RIMS Kôkyûroku Bessatsu
B51 (2014), 157–162

Symbolic generators of the Brauer group of diagonal cubic surfaces and their applications to zero-cycles

By

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Abstract

In this article, we first announce the result of the paper [10], in which we introduce the notion of uniform generators of the Brauer group of varieties, and consider the problem whether the Brauer group of diagonal cubic surfaces has such generators or not. The reader may refer to the paper [10] for proofs and details.

Secondly, as an application of such symbolic generators, we give an example of explicit calculations of the degree-zero part of the Chow group of zero-cycles on diagonal cubic surfaces over p-adic fields, along the same line as in [9].

§ 1. Background and known results

Let V be a variety over a field k. Grothendieck defined its cohomological Brauer group $\mathrm{Br}(V) := H^2_{\text{ét}}(V, \mathbb{G}_m)$ in his papers [4]. The group $\mathrm{Br}(V)$ plays an important role in studying the arithmetic and the geometry of V. For example, when k is a number field, it appears in the Brauer-Manin obstruction [6], which is used in constructing various counterexamples to the Hasse principle for rational points on V. For such studies, we want to answer the following two natural problems:

(1) Determine the structure of $\mathrm{Br}(V)$ as an abelian group.

(2) Find generators of $\mathrm{Br}(V)$ in terms of norm residue symbols.

Received March 31, 2013. Revised April 2, 2013, April 7, 2013 and April 11, 2013.
2010 Mathematics Subject Classification(s): Primary 14F22. Secondary 19F15, 11R34, 14J26.
Key Words: *Brauer groups, diagonal cubic surfaces, norm residue symbols*
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Let k be a field of characteristic zero containing a fixed primitive cubic root ζ of unity. We are concerned with the above problems for diagonal cubic surfaces V over k, that is, smooth projective surfaces defined by a homogeneous equation of the form

$$x^3 + by^3 + cz^3 + dt^3 = 0,$$

where $b, c, d \in k^*$. An original work in this direction was due to Manin [7]. Let V be a diagonal cubic surface of the form $x^3+y^3+z^3+dt^3 = 0$ for $d \in k^* \setminus (k^*)^3$, $\pi\colon V \to \operatorname{Spec} k$ be the structure morphism, and

$$\{\cdot,\cdot\}_3\colon K_2^M(k(V)) \to H^2(k(V), \mu_3^{\otimes 2}) \cong H^2(k(V), \mu_3) \hookrightarrow \operatorname{Br}(k(V))$$

be the norm residue symbol map. By the regularity of V, we can consider $\operatorname{Br}(V)$ as a subgroup of $\operatorname{Br}(k(V))$ in a natural way. Put $\operatorname{Br}(V)/\operatorname{Br}(k) := \operatorname{Br}(V)/\pi^* \operatorname{Br}(k)$. In this case, $\operatorname{Br}(V)/\operatorname{Br}(k) \cong (\mathbb{Z}/3\mathbb{Z})^2$,

$$\left\{d, \frac{x+\zeta y}{x+y}\right\}_3, \quad \left\{d, \frac{x+z}{x+y}\right\}_3 \tag{1.1}$$

are elements in $\operatorname{Br}(V)$ and their images in $\operatorname{Br}(V)/\operatorname{Br}(k)$ are generators of this group.

In the paper [10], we study these problems in a more general setting where the equation of V is of the forms $x^3 + y^3 + cz^3 + dt^3 = 0$ and $x^3 + by^3 + cz^3 + dt^3 = 0$. We announce the results in the next section.

§ 2. Results

First we state the following theorem, which gives an answer to the above problems (1) and (2) for the case $x^3 + y^3 + cz^3 + dt^3 = 0$.

Theorem 2.1 ([10], Theorem 4.1). *Let k be as above and V be the cubic surface over k defined by an equation $x^3 + y^3 + cz^3 + dt^3 = 0$, where c and $d \in k^*$. Assume that c, d, cd and d/c are not contained in $(k^*)^3$. Then we have the following:*

(1) *The group $\operatorname{Br}(V)/\operatorname{Br}(k)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$.*

(2) *The element*

$$e_1 = \left\{\frac{d}{c}, \frac{x+\zeta y}{x+y}\right\}_3 \in \operatorname{Br}(k(V))$$

is contained in $\operatorname{Br}(V)$.

(3) *The image of e_1 in $\operatorname{Br}(V)/\operatorname{Br}(k)$ is a generator of this group.*

Note that the claim (1) is due to [1] and a result similar to (2) appeared in [3]. For a proof of Theorem 2.1, see [10], §4.

