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Duality Theorem for Inductive Limit Group of Direct Product Type

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Let G be the inductive limit group of countable direct product groups $G(j) = \Pi_{k \leq j} G_k$, where G_k are non trivial type I locally compact groups.

In the previous paper [T], we proved a duality theorem for locally compact groups. That is, any locally compact group is isomorph to the group of so-called bi-representations on its dual space which is the set of all (equivalence classes of) unitary representations of the initial group.

Obviously, our G is not locally compact in general. But in this paper, we show that for the above G, analogous duality theorem holds too.

§ 1. Preliminary

We quote [TSH] for the definition of inductive limit group. At first we show a property of general inductive limit groups.

Lemma 1-1 Consider a set $\{K_j\}$ of countable locally compact groups $\{K_j\}$ such that

$\forall j, \quad K_j \subset K_{j+1}$ as a topological subgroup.

Let K be the inductive limit of $\{K_j\}$, and C be any compact set in K.

Then there exists n such that $C \subset K_n$.

Proof. Step 1. If there exists n such that K_n is open in $\forall K_m$ $(m > n)$, the assertion is obvious. Therefore we can assume that for $\forall n, \exists m > n$, K_n is not open in K_m.

Let the assertion fail, then we can take m as $C \cap (K_m - K_n) \neq \phi$.

If necessary, changing the numbering of groups, we can assume $\forall n$, $C \cap (K_n - K_{n-1}) \neq \phi$, and take a sequence $\{g_n\}$ as $g_n \in C \cap (K_n - K_{n-1})$.

Step 2. By induction on j, we construct a family $\{W_j\}$, where W_j is a neighborhood of e in K_j, satisfying

(1) $\quad \forall k < j, \quad g_j (g_k)^{-1} \notin W_1 W_2 \cdots W_{j-1} (W_j)^2$

(2) $\quad (W_j)^2 \cap K_{j-1} \subset W_{j-1}$

Since a locally compact subgroup of a topological group is closed, K_{j-1} is closed in K_j

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and $g_j (g_k)^{-1}$ dose not belong to K_{j-1}, we can select a neighborhood U of e in Kj as

$$\forall\ k < j,\quad g_j (g_k)^{-1} \notin K_{j-1}U\ (\supset W_1 W_2 \cdots W_{j-1}U).$$

Next we take neighborhood Wj of e in Kj satisfying (2) and $(W_j)^2 \subset U$.

Step 3. We put $W \equiv \cup^{\infty}_{j=1} W_1 W_2 \cdots W_j$.

We have shown that this set gives a neighborhood of e in K named Bamboo Shoot neighborhood .[TSH, Lemma 2.2.]

Here we remark that for any j, if $m < j$, obviously. $W_1 W_2 \cdots W_m \subset K_m$ can not contain $g_j (g_k)^{-1}$ ($\forall\ k < j$), and when $m \geqq j$, $g_m(g_k)^{-1}$ ($\forall\ k < j$) is not in $W_1 W_2 \cdots W_m$ from (1).

Step 4. Next we consider $W_1 W_2 \cdots W_m \cap K_j$ for the case $j \leqq m$. The condition (2) shows $W_1 W_2 \cdots W_m \cap K_j = W_1 W_2 \cdots W_m \cap K_{m-1} \cap K_j \subset$
$W_1 W_2 \cdots W_{m-2} (W_{m-1})^2 \cap K_{m-1} \cap K_j = W_1 W_2 \cdots W_{m-2} (W_{m-1})^2 \cap K_{m-3} \cap K_j \subset$
$W_1 W_2 \cdots W_{m-3} (W_{m-2})^2 \cap K_{m-2} \cap K_j = W_1 W_2 \cdots W_{m-3} (W_{m-2})^2 \cap K_{m-3} \cap K_j$
$\subset \cdots\cdots\cdots \subset W_1 W_2 \cdots W_{j-1} (W_j)^2$

The condition (1) leads us to

$\forall\ m$, $\forall\ k < j$, $g_j (g_k)^{-1} \notin W_1 W_2 \cdots W_{m-2} W_{m-1} W_m \cap K_j$

i.e. $g_j (g_k)^{-1} \notin W_1 W_2 \cdots W_{m-2} W_{m-1} W_m$.

