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Rationality problem for quasi-monomial actions

By

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Abstract

We give a short survey of the rationality problem for quasi-monomial actions which includes Noether's problem and the rationality problem for algebraic tori, and report some results on rationality problem in three recent papers Hoshi, Kang and Kitayama [HKKi], Hoshi, Kang and Kunyavskii [HKKu] and Hoshi and Yamasaki [HY].

Contents

In Section 1, we recall first the definitions of rationalities for field extensions, concerning which we give a survey in this article. In Subsection 1.1, we explain that the rationality problem for quasi-monomial actions includes Noether's problem, the rationality problem for algebraic tori and the rationality problem for Severi-Brauer varieties. In Subsection 1.2, we list known results for monomial actions in dimension two and three

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due to Hajja, Kang, Saltman, Hoshi, Rikuna, Prokhorov, Kitayama and Yamasaki. In Subsection 1.3, we mention some results on quasi-monomial actions due to Hoshi, Kang and Kitayama. A necessary and sufficient condition for the rationality under one-dimensional quasi-monomial actions and two-dimensional purely quasi-monomial actions will be given via norm residue 2-symbol.

In Section 2, we treat Noether's problem on rationality. Using the unramified Brauer groups, Saltman and Bogomolov were able to establish counter-examples to Noether's problem over algebraically closed field for non-abelian p-groups of order p^9 and p^6 respectively. We mention a result due to Hoshi, Kang and Kunyavskii which gives a necessary and sufficient condition for the non-vanishing of the unramified Brauer groups for groups of order p^5 where p is an odd prime number.

In Section 3, we consider the rationality problem for algebraic tori. In Subsection 3.1, we recall some results due to Voskresenskii, Kunyavskii, Endo and Miyata. A birational classification of the algebraic k-tori of dimension four and five due to Hoshi and Yamasaki will also be given. In Subsection 3.2, we give a detailed account of methods related to integral representations of finite groups.

§ 1. Rationality problem for quasi-monomial actions, generalities

§ 1.1. Definitions and examples

Definition 1.1. Let K/k and L/k be finitely generated extensions of fields.
(1) K is *rational* over k (for short, *k-rational*) if K is purely transcendental over k, i.e. $K \simeq k(x_1, \ldots, x_n)$ for some algebraically independent elements $x_1, \ldots, x_n$ over k.
(2) K is *stably k-rational* if $K(y_1, \ldots, y_m)$ is k-rational for some $y_1, \ldots, y_m$ such that $y_1, \ldots, y_m$ are algebraically independent over K.
(3) K and L are *stably k-isomorphic* if $K(y_1, \ldots, y_m) \simeq L(z_1, \ldots, z_n)$ for some algebraically independent elements $y_1, \ldots, y_m$ over K and $z_1, \ldots, z_n$ over L.
(4) K is *retract k-rational* if there exists a k-algebra A contained in K such that (i) K is the quotient field of A, (ii) there exist a non-zero polynomial $f \in k[x_1, \ldots, x_n]$ and k-algebra homomorphisms $\varphi\colon A \to k[x_1, \ldots, x_n][1/f]$ and $\psi\colon k[x_1, \ldots, x_n][1/f] \to A$ satisfying $\psi \circ \varphi = 1_A$.
(5) K is *k-unirational* if $k \subset K \subset k(x_1, \ldots, x_n)$ for some integer n.

It is not difficult to verify the following implications for an infinite field k:

$$k\text{-rational} \ \Rightarrow\ \text{stably } k\text{-rational} \ \Rightarrow\ \text{retract } k\text{-rational} \ \Rightarrow\ k\text{-unirational}.$$

Remark 1.2. In Saltman's original definition of retract k-rationality ([Sal82b, page 130], [Sal84b, Definition 3.1]), a base field k is required to be infinite in order to guarantee the existence of sufficiently many k-specializations (see also Theorem 2.1).

Definition 1.3. Let K/k be a finite extension of fields and $K(x_1,\ldots,x_n)$ be the rational function field over K with n variables $x_1,\ldots,x_n$. Let G be a finite subgroup of $\mathrm{Aut}_k(K(x_1,\ldots,x_n))$.
(1) The action of G on $K(x_1,\ldots,x_n)$ is called *quasi-monomial* if it satisfies the following three conditions:
(i) $\sigma(K)\subset K$ for any $\sigma\in G$;
(ii) $K^G=k$, where K^G is the fixed field under the action of G;
(iii) for any $\sigma\in G$ and any $1\le j\le n$,

$$\sigma(x_j)=c_j(\sigma)\prod_{i=1}^{n}x_i^{a_{ij}} \tag{1.1}$$

where $c_j(\sigma)\in K^\times$ and $[a_{ij}]_{1\le i,j\le n}\in GL_n(\mathbb{Z})$.
(2) The quasi-monomial action is called *purely quasi-monomial* action if $c_j(\sigma)=1$ for any $\sigma\in G$ and any $1\le j\le n$ in (iii).
(3) The quasi-monomial action is called *monomial* action if G acts trivially on K, i.e. $k=K$.
(4) The quasi-monomial action is called *purely monomial* action if it is purely quasi-monomial and monomial.

We have the following implications:

$$\begin{array}{ccc}
\text{quasi-monomial action} & \Leftarrow & \text{purely quasi-monomial action}\\
\Uparrow & & \Uparrow\\
\text{monomial action} & \Leftarrow & \text{purely monomial action.}
\end{array}$$

Although there are many variants and results on the rationality problem in algebraic geometry and invariant theory, we restrict ourselves to the following problem.

Question 1.4. Let K/k be a finite extension of fields and G be a finite group acting on $K(x_1,\ldots,x_n)$ by quasi-monomial k-automorphisms. Under what situation is the fixed field $K(x_1,\ldots,x_n)^G$ k-rational?

This problem includes Noether's problem and the rationality problem for algebraic tori (see Example 1.5 below). The reader is referred to Swan [Swa83], Manin and Tsfasman [MT86] and Colliot-Thélène and Sansuc [CTS07] for more general survey on the rationality problem, and also to Serre [Ser79, Ser02], Knus, Merkurjev, Rost and Tignol [KMRT98], Gille and Szamuely [GS06] and Berhuy [Ber10] for basic tools (e.g. Galois cohomology, Galois descent, Brauer groups, etc.) in this area.

Example 1.5 (Typical examples of quasi-monomial actions)**.**

(1) (Noether's problem). When G acts on $K(x_1, \ldots, x_n)$ by permutation of the variables $x_1, \ldots, x_n$ and trivially on K, i.e. $k = K$, the rationality problem of $K(x_1, \ldots, x_n)^G$ over k is called Noether's problem. We will discuss Noether's problem in Section 2.

(2) (Rationality problem for algebraic tori). When G acts on $K(x_1, \ldots, x_n)$ by purely quasi-monomial k-automorphisms and G is isomorphic to $\mathrm{Gal}(K/k)$, the fixed field $K(x_1, \ldots, x_n)^G$ is a function field of some algebraic torus defined over k and split over K (see Voskresenskii [Vos98, Chapter 2]). We will treat this in Section 3.

(3) (Rationality problem for Severi-Brauer varieties). Assume that G is isomorphic to $\mathrm{Gal}(K/k)$. Take $a_\sigma \in GL_{n+1}(K)$ for each $\sigma \in G$. Denote by $\bar{a}_\sigma$ the image of a_σ in the canonical map $GL_{n+1}(K) \to PGL_{n+1}(K)$. Consider the rational function fields $K(y_0, y_1, \ldots, y_n)$ and $K(x_1, \ldots, x_n)$ where $x_i = y_i/y_0$ for $1 \leq i \leq n$. For each $\sigma \in G$, a_σ induces a $\mathrm{Gal}(K/k)$-equivariant automorphism on $K(y_0, y_1, \ldots, y_n)$ and $K(x_1, \ldots, x_n)$ (note that elements of K in $K(y_0, \ldots, y_n)$ are acted through $\mathrm{Gal}(K/k)$). Assume furthermore that the map $\gamma : G \to PGL_n(K)$ defined by $\gamma(\sigma) = \bar{a}_\sigma$ is a 1-cocycle, i.e. $\gamma(\sigma\tau) = \gamma(\sigma) \cdot \sigma(\gamma(\tau))$. Then G induces an action on $K(x_1, \ldots, x_n)$. The fixed field $K(x_1, \ldots, x_n)^G$ is called a Brauer-field $F_{n,k}(\gamma)$, i.e. the function field of an n-dimensional Severi-Brauer variety over k associated to γ (see Roquette [Roq63, Roq64], Kang [Kan90]). If γ' is a 1-cocycle which is cohomologous to γ, it is easy to see that $F_{n,k}(\gamma) \simeq F_{n,k}(\gamma')$ over k; thus the Brauer-field $F_{n,k}(\gamma)$ depends only on the cohomology class $[\gamma] \in H^1(G, PGL_n(K))$. We can show that a Brauer-field over k is k-rational if and only if it is k-unirational (see Serre [Ser79, page 160]). If we assume that each column of a_σ has precisely one non-zero entry, then the action of G on $K(x_1, \ldots, x_n)$ becomes a quasi-monomial action.

Notation. Throughout this paper, S_n (resp. A_n, D_n, C_n) denotes the symmetric (resp. the alternating, the dihedral, the cyclic) group of degree n of order $n!$ (resp. $n!/2$, $2n$, n).

