherwise, $\{(f^{-1}(U) \times_U V) \cap V_j\}_{j\in J}$ gives a non-trivial quasi-compact open covering of V. For any different $j, j' \in J$, the intersection of $(f^{-1}(U) \times_U V) \cap V_j$ and $(f^{-1}(U) \times_U V) \cap V_{j'}$ is empty, because it is a quasi-compact open subset of V that contains no classical point. This contradicts the connectedness of V.

Hence, $f^{-1}(U) \times_U V \to V$ is an isomorphism, because it is a surjective open immersion. By faithfully flat descent, we see that $f^{-1}(U) \to U$ is an isomorphism. $\square$

Let (u, X_1) be the parameter on $\mathbf{X}_1(\mathfrak{p})$ under the identification (1.4). For rational numbers $0 \le a < b$, let $A(p^{-b}, p^{-a})$ denote an open annulus such that $A(p^{-b}, p^{-a})(\mathbf{C}) = \{x \in \mathbf{C} \mid a < v(x) < b\}$. The width of an open annulus $A(p^{-b}, p^{-a})$ means $b - a$. For a rational number $r \ge 0$, let $C[p^{-r}]$ be a circle such that $C[p^{-r}](\mathbf{C}) = \{x \in \mathbf{C} \mid v(x) = r\}$. A closed annulus in $A(p^{-b}, p^{-a})$ means an affinoid defined by $a' \le v(x) \le b'$ for the parameter x of $A(p^{-b}, p^{-a})$ and $a < a' < b' < b$.

Lemma 2.2. *We have the following:*
1. If $v(X_1) < 1/(q^2-1)$, we have $v(X_1) = v(u)/(q(q-1))$ and $v(u) < q/(q+1)$.
2. If $v(X_1) = 1/(q^2-1)$, we have $v(u) \ge q/(q+1)$.
3. If $v(X_1) > 1/(q^2-1)$, we have $v(X_1) = (1-v(u))/(q-1)$ and $v(u) < q/(q+1)$.
4. We have $v(X_1) < 1/(q-1)$.
5. We define $f\colon \mathbf{X}_1(\mathfrak{p}) \to A(p^{-\frac{1}{q-1}}, 1)$ by $(u, X_1) \mapsto X_1$. Then, f is an isomorphism.

Proof. We prove 1. By $v(X_1) < 1/(q^2-1)$, we have $v(u) = v(X_1^{q(q-1)})$ by Lemma 1.1 and hence $v(u) < q/(q+1)$. Therefore, the assertion 1 follows. The assertions 2 and 3 also follow immediately from Lemma 1.1. The assertion 4 follows from 3.

We prove 5. First we note that the inverse image under f of any closed affinoid is affinoid by the assertions 1, 2 and 3. We choose a rational number $0 < r < 1/(q-1)$. Set $C = C[p^{-r}]$. Now, we compute the reduction of the inverse image $f^{-1}C$ and the induced map $\bar{f}\colon \overline{f^{-1}C} \to \overline{C}$ by f. Let $\alpha \in \mathcal{O}_{\mathbf{C}}$ be an element such that $v(\alpha) = r$. We define β by

$$\beta = \begin{cases} \alpha^{q(q-1)} & \text{if } 0 < r \le 1/(q^2-1), \\ \varpi/\alpha^{q-1} & \text{if } 1/(q^2-1) < r < 1/(q-1). \end{cases}$$

On $f^{-1}C$, we set $u = \beta u_0$ and $X_1 = \alpha x_1$. Then, by Lemma 1.1, on $f^{-1}C$, the function u_0 is written as follows;

$$u_0 \equiv \begin{cases} -x_1^{q(q-1)} & (\mathrm{mod}\ 0+) \quad \text{if } 0 < r < 1/(q^2-1), \\ -x_1^{q(q-1)} - c x_1^{-(q-1)} & (\mathrm{mod}\ 0+) \quad \text{if } r = 1/(q^2-1), \\ -x_1^{-(q-1)} & (\mathrm{mod}\ 0+) \quad \text{if } 1/(q^2-1) < r < 1/(q-1), \end{cases} \tag{2.1}$$

with some unit $c \in \mathcal{O}_{\mathbf{C}}$. Hence, the reduction of $f^{-1}C$ is isomorphic to $\mathbb{G}_m = \mathrm{Spec}\, k^{\mathrm{ac}}[x_1^{\pm 1}]$ by (2.1). We also have $\bar{f}\colon \overline{f^{-1}C} \simeq \mathbb{G}_m \to \overline{C} \simeq \mathbb{G}_m = \mathrm{Spec}\, k^{\mathrm{ac}}[x_1^{\pm 1}]; x_1 \mapsto x_1$. Therefore, f is an isomorphism by Lemma 2.1. □

Let $\mathbf{X}$ be the affinoid in $\mathbf{X}_1(\mathfrak{p})$ defined by $v(X_1) = 1/(q^2-1)$. By the proof of Lemma 2.2, the reduction of $\mathbf{X}$ is isomorphic to $\mathbb{G}_m$. Then, the space $\mathbf{X}_1(\mathfrak{p})$ is a basic wide open space with the underlying affinoid $\mathbf{X}$ by Lemma 2.2. The covering $\mathcal{C}_1(\mathfrak{p}) = \{(\mathbf{X}_1(\mathfrak{p}), \mathbf{X})\}$ is a semi-stable covering. A semi-stable covering determines a semi-stable model by [IT1, Theorem 3.5]. The reduction of the semi-stable model of $\mathbf{X}_1(\mathfrak{p})$ determined by $\mathcal{C}_1(\mathfrak{p})$ is isomorphic to $\mathbb{P}^1$.

