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Quasi-coherent sheaves on algebraic moduli stacks of log structures

By

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Abstract

In this paper, after giving a review of Olsson's algebraic moduli stacks of log structures, we announce our results on quasi-coherent sheaves and their cohomologies on these algebraic stacks.

§ 1. Introduction

The logarithmic geometry (the theory of log schemes) is introduced by Fontaine-Illusie-Kato [2] to study, for example, degenerate fibers of semistable families. Then Olsson([7] and [8]) found a new approach to study log schemes. He introduced the algebraic stack $\mathcal{L}og_S$ associated to a log scheme S and gave a way to study log schemes via stacks. In this paper, we announce some results on quasi-coherent sheaves on the algebraic stacks $\mathcal{L}og_S$ presented in master's thesis of the author([5]).

After we fix the notation and the convention in Section 2, we give definitions of basic terms in logarithmic geometry in Section 3, for the convenience of the reader. Those who are interested in logarithmic geometry may refer to [2] and [6]. In Section 4, we give the definition of the algebraic stack $\mathcal{L}og_S$ associated to a log scheme S, following [7]. We also try to explain why we should consider the algebraic stack $\mathcal{L}og_S$ by explaining the relation to log geometry of Fontaine-Illusie-Kato. In Section 5, we explain the local structure of the algebraic stack $\mathcal{L}og_S$, which we need to explain our results. In Sections 6 and 7, we give statements of our results in [5]. In Section 6, we describe the category of quasi-coherent sheaves on $\mathcal{L}og_S$. An important point is that $\mathcal{L}og_S$ is covered by algebraic stacks $\mathcal{S}_{P/Q}$ introduced in Section 5, whose category of quasi-coherent sheaves

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is equivalent to a category of graded modules (see Proposition 6.5). In Section 7, we explain how to calculate the cohomologies of quasi-coherent sheaves on $\mathcal{L}og_S$.

§ 2. Notation and conventions

As for algebraic stacks, we follow the convention of [1]. An algebraic stack over a scheme S is a stack in groupoids over $(Sch/S)_{fppf}$ such that the diagonal $\Delta\colon \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable by algebraic spaces, and that there exists a surjective smooth morphism from a scheme $U \in \mathrm{Ob}((Sch/S)_{fppf})$ to $\mathcal{X}$. This definition might be different from that of other literatures. Especially, we do not assume the diagonal morphism of algebraic stacks to be quasi-compact and quasi-separated as in [4]. The main reason that we do not impose such condition is that the stacks $\mathcal{L}og_S$ introduced by Olsson for log schemes are not quasi-separated in general ([7, 3.17]).

When S is a scheme and P is a commutative monoid, we write $S[P]$ for the scheme $S \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}[P])$. By abuse of notation, we often use the same symbol for the corresponding morphisms in different categories. For example, let R be a ring and $\alpha\colon P \to Q$ be a morphism of monoids. Then we also denote by α the induced morphism of monoid rings $R[P] \to R[Q]$.

Let E be a fibered category over the category of S-schemes. We denote by E_T the fiber category over an S-scheme $f\colon T \to S$. For an object x in the fiber category E_T, we denote by (T, x) the corresponding object in the category E.

All monoids are commutative with unit in this paper.

§ 3. Log schemes

We review the definition of log schemes. In this section, we always assume that sheaves on schemes are defined in the étale topology.

Definition 3.1. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be schemes.

1. A *prelog structure* on $(X, \mathcal{O}_X)$ is a homomorphism of sheaves of monoids $\alpha\colon \mathcal{M}_X \to \mathcal{O}_X$.
2. A *morphism of prelog structures* on $(X, \mathcal{O}_X)$ is a homomorphism of sheaves of monoids $f\colon \mathcal{M}_X \to \mathcal{M}'_X$ such that the diagram

$$\begin{tikzcd} \mathcal{M}_X \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & \mathcal{M}'_X \arrow[dl, "\alpha'"] \\ & \mathcal{O}_X & \end{tikzcd}$$

   is commutative.

3. A *log structure* on $(X, \mathcal{O}_X)$ is a prelog structure such that the induced morphism $\alpha\colon \alpha^{-1}\mathcal{O}_X \to \mathcal{O}_X$ is an isomorphism.

4. A *log scheme* is a scheme X endowed with a log structure. We denote by $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$ the log structure on X.

5. A *morphism of log schemes* is a morphism $f\colon X \to Y$ of underlying schemes together with a morphism $f_M\colon f^{-1}\mathcal{M}_Y \to \mathcal{M}_X$ such that the diagram

$$\begin{tikzcd} f^{-1}\mathcal{M}_Y \arrow[r, "f_M"] \arrow[d, "f^{-1}\alpha_Y"'] & \mathcal{M}_X \arrow[d, "\alpha_X"] \\ f^{-1}\mathcal{O}_Y \arrow[r, "f_R"'] & \mathcal{O}_X \end{tikzcd}$$

is commutative, where f_R is the morphism of structure sheaves $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$.

For a log scheme S, we denote by $\underline{S}$ the underlying scheme.

There exists a functor $\mathcal{M} \mapsto \mathcal{M}^a$ from the category of prelog structures to the category of log structures which is the left adjoint of the inclusion functor. For a prelog structure $\mathcal{M}$, $\mathcal{M}^a$ is called the log structure associated to $\mathcal{M}$.

Definition 3.2. Let $\alpha\colon \mathcal{M} \to \mathcal{O}_X$ be a log structure on a scheme X. A *chart for* α is a morphism of prelog structures $\theta\colon P \to \mathcal{M}$ such that P is a constant sheaf of monoids and that $\theta^a\colon P^a \to \mathcal{M}$ is an isomorphism.

Definition 3.3.

1. An *integral monoid* is a monoid which can be embedded into an abelian group.

2. A *fine monoid* is a finitely generated integral monoid.

Definition 3.4.

1. A *fine log structure* is a log structure $\mathcal{M}$ which locally admits a chart $P \to \mathcal{M}$ such that P is a fine monoid.

2. A *fine log scheme* is a scheme X endowed with a fine log structure.

We can define log structures using other topologies. In [7, A.1], Olsson proved that there is a natural equivalence between the category of fine log structures on the fppf site and the category of fine log structures on the étale site. This result is used to prove that the fibered category $\mathcal{L}og_S$ is an algebraic stack in the fppf topology.

Definition 3.5.

1. A morphism of log schemes $f\colon X \to Y$ is said to be *strict* if the morphism of log structures $f^*\mathcal{M}_Y := (f^{-1}\mathcal{M}_Y)^a \to \mathcal{M}_X$ induced by the morphism $f_M : f^{-1}\mathcal{M}_Y \to \mathcal{M}_X$ is an isomorphism.

2. An *exact closed immersion* is a strict morphism whose underlying morphism is a closed immersion.

Definition 3.6.

