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Topics on diffusion semigroups on a path space with Gibbs measures

Dedicated to Professor Hideo Tamura on the occasion of his 60th birthday

By

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Abstract

In this paper, we give a summary of recent results on symmetric diffusion semigroups associated with classical Dirichlet forms on an infinite volume path space $C(\mathbf{R}, \mathbf{R}^d)$ with Gibbs measures. First, we discuss essential self-adjointness of diffusion operators (Dirichlet operators) associated with the Dirichlet forms. We also show the connection between the corresponding diffusion semigroup and the solution of a parabolic stochastic partial differential equation (=SPDE, in abbreviation) on $\mathbf{R}$. Next, we present some functional inequalities for the diffusion semigroup. As applications of these inequalities, we have the existence of a gap at the lower end of spectrum of the Dirichlet operator and the boundedness of the Riesz transforms.

§ 1. Introduction

In recent years, there has been a growing interest in infinite dimensional stochastic dynamics in several areas of Euclidean quantum field theory and statistical mechanics. Equilibrium states of such dynamics are described by Gibbs measures. The stochastic dynamics corresponding to these states is given by a diffusion semigroup. On some minimal domain of smooth functions, the infinitesimal generator of the semigroup coincides with the Dirichlet operator defined through a classical Dirichlet form of gradient type with a Gibbs measure. From an analytic point of view, it is very important to ask whether the extension of the Dirichlet operator restricted to the minimal domain is unique. As is well known, in the case of L^2-dynamics, the uniqueness is equivalent to essential self-adjointness on the minimal domain of the Dirichlet operators with the

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Gibbs measure considered. This kind of uniqueness problem on infinite dimensional state spaces has been studied by many researchers up to now. We refer to Eberle [4] and the references therein for a detailed review.

In this paper, we deal with diffusion semigroups of $P(\phi)_1$-quantum fields in infinite volume in the content of quantum field theory. The diffusion semigroups are defined through Dirichlet forms on an infinite volume path space $C(\mathbf{R},\mathbf{R}^d)$ with a Gibbs measure μ. The Gibbs measure μ is associated with the (formal) Hamiltonian

$$\mathcal{H}(w) := \frac{1}{2}\int_{\mathbf{R}} |w'(x)|^2 dx + \int_{\mathbf{R}} U(w(x))dx,$$

where $U : \mathbf{R}^d \to \mathbf{R}$ is an interaction potential function. The main purpose of this paper is to introduce the results of Kawabi-Röckner [11] on the above uniqueness problem and to We also discuss the connection between the diffusion semigroup and the solution of a parabolic SPDE called the time dependent Ginzburg-Landau type SPDE. Besides, we present some functional inequalities which will play important roles to develop harmonic analysis and potential theory on the path space $C(\mathbf{R},\mathbf{R}^d)$ with the Gibbs measure μ.

The organization of this paper is as follows: In Section 2, we give our framework and show essential self-adjointness on smooth cylinder functions of the Dirichlet operators. Our method in [11] is inspired by quite recent works by Da Prato-Tubaro [3] and Da Prato-Röckner [2] where an L^p-analysis of Kolmogorov operators in infinitely many variables is developed. They employ the theory of SPDE in an essential way and give a new approach to tackle such uniqueness problems. For our problem, we adopt their approach, however, with substantial necessary modifications. Hence we give an outline of the proof. We also show the logarithmic Sobolev inequality under the additional condition of convexity of U. As a consequence of this inequality, there is a gap at the lower end of the spectrum of the Dirichlet operator and $L^2(\mu)$-ergodicity of the corresponding semigroup holds. In Section 3, we study the Littlewood-Paley-Stein inequality and the Riesz transforms which play the fundamental roles in the Sobolev space theory.

§ 2. Essential self-adjointness of Dirichlet operators

Let us introduce some notations and objects we will be working with. First we define a weight function $\rho_r \in C^\infty(\mathbf{R},\mathbf{R}), r \in \mathbf{R}$, by $\rho_r(x) := e^{r\chi(x)}$, $x \in \mathbf{R}$, where $\chi \in C^\infty(\mathbf{R},\mathbf{R})$ is a positive symmetric convex function satisfying $\chi(x) = |x|$ for $|x| \geq 1$. We fix a constant $r > 0$ large enough and set $E = L^2_r(\mathbf{R},\mathbf{R}^d) := L^2(\mathbf{R},\mathbf{R}^d : \rho_{-2r}(x)dx)$. This space is a Hilbert space with the inner product defined by

$$(X,Y)_E := \int_{\mathbf{R}} \big(X(x),Y(x)\big)_{\mathbf{R}^d} \rho_{-2r}(x)dx, \quad X,Y \in E.$$

Moreover, we set $H := L^2(\mathbf{R}, \mathbf{R}^d)$ and denote by $\|\cdot\|_E$ and $\|\cdot\|_H$ the corresponding norms of E and H, respectively.

