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A Lefschetz trace formula for p^n-torsion étale cohomology: a resume

By

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This is a resume of our results ([5]) on a Lefschetz trace formula on varieties defined over a finite field $\mathbb{F}_q$ of characteristic p. It is a p^n-torsion version of a conjecture of Deligne which was originally formulated with ℓ-adic étale cohomology ($\ell \neq p$) and has been proved by Fujiwara in full generality ([2]).

We introduce some notations to state our results. We fix an algebraic closure k of $\mathbb{F}_q$. For an object $\mathcal{X}_0$ (*e.g.* scheme, sheaf on a scheme, morphism of schemes) over $\mathbb{F}_q$, $\mathcal{X}$ denotes the base change of $\mathcal{X}_0$ by the injection $\mathbb{F}_q \hookrightarrow k$. Let S be a scheme. For a morphism of S-schemes $b\colon V \to U \times_S U$, we put $b_1 = \mathrm{pr}_1 \circ b$ and $b_2 = \mathrm{pr}_2 \circ b$, where pr_1 (resp. pr_2) is the first (resp. second) projection of $U \times_S U$. The S-scheme $\mathrm{Fix}(b) = V \times_{U\times_S U} U$ is defined by the following cartesian diagram

$$\begin{tikzcd} \mathrm{Fix}(b) \arrow[r] \arrow[d] & U \arrow[d, "\Delta_{U/S}"] \\ V \arrow[r, "b"] & U \times_S U, \end{tikzcd}$$

where $\Delta_{U/S}$ is the diagonal morphism. Remark that if U and V are smooth over S, $db_1\colon b_1^*\Omega_{U/S} \to \Omega_{V/S}$ is zero and b_2 is étale, then $\mathrm{Fix}\, b$ is étale over S ([7, Cor. 17.13.6]). For an S-endomorpshim $f\colon U \to U$, we put $\mathrm{Fix}\, f = \mathrm{Fix}(f \times_S \mathrm{id}_U)$. Let U_0 and V_0 be $\mathbb{F}_q$-schemes and $b_0\colon V_0 \to U_0 \times_{\mathbb{F}_q} U_0$ an $\mathbb{F}_q$-morphism of schemes. We put $b^{(m)} = (\mathrm{Fr}_U^m \circ b_1, b_2)$, where Fr_U is the relative q-th power Frobenius morphism of U i.e. the base change of the absolute q-th power Frobenius morphism of U_0 by $\mathbb{F}_q \to k$.

First, we state a p-torsion version of Fujiwara's trace formula.

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Theorem 1 ([5, Corollary 3.2], [5, Theorem 6.1]). *Let U_0 and V_0 be separated $\mathbb{F}_q$-schemes of finite type and $\mathcal{F}_0$ a constructible étale $\mathbb{Z}/p$-sheaf on U_0.*

(1) *Let $f_0 \colon U_0 \to U_0$ be an automorphism of finite order and $m \geq 1$ an integer. Let $u_0 \colon (\mathrm{Fr}_{U_0}^m \circ f_0)^* \mathcal{F}_0 \to \mathcal{F}_0$ be an isomorphism of sheaves whose order is the same as that of f_0. Then we have the following equality*

$$\sum_i (-1)^i \mathrm{Tr}(u \circ (\mathrm{Fr}_U^m \circ f)^* \mid H_c^i(U, \mathcal{F})) = \sum_{P \in \mathrm{Fix}(\mathrm{Fr}_U^m \circ f)} \mathrm{Tr}(u_P \mid \mathcal{F}_P).$$

(2) *Let $b_0 \colon V_0 \to U_0 \times_{\mathbb{F}_q} U_0$ be a morphism of $\mathbb{F}_q$-schemes. We assume that $\mathcal{F}_0$ is smooth. Further we assume that there exist proper smooth $\mathbb{F}_q$-schemes X_0 and Y_0, and an $\mathbb{F}_q$-morphism $a_0 \colon Y_0 \to X_0 \times_{\mathbb{F}_q} X_0$ such that*

(a) *U_0 (resp. V_0) is an open $\mathbb{F}_q$-subscheme of X_0 (resp. Y_0), the diagram*

$$\begin{CD} V_0 @>{b_0}>> U_0 \times_{\mathbb{F}_q} U_0 \\ @VVV @VVV \\ Y_0 @>{a_0}>> X_0 \times_{\mathbb{F}_q} X_0 \end{CD}$$

is cartesian,

(b) *b_1 is proper, a_2 is étale, a is a closed immersion,*

(c) *$X \smallsetminus U$ is a Cartier divisor, and*

(d) *there exists a smooth constructible étale $\mathbb{Z}/p$-sheaf $\mathcal{G}_0$ on X_0 such that $\mathcal{G}_0|_{U_0} = \mathcal{F}_0$.*