1. A morphism of integral monoids $Q \to P$ is said to be *integral* if for any integral monoids Q' and any morphism $Q \to Q'$, the pushout $Q' \oplus_Q P$ is an integral monoid, where the pushout is taken in the category of monoids.

2. A morphism of log schemes $f\colon X \to Y$ is said to be *integral* if for any geometric point of X, the stalk of the morphism $f^*\mathcal{M}_Y \to \mathcal{M}_X$ is integral.

Definition 3.7. Let $f\colon X \to Y$ be a morphism of prelog schemes. We define the *sheaf of differentials* by the formula

$$\Omega^1_{X/Y} = (\Omega^1_{\underline{X}/\underline{Y}} \oplus (\mathcal{O}_X \otimes_{\mathbf{Z}} \mathcal{M}_X^{gp}))/R,$$

where R is the $\mathcal{O}_X$-submodule generated by sections of the form

$$(d\alpha_X(m), -\alpha_X(m) \otimes m) \quad \text{for} \quad m \in \mathcal{M}_X, \quad (0, 1 \otimes f_M(n)) \quad \text{for} \quad n \in f^*(\mathcal{M}_Y).$$

Recall that $\underline{X}$ (resp. $\underline{\mathrm{Y}}$) denotes the underlying scheme of the log scheme X (resp. Y).

§ 4. The algebraic stack $\mathcal{L}og_S$

In this section, we give a brief summary of results of Olsson [7]. We give a definition of the algebraic moduli stack of log structures $\mathcal{L}og_S$ for a log scheme S and explain how this algebraic stack is related to logarithmic geometry. As in the previous section, we denote by $\underline{S}$ the underlying scheme of S.

Throughout this and the following sections, we assume that S is a fine log scheme unless otherwise stated.

Definition 4.1 (See the paragraph above [7, 1.1])**.** We denote by $\mathcal{L}og_S$ the fibered category over the category of $\underline{S}$-schemes whose fiber over an $\underline{S}$-scheme $f\colon T \to \underline{S}$ is the groupoid of pairs $(\mathcal{N}, \eta)$, where $\mathcal{N}$ is a fine log structure on T and $\eta\colon f^*\mathcal{M}_S \to \mathcal{N}$ is a morphism of log structures. The morphism $h\colon (T, \mathcal{N}, \eta) \to (T', \mathcal{N}', \eta')$ is a morphism of log schemes over S such that $h^*\mathcal{N}' \to \mathcal{N}$ is an isomorphism.

Let T be an $\underline{S}$-scheme. By the definition of $\mathcal{L}og_S$, giving a morphism $T \to \mathcal{L}og_S$ over $\underline{S}$ is equivalent to giving a log structure $\mathcal{N}$ on T and a morphism of log schemes $(T, \mathcal{N}) \to S$.

Proposition 4.2 ([7, 1.1]). *The fibered category $\mathcal{L}og_S$ is an algebraic stack.*

A morphism of fine log schemes $f\colon X \to S$ naturally defines a representable morphism of algebraic stacks $\mathcal{L}og(f) : \mathcal{L}og_X \to \mathcal{L}og_S$ (see the paragraph at the beginning of [7, Section 4]), thus the association $S \mapsto \mathcal{L}og_S$ defines a 2-functor from the category of log schemes to the 2-category of algebraic stacks.

Let S be a fine log scheme and $\underline{S}$ be the underlying scheme of S. Let $\underline{f}\colon \underline{X} \to \underline{S}$ be a morphism of schemes. By the definition of $\mathcal{L}og_S$, there exists a one-to-one correspondence between the set of morphisms of stacks $\mathrm{Hom}_{\underline{S}}(\underline{X}, \mathcal{L}og_S)$ and the set of pairs $(\mathcal{M}, f)$ where $\mathcal{M}$ is a log structure on $\underline{X}$ and f is a morphism of log schemes $(\underline{X}, \mathcal{M}) \to S$ whose underlying morphism of schemes is $\underline{f}$. For a fine log scheme X, the morphism $\underline{X} \to \mathcal{L}og_X$ corresponding to the identity map $X \to X$ is known to be an open immersion [7, 3.19].

Proposition 4.3 ([7, 3.20]). *Let*

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

be a cartesian diagram in the category of fine log schemes. Then the induced diagram of algebraic stacks

$$\begin{tikzcd} \mathcal{L}og_Z \arrow[r] \arrow[d] & \mathcal{L}og_X \arrow[d] \\ \mathcal{L}og_Y \arrow[r] & \mathcal{L}og_S \end{tikzcd}$$

is also cartesian.

Definition 4.4. Let $\mathcal{P}$ be a property of representable morphisms of algebraic stacks. Let $f\colon X \to S$ be a morphism of fine log schemes.

1. We say that $f\colon X \to S$ has property $\mathcal{L}og(\mathcal{P})$ if the morphism of stacks $\mathcal{L}og(f)$ has property $\mathcal{P}$.

2. We say that $f\colon X \to S$ has property weakly $\mathcal{L}og(\mathcal{P})$ if the morphism of stacks $\underline{X} \to \mathcal{L}og_S$ corresponding to f has property $\mathcal{P}$.

For example, the property $\mathcal{P}$ could be the property of being étale, smooth, or flat. In these cases, we call the property $\mathcal{L}og(\mathcal{P})$ $\mathcal{L}$og étale, $\mathcal{L}$og smooth, and $\mathcal{L}$og flat respectively. If the property $\mathcal{P}$ is preserved by restricting to open substacks (which is the case when $\mathcal{P}$ is one of the above three properties), the property $\mathcal{L}og(\mathcal{P})$ implies the property weakly $\mathcal{L}og(\mathcal{P})$. Also, there are notions of log étale, log smooth and log flat morphisms of log schemes which is defined by K. Kato ([2] and [3]). Olsson proved ([7, 4.6]) the following equivalence among them.

Theorem 4.5 ([7, 4.6]). *Let $f\colon X \to S$ be a morphism of fine log schemes. Then the followings are equivalent:*

1. *The morphism f is $\mathcal{L}og$ smooth (resp. $\mathcal{L}og$ étale, $\mathcal{L}og$ flat).*
2. *The morphism f is weakly $\mathcal{L}og$ smooth (resp. weakly $\mathcal{L}og$ étale, weakly $\mathcal{L}og$ flat).*
3. *The morphism f is log smooth (resp. log étale, log flat) in the sense of K. Kato.*

§ 5. The local structure of $\mathcal{L}og_S$

Throughout this section, we fix a fine log scheme S and a global chart $\lambda : Q \to \mathcal{M}_S$ of S such that Q is a fine monoid. In this section, we explain a result in [7] concerning the local structure of the stack $\mathcal{L}og_S$. We denote the composite $Q \to \mathcal{O}_S$ of $\lambda : Q \to \mathcal{M}_S$ and the structure morphism $\mathcal{M}_S \to \mathcal{O}_S$ by λ, by abuse of notation. We use the same symbol λ to denote the morphism of schemes $\underline{S} \to \underline{S}[Q]$ induced by $\lambda : Q \to \mathcal{O}_S$. For a fine monoid P equipped with a morphism of monoids $l : Q \to P$, we use the same symbol l to denote the morphism of schemes $\underline{S}[P] \to \underline{S}[Q]$ induced by l.