§ 2.2. Group action on the reduction $\overline{\mathbf{X}}$

In this subsection, we give a description of the action of $(W_K \times D^\times)^0$ on the reduction $\overline{\mathbf{X}}$.

Let κ be an element such that $\kappa^{q^2-1} = \varpi$. On $\mathbf{X}$, we change a variable by $X_1 = \kappa x_1$ with $v(x_1) = 0$. Then, we identify the reduction $\overline{\mathbf{X}}$ with $\mathbb{G}_m = \mathrm{Spec}\ k^{\mathrm{ac}}[x_1^{\pm 1}]$.

Lemma 2.3. *Let $d \in \mathcal{O}_D^\times$. We write $\bar{d}$ for the image of d by the canonical surjection $\mathcal{O}_D^\times \longrightarrow (\mathcal{O}_D/\mathfrak{p}_D)^\times \simeq k_2^\times$. Then, the element d acts on the reduction $\overline{\mathbf{X}}$ as follows;*

$$\overline{\mathbf{X}} \to \overline{\mathbf{X}};\ x_1 \mapsto \bar{d}x_1.$$

Proof. By (1.9) and $j(X) \equiv X \pmod{(\varpi, u)}$, we obtain $d^* x_1 \equiv \bar{d}x_1 \pmod{0+}$. Hence, the required assertion follows. □

Remark 2.4. Let $\lambda \in (\mathcal{O}_K/\mathfrak{p}^n)^\times$. Let λ act on $\mathfrak{X}_1(\mathfrak{p}^n)$ by $[\lambda]\colon [(\mathcal{F}, \iota, P)] \mapsto [(\mathcal{F}, \iota, [\lambda]_{\mathcal{F}}(P))]$, which is called the diamond operator. This action of $[\lambda]$ corresponds to the action of $\lambda \in \mathcal{O}_K^\times \subset \mathcal{O}_D^\times$ given in the subsection 1.2. Hence, for $\lambda \in (\mathcal{O}_K/\mathfrak{p})^\times$, the action of $[\lambda]$ on the reduction $\overline{\mathbf{X}}$ is computed by Proposition 2.3.

In the following, we briefly recall an action of the Weil group W_K on the reduction of an affinoid over $\mathbf{C}$, and compute the action of W_K on the reduction $\overline{\mathbf{X}}$.

Let $\mathbf{Z}$ be a reduced affinoid over $\mathbf{C}$ with an action of W_K. For $P \in \mathbf{Z}(\mathbf{C})$, the image of P under the natural reduction map $\mathbf{Z}(\mathbf{C}) \to \overline{\mathbf{Z}}(k^{\mathrm{ac}})$ is denoted by $\overline{P}$. The action of W_K on $\overline{\mathbf{Z}}$ is a homomorphism $w_{\mathbf{Z}}\colon W_K \to \mathrm{Aut}(\overline{\mathbf{Z}})$ characterized by $\overline{\sigma(P)} = w_{\mathbf{Z}}(\sigma)(\overline{P})$ for $\sigma \in W_K$. For $\sigma \in W_K$, we simply write r_σ for $v(\sigma) \in \mathbb{Z}$ defined in the introduction.

Lemma 2.5. *Let $\sigma \in W_K$. Let κ be an element such that $\kappa^{q^2-1} = \varpi$. We write $\sigma(\kappa) = \zeta_\sigma \kappa$ with $\zeta_\sigma \in \mu_{q^2-1}(\mathbf{C})$. Then, we have*

$$w_{\mathbf{X}}(\sigma)\colon \overline{\mathbf{X}} \to \overline{\mathbf{X}};\ x_1 \mapsto \bar{\zeta}_\sigma x_1^{q^{-r_\sigma}}.$$

Proof. Let $P \in \mathbf{X}(\mathbf{C})$. By $X_1 = \kappa x_1$, we have $\kappa x_1(\sigma(P)) = X_1(\sigma(P)) = \sigma(X_1(P)) = \sigma(\kappa)\sigma(x_1(P))$. Hence, we acquire $x_1(\sigma(P)) = \zeta_\sigma \sigma(x_1(P))$. Since we have $\sigma(x_1(P)) \equiv x_1(P)^{q^{-r_\sigma}} \pmod{0+}$, we obtain $x_1(\sigma(P)) \equiv \zeta_\sigma x_1(P)^{q^{-r_\sigma}} \pmod{0+}$. Hence, the assertion follows. □

