RIMS Kôkyûroku Bessatsu
B51 (2014), 85–106

Singular homologies of non-archimedean analytic spaces and integrals along cycles: a research announcement

By

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Abstract

This article is an announcement of the author's results in [Mih]. We construct a new homology, which admits a canonical action of the absolute Galois group G_k of the base field k, for Berkovich's non-Archimedean analytic space. We also define a new integration of an overconvergent differential form along a cycle. The integration takes its value in Fontaine's p-adic period ring, and it induces a Galois equivariant pairing of the homology and the space of overconvergent differential forms. Moreover, we verify the homology and the integration satisfy many appropriate properties for homology and integration theories. We compute the homology of certain analytic spaces.

§ 0. Introduction

One of the aims of the author's paper [Mih] is to construct a new homology of Berkovich's non-Archimedean analytic space. The other one is to give an explicit definition of a new integration of an overconvergent differential form along a cycle in the sense of our homology. As results, we obtained a homology theory with Functoriality, Homotopy invariance, Long exact sequence for a space pair, Dimension axiom, Mayer–Vietoris exact sequence, Excision axiom, and Universal coefficient theorem (Theorem 1.13 (i)-(vii)), and an integration with Fundamental theorem of calculus (Proposition 2.17), Stokes' theorem (Theorem 2.19), Cauchy's integral theorem (Example 2.21), Residue theorem (Example 2.22), Cauchy's integral formula, and Cauchy–Goursat theorem (Example 2.23).

Received March 12, 2013. Revised March 29, 2013, May 19, 2013 and June 7, 2013.
2010 Mathematics Subject Classification(s): 11S80
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An integration is deeply related with p-adic Hodge theory in Number Theory. To begin with, we recall the classical Hodge theory. In the complex geometry, the Abel–Jacobi map $\mathrm{AJ}_C \colon C \to J(C)$ of a complex non-singular projective curve C with a base point $x_0 \in C$ is given in the following way: Let g be the genus of C. Take a basis $\omega_1, \ldots, \omega_g$ of holomorphic 1-forms $\mathrm{H}^0(C, \Omega_C)$. Denote by $\Lambda \subset \mathbb{C}^g \cong \mathbb{R}^{2g}$ the lattice generated by vectors of the form

$$\left(\int_\gamma \omega_i \right)_{i=1}^g \in \mathbb{C}^g$$

for a cycle $\gamma \in \mathrm{H}_1(C, \mathbb{Z}) \cong \mathbb{Z}^{2g}$. The non-canonical description of the Jacobian variety of C is given by the complex torus $J(C) := \mathbb{C}^g/\Lambda \cong \mathbb{R}^{2g}/\mathbb{Z}^{2g}$. For a point $x \in C$, the integral

$$\mathrm{AJ}_C(x) := \left(\int_\gamma \omega_i \right)_{i=1}^g \text{ modulo } \Lambda \in J(C)$$

for a path $\gamma \colon [0,1] \hookrightarrow C$ connecting x_0 and x is independent of γ, and hence induces a well-defined map

$$\begin{aligned} \mathrm{AJ}_C \colon C &\to J(C) \\ x &\mapsto \mathrm{AJ}_C(x). \end{aligned}$$

The Abel–Jacobi map $\mathrm{AJ}_C \colon C \to J(C)$ possesses ample information of the analytic structure of C. Indeed C can be reconstituted from $J(C)$ endowed with the polarization by the classical Torelli theorem originally verified in [Tor]. The integration identifies $\mathrm{H}^0(C, \Omega_C)$ with $\mathrm{F}^1\mathrm{H}^1(C, \mathbb{C}) \subset \mathrm{H}^1(C, \mathbb{C})$ for the Hodge filtration F, and the lattice data $\Lambda \cong \mathrm{H}_1(C, \mathbb{Z})$ corresponds to the integral structure $\mathrm{H}^1(C, \mathbb{Z}) \subset \mathrm{H}^1(C, \mathbb{C})$. The Hodge filtration associates a flag $(\mathrm{F}^1\mathrm{H}^1(C, \mathbb{C}), \mathrm{H}^1(C, \mathbb{C}))$ called the period $P(C)$ of C. There are extensions of Torelli theorem formulated using the notion of periods for several classes. Namely, consider a moduli of complex manifolds in certain good classes such as Abelian varieties and K3 surfaces. A period map from such a good moduli is injective. Thus a period reflects various analytic informations of complex manifolds. For more details, see [Gri] §7. A period of a variety is also important in Number Theory. The p-adic Hodge theory gives enormous contribution to the study of the Galois representation associated with the étale cohomology of a variety, and a period possesses non-trivial informations of the Galois representation itself and the geometry of the variety. Now as we explained through the example of the classical Abel–Jacobi map, an integration is one of the most simple ways to extract a period from an analytic variety. Therefore we expected a construction of a new integration might provide us a new method to extract an information of a period.

We use Berkovich's analytic space rather than an algebraic variety. Analytic Geometry often appears in modern Number Theory. One of the reasons why one needs the analysis is because several objects can be glued only on the rigid analytic topology even if they naturally arise in Algebraic Geometry. There are several notions of non-Archimedean analytic spaces nowadays, and they have distinct merits respectively. Berkovich's analytic space is easy to describe and study, and hence is used widely. For example, it appears in the proof of the local Langlands conjecture for $\mathrm{GL}_n(k)$ over a local field $k/\mathbb{Q}_p$ by Michael Harris and Richard Taylor in [HT]; in the study of Bruhat-Tits buildings by Berkovich in [Ber1] and by Bertrand Rémy, Amaury Thuillier, and Annette Werner in [RTW]; in the formulation of the relative p-adic Hodge theory comparing a relative (φ, Γ)-module and a local system by Kiran Sridhara Kedlaya and Ruochuan Liu in [KL1] and [KL2]; and so on.

The notion of an overconvergent structure of a rigid analytic geometry is introduced by Elmar Grosse-Klönne in [Klo]. An overconvergent analytic function is "an analytic function which converges on a wider subset". The overconvergence is an important property when we consider an integration. For example, observe the simplest case. Let D be the affinoid space associated with the Tate algebra

$$k\{T\} := \left\{ f = \sum_{i=0}^{\infty} f_i T^i \in k[[T]] \;\middle|\; \lim_{i \to \infty} |f_i| = 0 \right\}.$$

Then D is the closed unit disc. Although the differential $d/dT \colon T^i \mapsto iT^{i-1}$ is a contraction on $k\{T\}$, the integration $\int dT \colon T^i \mapsto (i+1)^{-1} T^{i+1}$ is not bounded or even not everywhere-defined on $k\{T\}$. An analytic function on the closed unit disc admits an antiderivative if and only if it extends to an open neighborhood of it, and the integration $\int dT$ is defined on the corresponding dense subalgebra

$$k\{T\}^{\dagger} := \left\{ f = \sum_{i=0}^{\infty} f_i T^i \in k[[T]] \;\middle|\; {}^{\exists} r \in (1, \infty), \ \lim_{i \to \infty} |f_i| r^i = 0 \right\}.$$

This is the ring of overconvergent analytic functions on the closed disc. Thus the notion of an overconvergence naturally arises when one considers an integration.

In §1, we give the definition of a new homology of Berkovich's non-Archimedean analytic space. We call it the analytic singular homology. The analytic singular homology is constructed in the same way as the singular homology of a topological space. The singular homology of a topological space is associated with the canonical functor from the simplicial category Δ to the category Top of topological spaces. In general, a functor from Δ to a category C associates a homological functor on C. In order to construct the analytic homology, we introduce a wide category C containing the category An_k of Berkovich's non-Archimedean analytic spaces as a full subcategory, and equip it with a functor $\Delta \to C$ induced by $\Delta \to \mathrm{Top}$. The analytic singular homology is the restriction H_* of the homological functor associated with $\Delta \to C$ on An_k. We verify appropriate properties of H_*. The absolute Galois group G_k of the base field acts on $\Delta \to C$ as natural equivalences, and it induces the action of G_k on H_*. The reason why we introduce such a wide category C is because An_k seems not to admit a canonical non-trivial functor from Δ equipped with a canonical non-trivial action of G_k.