Definition 5.1 (See [5, 4.1] and the paragraph above [7, 5.12]). Let P be a fine monoid equipped with a morphism of monoids $l : Q \to P$.

1. We denote by $\mathcal{S}_P$ the stack theoretic quotient of $\underline{S}[P]$ by the action of $\underline{S}[P^{gp}]$. It is denoted by $[\underline{S}[P]/\underline{S}[P^{gp}]]$ in [4]. The natural morphism $\underline{S}[P] \to \mathcal{S}_P$ is denoted by π. The morphism $l : \underline{S}[P] \to \underline{S}[Q]$ induces the morphism of stacks $\mathcal{S}_P \to \mathcal{S}_Q$. By abuse of notation, the induced morphism is also denote by l. The log structure on $\underline{S}[P]$ induced by the natural monoid morphism $P \to \mathbb{Z}[P]$ descends to $\mathcal{S}_P$ by the action of $\underline{S}[P^{gp}]$. We denote by $\mathcal{M}_{\mathcal{S}_P}$ the log structure on $\mathcal{S}_P$.
2. The algebraic stack $\mathcal{S}_{P/Q}$ is defined by the following cartesian diagram

$$\begin{tikzcd} \mathcal{S}_{P/Q} \arrow[r] \arrow[d] & \mathcal{S}_P \arrow[d, "l"] \\ \underline{S} \arrow[r, "\lambda"] & \mathcal{S}_Q, \end{tikzcd}$$

where the morphisms $\lambda : \underline{S} \to \mathcal{S}_Q$ is the composite

$$\underline{S} \xrightarrow{\lambda} \underline{S}[Q] \xrightarrow{\pi} \mathcal{S}_Q.$$

3. We denote by $\underline{S}_Q[P]$ the scheme defined by the following cartesian diagram

$$\begin{tikzcd} \underline{S}_Q[P] \arrow[r] \arrow[d] & \underline{S}[P] \arrow[d, "l"] \\ \underline{S} \arrow[r, "\lambda"] & \underline{S}[Q]. \end{tikzcd}$$

Note that $\mathcal{S}_{P/Q}$ and $\underline{S}_Q[P]$ depend not only on P and Q, but also on l and λ.

The morphisms $\pi : \underline{S}[P] \to \mathcal{S}_P$ and $\pi : \underline{S}[Q] \to \mathcal{S}_Q$ induce the morphism $\underline{S}_Q[P] \to \mathcal{S}_{P/Q}$ which we also denote by π.

The quotient stacks defined above have the following properties.

Lemma 5.2 ([5, 4.3]). *Let P be a fine monoid.*

1. *Let $\mathrm{pr} : \underline{S}[P \oplus P^{gp}] \to \underline{S}[P]$ be the canonical projection induced by the inclusion of monoids $P \to P \oplus P^{gp}$ to the first component which sends $x \in P$ to $(x, 0) \in P \oplus P^{gp}$. Let $a : \underline{S}[P \oplus P^{gp}] \to \underline{S}[P]$ be the morphism induced by the morphism of monoids $P \to P \oplus P^{gp}$ which sends $x \in P$ to $(x, x) \in P \oplus P^{gp}$. Then, the following diagram is cartesian*

$$\begin{tikzcd} \underline{S}[P \oplus P^{gp}] \arrow[r, "\mathrm{pr}"] \arrow[d, "a"'] & \underline{S}[P] \arrow[d, "\pi"] \\ \underline{S}[P] \arrow[r, "\pi"] & \mathcal{S}_P. \end{tikzcd}$$

2. *If P is a monoid equipped with an injective morphism of monoids $l : Q \to P$, the diagram*

$$\begin{tikzcd} (\underline{S}_Q[P])[P^{gp}/Q^{gp}] \arrow[r, "\mathrm{pr}"] \arrow[d, "a"'] & \underline{S}_Q[P] \arrow[d, "\pi"] \\ \underline{S}_Q[P] \arrow[r, "\pi"] & \mathcal{S}_{P/Q} \end{tikzcd}$$

is cartesian, where $\mathrm{pr} : (\underline{S}_Q[P])[P^{gp}/Q^{gp}] \to \underline{S}_Q[P]$ is the canonical projection induced by the inclusion of monoids $P \to P \oplus P^{gp}/Q^{gp}$ to the first component and $a : (\underline{S}_Q[P])[P^{gp}/Q^{gp}] \to \underline{S}_Q[P]$ is the morphism induced by the morphism of monoids $P \to P \oplus P^{gp}/Q^{gp}$ which sends $x \in P$ to $(x, x + Q^{gp}) \in P \oplus (P^{gp}/Q^{gp})$.

We can give the description of $\mathcal{S}_P$ and $\mathcal{S}_{P/Q}$ as fibered categories using log structures and morphisms.

For a sheaf of monoids $\mathcal{N}$, we denote by $\overline{\mathcal{N}}$ the quotient of $\mathcal{N}$ by the subsheaf of invertible elements.

Definition 5.3 ([7, 5.14, 5.20]). Let P be a monoid equipped with an injective morphism of monoids $l : Q \to P$.

1. We denote by $\mathcal{S}'_P$ the fibered category over the category of $\underline{S}$-schemes whose fiber over an $\underline{S}$-scheme $f : T \to \underline{S}$ is the groupoid of pairs $(\mathcal{N}, \gamma)$, where $\mathcal{N}$ is a fine log structure on T and $\gamma : P \to \overline{\mathcal{N}}$ is a morphism which fppf locally lifts to a chart of $\mathcal{N}$. A morphism $h : (T, \mathcal{N}, \gamma) \to (T', \mathcal{N}', \gamma')$ is a strict morphism of log schemes $(h, h_M) : (T, \mathcal{N}) \to (T', \mathcal{N}')$ such that the diagram
$$\begin{tikzcd} P \arrow[r, equal] \arrow[d, "h^{-1}\gamma'"'] & P \arrow[d, "\gamma"] \\ h^{-1}\overline{\mathcal{N}'} \arrow[r, "\overline{h_M}"] & \overline{\mathcal{N}} \end{tikzcd}$$
is commutative.