Let $\mathbb{Z}$ act on $k_2^\times$ by $n\colon \zeta \mapsto \zeta^{q^n}$ for $n \in \mathbb{Z}$. This action factors through the canonical surjection $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$. We consider the semidirect product $k_2^\times \rtimes \mathbb{Z}$ with respect to this action and the homomorphism

$$\Theta\colon (W_K \times D^\times)^0 \twoheadrightarrow k_2^\times \rtimes \mathbb{Z}; (\sigma, d\varphi^{-r_\sigma}) \mapsto (\bar{\zeta}_\sigma \bar{d}, r_\sigma).$$

Corollary 2.6. *Let the notation be as above. Furthermore, let* $k_2^\times \rtimes \mathbb{Z}$ *act on* $\overline{\mathbf{X}}$ *by* $(\zeta, r)\colon \overline{\mathbf{X}} \to \overline{\mathbf{X}}; x_1 \mapsto \zeta x_1^{q^{-r}}$ *for* $(\zeta, r) \in k_2^\times \rtimes \mathbb{Z}$. *Then, the group* $(W_K \times D^\times)^0$ *acts on* $\overline{\mathbf{X}}$ *through* Θ.

Proof. The required assertion follows from Lemma 2.3 and Lemma 2.5. □

§ 3. Stable covering of $\mathbf{X}_1(\mathfrak{p}^2)$

In this section, we calculate the stable reduction of $\mathbf{X}_1(\mathfrak{p}^2)$. In the subsection 3.1, we define an affinoid subspace $\mathbf{Y}_{1,1} \subset \mathbf{X}_1(\mathfrak{p}^2)$ and compute its reduction. The reduction of $\mathbf{Y}_{1,1}$ is defined by

$$x^q y - xy^q = 1$$

with genus $q(q-1)/2$. In the subsection 3.2, the space $\mathbf{X}_1(\mathfrak{p}^2)$ is proved to be a basic wide open space with the underlying affinoid $\mathbf{Y}_{1,1}$. Namely, the complement $\mathbf{X}_1(\mathfrak{p}^2)\backslash\mathbf{Y}_{1,1}$ is a disjoint union of open annuli. We also determine the width of each open annulus. In the subsection 3.3, we compute the action of $(W_K \times D^\times)^0$ on $\overline{\mathbf{Y}}_{1,1}$.

§ 3.1. The reduction of the affinoid $\mathbf{Y}_{1,1}$

In this subsection, we define the subspace $\mathbf{Y}_{1,1}$ in the Lubin-Tate curve $\mathbf{X}_1(\mathfrak{p}^2)$ and compute its reduction in Proposition 3.2.

Let (u, X_2) be the parameters on $\mathbf{X}_1(\mathfrak{p}^2)$ under the identification (1.4).

Definition 3.1. We define an affinoid subspace $\mathbf{Y}_{1,1} \subset \mathbf{X}_1(\mathfrak{p}^2)$ by the following conditions;

$$v(u) = \frac{1}{q+1},\ v(X_1) = \frac{q}{q^2-1},\ v(X_2) = \frac{1}{q(q^2-1)}.$$

In the following, we compute the reduction of the space $\mathbf{Y}_{1,1}$ defined above. We choose an element γ such that $\gamma^{q(q^2-1)} = \varpi$. On $\mathbf{Y}_{1,1}$, we change variables by $u = \gamma^{q(q-1)} u_0$, $X_1 = \gamma^{q^2} x_1^{-1}$, $X_2 = \gamma x_2$ with $v(u_0) = v(x_1) = v(x_2) = 0$. By $[\varpi]_{\mathrm{u}}(X_1) = 0$, $[\varpi]_{\mathrm{u}}(X_2) = X_1$, and Lemma 1.1, we acquire congruences

$$u_0 \equiv -x_1^{q-1} \pmod{0+}, \tag{3.1}$$

$$x_1^{-1} \equiv x_2^{q^2} + u_0 x_2^q \pmod{0+}. \tag{3.2}$$

By substituting (3.1) to the right hand side of (3.2), we have

$$1 \equiv x_1 x_2^{q^2} - (x_1 x_2)^q \pmod{0+}. \tag{3.3}$$

Finally, changing a variable by

$$t = \frac{1 + x_1 x_2}{x_2^q}, \tag{3.4}$$

we acquire $x_1 \equiv t^q \pmod{0+}$ by (3.3). Hence, by (3.4), the reduction $\overline{\mathbf{Y}}_{1,1}$ is defined by

$$x_2^q t - x_2 t^q = 1.$$

Hence, we have proved the following proposition.

Proposition 3.2. *Let the notation be as above. Then, the reduction of the affinoid space $\mathbf{Y}_{1,1}$ is defined by $x_2^q t - x_2 t^q = 1$.*

§ 3.2. Analysis of the space $\mathbf{X}_1(\mathfrak{p}^2)$

In this subsection, we analyze the wide open space $\mathbf{X}_1(\mathfrak{p}^2)$. As a result, we will prove that the space $\mathbf{X}_1(\mathfrak{p}^2)$ is a basic wide open space with the underlying affinoid $\mathbf{Y}_{1,1}$. In other words, the complement $\mathbf{X}_1(\mathfrak{p}^2)\backslash\mathbf{Y}_{1,1}$ is a disjoint union of open annuli.

