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Albanese varieties, Suslin homology and Rojtman's theorem

By

Thomas GEISSER*

Abstract

We recall the definition of the Albanese variety and of Suslin homology, and discuss generalizations of Rojtman's theorem, which states that the torsion of the Albanese variety is isomorphic to the torsion of the 0th Suslin homology for certain varieties over algebraically closed fields.

§ 1. Introduction

A classical theorem of Abel and Jacobi states that for a smooth projective curve C over an algebraically closed field k, two finite formal sums of points $D = \sum_i n_i p_i$ and $E = \sum_j m_j q_j$ with $n_i, m_j \in \mathbb{N}$ and $p_i, q_j \in C$ are the zeros and poles of a function f on C if and only if $\sum n_i = \sum m_j$ and the sums $\sum_i n_i[p_i]$ and $\sum_j m_j[q_j]$, taken in the Jacobian variety Jac_C, agree. In modern language, this means that there is an isomorphism between the degree zero part of the Chow group of zero-cycles and the rational points of the Jacobian variety

$$CH_0(C)^0 \xrightarrow{\sim} Jac_C(k).$$

For a smooth and proper scheme X of arbitrary dimension, the natural generalization is to replace the Jacobian variety by the Albanese variety (the universal object for morphisms from X to abelian varieties), and to study the Albanese map

$$alb_X : CH_0(X)^0 \to Alb_X(k).$$

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Supported by JSPS
*Graduate School of Mathematics, Nagoya University, Furocho, Nagoya 464-8602, Japan.
e-mail: `geisser@math.nagoya-u.ac.jp`

This map is surjective, but it was shown by Mumford that it cannot be an isomorphism in general: For smooth and proper surfaces, the Chow group on the left hand side is too big to be captured by an abelian variety in general. However, Rojtman [12] proved that alb_X induces an isomorphism of torsion subgroups away from the characteristic.

Rojtman's theorem has been generalized to smooth (open) schemes having a smooth projective model by Spiess-Szamuely [16]. Here the Chow group has to be replaced by Suslin homology, and the right hand side by Serre's Albanese variety, an extension of an abelian variety by a torus:

$$alb_X : {}_{\mathrm{tor}}H_0^S(X,\mathbb{Z}) \to {}_{\mathrm{tor}}Alb_X(k).$$

The same statements holds for the p-part in characteristic p under resolution of singularities ([4] based on [9]). Recently, this result was generalized to all normal schemes:

Theorem 1.1 ([5]). *Let X be a reduced normal scheme, separated and of finite type over an algebraically closed field k of characteristic $p \geq 0$. Then the Albanese map induces an isomorphism*

$${}_{\mathrm{tor}}H_0^S(X,\mathbb{Z}) \xrightarrow{\sim} {}_{\mathrm{tor}}Alb_X(k)$$

up to p-torsion groups, and $H_1^S(X,\mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z}$ is a p-torsion group. Under resolution of singularities, the restriction on the characteristic is unnecessary.

The theorem has been proved previously in characteristic 0 by Barbieri-Viale and Kahn [1, Cor.14.5.3]. In particular, this removes the hypothesis on the existence of a smooth projective model in the theorem of Spiess-Szamuely. The aim of this paper is to give an introduction to the ingredients of this theorem, and to sketch its proof.

Throughout the paper, we assume that the base field is algebraically closed.

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§ 2. Universal group schemes

We denote by Sch$/k$ the category of separated schemes of finite type over k. For a scheme X in Sch$/k$, it is natural to ask if there is a map from X to a group scheme over k which is universal for maps from X to a certain class of group schemes. Since this is well-defined only up to translation in the group scheme, we rigidify this by fixing a base point $x \in X$ and requiring that x maps to the origin in the group scheme. In general, there is no such universal group scheme, but in [14], Serre proved the following theorem:

Theorem 2.1 ([14, Theoreme 7]). *Let $\mathcal{C}$ be a category of reduced commutative group schemes over k closed under products and extensions with finite kernel. Assume* *that $\mathcal{C}$ does not contain the additive group $\mathbb{G}_a$. Then for every pointed reduced scheme (X, x), there is universal map to an object in $\mathcal{C}$.*

§ 2.1. The Albanese variety

The most important application of this is to take the categories of abelian varieties or semi-abelian varieties (a semi-abelian variety is an extension of an abelian variety A by a torus $T \cong \mathbb{G}_m^r$). In this case, the universal semi-abelian variety $u : X \to A_X$ is called the *Albanese variety* of X. Note that the image of X generates A_X in the sense that there is no semi-abelian subvariety containing the image of X (for otherwise, this subvariety would also have the universal property).