§ 1.2. Rationality problem for monomial actions

Monomial actions on $k(x_1, x_2)$ and $k(x_1, x_2, x_3)$ were investigated by Hajja, Kang, Saltman, Hoshi, Rikuna, Prokhorov, Kitayama, Yamasaki, etc. We list known results for (purely) monomial actions on $k(x_1, x_2)$ and $k(x_1, x_2, x_3)$.

Theorem 1.6 (Hajja [Haj87]). *Let k be a field and G be a finite group acting on $k(x_1, x_2)$ by monomial k-automorphisms. Then $k(x_1, x_2)^G$ is k-rational.*

Theorem 1.7 (Hajja and Kang [HK92, HK94], Hoshi and Rikuna [HR08]). *Let k be a field and G be a finite group acting on $k(x_1, x_2, x_3)$ by purely monomial k-automorphisms. Then $k(x_1, x_2, x_3)^G$ is k-rational.*

We define another terminology related to a quasi-monomial action.

Definition 1.8. Let G be a finite group acting on $K(x_1,\ldots,x_n)$ by quasi-monomial k-automorphisms with the conditions (i), (ii), (iii) in Definition 1.3 (1). Define a group homomorphism $\rho_{\underline{x}}: G \to GL_n(\mathbb{Z})$ by $\rho_{\underline{x}}(\sigma) = [a_{ij}]_{1\le i,j\le n} \in GL_n(\mathbb{Z})$ for any $\sigma \in G$ where the matrix $[a_{ij}]_{1\le i,j\le n}$ is given by $\sigma(x_j) = c_j(\sigma)\prod_{1\le i\le n} x_i^{a_{ij}}$ in (iii) of Definition 1.3 (1).

Proposition 1.9 ([HKKi, Proposition 1.12]). *Let G be a finite group acting on $K(x_1,\ldots,x_n)$ by quasi-monomial k-automorphisms. Then there exists a normal subgroup N of G satisfying the following conditions:* (i) $K(x_1,\ldots,x_n)^N = K^N(y_1,\ldots,y_n)$ *where each y_i is of the form $ax_1^{e_1}x_2^{e_2}\cdots x_n^{e_n}$ with $a\in K^\times$ and $e_i\in\mathbb{Z}$ (we may take $a=1$ if the action is a purely quasi-monomial action),* (ii) *G/N acts on $K^N(y_1,\ldots,y_n)$ by quasi-monomial k-automorphisms, and* (iii) *$\rho_{\underline{y}}: G/N \to GL_n(\mathbb{Z})$ is an injective group homomorphism where $\rho_{\underline{y}}$ is given as in Definition 1.8.*

Let G be a finite group acting on $k(x_1,x_2,x_3)$ by monomial k-automorphisms. Then we may assume that G is a subgroup of $GL_3(\mathbb{Z})$ by Proposition 1.9. The rationality problem of $k(x_1,x_2,x_3)^G$ is determined by G up to conjugacy in $GL_3(\mathbb{Z})$. There exist 73 conjugacy classes ($\mathbb{Z}$-classes) $[G]$ of finite subgroups G in $GL_3(\mathbb{Z})$. According to [BBNWZ78, Table 1], we denote by $[G_{i,j,k}]$ the k-th $\mathbb{Z}$-class of the j-th $\mathbb{Q}$-class of the i-th crystal system ($1\le i\le 7$) of $GL_3(\mathbb{Z})$. We define

$$\mathcal{N} := \big\{[G_{1,2,1}],[G_{2,3,1}],[G_{3,1,1}],[G_{3,3,1}],[G_{4,2,1}],[G_{4,2,2}],[G_{4,3,1}],[G_{4,4,1}]\big\}.$$

For two groups $G = G_{1,2,1}$, $G_{4,2,2}\in\mathcal{N}$, the rationality problem of $k(x_1,x_2,x_3)^G$ over k was completely solved by Saltman and Kang:

Theorem 1.10 (Saltman [Sal00, Theorem 0.1], Kang [Kan05, Theorem 4.4]). *Let k be a field with* char $k\neq 2$ *and $G_{1,2,1}=\langle\sigma\rangle\simeq C_2$ act on $k(x_1,x_2,x_3)$ by*

$$\sigma: x_1 \mapsto \tfrac{a_1}{x_1},\ x_2 \mapsto \tfrac{a_2}{x_2},\ x_3 \mapsto \tfrac{a_3}{x_3},\quad a_i\in k^\times,\ 1\le i\le 3.$$

Then $k(x_1,x_2,x_3)^{G_{1,2,1}}$ is not k-rational if and only if $[k(\sqrt{a_1},\sqrt{a_2},\sqrt{a_3}):k]=8$. If $k(x_1,x_2,x_3)^{G_{1,2,1}}$ is not k-rational, then it is not retract k-rational.

Theorem 1.11 (Kang [Kan04, Theorem 1.8]). *Let k be a field and $G_{4,2,2}=\langle\sigma\rangle \simeq C_4$ act on $k(x_1,x_2,x_3)$ by*

$$\sigma: x_1 \mapsto x_2 \mapsto x_3 \mapsto \tfrac{c}{x_1x_2x_3} \mapsto x_1,\quad c\in k^\times.$$

Then $k(x_1,x_2,x_3)^{G_{4,2,2}}$ is k-rational if and only if at least one of the following conditions is satisfied: (i) char $k=2$; (ii) $c\in k^2$; (iii) $-4c\in k^4$; (iv) $-1\in k^2$. *If $k(x_1,x_2,x_3)^{G_{4,2,2}}$ is not k-rational, then it is not retract k-rational.*

Theorem 1.12 (Hoshi, Kitayama and Yamasaki [HKY11], Yamasaki [Yam12]). *Let k be a field with* char $k \neq 2$ *and G be a finite subgroup of $GL_3(\mathbb{Z})$ acting on $k(x_1, x_2, x_3)$ by monomial k-automorphisms.*

(1) ([HKY11]) *If $G \notin \mathcal{N}$, then $k(x_1, x_2, x_3)^G$ is k-rational except for $G \in [G_{7,1,1}]$;*

(2) ([Yam12]) *If $G \in \mathcal{N}$, then $k(x_1, x_2, x_3)^G$ is not k-rational for some field k and coefficients $c_j(\sigma)$ as in Definition 1.8. If $k(x_1, x_2, x_3)^G$ is not k-rational, then it is not retract k-rational. Indeed, we can obtain the necessary and sufficient condition for the k-rationality of $k(x_1, x_2, x_3)^G$ in terms of k and $c_j(\sigma)$ for each $G \in \mathcal{N}$.*

For the exceptional case $G_{7,1,1} = \langle \tau, \lambda, \sigma \rangle \simeq A_4$, the problem can be reduced to the following actions:

$$\tau : x_1 \mapsto \tfrac{a}{x_1},\ x_2 \mapsto \varepsilon\tfrac{a}{x_2},\ x_3 \mapsto \varepsilon x_3, \quad \lambda : x_1 \mapsto \varepsilon\tfrac{a}{x_1},\ x_2 \mapsto \varepsilon x_2,\ x_3 \mapsto \tfrac{a}{x_3},$$
$$\sigma : x_1 \mapsto x_2,\ x_2 \mapsto x_3,\ x_3 \mapsto x_1$$

where $a \in k^\times$, $\varepsilon = \pm 1$. The following gives a partial answer to the problem though we do not know whether $k(x_1, x_2, x_3)^{G_{7,1,1}}$ is k-rational when $\varepsilon = -1$ and $[k(\sqrt{a}, \sqrt{-1}) : k] = 4$.

Theorem 1.13 ([HKY11, Theorem 1.7]). *Let k be a field with* char $k \neq 2$.
(1) *If $\varepsilon = 1$, then $k(x_1, x_2, x_3)^{G_{7,1,1}}$ is k-rational;*
(2) *If $\varepsilon = -1$ and $[k(\sqrt{a}, \sqrt{-1}) : k] \leq 2$, then $k(x_1, x_2, x_3)^{G_{7,1,1}}$ is k-rational.*

As a consequence of Theorem 1.12 and Theorem 1.13, we get the following:

Theorem 1.14 ([HKY11, Theorem 1.8]). *Let k be a field with* char $k \neq 2$ *and G be a finite group acting on $k(x_1, x_2, x_3)$ by monomial k-automorphisms. Then there exists $L = k(\sqrt{a})$ with $a \in k^\times$ such that $L(x_1, x_2, x_3)^G$ is L-rational. In particular, if k is a quadratically closed field, then $k(x_1, x_2, x_3)^G$ is always k-rational.*

Remark 1.15. Prokhorov [Pro10, Theorem 5.1] proved Theorem 1.14 when $k = L = \mathbb{C}$ using a technique of algebraic geometry (e.g. Segre embedding).

§ 1.3. Results in [HKKi] for quasi-monomial actions

In this subsection, we present some results about quasi-monomial actions of dimension ≤ 5 in Hoshi, Kang and Kitayama [HKKi]. It turns out that $K(x_1, \ldots, x_n)^G$ is not even k-unirational in general.

Proposition 1.16 ([HKKi, Proposition 1.13]). (1) *Let G be a finite group acting on $K(x)$ by purely quasi-monomial k-automorphisms. Then $K(x)^G$ is k-rational.*

(2) *Let G be a finite group acting on $K(x)$ by quasi-monomial k-automorphisms. Then $K(x)^G$ is k-rational except for the following case: There exists a normal subgroup*

N of G such that (i) $G/N = \langle\sigma\rangle \simeq C_2$, (ii) $K(x)^N = k(\alpha)(y)$ *with* $\alpha^2 = a \in K^\times$, $\sigma(\alpha) = -\alpha$ (*if* $\operatorname{char} k \neq 2$), *and* $\alpha^2 + \alpha = a \in K$, $\sigma(\alpha) = \alpha + 1$ (*if* $\operatorname{char} k = 2$), (iii) $\sigma \cdot y = b/y$ *for some* $b \in k^\times$.