Joining the results of Step 3 and Step 4 , we get $\forall\ j$, $\forall\ k < j$, $g_j (g_k)^{-1} \notin W$. But j and k are free . So $g_j (g_k)^{-1} \notin W$ for $\forall\ k \neq j$.

Step 5. After the result [TSH, Proposition 2.3], K is a topological group, so we have a symmetric open neighborhood V of e in K such that $V^2 \subset W$. And obtain $\forall\ k \neq j$, $g_j (g_k)^{-1} \notin V^2$. And from the symmetry of V this is the same as $Vg_j \cap Vg_k = \emptyset$.

Step 6. Now take an open covering $C \subset \cup_{g \in C} Vg$. Since C is compact, there exists a finite sub-covering as $C \subset \cup^{N}_{n} Vg'_n$ But all g_j's are belonging to C . So there exists a pair (gp, gq) contained in the same Vg'_n. That is,

$gp \in Vg'_n$, $gq \in Vg'_n$, i.e. $g'_n \in Vgp \cap Vgq$, so $Vgp \cap Vgq \neq \emptyset$.

This contradicts the conclusion of Step 5. q.e.d.

We consider a countable family $\{G_k\}$ (k = 1,2,.......) of non-trivial locally compact groups G_k. For finite number j , write $G(j) \equiv \Pi_{k \leqq j} G_k$. G(j) is imbeded into G(j+1) as a subgroup $\Pi_{k \leqq j} G_k \times \{e\}$.

By definition, the inductive limit group G of G(j)'s is equal to $\cup_j G(j) = \Pi'_k G_k$ (restricted direct product) as a set. And the topology of G is given by the following.

(*) A set E in G is open if and only if $\forall\ j$, $E \cap G(j)$ is open in G(j).

As in [TSH, Proposition 2.3.], by this topology, G becomes a topological group. So we can consider unitary representations of such a group.

Apply the analogous argument to the family $\{G_k\}$ $(k = j + 1, j + 2, \ldots\ldots)$ and get the inductive limit group $G[j]^{\vee}$ in the same.way. Then the following is easily shown.

Lemma 1-2 $G = G(j) \times G[j]^{\vee}$ as a topological group.

Proof. Omitted.

Definition 1-1. (Infinite tensor product of Hilbert spaces) For a given set of Hilbert spaces $\{H(\alpha)\}$, we consider a familly of vectors $\{v(\alpha) \mid \in H(\alpha),\ \| v(\alpha) \|_{\alpha} = 1\}$ (we call $v \equiv \otimes_{\alpha} v(\alpha)$, the reference vector). And we define an infinite product Hilbert space $H(v) \equiv \{\otimes_{\alpha} H(\alpha), v\}$, which is the completion of the space of linear combinations of symbols $u \equiv \otimes_{\alpha} u(\alpha)$ such that

$$\Sigma_{\alpha} \mid \| u(\alpha) \| - 1 \mid < \infty \quad \text{and} \quad \Sigma_{\alpha} \mid < u(\alpha), v(\alpha) > - 1 \mid < \infty,$$

with scalar product $< u, v > \ \equiv \ \Pi_{\alpha} < u(\alpha),\ v(\alpha) >$.

For properties of this tensor product, we quote [G], p.148.

Notations. Denote by Ω, the set of all unitary representations of G. The element of Ω, we use the notation as $\omega \equiv \{H(\omega), Tg(\omega)\}$, where $H(\omega)$ is the representation space of ω and $Tg(\omega)$ $(g \in G)$ the representation operators. For two representations ω_1, ω_2, $\omega_1 \sim_A \omega_2$ means ω_1 is unitary equivalent to ω_2 with the intertwining operator A.

A representation $\omega \equiv \{H(\omega), Tg(\omega)\}$ is called **cyclic**, if there exists a non-zero vector v in $H(\omega)$, such that the space of linear combinations of the set $\{Tg(\omega)v\ ;\ g \in G\}$ is dense in $H(\omega)$.

It is easily shown that an irreducible representation of G is cyclic.