In §2, we define an integration of an overconvergent differential form along a cycle in the sense of the analytic singular homology. We first recall the notion of overconvergence. An analytic space endowed with an overconvergent structure is called a dagger space. We construct an analytic singular homology of a dagger space in the same way as that of an analytic space. After then, we define an integral of a differential form, and verify the integrability of an overconvergent differential form. The basic properties of the integration follows immediately, and they imply that it is a non-Archimedean analogue of the Cauchy integral in the complex analysis.

§ 1. Analytic Singular Homology

In this section, we give an idea for the new homology of Berkovich's non-Archimedean analytic space over a complete valuation field of rank 1 with mixed characteristic $(0, p)$, and explain its basic properties. For the full descriptions of the proofs, see [Mih]. The new homology is an analogue of the singular homology of a topological space. First we recall the categorical construction of the singular homology. Secondly we apply it to a wide category containing the category of Berkovich's non-Archimedean analytic spaces. Finally we show several examples and properties.

§ 1.1. Homology associated with a simplicial functor

The singular homology is defined as a homology group of the singular chain complex, which is a chain complex defined by formal finite sums of singular simplices. A singular simplex on a topological space X is a continuous map from a standard simplex to X. This construction is purely topological, and it has a category-theoretical interpretation as follows: Denote by Top the category of topological spaces and continuous maps. A singular simplex on an object $X \in \mathrm{ob}(\mathrm{Top})$ is a morphism from a standard simplex to X, and a standard simplex is an object in the essential image of the canonical functor from the simplicial category Δ to Top. Here the simplicial category Δ is the category of sets of the form $\{0, \ldots, n\} \subset \mathbb{N}$ for $n \in \mathbb{N}$ and order-preserving maps with respect to the total order of natural numbers. In general, for a category C and a functor

$$\begin{aligned} S \colon \Delta &\to C \\ \{0, \ldots, n\} &\rightsquigarrow \Delta^n := S(\{0, \ldots, n\}), \end{aligned}$$

one can define the homology $\mathrm{H}_*^S(X, \mathbb{Z})$ of an object X of C to be the homology group of the chain complex $\mathrm{C}_*^S(X)$ whose degree n component is the Abelian group freely generated by the set $\mathrm{Hom}_C(\Delta^n, X)$. This construction of a homology is the generalization of that of the singular homology on Top; we obtain the singular homology on Top by applying the above general construction to Top and the canonical functor $\Delta \to \mathrm{Top}$. Furthermore, if a group G acts on the functor $S \colon \Delta \to C$, i.e. if a group homomorphism

$$\begin{aligned} \rho \colon G &\to \mathrm{Aut}(S) \\ g &\to (\rho(g) \colon S \xrightarrow{\sim} S) \end{aligned}$$

from G to the group $\mathrm{Aut}(S)$ of automorphisms of S is given, then the homology $\mathrm{H}_*^S(X, \mathbb{Z})$ admits a functorial G-action in the following way:

Definition 1.1. Let C be a small category, $S \colon \Delta \to C$ a functor, G a group, and $\rho \colon G \to \mathrm{Aut}(S)$ a group homomorphism. For an object $X \in \mathrm{ob}(C)$ and $n \in \mathbb{N}$, define $\rho'_{X,n} \colon G^{\mathrm{op}} \to \mathrm{Aut}_{\mathrm{Sets}}(\mathrm{Hom}(\Delta^n, X))$ as the group homomorphism

$$\begin{aligned} \rho'_{X,n} \colon G^{\mathrm{op}} &\to \mathrm{Aut}_{\mathrm{Sets}}(\mathrm{Hom}(\Delta^n, X)) \\ g &\to (\rho'_{X,n}(g) \colon \gamma \mapsto \gamma \circ \rho(g)(\{0, \ldots, n\})), \end{aligned}$$

where G^{op} is the group whose underlying set is that of G and whose multiplication $* \colon G^{\mathrm{op}} \times G^{\mathrm{op}} \to G^{\mathrm{op}}$ is given by setting $g * h := hg$ for each $g, h \in G^{\mathrm{op}}$. It induces an automorphism of the chain complex $\mathrm{C}_*^S(X)$ functorial on X, and thus it yields a functorial right group action $\rho_X \colon G^{\mathrm{op}} \to \mathrm{Aut}_{\mathrm{Ab}}(\mathrm{H}_*^S(X, \mathbb{Z}))$.

§ 1.2. Wide category

We would like to apply the construction above to Berkovich's non-Archimedean analytic spaces over k and the absolute Galois group G_k of k. See [Ber1] and [Ber2] for the definition and basic properties of Berkovich's non-Archimedean analytic spaces. We call Berkovich's analytic space an analytic space for short. We construct a functor from Δ to a wide category C containing the category of analytic spaces.

Definition 1.2. A commutative Banach k-algebra is a commutative k-algebra A endowed with a map $\| \cdot \| \colon A \to [0, \infty)$ satisfying the following:

(i) $\|a\| = 0$ if and only if $a = 0$ for any $a \in A$.

(ii) $\|a - b\| \leq \max\{\|a\|, \|b\|\}$ for any $a, b \in A$.

(iii) $\|ab\| \leq \|a\|\|b\|$ for any $a, b \in A$.

(iv) $\|ca\| = |c|\|a\|$ for any $a \in A$ and $c \in k$.

(v) The map

$$\begin{aligned} A \times A &\to [0,\infty) \\ (a,b) &\mapsto \|a-b\|, \end{aligned}$$

which is an ultrametric by the conditions (i) and (ii), is a complete metric.

Regard A as a topological k-algebra endowed with the complete ultrametric in the condition (v). A k-algebra homomorphism $\phi\colon A \to B$ between the underlying k-algebras of Banach k-algebras A and B is said to be bounded if there exists $D > 0$ such that $\|\phi(a)\| \leq D\|a\|$ for any $a \in A$. A morphism between Banach k-algebras is a bounded k-algebra homomorphism between their underlying k-algebras.

Lemma 1.3. *There is a category C equipped with fully faithful functors $\iota_1\colon An_k \to C$ from the category An_k of analytic spaces over k and and $\iota_2\colon Banach_k^{op} \to C$ from the opposite category $Banach_k^{op}$ of commutative Banach k-algebras such that the following hold:*

(i) The restrictions of the two functors on the opposite full subcategory Aff_k of k-affinoid algebras are naturally equivalent with each other.

(ii) The category C is universal in the categories equipped with functors from An_k and $Banach_k^{op}$ satisfying the condition (i). Namely, for a category C' and two functors $\iota'_1\colon An_k \to C'$ and $\iota'_2\colon Banach_k^{op} \to C'$ satisfying the condition (i), there exists a functor $\pi\colon C \to C'$ unique up to a natural equivalence such that the composition $\pi \circ \iota_i$ is naturally equivalent to ι'_i for each $i = 1, 2$.

The functor $S\colon \Delta \to C$ will be constructed in three steps: The first step is to construct the category Pol of polytopes in Euclidean spaces. Here by a technical reason, we only consider "rational" polytopes with "rational" affine maps in the sense of Definition 1.4. We do not use general polytopes and general affine maps, and hence we omit the adjective "rational". The second step is to define a canonical functor $R\colon \Delta \to \mathrm{Pol}$. The third step is to give a canonical functor $\mathrm{Pol} \to \mathrm{Banach}_k^{\mathrm{op}}$. Composing them, we will obtain the functor $S\colon \Delta \to \mathrm{Pol} \to \mathrm{Banach}_k^{\mathrm{op}} \hookrightarrow C$.

Definition 1.4. For $n \in \mathbb{N}$, a non-empty subset $P \subset \mathbb{R}^n$ is said to be a polytope if there is a finite subset $L \subset \mathbb{Q}^{n+1}$ such that P coincides with $\{(a_1,\ldots,a_n) \in \mathbb{R}^n \mid l_0 + l_1 a_1 + \cdots + l_n a_n \geq 0, {}^\forall(l_0,\ldots,l_n) \in L\}$. For polytopes $P_1 \subset \mathbb{R}^{n_1}$ and $P_2 \subset \mathbb{R}^{n_2}$, a map $\phi\colon P_1 \to P_2$ is said to be affine if there are a matrix $(F_{i,j})_{i,j} \in \mathrm{M}_{n_2,n_1}(\mathbb{Q})$ and a vector $(b_1,\ldots,b_{n_2}) \in \mathbb{Q}^{n_2}$ such that the map

$$\begin{aligned} \Phi\colon \mathbb{R}^{n_1} &\to \mathbb{R}^{n_2} \\ (a_1,\ldots,a_{n_1}) &\mapsto \left(b_1 + \sum_{j=1}^{n_1} F_{1,j}a_j, \ldots, b_{n_2} + \sum_{j=1}^{n_1} F_{n_2,j}a_j\right) \end{aligned}$$

satisfies $\Phi(P_1) \subset P_2$ and $\Phi|_{P_1} = \phi$. Denote by Pol the category of polytopes and affine maps.