For the exceptional case, $K(x)^G = k(\alpha)(y)^{G/N}$ *is k-rational if and only if the norm residue 2-symbol* $(a,b)_k = 0$ (*if* $\operatorname{char} k \neq 2$), *and* $[a,b)_k = 0$ (*if* $\operatorname{char} k = 2$).

Moreover, if $K(x)^G$ *is not k-rational, then the Brauer group* $\operatorname{Br}(k)$ *is not trivial and* $K(x)^G$ *is not k-unirational.*

For the definition of the norm residue 2-symbols $(a,b)_k$ and $[a,b)_k$, see [Dra83, Chapter 11].

Theorem 1.17 ([HKKi, Theorem 1.14]). *Let G be a finite group and G act on $K(x,y)$ by purely quasi-monomial k-automorphisms. Define*

$$N = \{\sigma \in G \mid \sigma(x) = x,\ \sigma(y) = y\},\ H = \{\sigma \in G \mid \sigma(\alpha) = \alpha \text{ for all } \alpha \in K\}.$$

Then $K(x,y)^G$ *is k-rational except possibly for the following situation:* (1) $\operatorname{char} k \neq 2$ *and* (2) $(G/N, HN/N) \simeq (C_4, C_2)$ *or* (D_4, C_2).

More precisely, in the exceptional situation we may choose $u, v \in k(x,y)$ *satisfying that* $k(x,y)^{HN/N} = k(u,v)$ (*and therefore* $K(x,y)^{HN/N} = K(u,v)$) *such that*

(i) *when* $(G/N, HN/N) \simeq (C_4, C_2)$, $K^N = k(\sqrt{a})$ *for some* $a \in k\backslash k^2$, $G/N = \langle\sigma\rangle \simeq C_4$, *then σ acts on* $K^N(u,v)$ *by* $\sigma : \sqrt{a} \mapsto -\sqrt{a}$, $u \mapsto \frac{1}{u}$, $v \mapsto -\frac{1}{v}$; *or*

(ii) *when* $(G/N, HN/N) \simeq (D_4, C_2)$, $K^N = k(\sqrt{a}, \sqrt{b})$ *is a biquadratic extension of k with* $a, b \in k\backslash k^2$, $G/N = \langle\sigma, \tau\rangle \simeq D_4$, *then σ and τ act on* $K^N(u,v)$ *by* $\sigma : \sqrt{a} \mapsto -\sqrt{a}$, $\sqrt{b} \mapsto \sqrt{b}$, $u \mapsto \frac{1}{u}$, $v \mapsto -\frac{1}{v}$, $\tau : \sqrt{a} \mapsto \sqrt{a}$, $\sqrt{b} \mapsto -\sqrt{b}$, $u \mapsto u$, $v \mapsto -v$.

For case (i), $K(x,y)^G$ *is k-rational if and only if the norm residue 2-symbol* $(a,-1)_k = 0$. *For case* (ii), $K(x,y)^G$ *is k-rational if and only if* $(a,-b)_k = 0$.

Moreover, if $K(x,y)^G$ *is not k-rational, then the Brauer group* $\operatorname{Br}(k)$ *is not trivial and* $K(x,y)^G$ *is not k-unirational.*

Saltman [Sal90a, Section 3] discussed the relationship of $K(M)^G$ and the embedding problem in Galois theory. We reformulate the two exceptional cases of Theorem 1.17 in terms of the embedding problem.

Proposition 1.18 ([HKKi, Proposition 4.3]). (1) *Let k be a field with* $\operatorname{char} k \neq 2$, $a \in k\backslash k^2$ *and* $K = k(\sqrt{a})$. *Let* $G = \langle\sigma\rangle$ *act on the rational function field* $K(u,v)$ *by*

$$\sigma : \sqrt{a} \mapsto -\sqrt{a},\ u \mapsto \tfrac{1}{u},\ v \mapsto -\tfrac{1}{v}.$$

Then $K(u,v)^G$ *is k-rational if and only if there exists a Galois extension L/k such that* $k \subset k(\sqrt{a}) \subset L$ *and* $\operatorname{Gal}(L/k)$ *is isomorphic to* C_4.

(2) *Let k be a field with* $\mathrm{char}\, k \neq 2$ *and* $a, b \in k \backslash k^2$ *such that* $[K : k] = 4$ *where* $K = k(\sqrt{a}, \sqrt{b})$. *Let* $G = \langle \sigma, \tau \rangle$ *act on the rational function field* $K(u, v)$ *by*

$$\sigma : \sqrt{a} \mapsto -\sqrt{a},\ \sqrt{b} \mapsto \sqrt{b},\ u \mapsto \tfrac{1}{u},\ v \mapsto -\tfrac{1}{v},$$
$$\tau : \sqrt{a} \mapsto \sqrt{a},\ \sqrt{b} \mapsto -\sqrt{b},\ u \mapsto u,\ v \mapsto -v.$$

Then $K(u, v)^G$ *is* k*-rational if and only if there exists a Galois extension* L/k *such that* $k \subset K \subset L$ *and* $\mathrm{Gal}(L/k)$ *is isomorphic to* D_4.

From the action of $G/N = \langle \sigma, \tau \rangle \simeq D_4$ on $K(x, y)$ in the exceptional case (ii) of Theorem 1.17, we obtain the following example.

Example 1.19 ([HKKi, Example 4.4])**.** Let k be a field with $\mathrm{char}\, k \neq 2$ and $K = k(\sqrt{a}, \sqrt{b})$ be a biquadratic extension of k. We consider the following actions of k-automorphisms of $D_4 = \langle \sigma, \tau \rangle$ on $K(x, y)$:

$$\sigma_a : \sqrt{a} \mapsto -\sqrt{a},\ \sqrt{b} \mapsto \sqrt{b},\ x \mapsto y,\ y \mapsto \tfrac{1}{x},$$
$$\sigma_b : \sqrt{a} \mapsto \sqrt{a},\ \sqrt{b} \mapsto -\sqrt{b},\ x \mapsto y,\ y \mapsto \tfrac{1}{x},$$
$$\sigma_{ab} : \sqrt{a} \mapsto -\sqrt{a},\ \sqrt{b} \mapsto -\sqrt{b},\ x \mapsto y,\ y \mapsto \tfrac{1}{x},$$
$$\tau_a : \sqrt{a} \mapsto -\sqrt{a},\ \sqrt{b} \mapsto \sqrt{b},\ x \mapsto y,\ y \mapsto x,$$
$$\tau_b : \sqrt{a} \mapsto \sqrt{a},\ \sqrt{b} \mapsto -\sqrt{b},\ x \mapsto y,\ y \mapsto x,$$
$$\tau_{ab} : \sqrt{a} \mapsto -\sqrt{a},\ \sqrt{b} \mapsto -\sqrt{b},\ x \mapsto y,\ y \mapsto x.$$

Define $L_{a,b} = K(x, y)^{\langle \sigma_a, \tau_b \rangle}$. Then
(1) $L_{a,b}$ is k-rational $\Leftrightarrow$ $L_{a,ab}$ is k-rational $\Leftrightarrow$ $(a, -b)_k = 0$;
(2) $L_{b,a}$ is k-rational $\Leftrightarrow$ $L_{b,ab}$ is k-rational $\Leftrightarrow$ $(b, -a)_k = 0$;
(3) $L_{ab,a}$ is k-rational $\Leftrightarrow$ $L_{ab,b}$ is k-rational $\Leftrightarrow$ $(a, b)_k = 0$.

In particular, if $\sqrt{-1} \in k$, then the obstructions to the rationality of the above fixed fields over k coincide, i.e. they are reduced to the same condition, $(a, b)_k = 0$.

On the other hand, consider the case $k = \mathbb{Q}$, $K = \mathbb{Q}(\sqrt{-1}, \sqrt{p})$ where p is a prime number with $p \equiv 1 \pmod 4$. Then $L_{a,b}$ and $L_{a,ab}$ are not $\mathbb{Q}$-rational because $(-1, -p)_{\mathbb{Q}} = (-1, -1)_{\mathbb{Q}} \neq 0$, while $L_{b,a}$, $L_{b,ab}$, $L_{ab,a}$ and $L_{ab,b}$ are $\mathbb{Q}$-rational.

The following gives an equivalent definition of purely quasi-monomial actions.

Definition 1.20. Let G be a finite group. A *G-lattice M* is a finitely generated $\mathbb{Z}[G]$-module which is $\mathbb{Z}$-free as an abelian group, i.e. $M = \bigoplus_{1 \leq i \leq n} \mathbb{Z} \cdot x_i$ with a $\mathbb{Z}[G]$-module structure. Let K/k be a field extension such that G acts on K with $K^G = k$. We define a purely quasi-monomial action of G on the rational function field $K(x_1, \ldots, x_n)$ by $\sigma \cdot x_j = c_j(\sigma) \prod_{1 \leq i \leq n} x_i^{a_{ij}} \in K(x_1, \ldots, x_n)$ when $\sigma \cdot x_j = \sum_{1 \leq i \leq n} a_{ij} x_i \in M$ and its fixed field is denoted by $K(M)^G$.