Remark. 1) Classically, the Albanese variety was the universal abelian variety and not the universal semi-abelian variety. But if X is proper, it is easy to see that the universal semi-abelian variety is in fact an abelian variety, so that the universal semi-abelian variety is a good generalization of the universal abelian variety from proper schemes to arbitrary schemes.

2) There is another object sometimes called Albanese variety, which is the universal object for *rational* maps (i.e. morphisms defined on a dense open subset) to abelian varieties. If X is smooth, the two concepts agree (as any rational map to a group scheme can be extended on a smooth scheme).

3) If X is not reduced, there is no universal homomorphism to semi-abelian varieties. For example, every map from the scheme $\operatorname{Spec} k[t]/t^2$ to $\mathbb{G}_m$ can be dominated by an isogeny $\mathbb{G}_m \to \mathbb{G}_m$, so there is no initial object.

If $\bar{X}$ is smooth projective variety, $D_1, \dots, D_r$ integral subschemes of codimension 1 and $X = \bar{X} - \bigcup D_i$, then Serre gave the following explicit construction of the Albanese variety A_X [15]:

Let I be the free abelian group with generators D_i, and J be the kernel of the map

$$I \to \operatorname{Pic}(\bar{X}) \to \operatorname{NS}(\bar{X}) := \operatorname{Pic}(\bar{X}) / \operatorname{Pic}^0(\bar{X})$$

sending a generator 1_{D_i} to the class of $\mathcal{O}(-D_i)$ in the Picard group $\operatorname{Pic}(\bar{X})$, with $\operatorname{Pic}^0(\bar{X})$ the subgroup of line bundles algebraically equivalent to 0. By definition of J, there is a map $\theta : J \to \operatorname{Pic}^0(\bar{X})$. Then J is a free abelian group of finite rank, and we let $S_X = \operatorname{Hom}(J, \mathbb{G}_m)$ be the torus with character group J. Let $\bar{X} \to A_{\bar{X}}$ be the Albanese variety of $\bar{X}$ (an abelian variety, since $\bar{X}$ is proper). From the duality of the Albanese variety and the Picard variety (for smooth and projective schemes), one has $\operatorname{Hom}(\mathbb{Z}, \operatorname{Pic}^0(\bar{X})) \cong \operatorname{Pic}^0(\bar{X}) \cong \operatorname{Ext}(A_{\bar{X}}, \mathbb{G}_m)$ and hence $\operatorname{Hom}(J, \operatorname{Pic}^0(\bar{X})) \cong \operatorname{Ext}(A_{\bar{X}}, S_X)$. Thus the element θ defines an extension of the abelian variety $A_{\bar{X}}$ by the torus S_X. This is the Albanese variety of X [15, Theoreme 1].

§ 2.2. Example: The maximal torus of a scheme

We can also apply Serre's theorem to the category $\mathcal{C}$ of tori (i.e. group schemes of the form $\mathbb{G}_m^r$), and obtain the maximal torus $v_X : X \to T_X$ of a scheme. We can recover the maximal torus from the Albanese variety:

Theorem 2.2. *Let X be a reduced scheme and $u_X : X \to A_X$ be the Albanese variety of X. Then the maximal torus of X and of A_X agree.*

Proof. From the universal property we obtain the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{v_X} & T_X \\ {\scriptstyle u_X}\downarrow & & {\scriptstyle \alpha}\downarrow \\ A_X & \xrightarrow{v_{A_X}} & T_{A_X}. \end{array}$$

The map α is necessarily surjective (because the image of X generates A_X and the image of A_X generates T_{A_X}). By the universal property of A_X, there is a diagonal map $\beta : A_X \to T_X$ with $\beta u_X = v_X$ (which is surjective because the image of X generates T_X). Then by the universal property of T_{A_X}, there is a map $\gamma : T_{A_X} \to T_X$ with $\gamma v_{A_X} = \beta$. Combining this, we get $v_X = \gamma v_{A_X} u_X = \gamma\alpha v_X$. Since the image of v_X generates T_X, we obtain that $\gamma\alpha = \mathrm{id}$. But α is surjective, so γ and α are mutually inverse isomorphisms. $\square$