Remark. Every affine map between polytopes is continuous, and therefore there is a canonical faithful functor $\mathrm{Pol} \hookrightarrow \mathrm{Top}$ which identifies Pol as a faithful subcategory of Top. Moreover, the canonical functor $\Delta \to \mathrm{Top}$ associating the singular homology of topological spaces uniquely factors through $\mathrm{Pol} \hookrightarrow \mathrm{Top}$.

Denote by $R \colon \Delta \to \mathrm{Pol}$ the induced functor. A complete valuation field k is said to be a local field if k is a discrete valuation field and if its residue field is a finite field. By a technical reason, if k is a local field, we compose R with the multiplication

$$\begin{array}{llll} q_k - 1 \colon & \mathrm{Pol} & \to & \mathrm{Pol} \\ & P & \rightsquigarrow & (q_k - 1)P \\ & \left(\begin{array}{ll} \phi \colon P_1 \to & P_2 \\ \quad t \mapsto & \phi(t) \end{array}\right) & \rightsquigarrow & \left(\begin{array}{ll} (q_k - 1)\phi \colon (q_k - 1)P_1 \to & (q_k - 1)P_2 \\ \qquad t & \mapsto (q_k - 1)\phi((q_k - 1)^{-1}t) \end{array}\right) \end{array}$$

by $q_k - 1$, where q_k is the cardinality of the finite residue field of k, and denote by R_k the composition. Otherwise, set $R_k := R$. For $n \in \mathbb{N}$, set $\Delta^n := R_k(\{0, \ldots, n\}) \in \mathrm{ob}(\mathrm{Pol})$.

Definition 1.5. For $n \in \mathbb{N}$, a polytope $P \subset \mathbb{R}^n$ is said to be thick if there is no $\mathbb{R}$-linear proper subspace $V \subset \mathbb{R}^n$ containing P.

Remark. Every polytope is isomorphic to a thick polytope.

Let us fix an algebraic closure k^{alg} of k. Set

$$\mathbb{Q}_k^\vee := \left\{ x \in \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Q}, k^{\mathrm{alg}\times}) \;\middle|\; x(1) \in k \right\},$$

where Ab denotes the category of Abelian groups. For $n \in \mathbb{N}$ and a thick polytope $P \subset \mathbb{R}^n$, consider a map

$$\begin{aligned} \| \cdot \|_{P,0} \colon \mathbb{Q}_k^{\vee n} &\to (0, \infty) \\ x = (x_1, \ldots, x_n) &\mapsto \|x\|_{P,0} := \sup \{ |x_1(t_1) \cdots x_n(t_n)| \mid (t_1, \ldots, t_n) \in P \cap \mathbb{Q}^n \}. \end{aligned}$$

Then the map

$$\begin{aligned} \| \cdot \|_P \colon k[\mathbb{Q}_k^{\vee n}] &\to [0, \infty) \\ \sum_{x \in \mathbb{Q}_k^{\vee n}} f_x x &\mapsto \sup_{x \in \mathbb{Q}_k^{\vee n}} |f_x| \|x\|_{P,0} \end{aligned}$$

is a multiplicative norm of a unital k-algebra. Denote by k_P the completion of the group algebra $k[\mathbb{Q}_k^{\vee n}]$ with respect to the norm $\| \cdot \|_P$. The embedding $k[\mathbb{Q}_k^{\vee n}] \hookrightarrow k^{\mathbb{Q}_k^{\vee n}}$ induces the natural identification

$$k_P = \left\{ (f_x)_{x \in \mathbb{Q}_k^{\vee n}} \in k^{\mathbb{Q}_k^{\vee n}} \;\middle|\; \lim_{x \in \mathbb{Q}_k^{\vee n}} |f_x| \|x\|_{P,0} = 0 \right\},$$

and we regard an element $(f_x)_{x \in \mathbb{Q}_k^\vee} \in k_P$ as a formal infinite sum $\sum_{x \in \mathbb{Q}_k^{\vee n}} f_x x$. Here the expression $\lim_{x \in \mathbb{Q}_k^{\vee n}} r(x) = 0$ for a map $r \colon \mathbb{Q}_k^{\vee n} \to [0, \infty)$ means that for any $\epsilon > 0$, there is a finite subset $F_\epsilon \subset \mathbb{Q}_k^{\vee n}$ such that $r(x) < \epsilon$ for any $x \in \mathbb{Q}_k^{\vee n} \backslash F_\epsilon$.

Remark. A character $x \in \mathbb{Q}_k^\vee$ is regarded as a function like an exponential map. For example, take a system $\epsilon \in \mathbb{Q}_k^\vee$ of roots of unity. For elements $a \in k$ and $b \in k^\times$, the formal sum $a + b\epsilon \in k[\mathbb{Q}_k^\vee]$ is analogous to the complex-valued function $a + b \exp(2\pi i t)$.

Lemma 1.6.

(i) For $n \in \mathbb{N}$ and a thick polytope $P \subset \mathbb{R}^n$, the natural action of G_k on $\mathbb{Q}_k^\vee$ induces a unitary representation $G_k \to Aut_{Banach_k}(k_P)$.

(ii) If there is an isomorphism $P_1 \to P_2$ of thick polytopes, then it naturally induces an isometric isomorphism $k_{P_2} \to k_{P_1}$.

(iii) The correspondence $P \rightsquigarrow k_P$ associating a thick polytope to a commutative Banach k-algebra is extended to a functor $k_\cdot \colon Pol \to Banach_k^{op} \colon P \rightsquigarrow k_P$ unique up to a natural equivalence.

(iv) The functor $k_\cdot \colon Pol \to Banach_k^{op}$ admits a canonical action of G_k.

Observe the case that a polynomial is the standard simplex $\Delta^n \in \mathrm{Top}$ for $n \in \mathbb{N}$. For convenience, we assume that the residue field of k is an infinite field, and omit the appearance of q_k. The polytope

$$\Delta^n = \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^n x_i = 1, x_i \geq 0, {}^\forall i = 0, \ldots, n \right\}$$

is not thick, and one has to take an isomorphism between Δ^n and a thick polytope. There is no canonical choice of such an isomorphism, but $n+1$ isomorphisms are given. Set

$$T^n := \left\{ (t_1, \ldots, t_n) \in \mathbb{R}^n \;\middle|\; \sum_{j=1}^n t_j \leq 1, t_j \geq 0, {}^\forall j = 1, \ldots, n \right\}.$$

For each $i = 0, \ldots, n$, the projection

$$\begin{aligned} p_i \colon \mathbb{R}^{n+1} &\to \mathbb{R}^n \\ (x_0, \ldots, x_n) &\mapsto (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \end{aligned}$$

induces an isomorphism $p_n \colon \Delta^n \to T^n$. Through the isomorphism, one has the isometric isomorphism

$$k_{\Delta^n} \cong k_{T^n} = \left\{ (f_x)_{x \in \mathbb{Q}_k^{\vee n}} \in k^{\mathbb{Q}_k^{\vee n}} \;\middle|\; \lim_{x \in \mathbb{Q}_k^{\vee n}} |f_x| \|x\|_{T^n, 0} = 0 \right\}$$

depending on the choice of $i = 0, \ldots, n$. In particular, consider the low dimensional cases. The thick polytopes T^0, T^1, T^2 are given as

$$T^0 = \{0\} = \mathbb{R}^0, T^1 = [0,1] \subset \mathbb{R},$$
$$T^2 = \left\{ (t_1, t_2) \in \mathbb{R}^2 \;\middle|\; 0 \leq t_1, 0 \leq t_2, t_1 + t_2 \leq 1 \right\},$$

and they associate the norms

$$\begin{aligned} \| \cdot \|_{T^0,0} \colon \mathbb{Q}_k^{\vee 0} = 1 &\to (0,\infty) \\ 1 &\mapsto 1 \end{aligned}$$

$$\begin{aligned} \| \cdot \|_{T^1,0} \colon \mathbb{Q}_k^{\vee} &\to (0,\infty) \\ x &\mapsto \max\{1, |x(1)|\} \end{aligned}$$

$$\begin{aligned} \| \cdot \|_{T^2,0} \colon \mathbb{Q}_k^{\vee 2} &\to (0,\infty) \\ (x_1, x_2) &\mapsto \max\{1, |x_1(1)|, |x_2(1)|\} \end{aligned}$$