2. We denote by $\mathcal{S}'_{P/Q}$ the fibered category over the category of $\underline{S}$-schemes whose fiber over an $\underline{S}$-scheme $f : T \to \underline{S}$ is the groupoid of triplets $(\mathcal{N}, \eta, \gamma)$, where $\mathcal{N}$ is a fine log structure on T, $\gamma : P \to \overline{\mathcal{N}}$ is a morphism which fppf locally lifts to a chart of $\mathcal{N}$, and $\eta : f^*\mathcal{M}_S \to \mathcal{N}$ is a morphism of log structures such that the diagram
$$\begin{tikzcd} Q \arrow[r, "l"] \arrow[d, "f^{-1}\lambda"'] & P \arrow[d, "\gamma"] \\ f^{-1}\overline{\mathcal{M}}_S \arrow[r, "\overline{\eta}"'] & \overline{\mathcal{N}} \end{tikzcd}$$
is commutative. A morphism $h : (T, \mathcal{N}, \eta, \gamma) \to (T', \mathcal{N}', \eta', \gamma')$ is a strict morphism of log schemes $(h, h_M) : (T, \mathcal{N}) \to (T', \mathcal{N}')$ over the log scheme $(S, \mathcal{M}_S)$ such that the diagram
$$\begin{tikzcd} P \arrow[r, equal] \arrow[d, "h^{-1}\gamma'"'] & P \arrow[d, "\gamma"] \\ h^{-1}\overline{\mathcal{N}'} \arrow[r, "\overline{h_M}"] & \overline{\mathcal{N}} \end{tikzcd}$$
is commutative.

Let $t : T \to \mathcal{S}_P$ be a morphism from a scheme T to the algebraic stack $\mathcal{S}_P$. Then, we obtain a pair $(t^*\mathcal{M}_{\mathcal{S}_P}, t^*\pi_P)$, where $t^*\mathcal{M}_{\mathcal{S}_P}$ is a fine log structure on T and the morphism $t^*\pi_P : P \to \overline{t^*\mathcal{M}}_{\mathcal{S}_P}$ which lifts fppf locally to a chart of $t^*\mathcal{M}_{\mathcal{S}_P}$. This induces the morphism of stacks $\mathcal{S}_P \to \mathcal{S}'_P$. Similarly, there are natural morphisms of stacks $\mathcal{S}_{P/Q} \to \mathcal{S}'_{P/Q}$.

Proposition 5.4 ([7, 5.14, 5.20]). *We continue to use the same assumptions as in Definition 5.3.*

1. *The natural morphism $\mathcal{S}_P \to \mathcal{S}'_P$ is an isomorphism of stacks.*
2. *The natural morhpism $\mathcal{S}_{P/Q} \to \mathcal{S}'_{P/Q}$ is an isomorphism of stacks.*

We shall identify the algebraic stacks $\mathcal{S}_P$ and $\mathcal{S}'_P$ by the natural isomorphism. There exists a natural morphism of algebraic stacks $\pi : \mathcal{S}_{P/Q} \to \mathcal{L}og_S$ which sends an object $(T, \mathcal{N}, \eta, \gamma) \in \mathrm{Ob}(\mathcal{S}_{P/Q})$ to $(T, \mathcal{N}, \eta) \in \mathrm{Ob}(\mathcal{L}og_S)$.

Proposition 5.5 ([7, 5.25]). *Let J be the set of isomorphism classes of pairs (P, l) where P is a fine monoid and $l : Q \to P$ is a morphism of monoids. Then the natural morphism*

$$\coprod_{(P,l)\in J} \mathcal{S}_{P/Q} \longrightarrow \mathcal{L}og_S$$

is representable, étale and surjective.

We explain relations between the algebraic stacks $\mathcal{S}_{P/Q}$ and log geometry. Suppose that the underlying scheme of S is an affine scheme $\mathrm{Spec}(R)$. Let us consider the morphism of affine schemes $\mathrm{Spec}(A) \to \underline{S}_Q[P]$. A morphism $\mathrm{Spec}(A) \to \underline{S}_Q[P]$ corresponds to the commutative diagram of morphisms of rings

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & R \arrow[d] \\ P \arrow[r] & A. \end{tikzcd}$$

This diagram defines a morphism of log schemes having a chart of the form $Q \to P$. Therefore, roughly speaking, we can say that the scheme $\underline{S}_Q[P]$ classifies morphisms of log schemes having a chart of the form $Q \to P$. However, there might exist infinitely many different morphisms from P to A which represent the same log structure on $\mathrm{Spec}(A)$. Thus the morphism $\underline{S}_Q[P] \to \mathcal{L}og_S$ is not étale in general. That is the reason why we take the stack theoretic quotient to define the algebraic stack $\mathcal{S}_{P/Q}$.

Recall that the objects of the fibered category $\mathcal{L}og_S$ corresponds to the morphisms of log schemes $T \to S$. The composite $\underline{S}_Q[P] \to \mathcal{S}_{P/Q} \xrightarrow{\pi} \mathcal{L}og_S$ is defined to be the morphism which converts the diagram above to the morphisms of log schemes. The fact that the morphism in Proposition 5.5 is surjective corresponds to the fact that every morphism of fine log schemes $T \to S$ fppf locally has a chart of the form $Q \to P$ for a fixed Q.

§ 6. The category $\mathcal{D}$

In this and the next section, we explain the author's results in [5]. In the previous section, we explained how the algebraic stack $\mathcal{L}og_S$ is covered by the algebraic stacks of the form $\mathcal{S}_{P/Q}$. By studying how those algebraic stacks are glued together, we have a way to describe quasi-coherent sheaves on $\mathcal{L}og_S$.

For this purpose, we shall introduce a category $\mathcal{D}_Q$ for a fine monoid Q. The objects of the category $\mathcal{D}_Q$ corresponds to the stacks $\mathcal{S}_{P/Q}$ in the covering. The morphisms of the category $\mathcal{D}_Q$ represent relations between these stacks.

For a monoid M, we write $\overline{M}$ for the quotient of M by all the invertible elements of M. We denote by **FMon** the category of fine monoids. When Q is a fine monod, we denote by ${}^{Q/}\mathbf{FMon}$ the under category, i.e. the category whose objects are morphism of fine monoids $Q \to P$.

We say a monoid M is sharp when $\overline{M}$ is isomorphic to M.

Definition 6.1. Let M be a monoid, S a subset of M, and E an M-set. The *localization of E by S* is a morphism of M-sets $\lambda_S \colon E \to S^{-1}E$ which satisfies the following universal property: For any morphism of M-sets $E \to E'$ such that each $s \in S$ acts bijectively on E', there is a unique morphism of M-sets $S^{-1}E \to E'$ which makes the diagram

$$\begin{tikzcd} E \arrow[r, "\lambda_S"] \arrow[dr] & S^{-1}E \arrow[d] \\ & E' \end{tikzcd}$$

commutative.

Definition 6.2. Let $f \colon M \to M'$ be a morphism of monoids. We say f is a localization if it is a localization of M-sets.

Definition 6.3. Let Q be a fine monoid.