We also introduce a suitable subspace of $C(\mathbf{R}, \mathbf{R}^d)$. For functions in $C(\mathbf{R}, \mathbf{R}^d)$, we set

$$\|w\|_{r,\infty} := \sup_{x \in \mathbf{R}} |w(x)| \rho_{-r}(x) \quad \text{for } r \in \mathbf{R},$$

and consider

$$\mathcal{C} := \bigcap_{r>0} \big\{ w \in C(\mathbf{R}, \mathbf{R}^d) \mid \|w\|_{r,\infty} < \infty \big\}.$$

Then it becomes a Fréchet space with the system of norms $\|\cdot\|_{r,\infty}$. We easily see that the inclusion $\mathcal{C} \subset E \cap C(\mathbf{R}, \mathbf{R}^d)$ is dense with respect to the topology of E. We endow $C(\mathbf{R}, \mathbf{R}^d)$ with the σ-field $\mathcal{B}$ generated by the point evaluation and denote by $\mathcal{P}(C(\mathbf{R}, \mathbf{R}^d))$ the class of all probability measures on the space $(C(\mathbf{R}, \mathbf{R}^d), \mathcal{B})$. For $T > 0$, we also denote by $\mathcal{B}_T$ and $\mathcal{B}_{T,c}$ the σ-fields of $C(\mathbf{R}, \mathbf{R}^d)$ generated by $\{w(x); -T \leq x \leq T\}$ and $\{w(x); x \leq -T, x \geq T\}$, respectively.

In this paper, we always assume the following three conditions on the potential function U:

(U1): $U \in C^1(\mathbf{R}^d, \mathbf{R})$ and there exists a constant $K_1 \in \mathbf{R}$ such that

$$\big(\nabla U(z_1) - \nabla U(z_2), z_1 - z_2\big)_{\mathbf{R}^d} \geq -K_1 |z_1 - z_2|^2, \quad z_1, z_2 \in \mathbf{R}^d.$$

(U2): There exist $K_2 > 0$ and $p > 0$ such that

$$|\nabla U(z)| \leq K_2 (1 + |z|^p), \qquad z \in \mathbf{R}^d.$$

(U3): $\lim_{|z| \to \infty} U(z) = \infty$.

(In [11], we impose slightly weaker conditions than the above conditions.) As examples of U, we can include the case

$$U(z) = \sum_{j=0}^{2m} a_j |z|^j, \ a_1 = 0, \ a_{2m} > 0, \ m \in \mathbf{N}.$$

Especially, we are interested in a square potential and a double-well potential. Those are, $U(z) = a|z|^2$ and $U(z) = a(|z|^4 - |z|^2)$, $a > 0$, respectively.

Now, we introduce a Gibbs measure. Consider the Schrödinger operator $H_U := -\frac{1}{2}\Delta + U$ on $L^2(\mathbf{R}^d, \mathbf{R})$, where $\Delta := \sum_{i=1}^d \partial^2 / \partial z_i^2$ is the d-dimensional Laplacian. Then condition **(U3)** assures that H_U has purely discrete spectrum and a complete set of eigenfunctions. We denote by $\lambda_0 (> \min U)$ the minimal eigenvalue and by Ω the corresponding normalized eigenfunction in $L^2(\mathbf{R}^d, \mathbf{R})$. It is called ground state and it decays exponentially.

Let $\mathcal{W}_{-T,z_1;T,z_2}$, $T > 0, z_1, z_2 \in \mathbf{R}^d$, be the path measure of Brownian bridge such that $w(-T) = z_1, w(T) = z_2$. We sometimes regard this measure as a probability measure on the space $(C(\mathbf{R}, \mathbf{R}^d), \mathcal{B})$ by considering $w(x) = z_1$ for $x \le -T$ and $w(x) = z_2$ for $x \ge T$. We define $\mu(A)$ for $A \in \mathcal{B}_T$, $T > 0$, by

$$\begin{aligned} \mu(A) := e^{2T\lambda_0} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} & \Omega(z_1)\Omega(z_2)p(2T, z_1, z_2) \\ & \times \mathbf{E}^{\mathcal{W}_{-T,z_1;T,z_2}} \Big[\exp\big(- \int_{-T}^{T} U(w(x))dx \big); A \Big] dz_1 dz_2, \end{aligned} \tag{2.1}$$

where $p(t, z_1, z_2)$ is the transition probability density of standard Brownian motion on $\mathbf{R}^d$. Then by the Feynman-Kac formula and the Markov property of Brownian motion, we can see that μ is well-defined as an element of $\mathcal{P}(C(\mathbf{R}, \mathbf{R}^d))$ and it satisfies the following DLR-equation for every $T > 0$ and μ-a.e. $\xi \in C(\mathbf{R}, \mathbf{R}^d)$:

$$\begin{aligned} & \mu(dw|\mathcal{B}_{T,c})(\xi) \\ & = Z_{T,\xi}^{-1} \exp\big(- \int_{-T}^{T} U(w(x))dx \big) \mathcal{W}_{-T,\xi(-T);T,\xi(T)}(dw), \end{aligned} \tag{2.2}$$

where $Z_{T,\xi}$ is a normalizing constant. See Proposition 2.7 in Iwata [5] for details. Although generally there exist other μ's in $\mathcal{P}(C(\mathbf{R}, \mathbf{R}^d))$ satisfying the DLR-equation (2.2), in this paper we only consider the Gibbs measure μ which has been constructed in (2.1).