With the aid of Theorem 1.17, we are able to show that $k(M)^G$ is k-rational whenever M is a decomposable G-lattice of $\mathbb{Z}$-rank 4. There are 710 G-lattices of $\mathbb{Z}$-rank 4 and the total number of decomposable ones is 415.

Theorem 1.21 ([HKKi, Theorem 1.16]). *Let k be a field, G be a finite group and M be a G-lattice with $\operatorname{rank}_{\mathbb{Z}} M = 4$ such that G acts on $k(M)$ by purely monomial k-automorphisms. If M is decomposable, i.e. $M = M_1 \oplus M_2$ as $\mathbb{Z}[G]$-modules where $1 \leq \operatorname{rank}_{\mathbb{Z}} M_1 \leq 3$, then $k(M)^G$ is k-rational.*

Proposition 1.22 ([HKKi, Proposition 5.2]). *Let G be a finite group and M be a G-lattice. Assume that $M = M_1 \oplus M_2 \oplus M_3$ as $\mathbb{Z}[G]$-modules where $\operatorname{rank}_{\mathbb{Z}} M_1 = \operatorname{rank}_{\mathbb{Z}} M_2 = 2$ and $\operatorname{rank}_{\mathbb{Z}} M_3 = 1$. Let k be a field, and let G act on $k(M)$ by purely monomial k-automorphisms. Then $k(M)^G$ is k-rational.*

Theorem 1.23 ([HKKi, Theorem 6.2, Theorem 6.4]). *Let G be a finite group and M be a G-lattice. Assume that* (i) *$M = M_1 \oplus M_2$ as $\mathbb{Z}[G]$-modules where $\operatorname{rank}_{\mathbb{Z}} M_1 = 3$ and $\operatorname{rank}_{\mathbb{Z}} M_2 = 2$,* (ii) *either M_1 or M_2 is a faithful G-lattice. Let k be a field, and let G act on $k(M)$ by purely monomial k-automorphisms. Then $k(M)^G$ is k-rational except the following situation: $\operatorname{char} k \neq 2$, $G = \langle \sigma, \tau \rangle \simeq D_4$ and $M_1 = \bigoplus_{1 \leq i \leq 3} \mathbb{Z} x_i$, $M_2 = \bigoplus_{1 \leq j \leq 2} \mathbb{Z} y_j$ such that $\sigma : x_1 \leftrightarrow x_2$, $x_3 \mapsto -x_1 - x_2 - x_3$, $y_1 \mapsto y_2 \mapsto -y_1$, $\tau : x_1 \leftrightarrow x_3$, $x_2 \mapsto -x_1 - x_2 - x_3$, $y_1 \leftrightarrow y_2$ where the $\mathbb{Z}[G]$-module structure of M is written additively.*

For the exceptional case, $k(M)^G$ is not retract k-rational. In particular, if $G = \langle \sigma, \tau \rangle \simeq D_4$ acts on the rational function field $k(x_1, x_2, x_3, x_4, x_5)$ by k-automorphisms

$$\begin{aligned}
\sigma &: x_1 \mapsto x_2,\ x_2 \mapsto x_1,\ x_3 \mapsto \tfrac{1}{x_1 x_2 x_3},\ x_4 \mapsto x_5,\ x_5 \mapsto \tfrac{1}{x_4}, \\
\tau &: x_1 \mapsto x_3,\ x_2 \mapsto \tfrac{1}{x_1 x_2 x_3},\ x_3 \mapsto x_1,\ x_4 \mapsto x_5,\ x_5 \mapsto x_4,
\end{aligned}$$

then $k(x_1, x_2, x_3, x_4, x_5)^G$ is not retract k-rational.

Remark 1.24. The exceptional case of Theorem 1.23 gives an example of purely monomial action of $G \simeq D_4$ whose invariant field $k(M)^G$ is not retract k-rational even over an algebraically closed field k (cf. Theorem 1.7 and Theorem 1.14).

We can deduce the exceptional case of Theorem 1.23 from the following theorem, where the action is not even quasi-monomial.

Theorem 1.25 ([HKKi, Theorem 6.3]). *Let k be a field with $\operatorname{char} k \neq 2$ and $G = \langle \rho \rangle \simeq C_2$ act on $k(x_1, x_2, x_3, x_4)$ by k-automorphisms defined as*

$$\rho : x_1 \mapsto -x_1,\ x_2 \mapsto \tfrac{x_4}{x_2},\ x_3 \mapsto \tfrac{(x_4 - 1)(x_4 - x_1^2)}{x_3},\ x_4 \mapsto x_4.$$

Then $k(x_1, x_2, x_3, x_4)^G$ is not retract k-rational.

§ 2. Noether's problem and unramified Brauer group

Let G be a finite group acting on the rational function field $k(x_g \mid g \in G)$ by k-automorphisms so that $g \cdot x_h = x_{gh}$ for any $g, h \in G$. Denote by $k(G)$ the fixed field $k(x_g \mid g \in G)^G$. Noether's problem asks whether $k(G)$ is rational (= purely transcendental) over k. It is related to the inverse Galois problem, to the existence of generic G-Galois extensions over k, and to the existence of versal G-torsors over k-rational field extensions (see Garibaldi, Merkurjev and Serre [GMS03, 33.1, page 86]). For example, Saltman proved the following theorem (see also [JLY02, Chapter 5]):

Theorem 2.1 (see Saltman [Sal82a, Sal82b] for details)**.** *Let G be a finite group. Assume that k is an infinite field. Then the following conditions are equivalent:*
(1) *$k(G)$ is retract k-rational;*
(2) *G has the lifting property over k;*
(3) *there exists a generic G-Galois extension* (*resp. generic G-polynomial*) *over k.*

We recall some known results on Noether's problem.

Theorem 2.2 (Fischer [Fis15], see also Swan [Swa83, Theorem 6.1])**.** *Let G be a finite abelian group with exponent e. Assume that* (i) *either* $\mathrm{char}\ k = 0$ *or* $\mathrm{char}\ k > 0$ *with* $\mathrm{char}\ k \not| e$, *and* (ii) *k contains a primitive e-th root of unity. Then $k(G)$ is k-rational.*

Kuniyoshi established the following theorem for p-groups (see [Kun56], Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955).

Theorem 2.3 (Kuniyoshi [Kun54, Kun55, Kun56])**.** *Let G be a p-group and k be a field with* $\mathrm{char}\ k = p > 0$. *Then $k(G)$ is k-rational.*

Swan [Swa69] showed that $\mathbb{Q}(C_{47})$ is not $\mathbb{Q}$-rational using Masuda's idea [Mas55, Mas68]. This is the first negative example to Noether's problem. After efforts of many mathematicians (e.g. Voskresenskii, Endo and Miyata), Noether's problem for abelian groups was solved by Lenstra [Len74]. The reader is referred to Swan's paper [Swa83] for a survey of this problem.

On the other hand, just a handful of results about Noether's problem are obtained when the groups are not abelian.

Theorem 2.4 (Chu and Kang [CK01])**.** *Let p be any prime number and G be a p-group of order $\leq p^4$ and of exponent e. If k is a field containing a primitive e-th root of unity, then $k(G)$ is k-rational.*

Theorem 2.5 (Serre [GMS03, Chapter IX]). *Let G be a group with a 2-Sylow subgroup which is cyclic of order ≥ 8 or the generalized quaternion Q_{16} of order 16. Then $\mathbb{Q}(G)$ is not $\mathbb{Q}$-rational.*

Theorem 2.6 (Chu, Hu, Kang and Prokhorov [CHKP08]). *Let G be a group of order 2^5 and of exponent e. If k is a field containing a primitive e-th root of unity, then $k(G)$ is k-rational.*

The notion of the unramified Brauer group of K over k, denoted by $\mathrm{Br}_{v,k}(K)$, was introduced by Saltman [Sal84a].

Definition 2.7 (Saltman [Sal84a, Definition 3.1], [Sal85, page 56]). Let K/k be an extension of fields. *The unramified Brauer group* $\mathrm{Br}_{v,k}(K)$ of K over k is defined by $\mathrm{Br}_{v,k}(K) = \bigcap_R \mathrm{Image}\{\mathrm{Br}(R) \to \mathrm{Br}(K)\}$ where $\mathrm{Br}(R) \to \mathrm{Br}(K)$ is the natural map of Brauer groups and R runs over all the discrete valuation rings R such that $k \subset R \subset K$ and K is the quotient field of R.

Lemma 2.8 (Saltman [Sal84a], [Sal85, Proposition 1.8], [Sal87]). *If K is retract k-rational, then the natural map $\mathrm{Br}(k) \to \mathrm{Br}_{v,k}(K)$ is an isomorphism. In particular, if k is an algebraically closed field and K is retract k-rational, then $\mathrm{Br}_{v,k}(K) = 0$.*

Theorem 2.9 (Bogomolov [Bog88, Theorem 3.1], Saltman [Sal90b, Theorem 12]). *Let G be a finite group and k be an algebraically closed field with $\gcd\{|G|, \operatorname{char} k\} = 1$. Then $\mathrm{Br}_{v,k}(k(G))$ is isomorphic to the group $B_0(G)$ defined by*

$$B_0(G) = \bigcap_A \mathrm{Ker}\{\mathrm{res} : H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(A, \mathbb{Q}/\mathbb{Z})\}$$

where A runs over all the bicyclic subgroups of G (a group A is called bicyclic if A is either a cyclic group or a direct product of two cyclic groups).