The theorem implies that the maximal torus T_X of X agrees with the torus in the maximal split quotient of the Albanese variety A_X of X. If X is smooth, we can see from the above description of the Albanese variety that the character group of T_X is the kernel J' of θ, as one sees from the diagram

$$\begin{array}{ccc} \theta \in \mathrm{Ext}(A_{\bar{X}}, S_X) & = & \mathrm{Hom}(J, \mathrm{Pic}^0(X)) \\ \downarrow & & \downarrow \\ 0 \in \mathrm{Ext}(A_{\bar{X}}, T_X) & = & \mathrm{Hom}(J', \mathrm{Pic}^0(X)). \end{array}$$

§ 2.3. The Albanese scheme

Serre observed that in order to get rid of the dependence on the base point, it is more natural not to consider universal maps to semi-abelian varieties, but to consider universal maps to torsors under semi-abelian varieties. Recall that a torsor under a group scheme G is a scheme Z together with an operation $\sigma : G \times Z \to Z$ such that the map $(\sigma, p_2) : G \times Z \to Z \times Z$ is an isomorphism This idea has been extended by Ramachandran [11], see also Kahn-Sujatha [8]. A *locally semi-abelian scheme* $\mathcal{A}$ is a commutative group scheme such that the connected component $\mathcal{A}^0$ is a semi-abelian variety, and $\pi_0(\mathcal{A})$ is a lattice D. Recall that for a scheme X/k, the scheme of components $\pi_0(X)$ of X is the spectrum of the largest etale extension of k in $\mathcal{O}_X(X)$. For example, if X is connected, $\pi_0(X)$ is just $\operatorname{Spec} k$. Ramachandran gave the following construction. Let Z_X be the sheaf on the big flat site $(\mathrm{Sch}/k)_{fl}$ associated to the free abelian group on the presheaf represented by X, $U \mapsto \mathbb{Z}[\operatorname{Hom}_k(U, X)]$. The *albanese scheme* is the universal object for morphisms from Z_X to sheaves represented by locally semi-abelian schemes. For reduced schemes of finite type over k, the Albanese scheme exists [11, Thm.1.1]. The image of $\mathrm{id}_X \in Z_X(X)$ in $\mathcal{A}_X(X)$ gives the universal morphism $u_X : X \to \mathcal{A}_X$, and the assignment $X \to \mathcal{A}_X$ is a covariant functor. It is also contravariant for finite flat maps [16]. There is an exact sequence

$$0 \to \mathcal{A}_X^0 \to \mathcal{A}_X \to D_X \to 0,$$

and D_X is the sheaf (for the flat topology) associated to the presheaf $T \mapsto \mathbb{Z}\operatorname{Hom}(T, \pi_0(X))$. For a connected scheme X with base point x_0, the connected component $\mathcal{A}_X^0$ is isomorphic to the usual Albanese variety A_X, because the map $X \to \mathcal{A}_X^0$, $x \mapsto u_X(x) - u_X(x_0)$ factors through A_X by the universal property.

§ 3. Suslin homology

Suslin homology is an analog of singular homology of topological spaces, and an important invariant of schemes, see [4], [17]. It is defined as follows: Let $\Delta^i = \operatorname{Spec} k[t_0, \ldots, t_i]/(1 - \sum_j t_j)$ be the algebraic i-simplex and $C_i(X)$ be the free abelian group on closed integral subschemes of $X \times_k \Delta^i$ which are finite and surjective over Δ^i. The alternating pull-back to faces $\Delta^{i-1} \subseteq \Delta^i$ defined by $t_j = 0$ makes this a complex of free abelian groups. For an abelian group A, the Suslin homology $H_i^S(X, A)$ with coefficients in A is the ith homology of the complex $C_*(X) \otimes A$. In particular, $H_0^S(X, \mathbb{Z})$ is the free abelian group $C_0(X)$ on the closed points of X modulo the relation defined by $Z \cap (X \times \{0\}) - Z \cap (X \times \{1\})$ for $Z \subseteq X \times \Delta^1$ finite and surjective over Δ^1. If X is proper, then $H_0^S(X, \mathbb{Z})$ agrees with the Chow group of zero cycles $CH_0(X)$, because then any closed $Z \subseteq X \times \Delta^1$ is automatically proper over Δ^1. But for non-proper schemes, Suslin homology is better behaved for our purposes. Let $H_0^S(X, \mathbb{Z})^0$ be the kernel of the canonical degree map $H_0^S(X, \mathbb{Z}) \to H_0^S(\pi_0(X), \mathbb{Z}) \cong D_X$.