An important observation is that in the result of Manin and Theorem 2.1, we can take generators *uniformly.* We briefly give a more precise description of this uniformity. Let c and d be indeterminates, $F = k(c, d)$, V be the cubic surface $x^3+y^3+cz^3+dt^3 = 0$ over F, and

$$e = e(c, d) := \left\{ \frac{d}{c}, \frac{x + \zeta y}{x + y} \right\}_3$$

be an element in $\mathrm{Br}(k(V))$. In fact, e is contained in $\mathrm{Br}(V)$ by Theorem 2.1. Let $P = (c_0, d_0)$ be a point in $k^* \times k^*$ with c_0, d_0, $c_0 d_0$ and d_0/c_0 not contained in $(k^*)^3$, and V_P the surface defined by $x^3+y^3+c_0z^3+d_0t^3 = 0$. If we want a symbolic generator of $\mathrm{Br}(V_P)/\mathrm{Br}(k)$, we can get it by specializing e at P. We denote this element by $\mathrm{sp}(e; P)$. For a precise definition of the specialization, see [10], §2. The Brauer group of a given family of varieties does not necessarily have such uniform generators.

The main result in the paper [10] is a result for general diagonal cubic surfaces with three parameters b, c and d. Unlike the result of Manin and Theorem 2.1, we show that there is no uniform generator in this situation. Let $F = k(b, c, d)$, where b, c, d are indeterminates over k, and let V be the projective cubic surface over F defined by the equation $x^3 + by^3 + cz^3 + dt^3 = 0$. For $P = (b_0, c_0, d_0) \in k^* \times k^* \times k^*$, let V_P be the projective cubic surface over k defined by the equation $x^3 + b_0y^3 + c_0z^3 + d_0t^3 = 0$. For $e \in \mathrm{Br}(V)$, we denote its specialization at P by $\mathrm{sp}(e; P) \in \mathrm{Br}(V_P)$. Put

$$\mathcal{P}_k = \{P \in k^* \times k^* \times k^* \mid \mathrm{Br}(V_P)/\mathrm{Br}(k) \cong \mathbb{Z}/3\mathbb{Z}\}.$$

Note ([1]) that $\mathrm{Br}(V_P)/\mathrm{Br}(k)$ is isomorphic to one of 0, $\mathbb{Z}/3\mathbb{Z}$ and $(\mathbb{Z}/3\mathbb{Z})^2$ and that Manin dealt with the last case. We also remark that for $P \in \mathcal{P}_k$, we can take a sum of symbols as a generator of $\mathrm{Br}(V_P)/\mathrm{Br}(k) \cong \mathbb{Z}/3\mathbb{Z}$ by a theorem of Merkurjev and Suslin [8]. However, unlike the result of Manin and Theorem 2.1, we do not know whether we can take *one* symbol as its generator.

The claim is the following:

Theorem 2.2 ([10], Corollary 5.3)**.** *Let k, F and V as above. Assume moreover* $\dim_{\mathbb{F}_3} k^*/(k^*)^3 \geq 2$. *Then there is no element $e \in \mathrm{Br}(V)$ satisfying the following condition*:

there exists a Zariski dense open subset $W \subset (\mathbb{G}_{m,k})^3$ such that

- $\mathrm{sp}(e; \cdot)$ *is defined on* $W(k) \cap \mathcal{P}_k$;
- *for all* $P \in W(k) \cap \mathcal{P}_k$, $\mathrm{sp}(e; P)$ *is a generator of* $\mathrm{Br}(V_P)/\mathrm{Br}(k)$.

Remark. By associating to each point $P \in \mathcal{P}_k$ the image under $P\colon \operatorname{Spec} k \to (\mathbb{G}_{m,k})^3$, we can consider $\mathcal{P}_k$ as a subset of $(\mathbb{G}_{m,k})^3$. Then the assumption $\dim_{\mathbb{F}_3} k^*/(k^*)^3 \geq 2$ is equivalent to the Zariski density of $\mathcal{P}_k \subset (\mathbb{G}_{m,k})^3$, which is essentially necessary to prove Theorem 2.2. We see that this assumption holds for various fields, for example,

- all finitely generated fields over $\mathbb{Q}(\zeta)$ and $\mathbb{Q}_p(\zeta)$ for any prime number p;
- all function fields of varieties of dimension ≥ 1.