Let (u, X_2) be the parameter on $\mathbf{X}_1(\mathfrak{p}^2)$ as before. By comparing the valuations of the main terms X_1, $X_2^{q^2}$, uX_2^q and ϖX_2 of the equation $X_1 = [\varpi]_{\mathrm{u}}(X_2)$, we define several subspaces in $\mathbf{X}_1(\mathfrak{p}^2)$ below.

Definition 3.3. Let the notation be as above.
1. Let $\mathbf{W}_0$ denote a subspace of $\mathbf{X}_1(\mathfrak{p}^2)$ defined by $v(X_1) = v(uX_2^q) < v(X_2^{q^2})$.
2. Let $\mathbf{W}_{k^\times}$ denote a subspace of $\mathbf{X}_1(\mathfrak{p}^2)$ defined by $v(X_2^{q^2}) = v(uX_2^q) < v(X_1)$.
3. Let $\mathbf{W}_\infty$ denote a subspace of $\mathbf{X}_1(\mathfrak{p}^2)\backslash\mathbf{Y}_{1,1}$ defined by $v(X_1) = v(X_2^{q^2})$.

We have $v(X_1) > v(u)$ on $\mathbf{W}_0 \cup \mathbf{W}_{k^\times}$. Hence, by considering $[\varpi]_{\mathrm{u}}(X_1) = 0$, we have $v(\varpi X_1) = v(uX_1^q) < v(X_1^{q^2})$ on the union.

Lemma 3.4. *Let the notation be as above. Then, we have $\mathbf{X}_1(\mathfrak{p}^2) = \mathbf{W}_0 \cup \mathbf{W}_{k^\times} \cup \mathbf{W}_\infty \cup \mathbf{Y}_{1,1}$.*

Proof. Let $(u, X_2) \in \mathbf{X}_1(\mathfrak{p}^2)\backslash\mathbf{Y}_{1,1}$. We consider $X_1 = [\varpi]_{\mathrm{u}}(X_2)$. Since we have $v(X_1) < 1/(q-1)$ by Lemma 2.2.4, we acquire $(u, X_2) \in \mathbf{W}_0 \cup \mathbf{W}_{k^\times} \cup \mathbf{W}_\infty$. Hence, the required assertion follows. □

Lemma 3.5. *Let the notation be as above.*
1. We have $v(u) < 1/(q+1)$ on $\mathbf{W}_{k^\times}$.
2. We have $v(X_2) < 1/(q(q^2-1))$ on $\mathbf{W}_\infty$.
3. If $v(X_2) < 1/q^2(q^2-1)$, we have $v(X_2) = v(u)/(q^3(q-1))$ and $v(u) < q/(q+1)$ on $\mathbf{W}_\infty$.
4. If $v(X_2) = 1/q^2(q^2-1)$, we have $v(u) \geq q/(q+1)$ on $\mathbf{W}_\infty$.
5. If $v(X_2) > 1/q^2(q^2-1)$, we have $v(X_2) = (1-v(u))/q^2(q-1)$ and $v(u) < q/(q+1)$ on $\mathbf{W}_\infty$.

Proof. We prove 1. As mentioned before, we have $v(X_1) = \frac{1-v(u)}{q-1}$ on $\mathbf{W}_{k^\times}$. By the definition of $\mathbf{W}_{k^\times}$, we have $v(X_2) = v(u)/(q(q-1))$ on $\mathbf{W}_{k^\times}$. By $v(X_2^{q^2}) < v(X_1)$, we obtain $v(u) < 1/(q+1)$ on $\mathbf{W}_{k^\times}$.

We prove 2 by contradiction. Assume that there exists a point $P \in \mathbf{W}_\infty$ such that $v(X_2(P)) \geq 1/(q(q^2-1))$. By $v(X_1(P)) = v(X_2(P)^{q^2})$, we acquire $v(X_1(P)) \geq q/(q^2-1)$. In particular, we have $v(\varpi X_1(P)) < v(X_1(P)^{q^2})$. Hence, by considering $[\varpi]_{\mathrm{u}}(X_1) = 0$, we acquire $v(\varpi X_1(P)) = v(u(P)X_1(P)^q)$ and hence $v(X_1(P)) = (1 - v(u(P)))/(q-1)$. By $v(X_1(P)) \geq q/(q^2-1)$, we obtain

$$v(u(P)) \leq 1/(q+1) \tag{3.5}$$

and $v(X_2(P)) = (1 - v(u(P)))/(q^2(q-1))$. In particular, we have $v(u(P)X_2(P)^q) < v(\varpi X_2(P))$. Hence, we acquire $v(X_2(P)^{q^2}) \leq v(u(P)X_2(P)^q)$ by considering $[\varpi]_{\mathrm{u}}(X_2) = X_1$. Therefore, we have $v(X_2(P)) = (1 - v(u(P)))/(q^2(q-1)) \leq v(u(P))/(q(q-1))$. This implies that $v(u(P)) \geq 1/(q+1)$. By (3.5), we acquire $v(u(P)) = 1/(q+1)$ and hence $v(X_1(P)) = q/(q^2-1)$ and $v(X_2(P)) = 1/(q(q^2-1))$. This means that $P \in \mathbf{Y}_{1,1}$. Since we have $\mathbf{W}_\infty \cap \mathbf{Y}_{1,1} = \emptyset$, this is contradiction.