Here we note that the Gibbs measure μ is supported on $\mathcal{C}$ by using the standard moment estimates of Brownian motion. Then by the continuity of the inclusion map of $\mathcal{C}$ into E, we can regard $\mu \in \mathcal{P}(E)$ by identifying it with its image measure under the inclusion map.

By virtue of the DLR-equation (2.2), the Gibbs measure μ is $C_0^\infty(\mathbf{R}, \mathbf{R}^d)$-quasi-invariant, i.e., $\mu(\cdot + k)$ and μ are mutually equivalent and

$$\mu(k + dw) = \Lambda(k, w)\mu(dw)$$

holds for every $k \in C_0^\infty(\mathbf{R}, \mathbf{R}^d)$. The Radon-Nikodym density $\Lambda(k, w)$ is represented by

$$\begin{aligned} \Lambda(k, w) = \exp\Big\{ \int_{\mathbf{R}} \Big(& U\big(w(x)\big) - U\big(w(x) + k(x)\big) \\ & - \frac{1}{2}|k'(x)|^2 + (w(x), \Delta_x k(x))_{\mathbf{R}^d} \Big) dx \Big\}, \end{aligned}$$

where $\Delta_x := d^2/dx^2$. Moreover, we have μ is translation invariant, and then by combining this with the fact that Ω decays exponentially, it holds that

$$\int_E \Big(\int_{\mathbf{R}} |w(x)|^{2m} \rho_{-2r}(x) dx \Big) \mu(dw)$$

$$\leq \frac{1}{r}\int_{\mathbf{R}^d} |z|^{2m}\Omega(z)^2 dz < \infty, \quad m \in \mathbf{N}, r > 0. \tag{2.3}$$

Now we define the space of smooth cylinder functions. We say a function $F : E \longrightarrow \mathbf{R}$ is in a class $\mathcal{FC}_b^\infty$ if there exist $n \in \mathbf{N}$, $\{\varphi_1, \cdots, \varphi_n\} \subset C_0^\infty(\mathbf{R}, \mathbf{R}^d)$ and a function $f \equiv f(\alpha_1, \ \cdots, \alpha_n) \in C_b^\infty(\mathbf{R}^n)$ such that

$$F(w) \equiv f(\langle w, \varphi_1\rangle, \cdots, \langle w, \varphi_n\rangle), \quad w \in E.$$

Here we use the notation $\langle w, \varphi\rangle := \int_{\mathbf{R}} (w(x), \varphi(x))_{\mathbf{R}^d} dx$ if the integral is absolutely converging. Note that $\mathcal{FC}_b^\infty$ is dense in $L^2(\mu)$. For $F \in \mathcal{FC}_b^\infty$, we also define the H-Fréchet derivative $D_H F : E \to H$ by

$$D_H F(w) := \sum_{j=1}^n \frac{\partial f}{\partial \alpha_j}(\langle w, \varphi_1\rangle, \cdots, \langle w, \varphi_n\rangle)\varphi_j.$$

We consider a pre-Dirichlet form $(\mathcal{E}, \mathcal{FC}_b^\infty)$ which is given by

$$\mathcal{E}(F, G) = \int_E \big(D_H F(w), D_H G(w)\big)_H \mu(dw), \quad F, G \in \mathcal{FC}_b^\infty.$$

Then by virtue of the $C_0^\infty(\mathbf{R}, \mathbf{R}^d)$-quasi-invariance, we have

$$\mathcal{E}(F, G) = \big(- \mathcal{L}_0 F, G\big)_{L^2(\mu)}, \quad F, G \in \mathcal{FC}_b^\infty, \tag{2.4}$$

where $\mathcal{L}_0$ is given by

$$\begin{aligned}\mathcal{L}_0 F(w) = &\sum_{i,j=1}^n \frac{\partial^2 f}{\partial \alpha_i \partial \alpha_j}\big(\langle w, \varphi_1\rangle, \cdots, \langle w, \varphi_n\rangle\big)\langle \varphi_i, \varphi_j\rangle \\ &+ \sum_{i=1}^n \frac{\partial f}{\partial \alpha_i}\big(\langle w, \varphi_1\rangle, \cdots, \langle w, \varphi_n\rangle\big) \cdot \big\{\langle w, \Delta_x \varphi_i\rangle - \langle \nabla U(w(\cdot)), \varphi_i\rangle\big\}.\end{aligned}$$

This means the operator $\mathcal{L}_0$ is the pre-Dirichlet operator which is associated with the pre-Dirichlet form $(\mathcal{E}, \mathcal{FC}_b^\infty)$. In particular, $(\mathcal{E}, \mathcal{FC}_b^\infty)$ is closable on $L^2(\mu)$. So we can define by $\mathcal{D}(\mathcal{E})$ the completion of $\mathcal{FC}_b^\infty$ with respect to $\mathcal{E}_1^{1/2}$-norm. We see that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form and the operator $\mathcal{L}_0$ is symmetric in $L^2(\mu)$.