We may call $B_0(G)$ the Bogomolov multiplier of G since $H^2(G, \mathbb{Q}/\mathbb{Z})$ is isomorphic to the Schur multiplier $H_2(G, \mathbb{Z})$ of G (see Karpilovsky [Kar87], Kunyavskii [Kun10]). Because of Theorem 2.9, we will not distinguish $B_0(G)$ and $\mathrm{Br}_{v,k}(k(G))$ when k is algebraically closed and $\gcd\{|G|, \operatorname{char} k\} = 1$.

Using the unramified Brauer groups, Saltman and Bogomolov are able to establish counter-examples to Noether's problem for non-abelian p-groups over algebraically closed field.

Theorem 2.10 (Saltman, Bogomolov). *Let p be any prime number and k be any algebraically closed field with $\operatorname{char} k \neq p$.*

(1) (Saltman [Sal84a]) *There exists a group G of order p^9 such that $B_0(G) \neq 0$. In particular, $k(G)$ is not retract k-rational. Thus $k(G)$ is not k-rational.*

(2) (Bogomolov [Bog88]) *There exists a group G of order p^6 such that $B_0(G) \neq 0$. Thus $k(G)$ is not (retract) k-rational.*

Example 2.11. (1) Let p be an odd prime number. If G is p-group of order $\leq p^4$ or 2-group of order $\leq 2^5$, then $B_0(G) = 0$ (see Theorem 2.4 and Theorem 2.6).

(2) Working on p-groups, Bogomolov [Bog88] developed a lot of techniques and interesting results on $B_0(G)$. Bogomolov claimed that

(i) ([Bog88, Lemma 4.11]) If G is a p-group with $B_0(G) \neq 0$ and $G/[G, G] \simeq C_p \times C_p$, then $p \geq 5$ and $|G| \geq p^7$;

(ii) ([Bog88, Lemma 5.6]) If G is a p-group of order $\leq p^5$, then $B_0(G) = 0$.

Because of (ii), Bogomolov proposed to classify all the groups G with $|G| = p^6$ satisfying $B_0(G) \neq 0$ ([Bog88, Remark 1, page 479]). (However, it turns out that (i) and (ii) are not correct, see Theorem 2.13 and Theorem 2.15.)

There are 267 non-isomorphic groups of order 2^6 (see Hall and Senior [HS64]).

Theorem 2.12 (Chu, Hu, Kang and Kunyavskii [CHKK10]). *Let $G = G(2^6, i)$ be the i-th group of order 2^6 in the database of GAP* [GAP] *$(1 \leq i \leq 267)$.*

(1) *$B_0(G) \neq 0$ if and only if $G = G(2^6, i)$ where $149 \leq i \leq 151$, $170 \leq i \leq 172$, $177 \leq i \leq 178$, or $i = 182$.*

(2) *If $B_0(G) = 0$ and k is an algebraically closed field with $\operatorname{char} k \neq 2$, then $k(G)$ is k-rational except possibly for groups $G = G(2^6, i)$ with $241 \leq i \leq 245$.*

It came as a surprise that Moravec [Mor12] disproved Example 2.11 (2) (i), (ii).

Theorem 2.13 (Moravec [Mor12, Section 5]). *If G is a group of order 3^5, then $B_0(G) \neq 0$ if and only if $G = G(3^5, i)$ with $28 \leq i \leq 30$, where $G(3^5, i)$ is the i-th group among groups of order 3^5 in the database of GAP* [GAP].

There exist 67 non-isomorphic groups of order 3^5. Moravec proves Theorem 2.13 by using computer calculations (GAP functions for computing $B_0(G)$ are available at his website `www.fmf.uni-lj.si/~moravec/b0g.g`).

Definition 2.14. Two p-groups G_1 and G_2 are called *isoclinic* if there exist group isomorphisms $\theta\colon G_1/Z(G_1) \to G_2/Z(G_2)$ and $\phi\colon [G_1, G_1] \to [G_2, G_2]$ such that $\phi([g, h]) = [g', h']$ for any $g, h \in G_1$ with $g' \in \theta(gZ(G_1))$, $h' \in \theta(hZ(G_1))$.

For a prime number p and a fixed integer n, let $G_n(p)$ be the set of all non-isomorphic groups of order p^n. In $G_n(p)$ consider an equivalence relation: two groups G_1 and G_2 are equivalent if and only if they are isoclinic. Each equivalence class of $G_n(p)$ is called an *isoclinism family.*

There exist

$$2p + 61 + \gcd\{4, p - 1\} + 2 \gcd\{3, p - 1\}$$

non-isomorphic groups of order p^5 ($p \geq 5$). For $p \geq 3$, there are precisely 10 isoclinism families $\Phi_1, \ldots \Phi_{10}$ for groups of order p^5 (see [Jam80, pages 619–621]).

Hoshi, Kang and Kunyavskii [HKKu] reached to the following theorem which asserts that the non-vanishing of $B_0(G)$ is determined by the isoclinism family of G when G is of order p^5 .

Theorem 2.15 ([HKKu, Theorem 1.12]). *Let p be any odd prime number and G be a group of order p^5. Then $B_0(G) \neq 0$ if and only if G belongs to the isoclinism family Φ_{10}. Each group G in the family Φ_{10} satisfies the condition $G/[G,G] \simeq C_p \times C_p$. There are precisely 3 groups in this family if $p = 3$. For $p \geq 5$, the total number of non-isomorphic groups in this family is*

$$1 + \gcd\{4, p-1\} + \gcd\{3, p-1\}.$$

Remark 2.16. For groups of order 2^6, there exist 27 isoclinism families (see [JNOB90, page 147]). We see that Theorem 2.12 can be rephrased in terms of the isoclinism family as follows. Let G be a group of order 2^6.

(1) $B_0(G) \neq 0$ if and only if G belongs to the 16th isoclinism family Φ_{16};

(2) If $B_0(G) = 0$ and k is an algebraically closed field with $\operatorname{char} k \neq 2$, then $k(G)$ is k-rational except possibly for groups G belonging to the 13th isoclinism family Φ_{13}.

By Theorem 2.12 and Theorem 2.15, we get the following theorem (this theorem is optimal by Theorem 2.4, Theorem 2.6 and Lemma 2.8).

Theorem 2.17 ([HKKu, Theorem 1.13]). *If $2^6 \mid n$ or $p^5 \mid n$ for some odd prime number p, then there exists a group G of order n such that $B_0(G) \neq 0$.*

The following result supplements Moravec's result (Theorem 2.13).

Theorem 2.18 (Chu, Hoshi, Hu and Kang [CHHK]). *Let G be a group of order 3^5 and of exponent e. If k is a field containing a primitive e-th root of unity and $B_0(G) = 0$, then $k(G)$ is k-rational except possibly for groups $G \in \Phi_7$, i.e. $G = G(3^5, i)$ with $56 \leq i \leq 60$.*

From observations in the proof of Theorem 2.15, we raised the following question.

Question 2.19 ([HKKu, Question 1.11]). Let G_1 and G_2 be isoclinic p-groups. Is it true that the fields $k(G_1)$ and $k(G_2)$ are stably k-isomorphic?

Recently, Moravec [Mor] (arXiv:1203.2422) announced that if G_1 and G_2 are isoclinic, then $B_0(G_1) \simeq B_0(G_2)$. Furthermore, Bogomolov and Böhning [BB, Theorem 3.2] (arXiv:1204.4747) announced that the answer to Question 2.19 is affirmative.

§ 3. Rationality problem for algebraic tori

§ 3.1. Low-dimensional cases

Let L be a finite Galois extension of k and $G = \mathrm{Gal}(L/k)$ be the Galois group of the extension L/k. Let $M = \bigoplus_{1 \le i \le n} \mathbb{Z} \cdot u_i$ be a G-lattice with a $\mathbb{Z}$-basis $\{u_1, \ldots, u_n\}$, i.e. finitely generated $\mathbb{Z}[G]$-module which is $\mathbb{Z}$-free as an abelian group. Let G act on the rational function field $L(x_1, \ldots, x_n)$ over L with n variables $x_1, \ldots, x_n$ by

$$\sigma(x_i) = \prod_{j=1}^{n} x_j^{a_{ij}}, \quad 1 \le i \le n \tag{3.1}$$

for any $\sigma \in G$, when $\sigma(u_i) = \sum_{j=1}^{n} a_{ij} u_j$, $a_{ij} \in \mathbb{Z}$. The field $L(x_1, \ldots, x_n)$ with this action of G will be denoted by $L(M)$.

Note that we changed the definition of the action of $\sigma \in G$ on $L(x_1, \ldots, x_n)$ from (1.1) by the following reason:

The category of G-lattices is anti-equivalent to the category of algebraic k-tori which split over L (see [Vos98, page 27, Example 6], [KMRT98, Proposition 20.17]). Indeed, if T is an algebraic k-torus, then the character group $X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ of T becomes G-lattice. Conversely, for a G-lattice M, there exists an algebraic k-torus T which splits over L such that $X(T)$ is isomorphic to M as a G-lattice.

The invariant field $L(M)^G$ may be identified with the function field of T and is always k-unirational (see [Vos98, page 40, Example 21]). Tori of dimension n over k correspond bijectively to the elements of the set $H^1(\mathcal{G}, GL_n(\mathbb{Z}))$ via Galois descent where $\mathcal{G} = \mathrm{Gal}(k_{\mathrm{s}}/k)$ since $\mathrm{Aut}(\mathbb{G}_m^n) = GL_n(\mathbb{Z})$. The k-torus T of dimension n is determined uniquely by the integral representation $h : \mathcal{G} \to GL_n(\mathbb{Z})$ up to conjugacy, and $h(\mathcal{G})$ is a finite subgroup of $GL_n(\mathbb{Z})$ (see [Vos98, page 57, Section 4.9]).