Then, for any integer $m \geq 1$ and any $u_0 \in \mathrm{Hom}(b_{01}^{(m)*}\mathcal{F}_0, b_{02}^* \mathcal{F}_0)$, we have the following equality

$$\sum_i (-1)^i \mathrm{Tr}(u_! \mid H_c^i(U, \mathcal{F})) = \sum_{P \in \mathrm{Fix}(b^{(m)})} \mathrm{Tr}(u_P \mid \mathcal{F}_P),$$

where $u_!$ is the composition

$$H_c^i(U, \mathcal{F}) \xrightarrow{b_1^{(m)*}} H_c^i(V, b_1^{(m)*}\mathcal{F}) \xrightarrow{u} H_c^i(V, b_2^* \mathcal{F}) \xrightarrow{b_{2!}} H_c^i(U, \mathcal{F}).$$

Remark that $\mathrm{Fix}(\mathrm{Fr}_U^m \circ f)$ is finite over k by Zink's lemma [6, Lemma 2.3] and $\mathrm{Fix}(b^{(m)})$ is finite étale over k since U and V are smooth over k, the differential of $b_1^{(m)}$ is zero and b_2 is étale.

Theorem 1 (1) is proved by using the Lefschetz trace formula for the Frobenius correspondence ([8, Fonct. L mod. ℓ Théorème 4.1]) and Deligne-Lusztig's method ([1, Section 3]). We sketch the proof of Theorem 1 (2). This is a generalization of the proof of [8, Fonct. L mod. ℓ^n, Théorème 4.1]. We put $\mathscr{G}_0' = \mathscr{I}_0(\mathcal{G}_0 \otimes \mathcal{O}_{X_0})$, where $\mathscr{I}_0$ is the ideal sheaf of definition of $X_0 \smallsetminus U_0$. Since $X \smallsetminus U$ is a Cartier divisor, $\mathscr{G}'$ is a locally free $\mathcal{O}_X$-module of finite rank and sits in the exact sequence

$$0 \longrightarrow j_!\mathcal{F} \longrightarrow \mathscr{G}' \xrightarrow{1-\Phi} \mathscr{G}' \longrightarrow 0,$$

where $\Phi\colon \mathscr{G}' \to \mathscr{G}'$ is the morphism induced by the p-th power map on $\mathcal{O}_X$. Then we can reduce the calculation of the trace of the endomorphism of the cohomology group of $\mathcal{F}$ to that of $\mathscr{G}'$. By applying the following trace formula to the trace, we obtain Theorem 1 (2).

Theorem 2 (Woods Hole formula, [5, Theorem 4.1]). *Let S be the spectrum of an artinian local ring, X and Y proper smooth schemes over S, and $a\colon Y \hookrightarrow X \times_S X$ a closed immersion over S. We assume that a_2 is étale and the homomorphism $da_1\colon a_1^*\Omega_{X/S} \to \Omega_{Y/S}$ is zero. Then, for any perfect complex $\mathscr{K}$ of $\mathcal{O}_X$-modules and any $u \in \mathrm{Hom}(a_1^*\mathscr{K}, a_2^*\mathscr{K})$, we have*

$$\sum_i (-1)^i \mathrm{Tr}(u_* \mid H^i(X,\mathscr{K})) = \sum_{\beta\in\pi_0(\mathrm{Fix}(a))} \mathrm{Tr}_{\beta/S}(\mathrm{Tr}(u_\beta \mid \mathscr{K}_\beta)),$$

where u_* is the composition

$$H^i(X,\mathscr{K}) \xrightarrow{a_1^*} H^i(Y, a_1^*\mathscr{K}) \xrightarrow{u} H^i(Y, a_2^*\mathscr{K}) \xrightarrow{a_{2*}} H^i(X,\mathscr{K}),$$

$\pi_0(\mathrm{Fix}(a))$ is the set of connected components of $\mathrm{Fix}(a)$, $\mathscr{K}_\beta$ (resp. u_β) is the pull-back of $\mathscr{K}$ (resp. u) by the immersion $i_\beta\colon \beta \hookrightarrow Y$ and $\mathrm{Tr}_{\beta/S}$ is the trace map $\Gamma(\beta,\mathcal{O}_\beta) \to \Gamma(S,\mathcal{O}_S)$.

Remark that $\mathrm{Fix}(a)$ is finite étale over S. Theorem 2 is a generalization of [9, Exp. III, Corollaire 6.12], and can be proved by using the Lefschetz-Verdier trace formula ([9, Exp. III, Théorème. 6.10]) and properties of residue symbols in [3, Ch. III, §9].

Secondly, we state a p^n-torsion version of Fujiwara's trace formula. At present, this requires more assumptions than Theorem 1.

For a perfect field K of characteristic p, we denote by $W_n(K)$ the ring of Witt vectors of K of length n. We write σ_0 for the Frobenius automorphism of $W_n(\mathbb{F}_q)$. For a scheme S, we write $\mathcal{O}_S$ for the structure sheaf of S. If S is of characteristic p, denote by $\Phi_{\mathcal{O}_S}$ the p-th power map on $\mathcal{O}_S$.