1. The *category $\mathcal{D}'_Q$* is defined as follows. The objects of $\mathcal{D}'_Q$ are sharp monoids in ${}^{Q/}\mathbf{FMon}$. A morphism of $\mathcal{D}'_Q$ is a morphism of sharp monoids $f \colon P \to P'$ in ${}^{Q/}\mathbf{FMon}$ such that there exists a localization $f' \colon P \to P''$ and an isomorphism $i \colon P' \to \overline{P''}$ in ${}^{Q/}\mathbf{FMon}$ which make the diagram

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "f"'] & P'' \arrow[d, "\pi"] \\ P' \arrow[r, "\sim", "i"'] & \overline{P''} \end{tikzcd}$$

commutative, where π is the natural projection.

2. The *category* $\mathcal{D}_Q$ is a small category equivalent to $\mathcal{D}'_Q$ whose objects are the isomorphism classes of sharp monoids in ${}^{Q/}\mathbf{FMon}$.

3. For any objects P and P' of $\mathcal{D}_Q$, we say $P' \leq P$ if and only if $\mathrm{Mor}_{\mathcal{D}_Q}(P, P')$ is not empty. Note that this gives a *structure of partially ordered set on the objects of* $\mathcal{D}_Q$, because any morphism of $\mathcal{D}'_Q$ is surjective.

4. Let P be an object of $\mathcal{D}_Q$. A *category* $\mathcal{D}_Q^P$ is a full subcategory of $\mathcal{D}_Q$ whose objects are the objects of $\mathcal{D}_Q$ which are smaller than P.

Since P is finitely generated, the number of objects of the category $\mathcal{D}_Q^P$ is finite.

Example 6.4. When $Q = 0$ and $P = \mathbb{N}^2$, the category $\mathcal{D}_Q^P$ has three objects $\overline{\mathbb{N}^2}$, $\overline{\mathbb{N} \oplus \mathbb{Z}}$, and $\overline{\mathbb{Z}^2}$. Let $\sigma\colon \mathbb{N}^2 \to \mathbb{N}^2$ be a morphism of monoids exchanging two components of $\mathbb{N}^2$. Let $\alpha\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{Z}$ be a morphism of monoids which sends $(x, y) \in \mathbb{N} \times \mathbb{N}$ to $(x, y) \in \mathbb{N} \times \mathbb{Z}$. We put $\beta = \alpha \circ \sigma$. We denote by γ the immersion $\mathbb{N} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$.

The category $\mathcal{D}'_Q$ can be visualized as follows

$$\begin{tikzcd} \overline{\mathbb{N}^2} \arrow[loop left, "\overline{\sigma}"] \arrow[r, shift left, "\overline{\alpha}"] \arrow[r, shift right, "\overline{\beta}"'] & \overline{\mathbb{N} \oplus \mathbb{Z}} \arrow[r, "\overline{\gamma}"] & \overline{\mathbb{Z}^2}. \end{tikzcd}$$

As in the previous section, we fix a fine log scheme S and a global chart $\lambda\colon Q \to \mathcal{M}_S$ of S such that Q is a fine monoid. In order to study quasi-coherent sheaves on $\mathcal{L}og_S$, first we study quasi-coherent sheaves on $\mathcal{S}_{P/Q}$.

By Lemma 5.2 (2), there exists a cartesian diagram

$$\begin{tikzcd} (\underline{S}_Q[P])[P^{gp}/Q^{gp}] \arrow[r, "\mathrm{pr}"] \arrow[d, "a"'] & \underline{S}_Q[P] \arrow[d, "\pi"] \\ \underline{S}_Q[P] \arrow[r, "\pi"] & \mathcal{S}_{P/Q}. \end{tikzcd}$$

Using the theory of fppf descent, we can express quasi-coherent sheaves on $\mathcal{S}_{P/Q}$ by the descent datum relative to the above cartesian diagram. The theory says that a quasi-coherent sheaf on $\mathcal{S}_{P/Q}$ can be canonically identified with a quasi-coherent sheaf on $\underline{S}_Q[P]$ which coincides on the "intersection" $(\underline{S}_Q[P])[P^{gp}/Q^{gp}]$.

For a ring R, the monoid ring $R[P]$ has a natural structure of P^{gp}-graded ring. This grading and the surjective morphism $R[P] \to R \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P]$ induces a well-defined P^{gp}/Q^{gp}-graded ring structure of the ring $R \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P]$. We regard the ring $R \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P]$ to be P^{gp}/Q^{gp}-graded by this induced structure.

Proposition 6.5. *Suppose that $\underline{S}$ is affine and let R be the ring such that $\mathrm{Spec}(R) = \underline{S}$. Let $l\colon Q \to P$ be an injective morphism of fine monoids. The category of quasi-coherent sheaves on the algebraic stack $\mathcal{S}_{P/Q}$ is canonically equivalent to the category of P^{gp}/Q^{gp}-graded $R \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P]$-modules.*

We denote by $\mathcal{L}og_S^P$ the open substack of $\mathcal{L}og_S$ which is the image of $\mathcal{S}_{P/Q}$ by the morphism $\mathcal{S}_{P/Q} \to \mathcal{L}og_S$.

Definition 6.6. Let $(Alg.stacks_{/\underline{S}})$ denote the category of algebraic stacks over $\underline{S}$. We define the fibered category $p\colon QCoh \to (Alg.stacks_{/\underline{S}})$ as follows:

1. An object of $QCoh$ is a pair $(\mathfrak{X}, \mathcal{F})$, where $\mathfrak{X}$ is an object of $(Alg.stacks_{/\underline{S}})$, and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module.

2. A morphism $(f, \varphi)\colon (\mathfrak{Y}, \mathcal{G}) \to (\mathfrak{X}, \mathcal{F})$ is a pair such that $f\colon \mathfrak{Y} \to \mathfrak{X}$ is a morphism of algebraic stacks over $\underline{S}$ and $\varphi \in \mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(f^*\mathcal{G}, \mathcal{F})$

The fiber category of $QCoh$ over an algebraic stack $\mathfrak{X}$ is the category $QCoh(\mathfrak{X})$ of quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules.

Definition 6.7. Let $\mathcal{S}\colon \mathcal{D}_Q \to (Alg.stacks_{/\underline{S}})^{\mathrm{opp}}$ be the covariant functor which sends an object P to the algebraic stack $\mathcal{S}_{P/Q}$ and sends a morphism $\alpha\colon P \to P'$ to the opposite of the natural morphism $\mathcal{S}_{P'/Q} \to \mathcal{S}_{P/Q}$ induced by α.

1. We denote by $\mathcal{E}_Q$ the fiber product $\mathcal{D}_Q \times_{(Alg.stacks_{/\underline{S}})^{\mathrm{opp}}} QCoh^{\mathrm{opp}}$. More specifically, an object of $\mathcal{E}_Q$ is a pair $(P, \mathcal{F})$, where P is an object of $\mathcal{D}_Q$ and $\mathcal{F}$ is a quasi-coherent sheaf on $\mathcal{S}_{P/Q}$.