In many applications, it is an important problem whether one has essential self-adjointness for $\mathcal{L}_0$, i.e., self-adjointness of the closure $(\overline{\mathcal{L}}_0, \mathrm{Dom}(\overline{\mathcal{L}}_0))$ of $(\mathcal{L}_0, \mathcal{FC}_b^\infty)$ in $L^2(\mu)$. The reason is that in general there are many lower bounded self-adjoint extensions $\widetilde{\mathcal{L}}_2$ of $\mathcal{L}_0$ in $L^2(\mu)$ which therefore define symmetric strongly continuous semigroups $\{e^{t\widetilde{\mathcal{L}}_2}\}_{t\geq 0}$ generated by them. In fact, there always exists one such extension called the Friedrich extension which is the operator corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. If $\mathcal{L}_0$ is essentially self-adjoint, there is hence only one such semigroup. Consequently, only one such dynamics associated with the Gibbs measure μ exists.

The following theorem is taken from Theorem 5.1 in [11]. We show that the semigroup is not only unique but also represented in terms of the solution of a parabolic SPDE (2.5) on $\mathbf{R}$.

Theorem 2.1. *The pre-Dirichlet operator $(\mathcal{L}_0, \mathcal{FC}_b^\infty)$ is essentially self-adjoint in $L^2(\mu)$. Furthermore, the semigroup $\{T_t\}_{t\geq 0}$ generated by $\overline{\mathcal{L}}_0$ is represented by*

$$T_t F(w) = \mathbf{E}\big[F(X_t^w)\big], \quad \mu\text{-a.s. } w, \quad t \geq 0.$$

Here $X^w = \{X_t^w(\cdot)\}_{t\geq 0}$ is the solution of the SPDE

$$dX_t(x) = \big\{\Delta_x X_t(x) - \nabla U(X_t(x))\big\}dt + \sqrt{2}dB_t(x), \quad x \in \mathbf{R},\ t > 0, \tag{2.5}$$

with initial datum $X_0 = w \in \mathcal{C}$, where $\{B_t\}_{t\geq 0}$ is an H-cylindrical Brownian motion over a probability space $(\Theta, \mathcal{F}, \mathbf{P})$.

Remark 1. We refer to Iwata [6] for the precise meaning of the (mild) solution to the SPDE (2.5). Here we collect some results on the SPDE (2.5).

(i) Under conditions **(U1)** and **(U2)**, (2.5) has a unique (mild) solution living in $C([0,\infty), \mathcal{C})$ for initial datum $w \in \mathcal{C}$. (See Theorems 5.1 and 5.2 in [6].)
(ii) By a usual coupling method for (2.5),

$$\big\|X_t^w - X_t^{w'}\big\|_E \leq e^{(K_1+2\lambda^2)t}\big\|w - w'\big\|_E, \qquad w, w' \in \mathcal{C}, \tag{2.6}$$

holds $\mathbf{P}$-almost surely. (See Lemma 2.1 in Kawabi [7].)

(iii) In addition, we impose condition **(U3)**. Then the Gibbs measure μ is a reversible measure of the solution of (2.5), that is,

$$\int_E \mathbf{E}\big[F(X_t^w)\big]G(w)\mu(dw) = \int_E F(w)\mathbf{E}\big[G(X_t^w)\big]\mu(dw), \quad F, G \in \mathcal{FC}_b^\infty.$$

(See Lemma 2.9 in [5].) Hence the transition semigroup of the solution of (2.5) can be extended to a strongly continuous semigroup on $L^p(\mu), p \geq 1$. We denote by $(\mathcal{L}_p, \mathrm{Dom}(\mathcal{L}_p))$ its infinitesimal generator. Then it holds that

$$(\mathcal{L}_0, \mathcal{FC}_b^\infty) \subset (\overline{\mathcal{L}}_0, \mathrm{Dom}(\overline{\mathcal{L}}_0)) \subset (\mathcal{L}_2, \mathrm{Dom}(\mathcal{L}_2)). \tag{2.7}$$

(See Proposition 3.1 and Remark 4.9 in [11].)

Proof. We only give a sketch of the proof in the following (see [11] for details). Since (2.7) and $\mathcal{FC}_b^\infty$ is dense in $L^2(\mu)$, it is sufficient to show

$$\mathcal{FC}_b^\infty \subset \mathrm{Range}(\lambda - \overline{\mathcal{L}}_0) \tag{2.8}$$

for some $\lambda > 0$. Note that showing (2.8) is equivalent to solving the elliptic problem

$$\lambda\Phi - \overline{\mathcal{L}}_0\Phi = F, \quad F \in \mathcal{F}\mathcal{C}_b^\infty. \tag{2.9}$$

Then by the Lumer-Phillips theorem (see Theorems 1.1 and 1.2 in [4]), we can see that (2.8) implies self-adjointness of $(\overline{\mathcal{L}}_0, \mathrm{Dom}(\overline{\mathcal{L}}_0))$ in $L^2(\mu)$. However it is not easy to consider $\overline{\mathcal{L}}_0$ directly. Hence we aim to insert a tractable space between $\mathcal{F}\mathcal{C}_b^\infty$ and $\mathrm{Dom}(\overline{\mathcal{L}}_0)$ which can be regarded as a domain of the Ornstein-Uhlenbeck operator.