Since $v(X_1) = v(X_2^{q^2})$ on $\mathbf{W}_\infty$, the assertions 3, 4 and 5 follow from 1, 2 and 3 of Lemma 2.2. $\square$

Proposition 3.6. *Let the notation be as above. Then, the space $\mathbf{W}_{k^\times}$ has $q-1$ connected components $\{\mathbf{W}_\zeta\}_{\zeta \in k^\times}$. Furthermore, the spaces $\{\mathbf{W}_a\}_{a \in k \cup \{\infty\}}$ are open annuli with width $1/(q(q^2-1))$.*

Proof. We choose a q^2-th root of ϖ, for which we write ϖ^{1/q^2}. We prove the required assertion for $\mathbf{W}_0$. On $\mathbf{W}_0$, we have

$$v(X_1) = \frac{1-v(u)}{q-1},\ v(X_2) = \frac{1-qv(u)}{q(q-1)},\ v(u) < 1/(q+1). \tag{3.6}$$

We consider $X_1^{-q} \times [\varpi]_{\mathrm{u}}(X_1) = 0$. By Lemma 1.1 and $v(X_1^{q(q-1)}), v(u\varpi) > v(X_2^{q(q-1)})$, we obtain

$$u \equiv -\varpi/X_1^{q-1} \pmod{v(X_2^{q(q-1)})+} \tag{3.7}$$

on $\mathbf{W}_0$. By $X_1^{q-1} \times ([\varpi]_{\mathrm{u}}(X_2) - X_1) = 0$, Lemma 1.1 and $v(u\varpi X_2^q) > v(X_2^{q^2})$, we have

$$X_1^q - uX_1^{q-1}X_2^q \equiv X_1^{q-1}X_2^{q^2} + \varpi X_1^{q-1}X_2 \pmod{v(X_1^{q-1}X_2^{q^2})+} \tag{3.8}$$

on $\mathbf{W}_0$. We set $t = X_1 + \varpi^{1/q}X_2$. By substituting (3.7) to the left hand side of (3.8), we acquire

$$t^q \equiv X_1^{q-1}X_2^{q^2} + \varpi X_1^{q-1}X_2 \pmod{v(X_1^{q-1}X_2^{q^2})+} \tag{3.9}$$

on $\mathbf{W}_0$. Hence, we have

$$v(t) \geq b = \min\{v(X_1^{q-1}X_2^{q^2}), v(\varpi X_1^{q-1}X_2)\}/q. \tag{3.10}$$

Note that, by (3.6), we have

$$v(X_1^{q-1}X_2^{q^2}) = \frac{2q-1-(q^2+q-1)v(u)}{q-1}, \quad v(\varpi X_1^{q-1}X_2) = \frac{2q^2-2q+1-q^2v(u)}{q(q-1)}$$

on $\mathbf{W}_0$. Hence, by the definition of b in (3.10), we have

$$v(\varpi^{(2q-3)/q}X_2^{q-2}) + 2b = \min\left\{\frac{2q^2-1}{q(q-1)} - \frac{3q^2-2}{q(q-1)}v(u), \frac{2q^3-3q+2}{q^2(q-1)} - \frac{q}{q-1}v(u)\right\}$$

on $\mathbf{W}_0$. We can directly check that the right hand side is greater than $v(X_1^{q-1}X_2^{q^2})$ on $\mathbf{W}_0$. Therefore, by (3.10), we obtain $v(\varpi^{(2q-3)/q}X_2^{q-2}t^2) > v(X_1^{q-1}X_2^{q^2})$. Therefore, we obtain

$$\begin{aligned}\varpi X_1^{q-1}X_2 = \varpi(-\varpi^{1/q}X_2+t)^{q-1}X_2 &\equiv \varpi\left(\varpi^{(q-1)/q}X_2^{q-1} - (q-1)\varpi^{(q-2)/q}X_2^{q-2}t\right)X_2 \\ &\equiv \varpi^{(2q-1)/q}X_2^{q} + \varpi^{2(q-1)/q}X_2^{q-1}t \pmod{v(X_1^{q-1}X_2^{q^2})+}\end{aligned}$$

on $\mathbf{W}_0$. Substituting this to the right hand side of (3.9), we acquire

$$t^q \equiv X_1^{q-1}X_2^{q^2} + \varpi^{(2q-1)/q}X_2^{q} + \varpi^{2(q-1)/q}X_2^{q-1}t \pmod{v(X_1^{q-1}X_2^{q^2})+} \tag{3.11}$$