Suslin homology appears in a wide variety of arithmetic applications. For example, Schmidt and Spiess [13] show that for a smooth variety over a finite field, the tame geometric abelianized fundamental group agrees with the finite group $H_0^S(X, \mathbb{Z})^0$. If X is not smooth, then a similar results is expected to hold for a modified version of Suslin homology [4]. Over an algebraically closed field, $H_1^S(X, \mathbb{Z}/m)$ surjects onto the tame abelian fundamental group modulo m, and this map is an isomorphism if the characteristic does not divide m or if resolution of singularities holds [7].

Another useful property is that over an algebraically closed field and with finite coefficients $\mathbb{Z}/m$ prime to the characteristic, Suslin-homology $H_i(X,\mathbb{Z}/m)$ is dual to etale cohomology $H^i_{\text{ét}}(X,\mathbb{Z}/m)$ [17].

Lemma 3.1. *The Albanese map $u_X : X \to \mathcal{A}_X$ induces a map from Suslin-homology to the Albanese scheme $H^S_0(X,\mathbb{Z}) \to \mathcal{A}_X(k)$ such that the following diagram commutes:*

$$\begin{tikzcd} 0 \arrow[r] & H_0^S(X,\mathbb{Z})^0 \arrow[r] \arrow[d] & H_0^S(X,\mathbb{Z}) \arrow[r] \arrow[d, "alb_X"'] & D_X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{A}_X^0(k) \arrow[r] & \mathcal{A}_X(k) \arrow[r] & D_X \arrow[r] & 0. \end{tikzcd} \tag{3.1}$$

§ 4. Rojtman's theorem

Rojtman proved in [12] that for a smooth and projective scheme over an algebraically closed field of characteristic 0, the Albanese map induces an isomorphism on torsion groups. A cohomological proof was later given by Bloch [2], and Milne proved the same statement for the p-part in characteristic p. The idea of the cohomological proof for smooth projective X is as follows (see [16]). For m prime to the characteristic of k, consider the following map of short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & H_1^S(X,\mathbb{Z})\otimes\mathbb{Z}/m \arrow[r] & H_1^S(X,\mathbb{Z}/m) \arrow[r] \arrow[d, equal] & {}_mH_0^S(X,\mathbb{Z}) \arrow[r] \arrow[d, "alb"'] & 0 \\ 0 \arrow[r] & C_m \arrow[r] & H^1_{\text{ét}}(X,\mathbb{Z}/m)^* \arrow[r] & \mathrm{Hom}_{GS}(\mu_m, \mathrm{Pic}_X^{0,red})^* \arrow[r] & 0 \end{tikzcd}$$

The middle vertical map is the above mentioned duality isomorphism of Suslin-Voevodsky [17]. The middle lower group is isomorphic to $\mathrm{Hom}_{GS}(\mu_m,\mathrm{Pic}_X)^*$, which is the dual of the group of homomorphisms of group schemes from the kernel μ_m of multiplication by m on $\mathbb{G}_m$ to the Picard scheme. The dual of the group of homomorphisms of group schemes of μ_m to the reduced part of the connected component of the Picard scheme $\mathrm{Hom}_{GS}(\mu_m,\mathrm{Pic}_X^{0,red})^*$is a quotient group of $\mathrm{Hom}_{GS}(\mu_m,\mathrm{Pic}_X)^*$, and by duality of Albanese and Picard variety, Cartier-Nishi duality gives

$$\mathrm{Hom}_{GS}(\mu_m, {}_m\mathrm{Pic}_X^{0,red})^* \cong \mathrm{Hom}_{GS}({}_mAlb_X,\mathbb{Z}/m)^* \cong {}_mAlb_X(k).$$