2. We denote by $\mathcal{E}_Q^P$ the fiber product $\mathcal{D}_Q^P \times_{(Alg.stacks_{/\underline{S}})^{\mathrm{opp}}} QCoh^{\mathrm{opp}}$.

Then we obtain structures of the cofibered categories $\mathcal{E}_Q \to \mathcal{D}_Q$ and $\mathcal{E}_Q^P \to \mathcal{D}_Q^P$.

Definition 6.8.

1. We say a morphism $f\colon a \to b$ in a category $\mathcal{C}$ is cocartesian with respect to a functor $F\colon \mathcal{C} \to \mathcal{D}$ if for every object c of $\mathcal{C}$, and every pair of morphisms $g\colon a \to c$ and

$u\colon F(b) \to F(c)$ such that $F(g) = u \circ F(f)$, there exists a unique morphism $h\colon b \to c$ such that $h \circ f = g$ and $F(h) = u$,

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[rr, "g"'] \arrow[d, dash, dashed] & b \arrow[r, "h"] \arrow[d, dash, dashed] & c \arrow[d, dash, dashed] \\
F(a) \arrow[r, "F(f)"] \arrow[rr, "F(g)"'] & F(b) \arrow[r, "u"] & F(c).
\end{tikzcd}$$

2. Let $F\colon \mathcal{C} \to \mathcal{D}$ and $F'\colon \mathcal{C}' \to \mathcal{D}$ be functors between categories. A functor $G\colon \mathcal{C} \to \mathcal{C}'$ such that $F = F' \circ G$ is a *cocartesian functor* if it takes cocartesian morphisms with respect to F to cocartesian morphisms with respect to F'. We denote by $\mathrm{Cocart}_{\mathcal{D}}(\mathcal{C}, \mathcal{C}')$ the category of cocartesian functors, with natural transformations as morphisms.

Using the categories defined above, we describe the category of quasi-coherent sheaves on $\mathcal{L}og_S$.

Theorem 6.9. *There exist equivalences of categories*

$$QCoh(\mathcal{L}og_S^P) \xrightarrow{\sim} Cocart_{\mathcal{D}_Q^P}(\mathcal{D}_Q^P, \mathcal{E}_Q^P)$$

and

$$QCoh(\mathcal{L}og_S) \xrightarrow{\sim} Cocart_{\mathcal{D}_Q}(\mathcal{D}_Q, \mathcal{E}_Q).$$

Theorem 6.9 is one of the main theorems of [5]. Theorem 6.9 implies that an object of $QCoh(\mathcal{L}og_S)$ is represented by a diagram of modules indexed by $\mathcal{D}_Q$ satisfying certain compatibility conditions.

§ 7. Calculus of cohomology

In this section, we study the cohomology of quasi-coherent sheaves on the algebraic stack $\mathcal{L}og_S^P$. First we construct a certain hypercovering of $\mathcal{L}og_S^P$, which is a generalization of a covering for which the cohomological descent works. Then we calculate the cohomology of $\mathcal{L}og_S^P$ by the theory of cohomological descent.

First we have the following vanishing result on the cohomologies of quasi-coherent sheaves on $\mathcal{S}_{P/Q}$.

Theorem 7.1. *Let $l\colon Q \to P$ be an injective morphism of fine monoids and $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{S}_{P/Q}$. Let $N = \bigoplus_{U \in P^{gp}/Q^{gp}} N_U$ be the P^{gp}/Q^{gp}-graded module corresponding to $\mathcal{F}$ via Proposition 6.5. If the underlying scheme $\underline{S}$ of S is affine,*

$$H^i(\mathcal{S}_{P/Q}, \mathcal{F}) = \begin{cases} 0 & (i > 0), \\ N_0 & (i = 0). \end{cases}$$

The main tool we use to compute the cohomologies on $\mathcal{L}og_S^P$ is the theory of cohomological descent. First, we construct a hypercovering of $\mathcal{L}og_S^P$. A hypercovering is a generalization of a covering, and is a special case of a simplicial augmentation where one has cohomological descent. Cohomological descent can be considered as a generalization of calculation of Čech cohomology.

Theorem 7.2. *There exists an étale hypercovering $X_\bullet \to \mathcal{L}og_S^P$ satisfying the following conditions:*

1. *The object X_i is of the form $\coprod_{k \in J_i} \mathcal{S}_{P'_k/Q}$ where $P'_k \in \mathcal{D}_Q^P$ and J_i is a finite set.*
2. *A morphism $X_i \to X_j$ is a morphism induced by a morphism $f\colon J_i \to J_j$ and open immersions $\alpha_k\colon \mathcal{S}_{P'_k/Q} \to \mathcal{S}_{P'_{f(k)}/Q}$ for each $k \in J_i$ induced from a morphism in $\mathcal{D}_Q^P$.*

Corollary 7.3. *Let $f\colon X_\bullet \to \mathcal{L}og_S^P$ be an étale hypercovering as in Theorem 7.2. We denote by f_p the morphism $X_p \to \mathcal{L}og_S^P$ induced by f. Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{L}og_S^P$. Then there is a spectral sequence*

$$E_1^{pq} = H^q(X_p, f_p^*\mathcal{F}) \Rightarrow H^{p+q}(\mathcal{L}og_S^P, \mathcal{F}),$$

which degenerates at E_2.

The degeneration of the spectral sequence is a consequence of the vanishing of cohomology for quasi-coherent sheaves on $\mathcal{S}_{P/Q}$ (Theorem 7.1).

The hypercovering $X_\bullet \to \mathcal{L}og_S^P$ is constructed by an inductive procedure. First we put $X_0 = \mathcal{S}_{P/Q}$. Suppose that an m-truncated simplicial object $X_\bullet^{\leq m}$ satisfying the conditions in the theorem is defined. Then we can construct an $(m+1)$-truncated simplicial object $X_\bullet^{\leq m+1}$ as follows:

1. For $i \leq m$, set $X_i^{\leq m+1} = X_i^{\leq m}$. We simply put $X_i = X_i^{\leq m+1} = X_i^{\leq m}$.
2. Take an étale covering $\coprod_{k \in J_{m+1}} \mathcal{S}_{P'_k/Q} \to (\operatorname{cosk}_m(X_\bullet^{\leq m}))_{m+1}$, where cosk_m is the coskelet functor on augmented m-truncated simplicial objects.
3. Modify the covering above and put $X_{m+1}^{\leq m+1} = \coprod_{k \in J_{m+1}} \mathcal{S}_{P'_k/Q}$ so that $X_\bullet^{\leq m+1}$ becomes an $(m+1)$-truncated simplicial object satisfying the conditions in the theorem.

The key point is that the algebraic stack $(\mathrm{cosk}_m(X_\bullet^{\leq m}))_{m+1}$ has an étale covering by finitely many algebraic stacks of the form $\mathcal{S}_{P'/Q}$ for each m.

If we regard the stacks $\mathcal{S}_{P/Q}$ as something like affine schemes, the process is an analogy of the following steps.