Now we introduce the Ornstein-Uhlenbeck operator. We fix a constant $\kappa > 2r^2$ and define the Ornstein-Uhlenbeck process $Y^w = \{Y_t^w(\cdot)\}_{t\geq 0}$ by the solution of the SPDE

$$dY_t(x) = (\Delta_x - \kappa)Y_t(x)dt + \sqrt{2}dB_t(x), \quad x \in \mathbf{R}, \ t > 0,$$

with initial datum $Y_0 = w \in E$. Next we introduce some function spaces on which the Ornstein-Uhlenbeck semigroup will act. We denote by $UC_{b,2}(E)$ the Banach space of all functions $F : E \to \mathbf{R}$ such that $\frac{F(\cdot)}{1+\|\cdot\|_E^2}$ is uniformly continuous and bounded. Endowed with the norm

$$\|F\|_{b,2} := \sup_{w\in E} \frac{|F(w)|}{1 + \|w\|_E^2} \ ,$$

$UC_{b,2}(E)$ is a Banach space. Moreover, $C_{b,2}^1(E)$ denotes the subspace of $UC_{b,2}(E)$ of those functions F which are continuously differentiable with

$$\|DF\|_{b,2} := \sup_{w\in E} \frac{\|DF(w)\|_E}{1 + \|w\|_E^2} < \infty \ ,$$

where $DF : E \to E$ means the E-Fréchet derivative of F. Note the relation $D_H F = \rho_{-r}(\cdot)DF$. We define the Ornstein-Uhlenbeck semigroup $\{R_t\}_{t\geq 0}$ by

$$R_t F(w) := \mathbf{E}\big[F(Y_t^w)\big], \qquad w \in E, \quad F \in UC_{b,2}(E).$$

R_t maps $UC_{b,2}(E)$ and $C_{b,2}^1(E)$ into themselves for all $t \geq 0$, respectively. Note that R_t is not strongly continuous in $UC_{b,2}(E)$. However, it is a π-semigroup in the sense of Da Prato and Priola (see [14, 15] for definition). Thus one can define its infinitesimal generator L through the resolvent

$$(\lambda - L)^{-1}F = \Psi_\lambda F := \int_0^\infty e^{-\lambda t} R_t F dt, \qquad F \in UC_{b,2}(E), \ \lambda > 0.$$

We call L the Ornstein-Uhlenbeck operator. Since the image of the resolvent is independent of $\lambda > 0$, we can set

$$\mathcal{D}(L, UC_{b,2}(E)) := R(\lambda, L)(UC_{b,2}(E)), \ \mathcal{D}(L, C_{b,2}^1(E)) := R(\lambda, L)(C_{b,2}^1(E)).$$

The following key proposition is taken from Propositions 4.5 and 4.6 in [11].

Proposition 2.1. (i) *$\mathcal{FC}_b^\infty \subset \mathcal{D}(L, C^1_{b,2}(E))$ and we have*

$$
\begin{aligned}
LF(w) = & \sum_{i,j=1}^n \frac{\partial^2 f}{\partial\alpha_i \partial\alpha_j}\big(\langle w,\varphi_1\rangle,\cdots,\langle w,\varphi_n\rangle\big)\langle\varphi_i,\varphi_j\rangle \\
& + \sum_{i=1}^n \frac{\partial f}{\partial\alpha_i}\big(\langle w,\varphi_1\rangle,\cdots,\langle w,\varphi_n\rangle\big)\cdot\big\langle w,(\Delta_x-\kappa)\varphi_i\big\rangle, \quad F\in\mathcal{FC}_b^\infty.
\end{aligned}
$$

(ii) *$\mathcal{D}(L, C^1_{b,2}(E)) \subset \mathrm{Dom}(\overline{\mathcal{L}}_0)$ and the following identity holds:*

$$
\overline{\mathcal{L}}_0 F = LF + (b, DF)_E, \quad F\in\mathcal{D}(L, C^1_{b,2}(E)),
$$

where $b:\mathrm{Dom}(b)\subset E\to E$ is a measurable mapping with $\mathrm{Dom}(b)=\mathcal{C}$ is defined by

$$
b(w)(\cdot) := \kappa w(\cdot) - \nabla U(w(\cdot)), \quad w\in\mathcal{C}.
$$

By the item (ii) of Proposition 2.1, we can rewrite the elliptic problem (2.9) as

$$
\lambda\Phi - L\Phi - \big(b(\cdot), D\Phi\big)_E = F. \tag{2.10}
$$

Finally, we are in a position to solve (2.9). It is sufficient to show that for $\lambda > K_1 + 2r^2$, (2.10) has a unique solution $\Phi\in\mathcal{D}(L, C^1_{b,2}(E))$ which is given by

$$
\Phi(w) := \int_0^\infty e^{-\lambda t}\mathbf{E}\big[F(X_t^w)\big]dt.
$$

See Proposition 5.3 in [11] for the detailed proof. However we note that condition **(U2)**, (2.3) and (2.6) work efficiently in the proof. This completes the proof of Theorem 2.1. □