Now one shows that the right square commutes (this is non-trivial), and that the groups $C_m = \mathrm{Hom}_{GS}(\mu_m, NS_X)^*$ vanish in the colimit over m because their order is bounded independently of m, so that the map $C_m \to C_{mn}$ becomes multiplication by n for large m. Hence the lower right map becomes an isomorphism in the colimit, and $H_1^S(X,\mathbb{Z})\otimes\mathbb{Q}/\mathbb{Z} = 0$. Spiess-Szuamely [16] showed that the same argument works for smooth X

even if X is not proper, and we showed that the same argument works for X proper and normal (by proving a duality theorem [3] replacing the duality theorem of Suslin-Voevodsky). Assuming resolution of singularities, this also holds for p-torsion, p the characteristic of k if X is smooth, because neither side changes if we replace X by a smooth and proper model [4]. The obvious generalization of all the above is to prove the theorem for arbitrary normal X. However, the above argument does not work, because it is not clear if the C_m are finite. To get around this, we work with (truncated) hypercoverings.

§ 4.1. Hypercoverings

A proper hypercovering $X_\bullet \to X$ is a simplicial scheme $X_\bullet$ such that the maps $X_{n+1} \to (\mathrm{cosk}_n X_\bullet)_{n+1}$ are proper and surjective. We only need truncated hypercoverings (i.e. we need X_n only for small n). For example, a 1-truncated hypercover of X is a diagram

$$X_1 \underset{\delta_1}{\overset{\delta_0}{\longrightarrow}} X_0 \xrightarrow{a} X$$

such that $a\delta_0 = a\delta_1$, such that $(\delta_0, \delta_1) : X_1 \to X_0 \times_X X_0 = (\mathrm{cosk}_0 X_\bullet)_1$ is proper and surjective, together with a section $s : X_0 \to X_1$ to δ_0 and δ_1.

Theorem 4.1. *Let X be a reduced, normal, connected variety, and $a : X_\bullet \to X$ be a 1-truncated proper hypercovering of X such that $X_0 \to X$ is generically etale. Then the Albanese scheme $\mathcal{A}_X$ of X is the largest locally semi-abelian scheme quotient of the sheaf $\mathcal{A}_{X_0}/d\mathcal{A}_{X_1}$, where $d = (\delta_0)_* - (\delta_1)_*$.*

The statement of the theorem is true even if X is only semi-normal, but can be wrong if X is not semi-normal.

Consider a 2-truncated simplicial scheme $X_\bullet$ together with their corresponding map of locally semi-abelian schemes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{A}^0_{X_2} & \longrightarrow & \mathcal{A}_{X_2} & \longrightarrow & D_{X_2} & \longrightarrow & 0 \\
 & & \downarrow & & {\scriptstyle d}\downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{A}^0_{X_1} & \longrightarrow & \mathcal{A}_{X_1} & \longrightarrow & D_{X_1} & \longrightarrow & 0 \\
 & & \downarrow & & {\scriptstyle d}\downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{A}^0_{X_0} & \longrightarrow & \mathcal{A}_{X_0} & \longrightarrow & D_{X_0} & \longrightarrow & 0
\end{array} \tag{4.1}$$

where d is the alternating sum of the maps induced by the face maps $X_i \to X_{i-1}$. Let $\mathrm{Tor}_i(\mathcal{A}_{X_\bullet}(k), \mathbb{Q}/\mathbb{Z})$ be the hyper-Tor with $\mathbb{Q}/\mathbb{Z}$-coefficients of the complex of abelian groups $\mathcal{A}_{X_\bullet}(k)$. We claim that the homology $H_0(D_{X_\bullet})$ is free. Indeed, we can assume that X is connected, choose a component $x_0 \subseteq X_0$, and have to show that $D_{X_1} \rightrightarrows$

$D_{X_0} \xrightarrow{a} D_X \cong \mathbb{Z}$ is exact in the middle. But the kernel of a is generated by $1_x - 1_{x_0}$ for connected components $x \subseteq X_0$. By hypothesis, $X_1 \to X_0 \times_X X_0$ is surjective, so we can find a connected component $x' \subseteq X_1$ which maps to x under the first projection, and to x_0 under the second projection.