Thus one obtains

$$k_{\Delta^0} \cong k, k_{\Delta^1} = \left\{ (f_x)_{x \in \mathbb{Q}_k^\vee} \in k^{\mathbb{Q}_k^\vee} \;\middle|\; \lim_{x \in \mathbb{Q}_k^\vee} |f_x| \max\{1, |x(1)|\} = 0 \right\},$$
$$k_{\Delta^2} = \left\{ (f_x)_{x \in \mathbb{Q}_k^{\vee 2}} \in k^{\mathbb{Q}_k^{\vee 2}} \;\middle|\; \lim_{x \in \mathbb{Q}_k^{\vee 2}} |f_x| \max\{1, |x_1(1)|, |x_2(1)|\} = 0 \right\}.$$

Definition 1.7. For a polytope P and an analytic space X, put $\mathrm{Hom}(P, X) := \mathrm{Hom}_C(\iota_2(k_P), \iota_1(X))$. We sometimes write $g \colon P \to X$ for an element $g \in \mathrm{Hom}(P, X)$.

Remark. For a polytope P and an affinoid space X, there is a functorial canonical bijective map between $\mathrm{Hom}(P, X)$ and $\mathrm{Hom}_{\mathrm{Banach}_k}(\mathrm{H}^0(X, O_X), k_P)$.

Definition 1.8. Denote by $S \colon \Delta \to C$ the composition of $R \colon \Delta \to \mathrm{Pol}$, $k_\cdot \colon \mathrm{Pol} \to \mathrm{Banach}_k$, and $\iota_2 \colon \mathrm{Banach}_k^{\mathrm{op}} \to C$.

Definition 1.9. For an analytic space X over k, define the analytic singular homology $\mathrm{H}_*(X, M)$ of X with coefficients in an Abelian group M as the homology group of the chain complex

$$\mathrm{C}_*(X, M) := \left(M^{\oplus \mathrm{Hom}(\Delta^0, X)} \leftarrow M^{\oplus \mathrm{Hom}(\Delta^1, X)} \leftarrow M^{\oplus \mathrm{Hom}(\Delta^2, X)} \leftarrow \cdots \right)$$

associated with the functor $S \colon \Delta \to C$. The analytic singular cohomology $\mathrm{H}^*(X, M)$ of X with coefficients in M is the cohomology group of the cochain complex

$$\mathrm{C}^*(X, M) := \left(M^{\mathrm{Hom}(\Delta^0, X)} \to M^{\mathrm{Hom}(\Delta^1, X)} \to M^{\mathrm{Hom}(\Delta^2, X)} \to \cdots \right)$$

associated with the functor $S\colon \Delta \to C$. The action of G_k on S induces a canonical action of G_k on $\mathrm{H}_*(X, M)$ and $\mathrm{H}^*(X, M)$ functorial on X and M.

Remark. Replacing C by the amalgamated sum $C^\dagger$ of the category Dg_k of dagger spaces and the opposite category $\mathrm{WBanach}_k^{\mathrm{op}}$ of weakly complete k-algebras over the opposite category $\mathrm{DgAff}_k^{\mathrm{op}}$ of dagger k-algebras, one obtains the analytic singular homology and the analytic singular cohomology of a dagger space in the same way. A dagger space is "an analytic space equipped with an overconvergent structure". We will explain the precise definitions of dagger spaces and dagger algebras in the next section.

Definition 1.10. For an analytic space X over k and a prime number l, we define the l-adic analytic singular homology $\mathrm{H}_*(X, \mathbb{Q}_l)$ of X as follows:

$$\mathrm{H}_*(X, \mathbb{Q}_l) := \varprojlim_{j \to \infty} \mathrm{H}_*(X, \mathbb{Z}/l^j\mathbb{Z}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

The compatible actions of G_k on $\mathrm{H}_*(X, \mathbb{Z}/l^j\mathbb{Z})$ for $j \in \mathbb{N}_+$ induce an l-adic representation of G_k on $\mathrm{H}_*(X, \mathbb{Q}_l)$ functorial on X.

§ 1.3. Examples and properties

Unfortunately there are few examples of analytic spaces whose analytic singular homologies have been calculated. Here are some of them. The calculations are done by similar ways as that for the singular homology of the closed unit disc $\{ a \in \mathbb{C} \mid |a| \leq 1 \}$ and the unit circle $\{ a \in \mathbb{C} \mid |a| = 1 \}$. For example, in order to verify that a given i-cycle is trivial, one has to construct an $(i+1)$-st morphism corresponding to a homotopy deformation in the calculation of a singular homology. For a singular homology, the multiplication by an additional parameter $t \in [0,1]$ is useful to construct a homotopy deformation. Similarly, we used the multiplication by $\underline{p}(t)$ for a fixed system $\underline{p} \in \mathbb{Q}_k^\vee$ of a power root of p. Note that the character $\underline{p}$ behaves like a path connecting $1 = \underline{p}(0)$ and $p = \underline{p}(1)$.

Example 1.11. Let X be the closed disc $M_k(k\{r^{-1}T\})$ of radius $r \in (0,\infty)^n$; the open disc $\bigcup_{0<s<r} M_k(k\{s^{-1}T\}) \subset M_k(k\{r^{-1}T\})$ of radius r; or the affine space $\mathbb{A}_k^n = \mathrm{Spec}(k[T_1, \ldots, T_n])^{\mathrm{an}} = \bigcup_{0<r} M_k(k\{r^{-1}T\})$ of dimension $n \in \mathbb{N}$. Then for a prime number l, there is a canonical G_k-equivariant isomorphism

$$\mathrm{H}_i(X, \mathbb{Q}_l) \cong \begin{cases} \mathbb{Q}_l \ (i = 0) \\ 0 \quad (i > 0) \end{cases}.$$

Example 1.12. Let X be the closed unit annulus $M_k(k\{T_1, T_1^{-1}\})$ or the algebraic group $\mathbb{G}_{m,k} = \mathrm{Spec}(k[T_1, T_1^{-1}])^{\mathrm{an}} = \bigcup_{0<s_1<r_1} M_k(k\{r_1^{-1}T_1, s_1 T_1^{-1}\})$ corresponding to the multiplicative group $k^\times$. Let l be a prime number. If $l \neq p$, there is a

canonical G_k-equivariant isomorphism

$$\mathrm{H}_i(X, \mathbb{Q}_l) \cong \begin{cases} \mathbb{Q}_l & (i = 0) \\ \mathbb{Q}_l(1) & (i = 1) \end{cases}.$$

If $l = p$, there are a canonical G_k-equivariant isomorphism

$$\mathrm{H}_0(X, \mathbb{Q}_p) \cong \mathbb{Q}_p$$

and a canonical exact sequence

$$0 \to \mathbb{Q}_p(1) \to \mathrm{H}_1(X, \mathbb{Q}_p) \to k \to 0$$

of p-adic Galois representations. The p-adic Galois representation $\mathrm{H}_1(X, \mathbb{Q}_p)$ is the canonical extension of k by $\mathbb{Q}_p(1)$ given by the surjective $\mathbb{Q}_p$-linear G_k-equivariant map

$$\begin{aligned} \{\ x \in \mathbb{Q}_k^\vee \mid |x(1) - 1| < 1\ \} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p &\twoheadrightarrow k \\ x \otimes 1 &\mapsto \log x(1), \end{aligned}$$

and if k is a local field, $\mathrm{H}_1(X, \mathbb{Q}_p)$ is a crystalline representation. Indeed, $\mathrm{H}_1(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{cris}}$ admits the G_k-invariant basis

$$\epsilon \otimes \frac{1}{\log \epsilon}, \left(\underline{\exp e_1} \otimes 1\right) - \left(\epsilon \otimes \frac{\log \underline{\exp e_1}}{\log \epsilon}\right), \ldots, \left(\underline{\exp e_n} \otimes 1\right) - \left(\epsilon \otimes \frac{\log \underline{\exp e_n}}{\log \epsilon}\right)$$

where ϵ is a system of p-power roots of unity, $n := \dim_{\mathbb{Q}_p} k$, $\{e_1, \ldots, e_n\} \subset k$ is a $\mathbb{Q}_p$-basis consisting of elements in the convergent domain of the exponential map, $\underline{\exp e_i}$ is a system of p-power roots of $\exp e_i \in \{\ a \in k \mid |a - 1| < 1\ \}$ for each $i = 1, \ldots, n$. Note that for a system $\underline{a}$ of p-power roots of $\{\ a \in k \mid |a - 1| < 1\ \}$, the logarithm $\log \underline{a} \in \mathrm{B}_{\mathrm{dR}}$ is contained in $\mathrm{B}_{\mathrm{cris}}$. We will recall the definition of the logarithm $\log \underline{a}$ in Definition 2.9.