As a corollary of Theorem 2.1, the Markov uniqueness also holds. See Eberle [4] for details. Here we say that a Dirichlet form $(\mathcal{E},\mathrm{Dom}(\mathcal{E}))$ in $L^2(\mu)$ is an extension of $(\mathcal{L}_0,\mathcal{FC}_b^\infty)$ if $\mathcal{FC}_b^\infty\subset\mathrm{Dom}(\mathcal{E})$ and

$$
\mathcal{E}(F,G) = \big(-\mathcal{L}_0F, G\big)_{L^2(\mu)} \quad \text{for any } F\in\mathcal{FC}_b^\infty, G\in\mathrm{Dom}(\mathcal{E}).
$$

Corollary 2.1. *$(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is the unique extension of $(\mathcal{L}_0,\mathcal{FC}_b^\infty)$.*

Remark 2. Let $(\mathcal{E},\mathrm{Dom}(\mathcal{E}))$ be an extension of $(\mathcal{L}_0,\mathcal{FC}_b^\infty)$ and we denote by $(\mathcal{L},\mathrm{Dom}(\mathcal{L}))$ the generator associated with $(\mathcal{E},\mathrm{Dom}(\mathcal{E}))$. Since

$$
\begin{aligned}
\mathrm{Dom}(\mathcal{L}) = \{F\in\mathrm{Dom}(\mathcal{E}) \,|\, & \text{ there exists a function } \Psi_F\in L^2(\mu) \text{ such that} \\
& \mathcal{E}(F,G) = \big(-\Psi_F, G\big)_{L^2(\mu)} \text{ for any } G\in\mathrm{Dom}(\mathcal{E})\},
\end{aligned}
$$

we can easily see $\mathcal{FC}_b^\infty\subset\mathrm{Dom}(\mathcal{L})$.

Before closing this section, we present some properties of the Dirichlet operator. Here we give the following gradient estimate formula for the diffusion semigroup $\{T_t\}_{t\geq 0}$ for our later use. It is obtained by the estimate (2.6). We refer to Proposition 2.4 in [7] for details.

Proposition 2.2. *Let $F \in \mathcal{D}(\mathcal{E})$. Then the following gradient estimate holds for any $t \in [0,\infty)$ and μ-a.e. $w \in E$:*

$$\big\|D_H(T_tF)(w)\big\|_H \leq e^{K_1 t} T_t\big(\|D_H F\|_H\big)(w). \tag{2.11}$$

This proposition leads us to the following logarithmic Sobolev inequality. See Theorem 1.2 in Kawabi [8] for the proof.

Theorem 2.2. *Assume $K_1 < 0$, that is, U is convex. Then for $F \in \mathcal{D}(\mathcal{E})$, the following logarithmic Sobolev inequality holds:*

$$\int_E F(w)^2 \log\Big(\frac{F(w)^2}{\|F\|^2_{L^2(\mu)}}\Big)\mu(dw) \leq -\frac{2}{K_1}\int_E \|D_H F(w)\|_H^2 \mu(dw). \tag{2.12}$$

By the Rothaus-Simon mass gap theorem, the logarithmic Sobolev inequality (2.12) implies that there is a spectral gap at the lower end of the spectrum of the Dirichlet operator, that is,

$$\mathrm{Spec}(-\mathcal{L}_2) \subset \{0\} \cup [-K_1, \infty).$$

As is well known, an application of the spectral theory gives the following $L^2(\mu)$-ergodicity of $\{T_t\}_{t\geq 0}$:

$$\big\|T_tF - \mu(F)\big\|_{L^2(\mu)} \leq e^{K_1 t}\|F - \mu(F)\|_{L^2(\mu)}, \quad t \geq 0,\ F \in L^2(\mu),$$

where $\mu(F) := \int_E F(w)\mu(dw)$.

§ 3. Littlewood-Paley-Stein inequality and Riesz transforms

In this section, we discuss the Littlewood-Paley-Stein inequality and the Riesz transforms on $C(\mathbf{R}, \mathbf{R}^d)$. The Littlewood-Paley-Stein inequality yields a characterization of L^p-norms in terms of the Poisson kernel. By E.M. Stein's pioneering work [19], the utility of this characterization can be seen in the theory of Hardy spaces as well as in that of Sobolev spaces. P.A. Meyer [13] proved this inequality for the Ornstein-Uhlenbeck semigroup on the Wiener space. It calls special attention that this inequality plays the fundamental role in the theory of the Malliavin calculus. After his work, many researchers studied this inequality by probabilistic approaches. Here we mention that Shigekawa-Yoshida [18] showed it for symmetric diffusion processes on general state spaces. In [18], they assumed the existence of a suitable core $\mathcal{A}$ which is not only a ring but also stable under the operation of the semigroup and the infinitesimal generator to employ Bakry-Emery's Γ_2-method in the proof, and established the Littlewood-Paley-Stein inequality under the condition that Γ_2 is bounded from below. We note that this condition is regarded as the lower boundedness of the Ricci curvature when we work on a usual complete Riemannian manifold. Moreover the gradient estimate formula (2.11) for the diffusion semigroup $\{T_t\}_{t\geq 0}$ and this condition are equivalent as long as there exists a good core $\mathcal{A}$. The readers are referred to Bakry [1] for details. However, it is not easy to check the existence of such a good core $\mathcal{A}$ when we consider problems of infinite dimensional diffusion processes, and if we work on the Heisenberg group, since the Ricci curvature is everywhere $-\infty$, we cannot apply this method. On the other hand, H.-Q. Li [12] recently established (2.11) for the heat semigroup on the Heisenberg group by using an explicit formula for the heat kernel. Hence we can see that the gradient estimate formula (2.11) is weaker than the lower boundedness of Γ_2.