For a given cyclic representation $\omega \equiv \{H(\omega), Tg(\omega)\}$, and any non-zero vector v, the function $\varphi(g) \equiv < Tg(\omega)v(\omega),\ v(\omega) >$ is continuous and satisfies the axiom of **positive definite property,**

(*) For any finite pairs $\{(g_j, c_j), g_j \in G, c_j \in \mathbf{C}\ \ j=1,2,3,\cdot\cdot,n\}$,

$$\Sigma^{n}_{j,k}\ \bar{c}_j c_k\ \varphi(g_j^{-1}g_k) \geqq 0.$$

We call this positive definite function as **associated to** ω.

Conversely, for any continuous positive definite function φ, we can construct a cyclic unitary representation ω which associates to φ.

If $\varphi(e)\ (= \| v \|^2) = 1$, this positive definite function φ is called **normalized.**

Of course a cyclic representation can have many positive definite functions associated to it.

Definition 1-2. (Fell-topology on the space of positive definite functions) Let Ω_p be the set of all normalized continuous positive definite functions on G. For any compact subset C in G, we consider semi-metrics $m_C(\varphi_1, \varphi_2) \equiv \sup_{g \in C}(|\varphi_1(g) - \varphi_2(g)|)$ and topology on Ω_p defined by these metrics.

Make running compact sets C, we obtain the topology τ on Ω_p generated by all m_C.
In this paper, we call this topology on Ω_p simply, as **Fell-topology.**

For the case where G is locally compact, this topology induces some important topology on the dual space of G. In our case, given G is not locally compact in general, but we can say the following.

Lemma 1-3 For our group G, Ω_p is compact convex in Fell-topology.

Proof. The convexity is trivial.
For any compact subset C, by Lemma 1-1 there exists an n such that $C \subset G(n)$. Now we consider the restriction φ_n of a given $\varphi \in \Omega_p$ onto $G(n)$, and obtain a continuous positive definite function on locally compact group $G(n)$. We denote the space of all normalized continuous positive definite functions on $G(n)$ by Ω_p^n.

General representation theory of locally compact groups taught us that Ω_p^n is compact under Fell-topology. If $n < m$, the restriction map $\kappa_{mn}: \Omega_p^m \ni \varphi_m \to \varphi_n \equiv \varphi_m|_{G(n)} \in \Omega_p^n$. is continuous and surjective for our group.

Take the compact convex set $\Omega^\wedge \equiv \prod_n \Omega_p^n$, then Ω_p is imbedded in $\Omega^\wedge$ by the continuous map $\kappa : \Omega_p \ni \varphi \to (\varphi_n(\equiv \kappa_n(\varphi) \equiv (\varphi|_{G(n)})))_n \in \Omega^\wedge$. By the definition of topology of Ω_p and $\Omega^\wedge$, this map must be open, that is, isomorphic.

Now we show that the image $\kappa(\Omega_p)$ is closed,

For this, it is enough to see that a ultra filter $\{\varphi_\alpha\}$ in Ω_p converges to an element of Ω_p. On $G(n)$, $\forall n$, $\varphi_\alpha|_{G(n)} \in \Omega_p^n$ converges to some φ_n, and $\kappa_{mn}(\varphi_m) \doteq \varphi_n$.
So there exists a φ satisfying $\kappa_n(\varphi) = \varphi_n$ as the compact uniform limit of φ_α's. It is easy to see that φ is positive definite.

We must show that φ is continuous. Now put

$$E(a, b) \equiv \{ g \in G, \mathrm{Re}(\varphi(g)) > a, \mathrm{Im}(\varphi(g)) > b \} \qquad (a, b \in \mathbb{R})$$

Since for any n, $E(a, b) \cap G(n) = \{ g \in G(n), \mathrm{Re}(\varphi_n(g)) > a, \mathrm{Im}(\varphi_n(g)) > b \}$ is open in $G(n)$, so $E(a, b)$ are open for any real a, b. This shows that φ is continuous. q.e.d.

Now we quote the following famous theorem by M.G.Krein and D.Mil'man.

Proposition 1-1 (External Point Theorem) Non-void convex subset in a locally convex space coincides with the closed convex envelope of the set of all its terminal points.

As a result of Lemma 1-3 and Proposition 1-1, we can confirm the following.

Proposition 1-2 (Extended I.M.Gel'fand-D.A.Raikov's Theorem) Any continuous positive definite function φ of G can be approached uniformaly on any compact set by linear combinations with positive coefficients of normalized positive definite functions associated to irreducible representations.