The analytic singular homology satisfies appropriate properties. They contain the properties called the axiom of homology theory.

Theorem 1.13. *Let X be an analytic space and M an Abelian group.*

(i) Functoriality: The analytic singular homology $H_*(X, M)$ is functorial on X and M.

(ii) Homotopy invariance: The canonical projection $X \times_k \mathbb{A}_k^1 \to X$ induces a G_k-equivariant isomorphism $H_*(X \times_k \mathbb{A}_k^1, M) \to H_*(X, M)$.

(iii) Long exact sequence for a space pair: A space pair is a pair (X, Y) of an analytic space X and an analytic domain $Y \subset X$. For a space pair (X, Y) and an Abelian

group M, denote by $H_*(X, Y, M)$ the homology group of the cokernel of the homomorphism $C_*(Y, M) \to C_*(X, M)$ associated with the embedding $Y \hookrightarrow X$. There is a canonical long exact sequence

$$H_{*+1}(X, Y, M) \to H_*(Y, M) \to H_*(X, M) \to H_*(X, Y, M) \to$$

of G_k-modules functorial on (X, Y) and M.

(iv) *Dimension axiom: There is a canonical G_k-equivariant isomorphism*

$$H_i(M_k(k), M) \cong \begin{cases} M & (i = 0) \\ 0 & (i > 0) \end{cases}$$

functorial on M.

(v) *Mayer–Vietoris exact sequence: Suppose k is algebraically closed. For Weierstrass domains $U, V \subset X$ with Int$(U/X) \cup$ Int$(V/X) = X$, the embeddings $U, V \hookrightarrow X$ induce a long exact sequence*

$$\to H_{*+1}(X, M) \to H_*(U \cap V, M) \to H_*(U, M) \oplus H_*(V, M) \to H_*(X, M) \to$$

of k-vector spaces functorial on M, where Int(U/X) and Int(V/X) are the relative interiors of the morphisms $U \hookrightarrow X$ and $V \hookrightarrow X$. Note that in this case, Int(U/X) and Int(V/X) coincide with the open subspaces of U and V corresponding to the interiors of their underlying topological spaces as subsets of that of X.

(vi) *Excision axiom: Suppose k is algebraically closed. For Weierstrass domains $X', A \subset X$ with Int$(X'/X) \cup$ Int$(A/X) = X$, the embeddings $X', A \hookrightarrow X$ induce an isomorphism*

$$H_*(X', X' \cap A, M) \cong H_*(X, A, M)$$

functorial on M.

(vii) *Universal coefficient theorem: There is a canonical exact sequence*

$$0 \to H_*(X, \mathbb{Z}) \otimes_{\mathbb{Z}} M \to H_*(X, M) \to \mathit{Tor}^{\mathbb{Z}}(H_{*-1}(X, \mathbb{Z}), M) \to 0$$

of G_k-modules functorial on X and M.

§ 2. Integration along a Cycle

Suppose that the base field k is a local field of mixed characteristic $(0, p)$. Then Fontaine's p-adic period ring $\mathrm{B}_{\mathrm{dR}}(k)$ associated with k is defined as a topological k^{ur}-algebra endowed with a complete discrete valuation whose residue field is the completion of the algebraic closure k^{alg} we fixed in the previous section, and it admits a canonical structure as a k^{alg}-algebra compatible with the reduction with respect to the complete discrete valuation. In fact it is known that the structure of $\mathrm{B}_{\mathrm{dR}}(k)$ as a k^{alg}-algebra is independent of the base field $k/\mathbb{Q}_p$. Namely, for local fields K and L over $\mathbb{Q}_p$, there is a canonical isomorphism $\mathrm{B}_{\mathrm{dR}}(K) \cong \mathrm{B}_{\mathrm{dR}}(L)$ as a k^{alg}-algebra. Therefore we denote by B_{dR} instead of $\mathrm{B}_{\mathrm{dR}}(k)$ for short. See [Fon] for more details. In this section, we briefly explain the idea of definition of the integration for an overconvergent differential form along a cycle in the sense of the analytic singular homology we defined in the previous section. Note that the notion of the overconvergence of an analytic function and a holomorphic differential form on an analytic space depends on the choice of its overconvergent structure, defined in [Klo]. Remark that a non-compact Stein space and the analytification of an algebraic variety admit the canonical overconvergent structures. Here we followed the definition of a Stein space by Berkovich in [Ber1] 5.1. Beware that there are several similar definitions of a Stein space. A Stein space is an analytic space X admitting an increasing sequence $W_0 \subset W_1 \subset \cdots \subset \bigcup W_i = X$ by an affinoid domains such that W_i is a Weierstrass domain of W_{i+1} for any $i \in \mathbb{N}$. For example, an affinoid space and an analytic space associated with an affine scheme is a Stein space. Strictly speaking, an analytic space with a fixed overconvergent structure is called a dagger space. First we briefly review dagger algebras and dagger spaces. See [MW] and [Klo] for details. Secondly we give the definition of the analytic singular homology of a dagger space in the same way as that of an analytic space. Thirdly we formulate the notion of the integrability of a differential form, and verify the integrability of an overconvergent differential form. Finally we see basic properties of the integration.

§ 2.1. Overconvergence

The overconvergent structure of an analytic space is given by a covering by affinoid dagger spaces. An affinoid dagger space is a locally ringed G-topological space associated with a normed algebra called an affinoid dagger algebra. An affinoid dagger algebra is a dense subalgebra of an affinoid algebra consisting of "analytic functions converging on a wider subset".

Definition 2.1. For $n \in \mathbb{N}$ and a parameter $r = (r_1, \ldots, r_n) \in (0, \infty)^n$, set

$$k\{r^{-1}T\}^\dagger = k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}^\dagger$$
$$:= \left\{ f = \sum_{I \in \mathbb{N}^n} f_I T_1^{I_1} \cdots T_n^{I_n} \in k[[T_1, \ldots, T_n]] \;\middle|\; {}^\exists \delta > 1, \lim_{|I| \to \infty} |f_I| \delta^{|I|} r_1^{I_1} \cdots r_n^{I_n} = 0 \right\}$$

and call it the Monsky-Washnitzer algebra of radius r. It is obviously a dense k-subalgebra of the Tate algebra $k\{r^{-1}T\}$, and we endow it with the restriction of the Gauss norm of $k\{r^{-1}T\}$.

Remark. The Monsky-Washnitzer algebra is an algebra of overconvergent power series. Namely, the restriction maps $k\{s^{-1}T\} \to k\{r^{-1}T\}$ for parameters $s > r$ induce a k-algebra isomorphism

$$\varinjlim_{s>r} k\{s^{-1}T\} \cong k\{r^{-1}T\}^\dagger,$$

where the left hand side is the direct limit of k-algebras. Every Monsky-Washnitzer algebra is a normed Noetherian k-algebra, and its ideal is closed with respect to the norm topology. Therefore the quotient of a Monsky-Washnitzer algebra by a proper ideal admits the quotient norm.

Definition 2.2. A normed k-algebra is said to be a dagger k-algebra if it admits an isomorphism with the quotient algebra of a Monsky-Washnitzer algebra by a proper ideal in the category of normed k-algebras. Denote by DgAff_k the opposite category of dagger k-algebras.