on $\mathbf{W}_0$. We choose an element $\beta \in \mathcal{O}_{\mathbf{C}}$ such that $\beta^q = \varpi^{(2q-1)/q} + \varpi^{2(q-1)/q}\beta$. Note that we have $v(\beta) = \frac{2q-1}{q^2}$ and $v(\varpi X_1^{q-1}X_2) = qv(\beta X_2)$ on $\mathbf{W}_0$. Hence, we have $v(\beta X_2) \geq b$ on $\mathbf{W}_0$. We put $t_1 = t - \beta X_2$. By (3.10), we obtain $v(t_1) \geq b$ on $\mathbf{W}_0$. By $v(p) + qb \geq 1 + qb \geq v(X_1^{q-1}X_2^{q^2})$ on $\mathbf{W}_0$, we acquire

$$t^q \equiv t_1^q + (\beta X_2)^q \pmod{v(X_1^{q-1}X_2^{q^2})+}$$

on $\mathbf{W}_0$. Substituting this and $t = t_1 + \beta X_2$ to (3.11) and using the equality $\beta^q = \varpi^{(2q-1)/q} + \varpi^{2(q-1)/q}\beta$, we acquire

$$t_1^q \equiv X_1^{q-1}X_2^{q^2} + \varpi^{2(q-1)/q}X_2^{q-1}t_1 \pmod{v(X_1^{q-1}X_2^{q^2})+} \tag{3.12}$$

on $\mathbf{W}_0$. We prove $v(t_1) = v(X_1^{q-1}X_2^{q^2})/q$ by contradiction. First, assume $v(t_1) < v(X_1^{q-1}X_2^{q^2})/q$. By (3.12), we have $v(t_1^q) = v(\varpi^{2(q-1)/q}X_2^{q-1}t_1)$. Hence, we obtain $v(t_1) = v(\varpi^{2/q}X_2)$. But, this contradicts the assumption $v(t_1) < v(X_1^{q-1}X_2^{q^2})/q$. Secondly, we assume $v(t_1) > v(X_1^{q-1}X_2^{q^2})/q$. By (3.12), we have $v(X_1^{q-1}X_2^{q^2}) = v(\varpi^{2(q-1)/q}X_2^{q-1}t_1)$ and hence

$$v(t_1) = \frac{2q-1}{q(q-1)} - \frac{q^2}{q-1}v(u).$$

But, this contradicts the assumption $qv(t_1) > v(X_1^{q-1}X_2^{q^2})$. Therefore, we have $v(t_1) = v(X_1^{q-1}X_2^{q^2})/q$ and hence $v(\varpi^{2(q-1)/q}X_2^{q-1}t_1) > v(t_1^q)$ on $\mathbf{W}_0$. Therefore, by (3.12), we obtain

$$t_1^q \equiv X_1^{q-1}X_2^{q^2} \pmod{v(X_1^{q-1}X_2^{q^2})+} \tag{3.13}$$

on $\mathbf{W}_0$. We set $f(X_1, X_2) = X_1 + (\varpi^{1/q} - \beta)X_2$ and

$$t_2 = X_1 X_2^q f(X_1, X_2)^{-1}. \tag{3.14}$$

By (3.13) and (3.14), we obtain

$$t_2^q \equiv X_1 \pmod{v(X_1)+} \tag{3.15}$$

on $\mathbf{W}_0$. Hence, we can define a morphism

$$f\colon \mathbf{W}_0 \to A(p^{-1/(q(q-1))}, p^{-1/(q^2-1)}); (u, X_2) \mapsto t_2 = X_1 X_2^q f(X_1, X_2)^{-1}.$$

We show that f is an isomorphism. First note that the inverse image under f of any closed affinoid is affinoid by (3.6) and (3.15). Let $1/(q^2-1) < r < 1/(q(q-1))$ be a rational number and $C = C[p^{-r}]$. By Lemma 2.1, it suffices to show that the induced map $\bar{f}\colon \overline{f^{-1}C} \to \overline{C}$ is an isomorphism. Let α be an element such that $v(\alpha) = r$. On $f^{-1}C$, we set $u = \varpi\alpha^{-q(q-1)}u_0$, $X_1 = \alpha^q x_1$, $X_2 = \alpha^q \varpi^{-(1/q)} x_2$ and $t_2 = \alpha t_0$. Then, by (3.7), (3.14) and (3.15), we acquire

$$u_0 \equiv -x_1^{-(q-1)}, \quad x_1 \equiv -x_2, \quad x_1 \equiv t_0^q \pmod{0+}.$$

Hence, the reduction of $f^{-1}C$ is isomorphic to $\mathbb{G}_m = \operatorname{Spec} k^{\mathrm{ac}}[t_0^{\pm 1}]$. Therefore, the required assertion follows.

We prove the required assertion for $\mathbf{W}_\infty$. By Lemma 3.5.2, we can define a morphism $f\colon \mathbf{W}_\infty \to A(p^{-1/(q(q^2-1))}, 1)$ by $(u, X_2) \mapsto X_2$. We show that f is an isomorphism. First note that the inverse image under f of any closed affinoid is affinoid by 3, 4 and 5 of Lemma 3.5. We choose a rational number $0 < r < 1/(q(q^2-1))$ and set $C = C[p^{-r}]$. By Lemma 2.1, it suffices to show that the induced map $\bar{f}\colon \overline{f^{-1}C} \to \overline{C}$ is an isomorphism.