By freeness of $H_0(D_{X_\bullet})$, the exact sequence of k-rational points

$$H_1(D_{X_\bullet}) \xrightarrow{\delta} \mathcal{A}^0_{X_0}(k)/d\mathcal{A}^0_{X_1}(k) \to \mathcal{A}_{X_0}(k)/d\mathcal{A}_{X_1}(k) \to H_0(D_{X_\bullet}) \to 0 \tag{4.2}$$

gives an isomorphism of abelian groups

$$_{\mathrm{tor}}\big(\mathcal{A}^0_{X_0}(k)/(d\mathcal{A}^0_{X_1}(k) + \operatorname{im}\delta)\big) \cong {}_{\mathrm{tor}}\big(\mathcal{A}_{X_0}(k)/d\mathcal{A}_{X_1}(k)\big).$$

A hypercover is called l-hyperenvelope, if for all points of the target of $X_{n+1} \to (\mathrm{cosk}_n X_\bullet)_{n+1}$ there is a point mapping to it such that the extension of residue fields is finite of order prime to l if $l \neq p$, and trivial if $l = p$, respectively. Such l-hyperenvelopes exist for $l \neq p$ using Gabber's refinement of de Jong's theorem, and under resolution of singularities if $l = p$. Using the result of Spiess-Szamuely, we can then prove:

Proposition 4.2. *Let $X_\bullet$ be a 2-truncated proper l-hyperenvelope of X which is contained as an open subscheme in a 2-truncated simplicial scheme $\bar{X}_\bullet$ consisting of smooth and projective schemes. Then we have an isomorphism*

$$H_1^S(X, \mathbb{Q}_l/\mathbb{Z}_l) \cong \mathrm{Tor}_1(\mathcal{A}_{X_\bullet}(k), \mathbb{Q}_l/\mathbb{Z}_l)$$

if either $l \neq p$ or if resolution of singularities exists.

Proposition 4.2 gives for *any* reduced semi-normal scheme a map of short exact sequences

$$\begin{array}{ccccc} H_1^S(X,\mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z} & \longrightarrow & H_1^S(X,\mathbb{Q}/\mathbb{Z}) & \longrightarrow & {}_{\mathrm{tor}}H_0^S(X,\mathbb{Z}) \\ \downarrow & & \| & & \downarrow \\ H_1(\mathcal{A}_{X_\bullet}(k)) \otimes \mathbb{Q}/\mathbb{Z} & \longrightarrow & \mathrm{Tor}_1(\mathcal{A}_{X_\bullet}(k), \mathbb{Q}/\mathbb{Z}) & \longrightarrow & {}_{\mathrm{tor}}(\mathcal{A}_{X_0}(k)/d\mathcal{A}_{X_1}(k)). \end{array} \tag{4.3}$$

Proof of Theorem 1.1: Using hypercohomological descent for Suslin homology [6], one can show that $H_1(D_{X_\bullet}, \mathbb{Z})$ is finite for normal X, and this implies that $H_1(\mathcal{A}_{X_\bullet}(k)) \otimes \mathbb{Q}/\mathbb{Z} = 0$ for normal X. Hence we obtain an isomorphism

$$_{\mathrm{tor}}H_0^S(X, \mathbb{Z}) \cong {}_{\mathrm{tor}}(\mathcal{A}_{X_0}(k)/d\mathcal{A}_{X_1}(k)).$$

Recall the short exact sequence (4.2)

$$0 \to \operatorname{im}\delta \to \mathcal{A}^0_{X_0}/d\mathcal{A}^0_{X_1}(k) \to \mathcal{A}_{X_0}(k)/d\mathcal{A}_{X_1}(k) \to H_0(D_{X_\bullet}) \to 0.$$

The finiteness of $H_1(D_{X_\bullet},\mathbb{Z})$ implies that $\operatorname{im}\delta$ is finite, so since $H_0(D_{X_\bullet})$ is torsion free, we obtain that ${}_{\mathrm{tor}}(\mathcal{A}_{X_0}(k)/d\mathcal{A}_{X_1}(k))$ is isomorphic to ${}_{\mathrm{tor}}\big(\mathcal{A}^0_{X_0}/d\mathcal{A}^0_{X_1}+\operatorname{im}\delta\big)(k)$ because taking k-valued points is exact. But $\mathcal{A}^0_{X_0}/d\mathcal{A}^0_{X_1}+\operatorname{im}\delta$ is the connected component of the largest locally semi-abelian scheme quotient of $\mathcal{A}_{X_0}/d\mathcal{A}_{X_1}$ which by Proposition 4.1 is the Albanese scheme of X.