1. Construct a covering of a scheme by affine schemes.
2. Then, construct a covering of the intersections of those affine schemes by affine schemes again.
3. If necessary, add redundant affine schemes to the covering, and continues the process inductively.

Let us explain the idea of the proof of the second step above. The proof is done by "translating" the meaning of the stacks to the corresponding statement in log geometry. For example, let us consider the algebraic stack $I_{P',P''}$ in the cartesian diagram

$$\begin{tikzcd} I_{P',P''} \arrow[r] \arrow[d] & \mathcal{S}_{P'/Q} \arrow[d] \\ \mathcal{S}_{P''/Q} \arrow[r] & \mathcal{L}og_S. \end{tikzcd}$$

Recall that the objects of $\mathcal{S}_{P/Q}$ have data of morphisms of log schemes represented by morphisms of monoids $Q \to P$. Therefore, the fiber product $I_{P',P''}$ have data of morphisms of log schemes which can be expressed by both of the morphisms $Q \to P'$ and $Q \to P''$. Let J be the set of triples (α, β, P''') where $P''' \in \mathcal{D}_Q$ and $\alpha\colon P' \to P'''$ and $\beta\colon P'' \to P'''$ are morphisms in $\mathcal{D}_Q$. Let $(\alpha, \beta)\colon \mathcal{S}_{P'''/Q} \to I_{P',P''}$ be the morphism defined by the universality of fiber product and the morphisms $\alpha\colon \mathcal{S}_{P'''/Q} \to \mathcal{S}_{P'/Q}$, $\beta\colon \mathcal{S}_{P'''/Q} \to \mathcal{S}_{P''/Q}$. We can prove that the morphism of algebraic stacks

$$\coprod_{(\alpha,\beta,P''') \in J} \mathcal{S}_{P'''/Q} \longrightarrow I_{P',P''}$$

is an étale covering.

§ 8. Examples

The following example shows that the cohomology of quasi-coherent sheaves on $\mathcal{L}og_S^P$ does not vanish in general.

Let $Q = 0$, $P = \mathbb{N}^2$, and $\underline{S} = \mathrm{Spec}(O)$ for a commutative ring O. Let S be a log scheme whose underlying scheme is $\underline{S}$ and the log structure is induced from the zero map $Q \to O$. We denote by σ the automorphism of P which exchanges the two components of $\mathbb{N}^2$. Then, a quasi-coherent sheaf on $\mathcal{L}og_S^P$ is represented by a descent datum $(\mathcal{F}, \varphi)$ with respect to the morphism $\mathcal{S}_P \to \mathcal{L}og_S^P$.

First, we take a 1-truncated étale hypercovering of $\mathcal{L}og_S^P$. We denote by I_P the algebraic stack $\mathcal{S}_P \times_{\mathcal{L}og_S} \mathcal{S}_P$. We denote by pr_0 and pr_1 the projection $I_P \to \mathcal{S}_P$ to the first and second components respectively. Set $\mathcal{A} = \mathcal{B} = \mathcal{S}_P$. We define the morphisms $\mathcal{A} \to I_P$ and $\mathcal{B} \to I_P$ by the following commutative diagrams.

$$
\begin{tikzcd}
\mathcal{A} \arrow[dr] \arrow[drr, "\mathrm{id}"] \arrow[ddr, "\mathrm{id}"'] & & \\
& I_P \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_0"] & \mathcal{S}_P \arrow[d] \\
& \mathcal{S}_P \arrow[r] & \mathcal{L}og_S
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{B} \arrow[dr] \arrow[drr, "\sigma"] \arrow[ddr, "\mathrm{id}"'] & & \\
& I_P \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_0"] & \mathcal{S}_P \arrow[d] \\
& \mathcal{S}_P \arrow[r] & \mathcal{L}og_S,
\end{tikzcd}
$$

where the morphism of algebraic stacks $\sigma\colon \mathcal{B} \to \mathcal{S}_P$ is the morphism induced by the morphism of monoids $\sigma\colon P \to P$. Let d_i be the composite $\mathcal{A} \coprod \mathcal{B} \to I_P \xrightarrow{\mathrm{pr}_i} \mathcal{S}_P$ for $i = 0, 1$, and $s_0\colon \mathcal{S}_P \to \mathcal{A} \coprod \mathcal{B}$ be the immersion induced by the identity morphism $\mathcal{S}_P \to \mathcal{A}$. Then the étale covering $\mathcal{A} \coprod \mathcal{B} \to I_P$ gives a 1-truncated étale hypercovering of $\mathcal{L}og_S^P$:

$$
\begin{tikzcd}
\mathcal{A} \coprod \mathcal{B} \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d] & \mathcal{S}_P \arrow[r] \arrow[d, equal] & \mathcal{L}og_S^P \arrow[d, equal] \\
I_P \arrow[r, shift left, "\mathrm{pr}_0"] \arrow[r, shift right, "\mathrm{pr}_1"'] & \mathcal{S}_P \arrow[r] & \mathcal{L}og_S^P.
\end{tikzcd}
$$

Let $X_\bullet \to \mathcal{L}og_S^P$ be the coskeleton of the 1-truncated hypercovering

$$
\begin{tikzcd}
\mathcal{A} \coprod \mathcal{B} \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & \mathcal{S}_P \arrow[r] & \mathcal{L}og_S^P .
\end{tikzcd}
$$

Then $X_\bullet \to \mathcal{L}og_S^P$ is a hypercovering satisfying the condition of Theorem 7.2.

We denote by Δ^+ the category with objects $[0], [1], [2], \ldots$ with $[n] = \{0, 1, 2, \ldots, n\}$ and morphisms $[n] \to [m]$ are the strictly increasing morphisms of the corresponding sets $\{0, 1, 2, \ldots, n\} \to \{0, 1, 2, \ldots, m\}$. We denote by $(\Delta^+/[n])_{\leq 1}$ the full subcategory of the over category $\Delta^+/[n]$ whose objects are the morphisms $[k] \to [n]$ of Δ^+ such that $k \leq 1$. The objects X_n are described by the formula

$$X_n = \varprojlim_{([k] \leftarrow [n]) \in (\Delta^+/[n])^{opp}_{\leq 1}} X_k,$$

where the projective limit is taken in the 2-category $(Alg.stacks_{/\mathcal{L}og_S^P})$. Calculating explicitely, we see that X_n is a finite disjoint union of $\mathcal{S}_P$'s and $\mathcal{S}_0$'s. The number of $\mathcal{S}_P$'s is 2^n, and the number of $\mathcal{S}_0$'s is $2^{(n+1)n/2} - 2^n$.