Let K/k be a separable field extension of degree n and L/k be the Galois closure of K/k. Let $G = \mathrm{Gal}(L/k)$ and $H = \mathrm{Gal}(L/K)$. The Galois group G may be regarded as a transitive subgroup of S_n. We have an exact sequence of $\mathbb{Z}[G]$-modules

$$0 \longrightarrow I_{G/H} \longrightarrow \mathbb{Z}[G/H] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

where $\varepsilon : \mathbb{Z}[G/H] \to \mathbb{Z}$ is the augmentation map. Taking the duals, we get an exact sequence of $\mathbb{Z}[G]$-modules

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[G/H] \longrightarrow J_{G/H} \longrightarrow 0.$$

In particular, $J_{G/H} = \mathrm{Hom}_{\mathbb{Z}}(I_{G/H}, \mathbb{Z})$ (called the Chevalley module) is of rank $n-1$. Furthermore, we get an exact sequence of algebraic k-tori

$$1 \longrightarrow R^{(1)}_{K/k}(\mathbb{G}_{m,K}) \longrightarrow R_{K/k}(\mathbb{G}_{m,K}) \xrightarrow{N_{K/k}} \mathbb{G}_m \longrightarrow 1.$$

Here $R_{K/k}(\mathbb{G}_{m,K})$ is the Weil restriction of $\mathbb{G}_{m,K}$ by the extension K/k and $R^{(1)}_{K/k}(\mathbb{G}_{m,K})$ is the norm one torus of K/k whose character group is $J_{G/H}$.

Write $J_{G/H} = \oplus_{1\leq i\leq n-1}\mathbb{Z}x_i$. Then the action of G on $L(J_{G/H}) = L(x_1,\ldots,x_{n-1})$ is nothing but (3.1).

All the 1-dimensional algebraic k-tori T, i.e. the trivial torus $\mathbb{G}_m$ and the norm one torus $R^{(1)}_{L/k}(\mathbb{G}_m)$ with $[L:k]=2$, are k-rational. A birational classification of the 2-dimensional algebraic tori and the 3-dimensional algebraic tori was given by Voskresenskii [Vos67] and Kunyavskii [Kun90] respectively.

Theorem 3.1 (Voskresenskii, Kunyavskii). *Let k be a field.*
(1) (Voskresenskii [Vos67]) *All the two-dimensional algebraic k-tori are k-rational. In particular, $K(x_1,x_2)^G$ is always k-rational if G is isomorphic to* $\mathrm{Gal}(K/k)$ *and G acts on $K(x_1,x_2)$ by purely quasi-monomial k-automorphisms.*
(2) (Kunyavskii [Kun90], see also Kang [Kan12, Section 1] for the last statement) *All the three-dimensional algebraic k-tori are k-rational except for the* 15 *cases in the list of* [Kun90, Theorem 1]. *For the exceptional* 15 *cases, they are not k-rational; in fact, they are even not retract k-rational.*

For 4-dimensional case, some birational invariants are computed by Popov [Pop98].

We list known results about the rationality problem for norm one tori $R^{(1)}_{K/k}(\mathbb{G}_m)$ of K/k.

When an extension K/k is Galois, the following theorem is known.

Theorem 3.2. *Let K/k be a finite Galois field extension and $G = \mathrm{Gal}(K/k)$.*
(1) (Endo and Miyata [EM74, Theorem 1.5], Saltman [Sal84b, Theorem 3.14]) $R^{(1)}_{K/k}(\mathbb{G}_m)$ *is retract k-rational if and only if all the Sylow subgroups of G are cyclic.*
(2) (Endo and Miyata [EM74, Theorem 2.3]) $R^{(1)}_{K/k}(\mathbb{G}_m)$ *is stably k-rational if and only if $G = C_m$ or $G = C_n \times \langle\sigma,\tau \mid \sigma^k = \tau^{2^d} = 1, \tau\sigma\tau^{-1} = \sigma^{-1}\rangle$ where $d\geq 1, k\geq 3$, n,k: odd, and* $\gcd\{n,k\}=1$.

When an extension K/k is non-Galois and separable, we take the Galois closure L/k of K/k and put $G = \mathrm{Gal}(L/k)$ and $H = \mathrm{Gal}(L/K)$. Then we have:

Theorem 3.3 (Endo [End11, Theorem 2.1]). *Assume that $G = \mathrm{Gal}(L/k)$ is nilpotent. Then $R^{(1)}_{K/k}(\mathbb{G}_m)$ is not retract k-rational.*

Theorem 3.4 (Endo [End11, Theorem 3.1]). *Assume that the Sylow subgroups of $G = \mathrm{Gal}(L/k)$ are all cyclic. Then $R^{(1)}_{K/k}(\mathbb{G}_m)$ is retract k-rational, and the following conditions are equivalent:*
(i) $R^{(1)}_{K/k}(\mathbb{G}_m)$ *is stably k-rational;*
(ii) *$G = D_n$ with n odd ($n\geq 3$) or $G = C_m \times D_n$ where m,n are odd, $m,n\geq 3$,*

$\gcd\{m,n\}=1$, *and* $H=\mathrm{Gal}(L/K)\leq D_n$ *is of order* 2*;*
(iii) $H=C_2$ *and* $G\simeq C_r\rtimes H$, $r\geq 3$ *odd, where* H *acts non-trivially on* C_r*.*

Theorem 3.5 (Endo [End11, Theorem 4.1], see also [End11, Remark 4.2])**.** *Assume that* $\mathrm{Gal}(L/k)=S_n$, $n\geq 3$, *and* $\mathrm{Gal}(L/K)=S_{n-1}$ *is the stabilizer of one of the letters in* S_n*.*
(1) $R^{(1)}_{K/k}(\mathbb{G}_m)$ *is retract* k*-rational if and only if* n *is a prime;*
(2) $R^{(1)}_{K/k}(\mathbb{G}_m)$ *is* (*stably*) k*-rational if and only if* $n=3$*.*

Theorem 3.6 (Endo [End11, Theorem 4.4])**.** *Assume that* $\mathrm{Gal}(L/k)=A_n$, $n\geq 4$, *and* $\mathrm{Gal}(L/K)=A_{n-1}$ *is the stabilizer of one of the letters in* A_n*.*
(1) $R^{(1)}_{K/k}(\mathbb{G}_m)$ *is retract* k*-rational if and only if* n *is a prime.*
(2) *For some positive integer* t, $[R^{(1)}_{K/k}(\mathbb{G}_m)]^{(t)}$ *is stably* k*-rational if and only if* $n=5$, *where* $[R^{(1)}_{K/k}(\mathbb{G}_m)]^{(t)}$ *is the product of* t *copies of* $R^{(1)}_{K/k}(\mathbb{G}_m)$*.*

A birational classification of the algebraic k-tori of dimension 4 is given by Hoshi and Yamasaki [HY]. Note that there are 710 $\mathbb{Z}$-classes forming 227 $\mathbb{Q}$-classes in $GL_4(\mathbb{Z})$.

Theorem 3.7 ([HY, Theorem 1.8])**.** *Let* L/k *be a Galois extension and* $G\simeq\mathrm{Gal}(L/k)$ *be a finite subgroup of* $GL_4(\mathbb{Z})$ *which acts on* $L(x_1,x_2,x_3,x_4)$ *via* (3.1)*.*
(i) $L(x_1,x_2,x_3,x_4)^G$ *is stably* k*-rational if and only if* G *is conjugate to one of the* 487 *groups which are not in* [HY, Tables 2, 3 and 4]*.*
(ii) $L(x_1,x_2,x_3,x_4)^G$ *is not stably but retract* k*-rational if and only if* G *is conjugate to one of the* 7 *groups which are given as in* [HY, Table 2]*.*
(iii) $L(x_1,x_2,x_3,x_4)^G$ *is not retract* k*-rational if and only if* G *is conjugate to one of the* 216 *groups which are given as in* [HY, Tables 3 and 4]*.*

Let F_{20} be the Frobenius group of order 20. By Theorem 3.7, we have:

Theorem 3.8 ([HY, Theorem 1.9])**.** *Let* K/k *be a separable field extension of degree* 5 *and* L/k *be the Galois closure of* K/k*. Assume that* $G=\mathrm{Gal}(L/k)$ *is a transitive subgroup of* S_5 *which acts on* $L(x_1,x_2,x_3,x_4)$ *via* (3.1), *and* $H=\mathrm{Gal}(L/K)$ *is the stabilizer of one of the letters in* G*.*
(1) $R^{(1)}_{K/k}(\mathbb{G}_m)$ *is stably* k*-rational if and only if* $G\simeq C_5$, D_5 *or* A_5*;*
(2) $R^{(1)}_{K/k}(\mathbb{G}_m)$ *is not stably but retract* k*-rational if and only if* $G\simeq F_{20}$ *or* S_5*.*

Theorem 3.8 is already known except for the case of A_5 (see Theorems 3.2, 3.4, 3.5 and 3.6). Stable k-rationality of $R^{(1)}_{K/k}(\mathbb{G}_m)$ for the case A_5 is asked by S. Endo in [End11, Remark 4.6]. By Theorems 3.6 and 3.8, we get:

Corollary 3.9 ([HY, Corollary 1.10])**.** *Let* K/k *be a non-Galois separable field extension of degree* n *and* L/k *be the Galois closure of* K/k*. Assume that* $\mathrm{Gal}(L/k)=$

A_n, $n \geq 4$, and $\mathrm{Gal}(L/K) = A_{n-1}$ *is the stabilizer of one of the letters in A_n. Then $R^{(1)}_{K/k}(\mathbb{G}_m)$ is stably k-rational if and only if $n = 5$.*

There are 6079 $\mathbb{Z}$-classes forming 955 $\mathbb{Q}$-classes in $GL_5(\mathbb{Z})$. A birational classification of the algebraic k-tori of dimension 5 is given as follows:

Theorem 3.10 ([HY, Theorem 1.11])**.** *Let L/k be a Galois extension and $G \simeq$* $\mathrm{Gal}(L/k)$ *be a finite subgroup of $GL_5(\mathbb{Z})$ which acts on $L(x_1, x_2, x_3, x_4, x_5)$ via* (3.1).
(i) *$L(x_1, x_2, x_3, x_4, x_5)^G$ is stably k-rational if and only if G is conjugate to one of the* 3051 *groups which are not in* [HY, Tables 11, 12, 13, 14 and 15].
(ii) *$L(x_1, x_2, x_3, x_4, x_5)^G$ is not stably but retract k-rational if and only if G is conjugate to one of the* 25 *groups which are given as in* [HY, Table 11].
(iii) *$L(x_1, x_2, x_3, x_4, x_5)^G$ is not retract k-rational if and only if G is conjugate to one of the* 3003 *groups which are given as in* [HY, Tables 12, 13, 14 and 15].