Let α be an element such that $v(\alpha) = r$. We define β as follows;

$$\beta = \begin{cases} \alpha^{q^3(q-1)} & \text{if } 0 < r \le 1/(q^2(q^2-1)), \\ \varpi/\alpha^{q^2(q-1)} & \text{if } 1/(q^2(q^2-1)) < r < 1/(q(q^2-1)). \end{cases}$$

On $f^{-1}C$, we change variables by $u = \beta u_0$, $X_1 = \alpha^{q^2} x_1$ and $X_2 = \alpha x_2$. Then, on $f^{-1}C$, we obtain the same relationships between u_0 and x_1 as (2.1). Furthermore, by Lemma 1.1, we acquire $x_1 \equiv x_2^{q^2} \pmod{0+}$. Hence, the reduction of $f^{-1}C$ is isomorphic to

$\mathbb{G}_m = \operatorname{Spec} k^{\mathrm{ac}}[x_2^{\pm 1}]$ and the map $\bar{f}\colon \overline{f^{-1}C} \to \overline{C} \simeq \mathbb{G}_m = \operatorname{Spec} k^{\mathrm{ac}}[x_2^{\pm 1}]$ is given by $x_2 \mapsto x_2$. Hence, the required assertion follows.

We prove the required assertion for $\mathbf{W}_{k^\times}$. We choose a $(q-1)$-th root of ϖ, for which we write $\varpi^{1/(q-1)}$. Note that we have $v(X_1) = (1-v(u))/(q-1)$ and $v(X_2) = v(u)/(q(q-1))$. By $X_1^{-q} \times [\varpi]_{\mathrm{u}}(X_1) = 0$ and Lemma 1.1, we obtain

$$u \equiv -\varpi/X_1^{q-1} \pmod{v(u)+}. \tag{3.16}$$

By $X_1^q \times ([\varpi]_{\mathrm{u}}(X_2) - X_1) = 0$, Lemma 1.1 and Lemma 3.5.1, we acquire

$$(X_1 X_2^q)^q + u(X_1X_2)^q \equiv 0 \pmod{q/(q-1)+} \tag{3.17}$$

on $\mathbf{W}_{k^\times}$. Note that

$$v(u) + qv(X_1X_2) = v(u) + q\left(\frac{1}{q-1} - \frac{v(u)}{q}\right) = \frac{q}{q-1}$$

on $\mathbf{W}_{k^\times}$. Hence, by substituting (3.16) to (3.17), we obtain

$$(X_1X_2^q)^q - \varpi X_1 X_2^q \equiv 0 \pmod{q/(q-1)+}$$

on $\mathbf{W}_{k^\times}$. Hence, on $\mathbf{W}_{k^\times}$, we acquire

$$X_1X_2^q \equiv \zeta\varpi^{1/(q-1)} \pmod{1/(q-1)+} \tag{3.18}$$

with some $\zeta \in \mu_{q-1}(\mathcal{O}_K)$. We write $\mathbf{W}_\zeta$ for the subspace of $\mathbf{W}_{k^\times}$ on which we have (3.18). We define a morphism $f\colon \mathbf{W}_\zeta \to A(p^{-1/(q(q^2-1))}, 1)$ by $(u, X_2) \mapsto X_2$. Then, we can prove that f is an isomorphism in the same way as the case for $\mathbf{W}_\infty$. We omit the details. □

By Proposition 3.2 and Proposition 3.6, the space $\mathbf{X}_1(\mathfrak{p}^2)$ is a basic wide open space with the underlying affinoid $\mathbf{Y}_{1,1}$. Since the reduction of $\mathbf{Y}_{1,1}$ has positive genus by Proposition 3.2, the covering $\mathcal{C}_1(\mathfrak{p}^2) = \{(\mathbf{X}_1(\mathfrak{p}^2), \mathbf{Y}_{1,1})\}$ is a stable covering. The reduction of the stable model of $\mathbf{X}_1(\mathfrak{p}^2)$ determined by $\mathcal{C}_1(\mathfrak{p}^2)$ is isomorphic to $\overline{\mathbf{Y}}_{1,1}^{\mathrm{c}}$.

§ 3.3. Group action on the reduction $\overline{\mathbf{Y}}_{1,1}$

In this subsection, we calculate the action of $(W_K \times D^\times)^0$ on the reduction $\overline{\mathbf{Y}}_{1,1}$. First, we compute the action of $\mathcal{O}_D^\times$ on the reduction $\overline{\mathbf{Y}}_{1,1}$.

Lemma 3.7. *Let $d \in \mathcal{O}_D^\times$. We write $\bar{d}$ for the image of d by the canonical surjection $\mathcal{O}_D^\times \longrightarrow (\mathcal{O}_D/\mathfrak{p}_D)^\times \simeq k_2^\times$. Then, the element d acts on the reduction $\overline{\mathbf{Y}}_{1,1}$ as follows;*

$$\overline{\mathbf{Y}}_{1,1} \to \overline{\mathbf{Y}}_{1,1};\ (x_2, t) \mapsto (\bar{d}x_2, \bar{d}^{-q}t).$$

Proof. By (1.9) and $j(X) \equiv X \pmod{(\varpi, u)}$, we obtain

$$d^* x_1 \equiv \bar{d}^{-1} x_1,\ d^* x_2 \equiv \bar{d} x_2 \pmod{0+}.$$

By (3.4), we have $d^* t \equiv \bar{d}^{-q} t \pmod{0+}$. Hence, the required assertion follows. □

Remark 3.8. Let $\lambda \in (\mathcal{O}_K/\mathfrak{p}^2)^\times$. Then, the action of the diamond operator $[\lambda]$ on $\overline{\mathbf{Y}}_{1,1}$ is calculated by Lemma 3.7 by the same reason as Remark 2.4.