Remark. The completion of a dagger algebra is an affinoid algebra.

Definition 2.3. A dagger space is defined in the same way as an analytic space in [Ber2] replacing affinoid spaces by affinoid dagger spaces.

Remark.

(i) A dagger space is a G-locally ringed G-topological space $X^\dagger = (X_G^\dagger, O_{X^\dagger})$ endowed with an equivalent class of a net τ of G-admissible subsets of the underlying topological space $|X^\dagger|$ in the sense of [Ber2] with a suitable property. A dagger space is not formulated as a locally ringed G-topological space G-locally isomorphic to an affinoid dagger space. It is because of the same reason as that for an analytic space. The weak G-topology of an affinoid dagger space is not saturated and the collection of affinoid domains is not closed under unions. Therefore an analytic domain of a dagger space is not weakly G-admissible, and a morphism between analytic spaces is not weakly G-continuous in general. This is why one has to consider another slightly finer G-topology for a non-affinoid analytic space. See [BGR] for details of G-topological spaces.

(ii) The completion of a dagger algebra preserves the underlying topological spaces of the spectra. The weak G-topology of an affinoid dagger space is slightly weaker than that of the corresponding affinoid space.

Definition 2.4. The completion functor from the category of dagger algebras induces a functor from the category Dg_k of dagger spaces to An_k preserving the underlying topological spaces. When an analytic space X is associated to a dagger space $X^\dagger$, then we call the structure sheaf $O_{X^\dagger}$ of $X^\dagger$ an overconvergent structure of X. Denote by $\Omega^\dagger_X = \Omega_{X^\dagger}$ the conormal sheaf of the diagonal embedding $\Delta\colon X^\dagger \hookrightarrow X^\dagger \times_k X^\dagger$, and call it the sheaf of overconvergent differential forms.

Remark. In the situation above, the overconvergent sheaves $O_{X^\dagger}$ and $\Omega_{X^\dagger}$ are naturally embedded in the sheaves O_X and Ω_X respectively restricted on the G-topology of $X^\dagger$ through the identification of the underlying topological spaces $|X| \cong |X^\dagger|$.

§ 2.2. Homology of a dagger space

As we remarked in the construction of the analytic singular homology of an analytic space, the same works for a dagger space. We define the overconvergent structure of a polytope and the overconvergence of a morphism from a polytope to a dagger space. Using them, we formalize the canonical functor from Δ to a wide category containing the category of dagger spaces. It associates a homology of a dagger space.

Definition 2.5. A normed k-algebra A is said to be weakly complete if for $m \in \mathbb{N}$ and non-zero elements $a_1, \ldots, a_m \in A\backslash\{0\}$, the k-algebra homomorphism $k[T_1, \ldots, T_m] \to A\colon T_i \mapsto a_i$ is uniquely extended to a bounded k-algebra homomorphism $k\{\|a_1\|^{-1}T_1, \ldots, \|a_m\|^{-1}T_m\}^\dagger \to A$. Denote by $\mathrm{WBanach}_k$ the category of weakly complete k-algebras.

Remark.

(i) Every dagger algebra and every Banach k-algebra are weakly complete.

(ii) The forgetful functor from $\mathrm{WBanach}_k$ to the category of normed k-algebras admits the left adjoint, and call it the weak completion functor. The weak completion of a normed k-algebra is a dense weakly complete k-subalgebra of the completion.

Definition 2.6. For $n \in \mathbb{N}$ and a thick polytope $P \subset \mathbb{R}^n$, denote by $k^\dagger_P \subset k_P$ the weak completion of the dense k-subalgebra $k[\mathbb{Q}^{\vee n}_k] \subset k_P$.

Lemma 2.7. *If there is an isomorphism $P_1 \to P_2$ between two thick polytopes, then it naturally induces the isometric isomorphism $k^\dagger_{P_2} \to k^\dagger_{P_1}$.*

Lemma 2.8. *The correspondence $P \rightsquigarrow k^\dagger_P$ which associates a thick polytope to a weakly complete k-algebra is extended to a functor $k^\dagger_\cdot\colon Pol \to WBanach_k$ unique up to a natural equivalence.*

We regard the weakly complete k-algebra $k^\dagger_P$ as "the overconvergent structure" of a polytope P. This allows us to define the notion of an overconvergent morphism from a polytope to a dagger space in the following way: Replacing the categories An_k, Banach_k, and Aff_k by Dg_k, $\mathrm{WBanach}_k$, and DgAff_k respectively, we obtain a wide category $C^\dagger$ corresopondind to C. An overconvergent morphism from a polytope to a dagger space is defined as a morphism between the images of them in $C^\dagger$. The functor $\mathrm{Pol} \to \mathrm{WBanach}_k \colon P \rightsquigarrow k_P^\dagger$ induces the functor $S^\dagger \colon \Delta \to C^\dagger$, and it associates the analytic singular homology of a dagger space. The completion functor induces the canonical faithful functor $C^\dagger \to C$ compatible with S and $S^\dagger$, and hence it gives the canonical homomorphism

$$\mathrm{H}_*(X^\dagger, M) \to \mathrm{H}_*(X, M)$$

for a dagger space $X^\dagger$, the analytic space X associated to $X^\dagger$, and an Abelian group M. We do not know whether the homomorphism is injective (surjective) or not.

§ 2.3. Convergence of the integral

We want to define the integral of an overconvergent differential n-form ω on a dagger space $X^\dagger$ along an overconvergent morphism $\gamma \colon \Delta^n \to X^\dagger$ by the equality

$$\int_\gamma \omega := \int_{\Delta^n} \gamma^* \omega.$$

Thus in order to define the integral of an overconvergent differential form ω along a cycle $[\gamma]$, it suffices to define the "pull-back" $\gamma^*\omega$ and the integral of $\gamma^*\omega$ over Δ^n satisfying Stokes' theorem. The definition of the integration for a general dimention n is a little complicated, and hence we only deal with the case $n = 1$. By a technical reason, we identify the standard simplex Δ^1 with the interval $[0, q_k - 1]$, where q_k is the cardinality of the finite residue field of k.

Definition 2.9. Denote by $O \subset k^{\mathrm{alg}}$ the ring of integers. Define the logarithm $\log \colon \mathbb{Q}_k^\vee \to \mathrm{B}_{\mathrm{dR}}$ as the composition of the inclusion $\mathbb{Q}_k^\vee \hookrightarrow \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Q}, k^{\mathrm{alg}\times})$, the restriction $\mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Q}, k^{\mathrm{alg}\times}) \to \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}[p^{-1}], k^{\mathrm{alg}\times})$, the isomorphisms $\mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}[p^{-1}], k^{\mathrm{alg}\times}) \cong \varprojlim_{z \mapsto z^p} k^{\mathrm{alg}\times} \cong \mathrm{Frac}(\varprojlim_{\mathrm{Frob}} O/p)^\times$, and the extension $\log \colon \mathrm{Frac}(\varprojlim_{\mathrm{Frob}} O/p)^\times \to \mathrm{B}_{\mathrm{dR}}$ of the monoid homomorphism

$$\begin{aligned} \log \colon \varprojlim_{\mathrm{Frob}} O/p &\to \mathrm{B}_{\mathrm{dR}} \\ a &\mapsto \sum_{i=1}^{\infty} \frac{1}{i} \left(1 - \frac{[a]}{a(1)} \right)^i \end{aligned}$$

with respect to the multiplicative monoid structure of $\varprojlim_{\mathrm{Frob}} O/p$, where $[a]$ denotes the Teichmüller lift $(a, 0, 0, \ldots) \in W(\varprojlim_{\mathrm{Frob}} O/p) \subset \mathrm{B}_{\mathrm{dR}}$. For details of the isomorphisms and the logarithm above, see [Fon].

Lemma 2.10. *For $n \in \mathbb{N}$ and a thick polytope $P \subset \mathbb{R}^n$, consider the formal derivation*

$$d\colon k_P^\dagger \to \bigoplus_{i=1}^n k_P^\dagger \otimes_k B_{dR} dt_i$$

$$f = \sum_{x \in \mathbb{Q}_k^{\vee n}} f_x x \mapsto df := \sum_{x \in \mathbb{Q}_k^{\vee n}} \sum_{i=1}^n f_x x \otimes \log x_i dt_i.$$

The infinite sum in the definition of the formal derivation d makes sense by the condition that an element of $k_P^\dagger$ is of the form $f = \sum_{I \in \mathbb{N}^m} g_I x_1^{I_1} \cdots x_m^{I_m}$ for $m \in \mathbb{N}$ and $x_1, \ldots, x_m \in \mathbb{Q}_k^{\vee n}$.