Theorem 3.11 ([HY, Theorem 1.13])**.** *Let K/k be a separable field extension of degree* 6 *and L/k be the Galois closure of K/k. Assume that $G =$* $\mathrm{Gal}(L/k)$ *is a transitive subgroup of S_6 which acts on $L(x_1, x_2, x_3, x_4, x_5)$ via* (3.1), *and $H =$* $\mathrm{Gal}(L/K)$ *is the stabilizer of one of the letters in G. Then $R^{(1)}_{K/k}(\mathbb{G}_m)$ is stably k-rational if and only if $G \simeq C_6$, S_3 or D_6. Moreover, if $R^{(1)}_{K/k}(\mathbb{G}_m)$ is not stably k-rational, then it is not retract k-rational.*

In Theorems 3.7, 3.8, 3.10 and 3.11, when the field is stably k-rational, we do not know whether $L(M)^G$ is k-rational except for few cases (see [Vos98, Chapter 2]).

§ 3.2. Strategy of the proof: the flabby class $[M]^{fl}$ of a G-lattice M

Let M be a G-lattice. For the rationality problem of $L(M)^G$ over k, the flabby class $[M]^{fl}$ of M plays a crucial role as follows (see Voskresenskii [Vos98, Section 4.6], Lorenz [Lor05, Section 9.5]):

Theorem 3.12 (Endo and Miyata, Voskresenskii, Saltman)**.** *Let L/k be a finite Galois extension with Galois group $G =$* $\mathrm{Gal}(L/k)$ *and M, M' be G-lattices.*
(1) (Endo and Miyata [EM73, Theorem 1.6])
$[M]^{fl} = 0$ if and only if $L(M)^G$ is stably k-rational.
(2) (Voskresenskii [Vos74, Theorem 2])
$[M]^{fl} = [M']^{fl}$ if and only if $L(M)^G$ and $L(M')^G$ are stably k-isomorphic.
(3) (Saltman [Sal84b, Theorem 3.14])
$[M]^{fl}$ is invertible if and only if $L(M)^G$ is retract k-rational.

Unfortunately, the flabby class $[M]^{fl}$ is useful to verify the stable k-rationality and the retract k-rationality of $L(M)^G$ but useless to verify the k-rationality.

In order to give the definition of the flabby class $[M]^{fl}$ of G-lattice M (Definition 3.16), we prepare some terminology.

Definition 3.13. Let M be a G-lattice.
(1) M is called a *permutation* G-lattice if M has a $\mathbb{Z}$-basis permuted by G, i.e. $M \simeq \oplus_{1\leq i\leq m}\mathbb{Z}[G/H_i]$ for some subgroups $H_1,\ldots,H_m$ of G.
(2) M is called a *stably permutation* G-lattice if $M \oplus P \simeq P'$ for some permutation G-lattices P and P'.
(3) M is called *invertible* if it is a direct summand of a permutation G-lattice, i.e. $P \simeq M \oplus M'$ for some permutation G-lattice P and a G-lattice M'.
(4) M is called *coflabby* if $H^1(H,M)=0$ for any subgroup H of G.
(5) M is called *flabby* if $\widehat{H}^{-1}(H,M)=0$ for any subgroup H of G where $\widehat{H}$ is the Tate cohomology.

It is not difficult to verify the following implications:

permutation $\Rightarrow$ stably permutation $\Rightarrow$ invertible $\Rightarrow$ flabby and coflabby.

Definition 3.14 (see [EM74, Section 1], [Vos98, Section 4.7])**.** Let $\mathcal{C}(G)$ be the category of all G-lattices. Let $\mathcal{S}(G)$ be the full subcategory of $\mathcal{C}(G)$ of all permutation G-lattices and $\mathcal{D}(G)$ be the full subcategory of $\mathcal{C}(G)$ of all invertible G-lattices. Let

$$\mathcal{H}^i(G)=\{M\in\mathcal{C}(G)\mid \widehat{H}^i(H,M)=0 \text{ for any } H\leq G\}\ (i=\pm 1)$$

be the class of "$\widehat{H}^i$-vanish" G-lattices where $\widehat{H}^i$ is the Tate cohomology. Then one has the inclusions $\mathcal{S}(G)\subset\mathcal{D}(G)\subset\mathcal{H}^i(G)\subset\mathcal{C}(G)$ $(i=\pm 1)$.

Definition 3.15 (The commutative monoid $\mathcal{T}(G)=\mathcal{C}(G)/\mathcal{S}(G)$)**.** We say that two G-lattices M_1 and M_2 are *similar* if there exist permutation G-lattices P_1 and P_2 such that $M_1\oplus P_1\simeq M_2\oplus P_2$. We denote the set of similarity classes $\mathcal{C}(G)/\mathcal{S}(G)$ by $\mathcal{T}(G)$ and the similarity class of M by $[M]$. $\mathcal{T}(G)$ becomes a commutative monoid with respect to the sum $[M_1]+[M_2]:=[M_1\oplus M_2]$ and the zero $0=[P]$ where $P\in\mathcal{S}(G)$.

Definition 3.16. For a G-lattice M, there exists a short exact sequence of G-lattices $0\to M\to P\to F\to 0$ where P is permutation and F is flabby which is called a *flasque resolution* of M (see Endo and Miyata [EM74, Lemma 1.1], Colliot-Thélène and Sansuc [CTS77, Lemma 3]). The similarity class $[F]\in\mathcal{T}(G)$ of F is determined uniquely and is called *the flabby class* of M. We denote the flabby class $[F]$ of M by $[M]^{fl}$. We say that $[M]^{fl}$ is invertible if $[M]^{fl}=[E]$ for some invertible G-lattice E.

Theorem 3.17 (Colliot-Thélène and Sansuc [CTS77, Corollaire 1])**.** *Let G be a finite group. The following conditions are equivalent:*

(i) $[J_G]^{fl}$ *is coflabby;*
(ii) *any Sylow subgroup of* G *is cyclic or generalized quaternion* Q_{4n} *of order* $4n$ $(n \geq 2)$*;*
(iii) *any abelian subgroup of* G *is cyclic;*
(iv) $H^3(H, \mathbb{Z}) = 0$ *for any subgroup* H *of* G.

Theorem 3.18 (Endo and Miyata [EM82, Theorem 2.1]). *Let* G *be a finite group. The following conditions are equivalent:*
(i) $\mathcal{H}^1(G) \cap \mathcal{H}^{-1}(G) = \mathcal{D}(G)$*, i.e. any flabby and coflabby* G*-lattice is invertible;*
(ii) $[J_G \otimes_{\mathbb{Z}} J_G]^{fl} = [[J_G]^{fl}]^{fl}$ *is invertible;*
(iii) *any* p*-Sylow subgroup of* G *is cyclic for odd* p *and cyclic or dihedral* (*including Klein's four group*) *for* $p = 2$.

Note that $H^1(H, [J_G]^{fl}) \simeq H^3(H, \mathbb{Z})$ for any subgroup H of G (see [Vos70, Theorem 7] and [CTS77, Proposition 1]) and $[J_G]^{fl} = [J_G \otimes_{\mathbb{Z}} J_G]$ (see [EM82, Section 2]).

For G-lattice M, it is not difficult to see

$$\begin{array}{ccccccc} \text{permutation} & \Rightarrow & \text{stably permutation} & \Rightarrow & \text{invertible} & \Rightarrow & \text{flabby and coflabby} \\ & & \Downarrow & & \Downarrow & & \\ & & [M]^{fl} = 0 \text{ in } \mathcal{T}(G) & \Rightarrow & [M]^{fl} \text{ is invertible in } \mathcal{T}(G). & & \end{array}$$

The above implications in each step cannot be reversed. Swan [Swa60] gave an example of Q_8-lattice M of rank 8 which is not permutation but stably permutation: $M \oplus \mathbb{Z} \simeq \mathbb{Z}[Q_8] \oplus \mathbb{Z}$. This also indicates that the direct sum cancellation fails. Colliot-Thélène and Sansuc [CTS77, Remarque R1], [CTS77, Remarque R4] gave examples of S_3-lattice M of rank 4 which is not permutation but stably permutation: $M \oplus \mathbb{Z} \simeq \mathbb{Z}[S_3/\langle\sigma\rangle] \oplus \mathbb{Z}[S_3/\langle\tau\rangle]$ where $S_3 = \langle\sigma, \tau\rangle$ and of F_{20}-lattice $[J_{F_{20}/C_4}]^{fl}$ of the Chevalley module J_{F_{20}/C_4} of rank 4 which is not stably permutation but invertible (see also Theorem 3.21 and Theorem 3.25 (ii), (iv) and (v)). By Theorem 3.2 (1), Theorem 3.12 (2) and Theorem 3.17, the flabby class $[J_{Q_8}]^{fl}$ of the Chevalley module J_{Q_8} of rank 7 is not invertible but flabby and coflabby (we may take $[J_{Q_8}]^{fl}$ of rank 9, see [HY, Example 7.3]). The inverse direction of the vertical implication holds if M is coflabby (see [HY, Lemma 2.11]).