In Ω, there exist three relations , 1) unitary equivaence, 2) direct sum, 3) tensor product. Using these relations we define the following .

Definition 1-3. (Birepresentation) An operator field $U \equiv \{U(\omega)\}$ over Ω, where $U(\omega)$ is a bounded operator in $H(\omega)$, is called a **birepresentation** when

(1) $\forall\ \omega_1, \omega_2 \in \Omega$, if $\omega_1 \sim_A \omega_2$ then $U(\omega_1) = A^{-1}U(\omega_2)A$,

(2) $\forall\ \omega_1, \omega_2 \in \Omega$, $U(\omega_1 \oplus \omega_2) = U(\omega_1) \oplus U(\omega_2)$,

(3) $\forall\ \omega_1, \omega_2 \in \Omega$, $U(\omega_1 \otimes \omega_2) = U(\omega_1) \otimes U(\omega_2)$,

(4) $\forall\ \omega \in \Omega$, $U(\omega) \neq 0$.

In [T], to prove duality theorem for locally compact groups, in the definition of birepresentation, conditions (1)-(4) were enough, but in this paper we must add the following condition:

(5) $U(\omega)$ is weak continuous (w-continuous) on Ω_p with respect to Fell-topology.

This means that if $\Omega_p \ni \varphi$ is given as $\varphi(g) \equiv \langle T_g(\omega)v(\omega), v(\omega)\rangle$,then $\forall g_0 \in G$, the function $\varphi \rightarrow U(\varphi)(g_0) \equiv \langle T_{g_0}(\omega)U(\omega)v(\omega), v(\omega)\rangle$ is continuous on Ω_p.

For any $g \in G$, operator field $T_g \equiv \{T_g(\omega)\}$ over Ω gives a birepresentation.

§ 2. Unitary representations

Let $\omega_k \equiv \{H^k, T^k g\}$ be a unitary representation of group G_k for each k.
We consider the Hilbert space $H = \{\otimes_k H^k, v \equiv \otimes_k v_k\}$, where $v_k \in H^k$ ($\forall$ k, $\|v_k\| = 1$) and v is a reference vector in H.

For any element $g = \{g_k\}$ in G, $g_k = e$ except finite k's, so the operator $T_g \equiv \otimes_k T^k g$ can be defined as a unitary operator on H and $\omega \equiv \{H, T_g\}$ is an algebraic representation of G. It is easy to see that $G \ni g \rightarrow T_g$ is weak continuous. So ω gives a unitary representation of G.

Definition 2-1 We call the above $\omega \equiv \{H, Tg\}$, a **direct product type representation (DPR)**. And denote it as $\omega (\equiv \omega(v)) = \{\hat{\otimes}_k \omega_k, v \equiv \otimes_k v_k\}$, where $\hat{\otimes}_k$ means multiple of outer tensor products operation. (The notation $\hat{\otimes}$ shows outer tensor product.)

And we denote the set of direct sums of DPR's of G by Ω_D.

Definition 2-2 For a (DPR) $\omega(v) = \{\hat{\otimes}_k \omega_k, v \equiv \otimes_k v_k\}$, if ω_k are the trivial representation of Gk except finite k's, that is, there is a finite subset S in **N** such that $\omega_k = I_k$ (the trivial representation) for $k \notin S$, we call this direct product type representation **of finite type (FT)**.

Especially if $S = \{k\}$ is a one point set, this $\omega(v)$ is called **single type of index k**. And we show the set of all index k single type representations by $\Omega(k)$.

Easy to see that for FT-representation, every reference vector gives the same Hilbert space. Therefore hereafter, we use the notation for FT-representation without reference vector.

An index k single type representation is of the following form:

$$\omega = (\hat{\otimes} I) \hat{\otimes} \omega(k) \hat{\otimes} (\hat{\otimes} I) \quad (\omega(k) \equiv \{H(k), T(k)g\} \in \Omega(Gk)).$$

It is easy to see that by the correspondence $\omega(v) \to \omega(k)$, we can see the set of all index k single type representations as the set of all representations of Gk. So we can identify $\Omega(k)$ to the weak dual $\Omega(Gk)$ of Gk.