For $m \in \mathbb{N}$, set $\Omega_{P^\dagger}^m := \bigwedge_{k_P^\dagger}^m (\bigoplus_{i=1}^n k_P^\dagger dt_i) \otimes_k \mathrm{B_{dR}}$ and denote by $d_m\colon \Omega_{P^\dagger}^m \to \Omega_{P^\dagger}^{m+1}$ the wedge product of d. Call $(\Omega_{P^\dagger}^*, d_*)$ the de Rham complex associated to P.

Lemma 2.11. *If there is an isomorphism $P_1 \to P_2$ between two thick polytopes, then it naturally induces an isomorphism $(\Omega_{P_2^\dagger}^*, d_*) \to (\Omega_{P_1^\dagger}^*, d_*)$ of chain complexes of free modules over the identified weakly complete k-algebras $k_{P_2}^\dagger \cong_k k_{P_1}^\dagger$.*

Lemma 2.12. *The correspondence $P \rightsquigarrow (\Omega_{P^\dagger}^*, d_*)$ which associates a thick polytope P to a chain complex of a finite free $k_P^\dagger$-module is extended to a functor $(\Omega_{\cdot^\dagger}^*, d_*)\colon Pol \to C^*(Ab)$ endowed with the structure of finite free $k_\cdot^\dagger$-modules. Here $C^*(Ab)$ denotes the category of cochain complexes in Ab, and the functor $(\Omega_{\cdot^\dagger}^*, d_*)$ is unique up to a natural equivalence. If one fixes a representative of the natural equivalence class of the functor $k_\cdot^\dagger$, then there is a canonical representative of the natural equivalence class of the functor $(\Omega_{\cdot^\dagger}^*, d_*)\colon Pol \to C^*(Ab)$.*

Example 2.13. Recall that for $n \in \mathbb{N}$, the polytope $\Delta^n = R_k(\{0, \ldots, n\})$ is the subset of $\mathbb{R}^{n+1}$ of the form

$$\Delta^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; t_0 + \cdots + t_n = N_k \right\},$$

where N_k is 1 when k is not a local field and is $q_k - 1$ when k is a local field. The degree n component of the de Rham complex associated with Δ^n is computed as follows:

$$\begin{aligned}\Omega_{\Delta^{n\dagger}}^n &= \frac{\bigoplus_{i=0}^n k_{\Delta^n}^\dagger \otimes_k \mathrm{B_{dR}} dt_0 \wedge \cdots \wedge dt_{i-1} \wedge dt_{i+1} \wedge \cdots \wedge dt_n}{(dt_0 \wedge \cdots \wedge dt_{i_1-1} \wedge dt_{i_1+1} \wedge \cdots \wedge dt_n - dt_0 \wedge \cdots \wedge dt_{i_2-1} \wedge dt_{i_2+1} \wedge \cdots \wedge dt_n)} \\ &\cong_{k_{\Delta^n}^\dagger} k_{\Delta^n}^\dagger \otimes_k \mathrm{B_{dR}} dt_1 \wedge \cdots \wedge dt_n\end{aligned}$$

Lemma 2.14. *For $n, m \in \mathbb{N}$, there is a non-trivial canonical map*

$$\begin{aligned} Hom(\Delta^n, X^\dagger) \times H^0(X^\dagger, \Omega_{X^\dagger}^m) &\to \Omega_{\Delta^{n\dagger}}^m \\ (\gamma, \omega) &\mapsto \gamma^* \omega \end{aligned}$$

compatible with the derivations and functorial on a dagger space $X^\dagger$. We call it the pull-back morphism.

Next we define the integral of an element of $\Omega^1_{[0,q_k-1]^\dagger}$ over $[0, q_k - 1]$. Since $\Omega^1_{[0,q_k-1]^\dagger} = k^\dagger_{[0,q_k-1]} \otimes_k \mathrm{B}_{\mathrm{dR}} dt_1$, it is reduced to the construction of an integration $\int_0^{q_k-1} dt_1 \colon k^\dagger_{[0,q_k-1]} \to \mathrm{B}_{\mathrm{dR}}$.

Definition 2.15. For an element $f = \sum_{x \in \mathbb{Q}_k^\vee} f_x x \in k_{[0,q_k-1]}$, we define its integral $\int_0^{q_k-1} f dt_1$ by

$$\int_0^{q_k-1} f dt_1 := \sum_{\log x \neq 0} f_x \frac{x(q_k-1)-1}{\log x} + (q_k - 1) \sum_{\log x = 0} f_x$$

if the infinite sum in the right hand side converges in B_{dR}. We say that f is integrable if the above integral is defined.

Remark. Recall that a character $x \in \mathbb{Q}_k^\vee$ is regarded as a function like an exponential map. The definition of the integral

$$\int_0^{q_k-1} x(t_1) dt_1 = \frac{x(q_k-1)-1}{\log x} = \frac{x(q_k-1)-x(0)}{\log x} \in \mathrm{B}_{\mathrm{dR}}$$

is analogous to the integral

$$\int_0^r \exp(\alpha t_1) dt_1 = \frac{\exp(\alpha r) - \exp(0)}{\alpha} \in \mathbb{C}$$

of the exponential map $t \mapsto \exp(\alpha t)$ for $\alpha \in \mathbb{C}^\times$ and $r \in \mathbb{R}$.

Theorem 2.16. *Every element $f \in k^\dagger_{[0,q_k-1]}$ is integrable. The integration induces a B_{dR}-linear G_k-equivariant homomorphism*

$$\int_0^{q_k-1} \colon \Omega^1_{[0,q_k-1]^\dagger} \to B_{dR}$$
$$f \otimes b dt_1 \mapsto \int_0^{q_k-1} f \otimes b dt_1 := b \int_0^{q_k-1} f dt_1.$$

§ 2.4. Properties and calculations of the integral

We show that the integration satisfies appropriate properties which is desired against a good integration. These properties imply that the integration is a non-Archimedean analogue of the Cauchy integral in the complex analysis.

Proposition 2.17 (Fundamental theorem of calculus). *For an element $f \in k^{\dagger}_{[0,q_k-1]}$, the equality*

$$\int_0^{q_k-1} df = f(q_k - 1) - f(0) \in k \subset B_{dR}$$

holds.

Definition 2.18. For a dagger space $X^\dagger$ and $n \in \mathbb{N}$, define the pairing $\int$ as follows:

$$\int : \mathrm{Hom}(\Delta^n, X^\dagger) \times \mathrm{H}^0(X^\dagger, \Omega^n_{X^\dagger}) \to \mathrm{B}_{\mathrm{dR}}$$
$$(\gamma, \omega) \mapsto \int_\gamma \omega := \int_{\Delta^n} \gamma^* \omega.$$

By the universality of the free Abelian group $\mathbb{Z}^{\oplus \mathrm{Hom}(\Delta^n, X^\dagger)}$, it induces a $\mathbb{Z}$-linear G_k-equivariant homomorphism

$$\int : \mathbb{Z}^{\oplus \mathrm{Hom}(\Delta^n, X^\dagger)} \otimes_{\mathbb{Z}} \mathrm{H}^0(X^\dagger, \Omega^n_{X^\dagger}) \to \mathrm{B}_{\mathrm{dR}}$$
$$\gamma \otimes \omega \mapsto \int_\gamma \omega,$$

where G_k acts trivially on the cohomology $\mathrm{H}^0(X^\dagger, \Omega^n_{X^\dagger})$.

Theorem 2.19 (Stokes' theorem). *For a dagger space $X^\dagger$, $n \in \mathbb{N}$, an overconvergent $(n-1)$-form $\omega \in H^0(X^\dagger, \Omega^{n-1}_{X^\dagger})$, and an element $\gamma \in \mathbb{Z}^{\oplus Hom(\Delta^n, X^\dagger)}$, the equality*

$$\int_\gamma d\omega = \int_{\partial\gamma} \omega$$

holds, where $\partial \colon \mathbb{Z}^{\oplus Hom(\Delta^n, X^\dagger)} \to \mathbb{Z}^{\oplus Hom(\Delta^{n-1}, X^\dagger)}$ is the n-th derivation of the chain complex $C_*(X^\dagger, \mathbb{Z})$.