First, we review the result of Kawabi-Miyokawa [10] which is an extension of [18]. In [10], the Littlewood-Paley-Stein inequality is proved under the gradient estimate formula (2.11) for the diffusion semigroup $\{T_t\}_{t\geq 0}$.

For $\alpha > 0$, we denote by $\{Q_t^{(\alpha)}\}_{t\geq 0}$ the α-order subordination of $\{P_t\}$ on $L^p(\mu)$. Let $-\sqrt{\alpha - L_p}$ be the infinitesimal generator of $\{Q_t^{(\alpha)}\}$ on $L^p(\mu)$. For $F \in L^2 \cap L^p(\mu)$, we define

$$g_F^{\rightarrow}(w,t) := \left|\frac{\partial}{\partial t}(Q_t^{(\alpha)}F)(w)\right|, \qquad g_F^{\uparrow}(w,t) := \left\|D_H Q_t^{(\alpha)}F(w)\right\|_H,$$
$$g_F(w,t) := \sqrt{(g_F^{\rightarrow}(w,t))^2 + (g_F^{\uparrow}(w,t))^2}\ .$$

Then the Littlewood-Paley G-functions are defined by

$$G_F^{\rightarrow}(w) := \left(\int_0^\infty t g_F^{\rightarrow}(w,t)^2\,dt\right)^{1/2},$$
$$G_F^{\uparrow}(w) := \left(\int_0^\infty t g_F^{\uparrow}(w,t)^2\,dt\right)^{1/2},$$
$$G_F(w) := \left(\int_0^\infty t g_F(w,t)^2\,dt\right)^{1/2}.$$

Now we are in a position to present the Littlewood-Paley-Stein inequality. See Theorem 1.2 in [10] for details.

Theorem 3.1. *For any $1 < p < \infty$ and $\alpha > K_1 \vee 0$, there exists a constant $C_p > 0$ depending only on p such that the following inequalities hold for $F \in L^2 \cap L^p(\mu)$:*

$$\|G_F\|_{L^p(\mu)} \leq C_p \|F\|_{L^p(\mu)},$$
$$\|F\|_{L^p(\mu)} \leq C_p \|G_F^{\rightarrow}\|_{L^p(\mu)}.$$

Next, we discuss the boundedness of the Riesz transforms as an application of the Littlewood-Paley-Stein inequality. For $\alpha > 0$, we define the Riesz transform $R_\alpha(\mathcal{L})$ by

$$R_\alpha(\mathcal{L})F := D_H(\alpha - \mathcal{L})^{-1/2}F, \qquad F \in \mathcal{FC}_b^\infty.$$

(In the sequel of this section, we usually denote the generator $\mathcal{L}_p$ in $L^p(\mu)$ by $\mathcal{L}$ for simplicity.) It is a fundamental and important problem in harmonic analysis and potential theory to establish the boundedness of $R_\alpha(\mathcal{L})$ on $L^p(\mu)$ for all $p > 1$ and for some $\alpha > 0$, and we note that the boundedness of $R_\alpha(\mathcal{L})$ yields the Meyer equivalence of first order Sobolev norms.

To show the boundedness of $R_\alpha(\mathcal{L})$, it is necessary to have the intertwining property of the diffusion semigroup $\{T_t\}_{t\geq 0}$ and a semigroup $\{\overrightarrow{T}_t\}_{t\geq 0}$ acting on H-valued functions (see Shigekawa [17] for details). Here we replace the conditions **(U1)** and **(U2)** by the following two conditions:

(U1)': $U \in C^2(\mathbf{R}^d, \mathbf{R})$ and there exists a constant $K_1 \in \mathbf{R}$ such that

$$\nabla^2 U(z) \geq -K_1, \qquad z \in \mathbf{R}^d.$$

(U2)': There exist $K_2 > 0$ and $p > 0$ such that

$$|\nabla U(z)| + |\nabla^2 U(z)|_{\mathbf{R}^d \otimes \mathbf{R}^d} \leq K_2(1 + |z|^p), \qquad z \in \mathbf{R}^d.$$