Theorem 2.2 is a direct corollary of the following:

Theorem 2.3 ([10], Theorem 5.1)**.** *Let k, F and V be as above. Then*

$$\operatorname{Br}(V)/\operatorname{Br}(F) = 0.$$

We briefly explain the strategy of our proof of Theorem 2.3. Fix an algebraic closure $\overline{F}$ of F and put $\overline{V} = V \times_F \overline{F}$. The starting point is the following exact sequence:

$$0 \to \operatorname{Br}(V)/\operatorname{Br}(F) \to H^1(F, \operatorname{Pic}(\overline{V})) \xrightarrow{d^{1,1}} H^3(F, \overline{F}^*), \tag{2.1}$$

which is derived by the Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(F, H^q_{\text{ét}}(\overline{V}, \mathbb{G}_m)) \Rightarrow H^{p+q}_{\text{ét}}(V, \mathbb{G}_m)$$

and the rationality of $\overline{V}$. Secondly, we show $H^1(F, \operatorname{Pic}(\overline{V})) \cong \mathbb{Z}/3\,\mathbb{Z}$ by computing the action of $\operatorname{Gal}(\overline{F}/F)$ on $\operatorname{Pic}(\overline{V})$, which is generated by the classes of the well-known 27 lines on $\overline{V}$. Thus, to prove the theorem, we have to prove that the image of a generator ϕ of $H^1(F, \operatorname{Pic}(\overline{V}))$ under $d^{1,1}$ is not zero. Thirdly, using a description of $d^{1,1}$ due to Kresch and Tschinkel [5], we obtain the image $d^{1,1}(\phi)$ in an explicit form. Finally, we construct a group G satisfying

- there exists a homomorphism $f : H^3(F, \overline{F}^*) \to G$, and
- the image $f(d^{1,1}(\phi))$ is nonzero in G

by using residue maps of Galois cohomology. These steps complete the proof of Theorem 2.3.

Remark. Let V/k be a diagonal cubic surface. If the base field k is of cohomological dimension less than or equal to two, we have $\operatorname{Br}(V)/\operatorname{Br}(k) \cong H^1(k, \operatorname{Pic}(\overline{V}))$ by the above exact sequence (2.1) for V/k. This isomorphism also holds when V has a k-rational point. Note that the surfaces appearing in the result of Manin and Theorem 2.1 have a rational point. However, our V/F does not satisfy these conditions, which makes the problem more complicated. As far as we know, this would be the first example of computation of Brauer groups for such varieties.

§ 3. An application to zero-cycles

Let k be a p-adic field and V be a diagonal cubic surface over k. We are concerned with computing $A_0(V)$, the degree-zero part of the Chow group of zero-cycles on V. For results on diagonal cubic surfaces $V : x^3 + y^3 + z^3 + dt^3 = 0$ over k with $p \neq 3$, see [2] and [9]. Saito and Sato [9] recently proved a certain unramifiedness theorem for Brauer groups. Using this theorem and the symbolic generators (1.1), they computed $A_0(V)$ for such V with $p = 3$. Their result is the following:

Theorem 3.1 ([9], Theorem 4.1.1 (2))**.** *Let k be a finite extension of $\mathbb{Q}_3$ containing a primitive cubic root ζ of unity, $d \in k^*$ and V be the projective surface over k defined by $x^3 + y^3 + z^3 + dt^3 = 0$. If $\mathrm{ord}_k(d) \equiv 1 \mod 3$, then $A_0(V) \cong (\mathbb{Z}/3\mathbb{Z})^2$.*

As an application of Theorem 2.1, we compute $A_0(V)$ for $V : x^3+y^3+cz^3+dt^3 = 0$ by using the same method as in [9]. The result is the following:

Theorem 3.2. *Let k be a finite extension of $\mathbb{Q}_3$ containing a primitive cubic root ζ of unity, $c, d \in k^*$ and V be the projective surface over k defined by $x^3+y^3+cz^3+dt^3 = 0$. Assume that*

- *c, d, cd, $c/d \notin (k^*)^3$,*
- *$\mathrm{ord}_k(c-1)$ is greater than the absolute ramification index e of k,*
- *$\mathrm{ord}_k(d) \equiv 1 \mod 3$.*

Then $A_0(V) \cong \mathbb{Z}/3\mathbb{Z}$.

Remark. A proof of the above theorem will be written in a forthcoming paper. In our proof, the assumption $\mathrm{ord}_k(c-1) > e$ is essential, since this enables us to exclude difficulties caused by the fact $c \neq 1$ and reduce to the same situation as in [9]. If $\mathrm{ord}_k(c-1) \leq e$, we cannot ignore the difference between c and 1, at least in our method, which seems to make the computation of $A_0(V)$ more difficult.

Acknowledgement

The author would like to thank the referee for reading the paper carefully and giving valuable comments.

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