Theorem 3.2 (resp. Theorem 3.4, Theorem 3.5, Theorem 3.6) can be obtained by Theorem 3.19 and Theorem 3.20 (resp. Theorem 3.21, Theorem 3.22, Theorem 3.23) by using the interpretation as in Theorem 3.12.

Theorem 3.19 (Endo and Miyata [EM74, Theorem 1.5]). *Let* G *be a finite group. The following conditions are equivalent:*
(i) $[J_G]^{fl}$ *is invertible;*
(ii) *all the Sylow subgroups of* G *are cyclic;*
(iii) $\mathcal{H}^{-1}(G) = \mathcal{H}^1(G) = \mathcal{D}(G)$*, i.e. any flabby* (*resp. coflabby*) G*-lattice is invertible.*

Theorem 3.20 ([EM74, Theorem 2.3], see also [CTS77, Proposition 3]). *Let G be a finite group. The following conditions are equivalent:*
(i) $[J_G]^{fl}=0$;
(ii) $[J_G]^{fl}$ *is of finite order in* $\mathcal{T}(G)$;
(iii) *all the Sylow subgroups of G are cyclic and* $H^4(G,\mathbb{Z})\simeq \widehat{H}^0(G,\mathbb{Z})$;
(iv) $G=C_m$ *or* $G=C_n\times\langle\sigma,\tau\mid \sigma^k=\tau^{2^d}=1,\tau\sigma\tau^{-1}=\sigma^{-1}\rangle$ *where* $d\geq 1, k\geq 3$, n,k: *odd, and* $\gcd\{n,k\}=1$;
(v) $G=\langle s,t\mid s^m=t^{2^d}=1, tst^{-1}=s^r, m: odd,\ r^2\equiv 1 \pmod{m}\rangle$.

Theorem 3.21 (Endo [End11, Theorem 3.1]). *Let G be a non-abelian group. Assume that Sylow subgroups of G are all cyclic. Let H be a non-normal subgroup of G which contains no normal subgroup of G except $\{1\}$. (By Theorem 3.19, $[J_{G/H}]^{fl}$ is invertible.) The following conditions are equivalent:*
(i) $[J_{G/H}]^{fl}=0$;
(ii) $[J_{G/H}]^{fl}$ *is of finite order in* $\mathcal{T}(G)$;
(iii) $G=D_n$ *with n odd* ($n\geq 3$) *or* $G=C_m\times D_n$ *where m,n are odd,* $m,n\geq 3$, $\gcd\{m,n\}=1$, *and* $H\leq D_n$ *is of order* 2;
(iv) $H=C_2$ *and* $G\simeq C_r\rtimes H$, $r\geq 3$ *odd, where H acts non-trivially on* C_r.

Theorem 3.22 (Endo [End11, Theorem 4.3], see also [End11, Remark 4.2]). *Let $n\geq 3$ be an integer.*
(1) $[J_{S_n/S_{n-1}}]^{fl}$ *is invertible if and only if n is a prime.*
(2) $[J_{S_n/S_{n-1}}]^{fl}=0$ *if and only if* $n=3$.
(3) $[J_{S_n/S_{n-1}}]^{fl}$ *is of finite order in $\mathcal{T}(G)$ if and only if* $n=3$.

Theorem 3.23 (Endo [End11, Theorem 4.5]). *Let $n\geq 4$ be an integer.*
(1) $[J_{A_n/A_{n-1}}]^{fl}$ *is invertible if and only if n is a prime.*
(2) $[J_{A_n/A_{n-1}}]^{fl}$ *is of finite order in $\mathcal{T}(G)$ if and only if* $n=5$.

We conclude this article, by introducing results on G-lattices of rank 4 and 5.

Definition 3.24 (The G-lattice M_G). Let G be a finite subgroup of $GL_n(\mathbb{Z})$. The G-lattice M_G of rank n is defined to be the G-lattice with a $\mathbb{Z}$-basis $\{u_1,\ldots,u_n\}$ on which G acts by $\sigma(u_i)=\sum_{j=1}^n a_{ij}u_j$ for any $\sigma=[a_{ij}]\in G$.

For $2\leq n\leq 4$, the GAP code (n,i,j,k) of a finite subgroup G of $GL_n(\mathbb{Z})$ stands for the k-th $\mathbb{Z}$-class of the j-th $\mathbb{Q}$-class of the i-th crystal system of dimension n as in [BBNWZ78, Table 1] and [GAP].

A birational classification of the k-tori of dimension 4 and 5 (Theorem 3.7 and Theorem 3.10) can be obtained by Theorem 3.25 and Theorem 3.26 respectively.

Theorem 3.25 ([HY, Theorem 1.25]). *Let G be a finite subgroup of $GL_4(\mathbb{Z})$ and M_G be the G-lattice as in Definition 3.24.*
(i) $[M_G]^{fl}=0$ *if and only if G is conjugate to one of the* 487 *groups which are not in* [HY, Tables 2, 3 and 4].
(ii) $[M_G]^{fl}$ *is not zero but invertible if and only if G is conjugate to one of the* 7 *groups which are given as in* [HY, Table 2].
(iii) $[M_G]^{fl}$ *is not invertible if and only if G is conjugate to one of the* 216 *groups which are given as in* [HY, Tables 3 and 4].
(iv) $[M_G]^{fl}=0$ *if and only if* $[M_G]^{fl}$ *is of finite order in* $\mathcal{T}(G)$.
(v) *For* $G\simeq S_5$ *of the GAP code* $(4,31,5,2)$ *in* (ii), *we have* $-[M_G]^{fl}=[J_{S_5/S_4}]^{fl}\neq 0$.
(vi) *For* $G\simeq F_{20}$ *of the GAP code* $(4,31,1,4)$ *in* (ii), *we have* $-[M_G]^{fl}=[J_{F_{20}/C_4}]^{fl}\neq 0$.

Theorem 3.26 ([HY, Theorem 1.26]). *Let G be a finite subgroup of $GL_5(\mathbb{Z})$ and M_G be the G-lattice as in Definition 3.24.*
(i) $[M_G]^{fl}=0$ *if and only if G is conjugate to one of the* 3051 *groups which are not in* [HY, Tables 11, 12, 13, 14 and 15].
(ii) $[M_G]^{fl}$ *is not zero but invertible if and only if G is conjugate to one of the* 25 *groups which are given as in* [HY, Table 11].
(iii) $[M_G]^{fl}$ *is not invertible if and only if G is conjugate to one of the* 3003 *groups which are given as in* [HY, Tables 12, 13, 14 and 15].
(iv) $[M_G]^{fl}=0$ *if and only if* $[M_G]^{fl}$ *is of finite order in* $\mathcal{T}(G)$.

Remark 3.27. (1) By the interpretation as in Theorem 3.12, Theorem 3.25 (v), (vi) claims that the corresponding two tori T and T' of dimension 4 are not stably k-rational and are not stably k-isomorphic each other but the torus $T\times T'$ of dimension 8 is stably k-rational.
(2) When $[M]^{fl}$ is invertible, the inverse element of $[M]^{fl}$ is $-[M]^{fl}=[[M]^{fl}]^{fl}$. Hence Theorem 3.25 (v) also claims that $[[M_G]^{fl}]^{fl}=[J_{S_5/S_4}]^{fl}$ and $[[J_{S_5/S_4}]^{fl}]^{fl}=[M_G]^{fl}$.

Finally, we give an application of the results in [HY] which provides the smallest example exhibiting the failure of the Krull-Schmidt theorem for permutation G-lattices (see Dress's paper [Dre73, Proposition 9.6]):

Proposition 3.28 ([HY, Proposition 6.7]). *Let D_6 be the dihedral group of order* 12 *and* $\{1\}$, $C_2^{(1)}$, $C_2^{(2)}$, $C_2^{(3)}$, C_3, V_4, C_6, $S_3^{(1)}$, $S_3^{(2)}$ *and D_6 be the conjugacy classes of subgroups of D_6. Then the following isomorphism of permutation D_6-lattices holds:*

$$\mathbb{Z}[D_6]\oplus\mathbb{Z}[D_6/V_4]^{\oplus 2}\oplus\mathbb{Z}[D_6/C_6]\oplus\mathbb{Z}[D_6/S_3^{(1)}]\oplus\mathbb{Z}[D_6/S_3^{(2)}]$$
$$\simeq\mathbb{Z}[D_6/C_2^{(1)}]\oplus\mathbb{Z}[D_6/C_2^{(2)}]\oplus\mathbb{Z}[D_6/C_2^{(3)}]\oplus\mathbb{Z}[D_6/C_3]\oplus\mathbb{Z}^{\oplus 2}.$$

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