Now we explain how to establish this property. We denote by $\mathcal{FC}_b^\infty(H)$ the set of H-valued smooth cylinder functions on E represented by $\sum_{k=1}^m F_k e_k, m \geq 0$, where $F_k \in \mathcal{FC}_b^\infty, e_k \in C_0^\infty(\mathbf{R}, \mathbf{R}^d)$. For $\theta = \sum_{k=1}^m F_k e_k \in \mathcal{FC}_b^\infty(H)$ with $F_k(w) = f_k(\langle w, \varphi_1\rangle, \ldots, \langle w, \varphi_n\rangle)$, we define a second order differential operator $\overrightarrow{\mathcal{L}}$ acting on $\mathcal{FC}_b^\infty(H)$ by

$$\begin{aligned}
\overrightarrow{\mathcal{L}}\theta(w)(x) &:= \sum_{i,j=1}^n \sum_{k=1}^m \frac{\partial^2 f_k}{\partial\alpha_i \partial\alpha_j}\big(\langle w, \varphi_1\rangle, \cdots, \langle w, \varphi_n\rangle\big)\langle \varphi_i, \varphi_j\rangle\, e_k(x) \\
&+ \sum_{i=1}^n \sum_{k=1}^m \frac{\partial f_k}{\partial \alpha_i}\big(\langle w, \varphi_1\rangle, \cdots, \langle w, \varphi_n\rangle\big)\big\{\langle w, \Delta_x\varphi_i\rangle - \langle \nabla U(w(\cdot)), \varphi_i\rangle\big\}\, e_k(x) \\
&+ \sum_{k=1}^m f_k\big(\langle w, \varphi_1\rangle, \cdots, \langle w, \varphi_n\rangle\big) \cdot \Big(\Delta_x e_k(x) - \nabla^2 U(w(\cdot))[e_k(x)]_{\mathbf{R}^d}\Big), \quad x \in \mathbf{R}.
\end{aligned}$$

Note that condition **(U2)'** leads us to $\overrightarrow{\mathcal{L}}\theta \in L^2(\mu; H)$. Next we define a bi-linear form $\overrightarrow{\mathcal{E}}$ by

$$\overrightarrow{\mathcal{E}}(\theta, \eta) := \big(-\overrightarrow{\mathcal{L}}\theta, \eta\big)_{L^2(\mu;H)}, \qquad \theta, \eta \in \mathcal{FC}_b^\infty(H).$$

Then by condition **(U1)'**, we have

$$\overrightarrow{\mathcal{E}}(\theta, \theta) \geq -K_1\|\theta\|_{L^2(\mu;H)}^2, \qquad \theta \in \mathcal{FC}_b^\infty(H),$$

and thus there exists the Friedrichs extension of $(\overrightarrow{\mathcal{L}}, \mathcal{FC}_b^\infty(H))$ in $L^2(\mu; H)$. We denote it by $(\overrightarrow{\mathcal{L}}, \mathrm{Dom}(\overrightarrow{\mathcal{L}}))$. We define by $(\overrightarrow{\mathcal{E}}, \mathcal{D}(\overrightarrow{\mathcal{E}}))$ the minimal extension of $(\overrightarrow{\mathcal{E}}, \mathcal{FC}_b^\infty(H))$ and by $\{\overrightarrow{T}_t\}_{t\geq 0}$ the symmetric strongly continuous semigroup on $L^2(\mu; H)$ generated by $(\overrightarrow{\mathcal{L}}, \mathrm{Dom}(\overrightarrow{\mathcal{L}}))$.

Now we are in a position to present the following intertwining property.

Lemma 3.1. *For $F \in \mathcal{D}(\mathcal{E})$, it holds that*

$$D_H T_t F = \overrightarrow{T}_t D_H F, \qquad t \geq 0. \tag{3.1}$$

Proof. We easily see that the generator version of the intertwining property $D_H \mathcal{L} F = \overrightarrow{\mathcal{L}} D_H F$ holds for $F \in \mathcal{FC}_b^\infty$. Since we have already obtained essential self-adjointness of our Dirichlet operator $(\mathcal{L}_0, \mathcal{FC}_b^\infty)$ in Theorem 2.1, we can apply Theorems 2.1 and 3.2 in Shigekawa [17]. Then we have our desired equality (3.1). □

Finally, by using Theorem 3.1 and Lemma 3.1, we have the following theorem. For the detailed proof, see the forthcoming paper Kawabi [9].

Theorem 3.2. *Under conditions* **(U1)'**, **(U2)'** *and* **(U3)**, *the Riesz transform $R_\alpha(\mathcal{L})$ is bounded on $L^p(\mu)$ for all $p > 1$ and $\alpha > K_1 \vee 0$. That is, there exists a positive constant C_p depending only on p such that*

$$\|R_\alpha(\mathcal{L})F\|_{L^p(\mu)} \leq C_p \|F\|_{L^p(\mu)}, \qquad F \in \mathcal{FC}_b^\infty.$$

Consequently, the Sobolev norm $\|F\|_{L^p(\mu)} + \|D_H F\|_{L^p(\mu;H)}$ is equivalent to the Sobolev norm $\|(1-\mathcal{L})^{1/2}F\|_{L^p(\mu)}$.

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