Theorem 3 ([5, Theorem 7.1]). *Let U_0 and V_0 be smooth $\mathbb{F}_q$-schemes, $b_0\colon V_0 \to U_0 \times_{\mathbb{F}_q} U_0$ an $\mathbb{F}_q$-morphism, and $\mathcal{F}_0$ a locally free constructible étale $\mathbb{Z}/p^n$-sheaf on U_0. We assume that there exist proper smooth $W_n(\mathbb{F}_q)$-schemes $\mathscr{X}_0$ and $\mathscr{Y}_0$, a Cartier divisor $\mathscr{D}_0$ of $\mathscr{X}_0$ which is flat over $W_n(\mathbb{F}_q)$, a $W_n(\mathbb{F}_q)$-morphism $\widetilde{a}_0\colon \mathscr{Y}_0 \to \mathscr{X}_0 \times_{W_n(\mathbb{F}_q)} \mathscr{X}_0$, and a morphism $\Phi_{\mathcal{O}_{\mathscr{X}_0}}\colon \mathcal{O}_{\mathscr{X}_0} \to \mathcal{O}_{\mathscr{X}_0}$ such that*

(a) *when we put* $X_0 = \mathscr{X}_0 \times_{W_n(\mathbb{F}_q)} \mathbb{F}_q$, $Y_0 = \mathscr{Y}_0 \times_{W_n(\mathbb{F}_q)} \mathbb{F}_q$ *and define* a_0 *such that the diagram*

$$\begin{tikzcd} \mathscr{Y}_0 \arrow[r, "\widetilde{a_0}"] & \mathscr{X}_0 \times_{W_n(\mathbb{F}_q)} \mathscr{X}_0 \\ Y_0 \arrow[u] \arrow[r, "a_0"] & X_0 \times_{\mathbb{F}_q} X_0 \arrow[u] \end{tikzcd}$$

is cartesian, then $(U_0, V_0, b_0, X_0, Y_0, a_0)$ *satisfies the condition* (a) *in* Theorem 1 (2),

(b) b_1 *is proper,* $\widetilde{a}_2$ *is étale,* $\widetilde{a}$ *is a closed immersion,*

(c) $\mathscr{D}_0$ *is a lift of* $X_0 \smallsetminus U_0$ *to* $W_n(\mathbb{F}_q)$,

(d) *the diagrams*

$$\begin{tikzcd} W_n(\mathbb{F}_q) \arrow[r, "\sigma_0"] \arrow[d] & W_n(\mathbb{F}_q) \arrow[d] \\ \mathcal{O}_{\mathscr{X}_0} \arrow[r, "\Phi_{\mathcal{O}_{\mathscr{X}_0}}"] & \mathcal{O}_{\mathscr{X}_0} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{O}_{\mathscr{X}_0} \arrow[r, "\Phi_{\mathcal{O}_{\mathscr{X}_0}}"] \arrow[d] & \mathcal{O}_{\mathscr{X}_0} \arrow[d] \\ \mathcal{O}_{X_0} \arrow[r, "\Phi_{\mathcal{O}_{X_0}}"] & \mathcal{O}_{X_0} \end{tikzcd}$$

commute,

(e) *the inclusion* $\Phi_{\mathcal{O}_{\mathscr{X}_0}}(\mathscr{I}_0) \subset \mathscr{I}_0$ *holds, where* $\mathscr{I}_0$ *is the defining ideal of* $\mathscr{D}_0$,

(f) *there exists a locally free constructible étale* $\mathbb{Z}/p^n$*-sheaf* $\mathcal{G}_0$ *on* X_0 *such that* $\mathcal{G}_0|_{U_0} = \mathcal{F}_0$, *and*

(g) $H^i_c(U, \mathcal{F})$ *(resp.* $H^i(\mathscr{X}, \mathcal{G} \otimes_{\mathbb{Z}/p^n} \mathscr{I})$*) is free over* $\mathbb{Z}/p^n$ *(resp.* $W_n(k)$*) for any* i.

Then there exists an integer M *such that, for any integer* $m \geq M$ *and any homomorphism* $u_0 \in \mathrm{Hom}(b_{01}^{(m)*}\mathcal{G}_0, b_{02}^*\mathcal{G}_0)$, *we have the following equality*

$$\sum_i (-1)^i \mathrm{Tr}(u_! \mid H^i_c(U, \mathcal{F})) = \sum_{P \in \mathrm{Fix}(b^{(m)})} \mathrm{Tr}(u_P \mid \mathcal{F}_P).$$

We note that the integer M in Theorem 3 depends on the sheaf $\mathcal{F}_0$. We need the assumption on existence of $\mathscr{X}_0, \mathscr{Y}_0, \mathscr{D}_0, \widetilde{a_0}$ and $\Phi_{\mathcal{O}_{\mathscr{X}_0}}$ in order to use the same argument used in the proof of Theorem 1 (2), and the assumption (g) in order to compute the trace in the category of $\mathbb{Z}/p^n$-modules, not in that of perfect complexes of $\mathbb{Z}/p^n$-modules. If X_0 is a curve, then the assumption (e) automatically holds ([4, Lemma 1.1.2]).

The proof of Theorem 3 is similar to that of Theorem 1 (2).

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