Question: We saw that the Albanese variety is related to Suslin homology. Is there a similar motivic description for the maximal torus of X?

§ 5. Curves

Let E be an elliptic curve and p be a closed point of E. Let N be the variety obtained by glueing the points 0 and p of E. A 2-truncated hypercovering of N is given by

$$E\times_N E\times_N E\to E\times_N E\overset{\delta_0,\delta_1}{\longrightarrow} E\to N.$$

The middle term is isomorphic to $E\cup x\cup y$ where x and y correspond to the points $(0,p)$ and $(p,0)$ in the product, respectively. Similarly, the term on the left is isomorphic to E and 6 points corresponding to triples (x,y,z) with $x,y,z\in\{0,p\}$ and not all equal. The Albanese schemes are

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] \arrow[d, equal] & \mathcal{A}_2 \arrow[r] \arrow[d] & \mathbb{Z}^7 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E \arrow[r] \arrow[d, "0"'] & \mathcal{A}_1 \arrow[r] \arrow[d, "\delta_1"', "\delta_0"] & \mathbb{Z}^3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E \arrow[r] & \mathcal{A}_0 \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

A calculation shows that $H_1(D_\bullet)=\mathbb{Z}$, and the sequence (4.2) becomes

$$\mathbb{Z}\overset{\delta}{\to}E\to H_0(\mathcal{A}_\bullet(k))\to\mathbb{Z}\to 0,$$

with δ sending 1 to $p-0$ on E. Now assume that p is not torsion. Then the Albanese scheme of N is isomorphic to $\mathbb{Z}$, because it is the largest locally semi-abelian scheme quotient of $\mathcal{A}_{X_0}$ modulo the subabelian variety generated by $\langle p\rangle$. In particular, its torsion is trivial. The corank of $H_1^S(N,\mathbb{Q}/\mathbb{Z})$ and of ${}_{\mathrm{tor}}H_0^S(N,\mathbb{Z})$ is 3, in particular ${}_{\mathrm{tor}}H_0^S(N,\mathbb{Z})$ is not isomorphic to the torsion of the Albanese variety.

However, the corank of ${}_{\mathrm{tor}}H_0(\mathcal{A}_\bullet(k))$ is also 3, and ${}_{\mathrm{tor}}H_0^S(N,\mathbb{Z})$ is isomorphic to the torsion of the quotient of abelian groups $H_0(\mathcal{A}_\bullet(k))=\mathcal{A}_{X_0}(k)/d\mathcal{A}_{X_1}(k)\cong\big(\mathcal{A}^0_{X_0}/d\mathcal{A}^0_{X_1}\big)(k)/\operatorname{im}\delta$. In other words, taking the quotient in the category of locally semi-abelian schemes and then taking rational points does not give the correct answer, but taking rational points and then the quotient in the category of abelian groups does. More generally,

Theorem 5.1. *Let X be a reduced semi-normal curve. Then the Albanese map induces an isomorphism*

$$H_0^S(X,\mathbb{Z}) \cong \mathcal{A}_{X_0}(k)/d\mathcal{A}_{X_1}(k).$$

The right hand group is isomorphic to $\mathcal{A}_{\tilde{X}}(k)/\delta H_1(D_{X_\bullet})$, for $\tilde{X}$ the normalization of X, and $H_1(D_{X_\bullet})$ has the same rank as $H^1_{\text{ét}}(X,\mathbb{Z})$.

Question: Does the analog statement hold in higher dimensions, i.e. is the surjection of the right hand side of (4.3)

$${}_{\text{tor}}H_0^S(X,\mathbb{Z}) \twoheadrightarrow {}_{\text{tor}}(\mathcal{A}_{X_0}(k)/d\mathcal{A}_{X_1}(k))$$

an isomorphism for any reduced semi-normal scheme X?

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