Corollary 2.20. *The pairing in the previous proposition induces a well-defined G_k-equivariant pairing*

$$\int : H_*(X^\dagger, \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(H^0(X^\dagger, \Omega^{\cdot}_{X^\dagger})) \to B_{dR}$$
$$\gamma \otimes \omega \mapsto \int_\gamma \omega.$$

If $X^\dagger$ is a smooth Stein space, then there is a canonical isomorphism $H^*(H^0(X^\dagger, \Omega^{\cdot}_{X^\dagger})) \cong_k H^*_{dR}(X^\dagger)$ through the Hodge to de Rham spectral sequence by [Kie] 2.4.2, where $H^*_{dR}(X^\dagger)$

is the hypercohomology of the overconvergent de Rham complex $(\Omega^{\cdot}_{X^\dagger}, d.)$. *Therefore the above pairing induces a* G_k*-equivariant canonical pairing*

$$\int : H_*(X^\dagger, \mathbb{Z}) \otimes_{\mathbb{Z}} H^*_{dR}(X^\dagger) \to B_{dR}.$$

Example 2.21 (Cauchy's integral theorem). The integration

$$\int dT \colon \mathrm{H}_1(\mathbb{A}^{1\dagger}_k, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{H}^0(\mathbb{A}^{1\dagger}_k, O^\dagger_{\mathbb{A}^1}) \to \mathrm{B}_{\mathrm{dR}}$$
$$\gamma \otimes f \mapsto \int_\gamma f dT$$

is zero because $\mathrm{H}_1(\mathbb{A}^{1\dagger}_k, \mathbb{Z}) = 0$. In particular for a system $\epsilon \in \mathbb{Q}^\vee_k$ of roots of unity, the integration

$$\int_{\gamma_\epsilon} dT \colon \mathrm{H}^0(\mathbb{A}^{1\dagger}_k, O_{\mathbb{A}^1_k}{}^\dagger) \to \mathrm{B}_{\mathrm{dR}}$$
$$f \mapsto \int_{\gamma_\epsilon} f dT$$

is zero, where $\gamma_\epsilon \colon \Delta^1 \cong [0, q_k - 1] \to \mathbb{A}^{1\dagger}_k$ is the morphism associated with $\epsilon^{1/(q_k-1)} \in \mathbb{Q}^\vee_k \subset k^\dagger_{[0,q_k-1]}$. Note that since $q_k - 1$ is the cardinality of the subgroup of $k^\times$ consisting of roots of unity of order coprime with p, the character $\epsilon^{1/(q_k-1)}$ in the multiplicative $\mathbb{Q}$-vector space $\mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Q}, k^{\mathrm{alg}\times})$ is contained in $\mathbb{Q}^\vee_k$.

Example 2.22 (Residue theorem). Let $\epsilon \in \mathbb{Q}^\vee_k$ be a system of roots of unity, and denote by $\gamma_\epsilon \colon \Delta^1 \to \mathbb{G}^\dagger_{m,k}$ the closed morphism associated with $\epsilon^{1/(q_k-1)} \in (k^\dagger_{[0,q_k-1]})^\times$. For a Laurent series $f = \sum_{i \in \mathbb{Z}} f_i T^i \in \mathrm{H}^0(\mathbb{G}^\dagger_{m,k}, O^\dagger_{\mathbb{G}_{m,k}}) = \mathrm{H}^0(\mathbb{G}_{m,k}, O_{\mathbb{G}_{m,k}})$, the equality

$$\frac{1}{\log \epsilon} \int_{\gamma_\epsilon} f dT = f_{-1}$$

holds. Indeed, one has

$$\frac{1}{\log \epsilon} \int_{\gamma_\epsilon} f dT = \frac{1}{\log \epsilon} \int_{[0,q_k-1]} f\left(\epsilon^{\frac{1}{q_k-1}}\right) d\left(\epsilon^{\frac{1}{q_k-1}}\right)$$
$$= \frac{1}{\log \epsilon} \int_{[0,q_k-1]} \sum_{i \in \mathbb{Z}} f_i \epsilon^{\frac{i}{q_k-1}}(t_1) \left(\log \epsilon^{\frac{1}{q_k-1}} \epsilon^{\frac{1}{q_k-1}}(t_1) dt_1\right) = \frac{\log \epsilon^{\frac{1}{q_k-1}}}{\log \epsilon} \sum_{i \in \mathbb{Z}} f_i \int_{[0,q_k-1]} \epsilon^{\frac{i+1}{q_k-1}}(t_1) dt_1$$
$$= \frac{1}{q_k - 1} \left(\sum_{i \in \mathbb{Z} \setminus \{-1\}} f_i \frac{\epsilon(i+1) - 1}{\log \epsilon^{\frac{i+1}{q_k-1}}} + (q_k - 1) f_{-1} \right) = \frac{1}{\log \epsilon} \sum_{i \in \mathbb{Z} \setminus \{-1\}} f_i \frac{1-1}{i+1} + f_{-1}$$
$$= f_{-1}.$$

Example 2.23 (Cauchy's integral formula, Cauchy-Goursat theorem)**.** In the situation above, for an entire function $f = \sum_{i \in \mathbb{N}} f_i T^i \in \mathrm{H}^0(\mathbb{A}_k^{1\dagger}, O_{\mathbb{A}_k^1}^\dagger) = \mathrm{H}^0(\mathbb{A}_k^1, O_{\mathbb{A}_k^1})$, the equality

$$\frac{1}{\log \epsilon} \int_{\gamma_\epsilon} \frac{f}{(T-a)^{i+1}} dT = \frac{d^i f}{dT^i}(a)$$

holds for $a \in k$ and $i \in \mathbb{N}$.

Example 2.24. Take an element $q \in k$ with $0 < |q| < 1$. Let $\epsilon, \underline{q} \in \mathbb{Q}_k^\vee$ be systems of roots of unity and q, and denote by $\gamma_\epsilon, \gamma_{\underline{q}} \colon \Delta^{1\dagger} \to \mathbb{G}_{m,k}^\dagger$ the morphism associated with $\epsilon^{1/(q_k - 1)}, \underline{q} \in (k_{[0,q_k-1]}^\dagger)^\times$ respectively. Denote by $\gamma'_\epsilon, \gamma'_{\underline{q}} \colon \Delta^{1\dagger} \to \mathbb{G}_{m,k}^\dagger$ the compositions of γ_ϵ and $\gamma_{\underline{q}}$ respectively with the canonical projection $\mathbb{G}_{m,k}^\dagger \twoheadrightarrow \mathbb{G}_{m,k}^\dagger / q^{\mathbb{Z}}$. They are obviously cycles. Let $\omega \in \mathrm{H}^0(\mathbb{G}_{m,k}^\dagger / q^{\mathbb{Z}}, \Omega_{\mathbb{G}_{m,k}/q^{\mathbb{Z}}}^\dagger) = \mathrm{H}^0(\mathbb{G}_{m,k}/q^{\mathbb{Z}}, \Omega_{\mathbb{G}_{m,k}/q^{\mathbb{Z}}})$ be the volume form whose pull-back by the canonical projection $\mathbb{G}_{m,k}^\dagger \twoheadrightarrow \mathbb{G}_{m,k}^\dagger / q^{\mathbb{Z}}$ is $dT/T \in \mathrm{H}^0(\mathbb{G}_{m,k}^\dagger, \Omega_{\mathbb{G}_{m,k}}^\dagger) = \mathrm{H}^0(\mathbb{G}_{m,k}, \Omega_{\mathbb{G}_{m,k}})$. Then the equality

$$\begin{pmatrix} \int_{\gamma'_\epsilon} \omega \\ \int_{\gamma'_{\underline{q}}} \omega \end{pmatrix} = \begin{pmatrix} \log \epsilon \\ (q_k - 1) \log \underline{q} \end{pmatrix}$$

holds, and thus the integration gives Fontaine's p-adic periods of the Tate curve $\mathbb{G}_{m,k}/q^{\mathbb{Z}}$.

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