Let $(\mathcal{F}, \varphi)$ be a descent datum with respect to the morphism $\mathcal{S}_P \to \mathcal{L}og_S^P$ corresponding to a quasi-coherent sheaf on $\mathcal{L}og_S^P$. Set $R = O \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P] = O[P]$. Let M be a $P^{gp}/Q^{gp} = P^{gp}$-graded R-module corresponding to $\mathcal{F}$ via Proposition 6.5. We denote by M^+ the R-module $M \otimes_R O[P^{gp}]$, which is a localization of M by homogenous elements of $R = O[P]$. Note that M^+ naturally inherits grading of M. Let M_0 be the term of degree 0 of M. Then φ corresponds to an automorphism of the O-modules $\tilde{\sigma}\colon M \to M$ satisfying the conditions that $\tilde{\sigma}(xm) = \sigma(x)\tilde{\sigma}(m)$ for any $x \in P$ and $m \in M$, $\tilde{\sigma}^2 = \mathrm{id}$, the morphism $\tilde{\sigma} \otimes \mathrm{id}\colon M^+ \to M^+$ is an identity at degree 0, and $\tilde{\sigma}$ restricted to M_i is an isomorphism to $M_{\tilde{\sigma}(i)}$ for each $i \in P^{gp}$. This fact follows from the proof of Theorem 6.9.

The cochain complex associated to the hypercovering of $\mathcal{L}og_S^P$ constructed above has the form

$$0 \to M_0 \to \bigoplus_{j=0}^{2} M_0 \to \bigoplus_{j=0}^{4} M_0 \oplus (\bigoplus M_0^+) \to \bigoplus_{j=0}^{8} M_0 \oplus (\bigoplus M_0^+) \to \cdots . \tag{8.1}$$

The second morphism is the morphism $(\mathrm{id}, \tilde{\sigma})\colon M_0 \to \bigoplus_{j=0}^{2} M_0$. By Corollary 7.3, we have

$$H^0(\mathcal{L}og_S^P, (\mathcal{F}, \varphi)) = (M_0)^{\tilde{\sigma}},$$

where $(M_0)^{\tilde{\sigma}}$ consists of elements of M_0 fixed by the automorphism $\tilde{\sigma}$.

When $M \otimes_R O[P^{gp}] = 0$, the calculation of cohomology is relatively simple because the quasi-coherent sheaf is 0 on the $\mathcal{S}_0$'s. In this case, the cochain complex (8.1) can be simplified as follows:

$$0 \longrightarrow M_0 \longrightarrow \bigoplus_{j=0}^{2} M_0 \longrightarrow \bigoplus_{j=0}^{4} M_0 \longrightarrow \bigoplus_{j=0}^{8} M_0 \longrightarrow \cdots . \tag{8.2}$$

This cochain complex is homotopic to the cochain complex

$$0 \longrightarrow M_0 \xrightarrow{-\mathrm{id}+\tilde{\sigma}} M_0 \xrightarrow{\mathrm{id}+\tilde{\sigma}} M_0 \xrightarrow{-\mathrm{id}+\tilde{\sigma}} M_0 \xrightarrow{\mathrm{id}+\tilde{\sigma}} \cdots . \tag{8.3}$$

The cohomology $H^i(\mathcal{L}og_S^P, (\mathcal{F}, \varphi))$ is the i-th cohomology of the cochain complex above. The cochain complex above is cyclic. Especially, when 2 is not invertible in O, there exists a quasi-coherent sheaf $(\mathcal{F}, \varphi)$ such that $H^i(\mathcal{L}og_S^P, (\mathcal{F}, \varphi))$ is not 0 for arbitrary large i. When 2 is invertible in O, we see that $H^i(\mathcal{L}og_S^P, (\mathcal{F}, \varphi)) = 0$ for $i > 0$.

In fact, when 2 is invertible in O, we can prove $H^i(\mathcal{L}og_S^P, (\mathcal{F}, \varphi)) = 0$ for any $(\mathcal{F}, \varphi)$ and $i > 0$. Let M' be the kernel of the morphism of graded modules $M \to M \otimes_R O[P^{gp}]$. Since the morphism $\tilde{\sigma}$ preserves the kernel, the pair $(\mathcal{F}', \varphi)$ defines a quasi-coherent sheaf on $\mathcal{L}og_S^P$, where $\mathcal{F}'$ is the quasi-coherent sheaf on $\mathcal{S}_P$ corresponding to the P^{gp}-graded

R-module M'. The pair $(\mathcal{F}/\mathcal{F}', \varphi)$ also defines a quasi-coherent sheaf on $\mathcal{L}og_S^P$. The short exact sequence

$$0 \longrightarrow (\mathcal{F}', \varphi) \longrightarrow (\mathcal{F}, \varphi) \longrightarrow (\mathcal{F}/\mathcal{F}', \varphi) \longrightarrow 0$$

induces the long exact sequence of cohomology. The cohomology of $(\mathcal{F}', \varphi)$ vanishes by the same argument as in the previous paragraph. As for $(\mathcal{F}/\mathcal{F}', \varphi)$, the localization $M/M' \to (M/M') \otimes_R O[P^{gp}]$ is injective. Therefore it is enough to prove the assertion for the pair $(\mathcal{F}, \varphi)$ such that the natural morphism $M \to M \otimes_R O[P^{gp}]$ is injective. When 2 is invertible in O and M satisfies the condition that the localization $M \to M \otimes_R O[P^{gp}]$ is injective, we can prove that the cochain complex (8.1) is homotopic to the cochain complex

$$0 \longrightarrow (M_0)^{\tilde{\sigma}} \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots . \tag{8.4}$$

This example is summarized as follows.

Proposition 8.1. *Let S be a log scheme whose underlying scheme is $\underline{S} = \mathrm{Spec}(O)$ and the log structure is trivial. Let $P = \mathbb{N}^2$ and σ be the automorphism of P which exchanges the two components of $\mathbb{N}^2$. Let $(\mathcal{F}, \varphi)$ be a descent datum with respect to the morphism $\mathcal{S}_P \to \mathcal{L}og_S^P$ corresponding to a quasi-coherent sheaf on $\mathcal{L}og_S^P$. Let M be a P^{gp}-graded $R = O[P]$-module corresponding to $\mathcal{F}$ via Proposition 6.5 and $\tilde{\sigma} \colon M \to M$ be an automorphism of the O-modules corresponding to φ. Let M_0 be the O-submodule of M consisting of the elements of degree 0.*

Then, we have

$$H^0(\mathcal{L}og_S^P, (\mathcal{F}, \varphi)) = (M_0)^{\tilde{\sigma}}.$$

When 2 is invertible in the ring O,

$$H^i(\mathcal{L}og_S^P, (\mathcal{F}, \varphi)) = 0$$

for every quasi-coherent sheaf $(\mathcal{F}, \varphi)$ and $i > 0$.

When 2 is not invertible in O, there exists a quasi-coherent sheaf $(\mathcal{F}, \varphi)$ such that $H^i(\mathcal{L}og_S^P, (\mathcal{F}, \varphi))$ is not 0 for arbitrary large i.

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