Secondly, we compute the action of W_K on the component $\overline{\mathbf{Y}}_{1,1}$.

Lemma 3.9. *Let $\sigma \in W_K$. We choose γ as in the subsection 3.1. We write $\sigma(\gamma) = \xi_\sigma \gamma$ with $\xi_\sigma \in \mu_{q(q^2-1)}(\mathbf{C})$. Then, we have*

$$w_{\mathbf{Y}_{1,1}}(\sigma) \colon \overline{\mathbf{Y}}_{1,1} \to \overline{\mathbf{Y}}_{1,1};\ (x_2, t) \mapsto (\bar{\xi}_\sigma x_2^{q^{-r_\sigma}}, \bar{\xi}_\sigma^{-q} t^{q^{-r_\sigma}}).$$

Proof. We use the same notation as in the subsection 3.1. Let $P \in \mathbf{Y}_{1,1}(\mathbf{C})$. Since we set $X_2 = \gamma x_2$, we have $\gamma x_2(\sigma(P)) = X_2(\sigma(P)) = \sigma(X_2(P)) = \sigma(\gamma)\sigma(x_2(P))$. Hence, by $\sigma(x_2(P)) \equiv x_2(P)^{q^{-r_\sigma}} \pmod{0+}$, we acquire $x_2(\sigma(P)) \equiv \xi_\sigma x_2(P)^{q^{-r_\sigma}} \pmod{0+}$. In the same way, we obtain $x_1(\sigma(P)) \equiv \xi_\sigma^{-1} x_1(P)^{q^{-r_\sigma}} \pmod{0+}$. By (3.4), acquire $t(\sigma(P)) \equiv \xi_\sigma^{-q} t(P)^{q^{-r_\sigma}} \pmod{0+}$. Hence, the assertion follows. □

Let the notation be as in Lemma 3.7 and Lemma 3.9. Let $k_2^\times \rtimes \mathbb{Z}$ be the semidirect product as in the previous section. We consider the homomorphism

$$\Theta' \colon (W_K \times D^\times)^0 \twoheadrightarrow k_2^\times \rtimes \mathbb{Z}; (\sigma, d\varphi^{-r_\sigma}) \mapsto (\bar{\xi}_\sigma \bar{d}, r_\sigma).$$

Corollary 3.10. *Let the notation be as above. Furthermore, let $k_2^\times \rtimes \mathbb{Z}$ act on $\overline{\mathbf{Y}}_{1,1}$ by $(\xi, r) \colon \overline{\mathbf{Y}}_{1,1} \to \overline{\mathbf{Y}}_{1,1}; (x_2, t) \mapsto (\xi x_2^{q^{-r}}, \xi^{-q} t^{q^{-r}})$ for $(\xi, r) \in k_2^\times \rtimes \mathbb{Z}$. Then, the group $(W_K \times D^\times)^0$ acts on $\overline{\mathbf{Y}}_{1,1}$ through Θ'.*

Proof. The required assertion follows from Lemma 3.7 and Lemma 3.9. □

References

[CM] R. Coleman and K. McMurdy, *Stable reduction of* $X_0(p^3)$, Algebra and Number Theory **4** (2010), no. 4 357–431, with an appendix by Everette W. Howe.

[DR] P. Deligne and M. Rapoport, *Les Schémas de modules de courbes elliptiques*, Lecture Notes in Math. **349** (1973), 143–316.

[Dr] V. G. Drinfeld, *Elliptic modules*, Mat. Sb. (N.S.) **94**(136) (1974), 594-627.

[Ed] B. Edixhoven, *Minimal resolution and the stable reduction of* $X_0(N)$, Ann. Inst. Fourier(Grenoble) **40** (1990), no. 1, 31-67.

[GH] B. Gross and M. Hopkins, *Equivariant vector bundles on the Lubin-Tate moduli space*, Topology and Representation Theory (Evanston, IL, 1992), Contemp. Math. **158** (1994), 23–88.

[Ha] M. Hazewinkel, *Formal groups and applications*, Academic Press, 1978.

[Hu] R. Huber, *Étale cohomology of rigid analytic varieties and adic spaces*, Aspects of Mathematics, E30, Friedr. Vieweg & Sohn, Braunschweig, 1996.

[IT1] N. Imai and T. Tsushima, *Cohomology of rigid curves with semi-stable coverings*, to appear in Asian J. Math., available at arXiv:1109.2824.

[IT2] N. Imai and T. Tsushima, *Stable models of Lubin-Tate curves with level three*, preprint, arXiv:1111.1893.