Now we consider a DPR $\omega(v) = \{\hat{\otimes}_k \omega_k, v \equiv \otimes_k v_k\}$, FT-representation $\omega = (\hat{\otimes} I) \hat{\otimes} \omega(k) \hat{\otimes} (\hat{\otimes} I)$ and thier innner tensor product $\omega \otimes \omega(v)$.

Take any normalized vector u in $H(k)$, then we get

$$\omega \otimes \omega(v) = \{ \hat{\otimes}_{j<k} \omega_j \hat{\otimes} (\omega(k) \otimes \omega_k) \hat{\otimes}_{j>k} \omega_j, \otimes_{j<k} v_j \otimes (u \otimes v_k) \otimes_{j>k} v_j \}$$

Corresponding to arbitrarily given DPR $\omega(v) = \{\hat{\otimes}_k \omega_k, v \equiv \otimes_k v_k\}$ and finite subset S in **N**, we can consider a finite type DPR

$$\omega(v)_S \equiv \{ (\hat{\otimes}_k \omega_k (k \in S)) \hat{\otimes} (\hat{\otimes}_k I_k (k \notin S)), (\otimes_k v_k (k \in S)) \otimes (\otimes 1)) \},$$

and the representation

$$(\omega(v)_S)^\wedge \equiv \{ (\hat{\otimes}_k I_k (k \in S)) \hat{\otimes} (\hat{\otimes}_k \omega_k (k \notin S)), (\otimes_k 1) \otimes (\otimes_k v_k (k \notin S)) \} .$$

Then $\omega(v) = \omega(v)_S \otimes (\omega(v)_S)^\wedge$. ($\otimes$ shows inner tensor product).

Definition 2-3 The case that for any k, $\omega_k = \mathfrak{R}_k$ (the right regular representation of Gk), we call such $\omega(v)$ the **full regular representation** of G., and denote it by $\mathfrak{R}(v)$.

As well known, for a locally compact group its regular representation is unique up to unitary equivalence, but in our present case there exist many $\mathfrak{R}(v)$' s depending on the reference vectors $v \equiv \otimes_k v_k$, and in general they are not equivalent mutually.

Example 2-1. Consider the case where all Gk are compact. $\mathfrak{R}_k$ has trivial component I_k with multiplicity 1. Denote the normalized vector in the component I_k as 1_k. Take another irreducible component ω_k, and a normalized vector v_k in $H(\omega_k)$.

$\forall k$, $1_k \perp v_k$, so the reference vectors $\mathbf{1} \equiv \otimes_k 1_k$ and $v \equiv \otimes_k v_k$ can not be in the same representation space. $\mathfrak{R}(\mathbf{1})$ contains trivial representation on **1**, but $\mathfrak{R}(v)$ can not contain **1** , and it has no trivial component. That is, $\mathfrak{R}(\mathbf{1})$ and . $\mathfrak{R}(v)$ are not mutually equivalent.

Definition 2-4 If a unitary representation $\omega \equiv \{H, Tg\}$ satisfies the following, we call this representation of **quasi-direct product type**

(*) For any j , $\omega = (\hat{\otimes}_{k \leq j}\, \omega_k) \hat{\otimes}\, \omega[j]$. Here ω_k is a representation of Gk and $\omega[j]$ is of $G[j]^\vee$

Of course DPR is quasi-direct type representation. But I don't know conditions under which a quasi-direct type representation is DPR.

Lemma 2-1 For two topological groups H_1, H_2, and a unitary representation $\omega \equiv \{H(\omega), Tg\}$ of $H \equiv H_1 \times H_2$, if the restriction of ω to H_1 contains some irreducible representation $D \equiv \{H(D), Tg(D)\}$ as a discrete component, then ω contains subrepresentstion $D \hat{\otimes} D[2]$, where $D[2]$ is a representation of H_2.

Proof. Take the maximal subspace $H(D)^\vee$ of $H(\omega)$ on which multiple of D acts. Then $H(D)^\vee$ is invariant under $\{Tg \mid g \in H\}$, and any $\{Tg \mid g \in H_2\}$ commmutes with operators of $\Sigma^{\oplus} D$. Since D is irreducible, so the space $H(D)^\vee$ is of the form $H(D) \otimes H(2)$, and the restriction of ω to H_2 on $H(D)^\vee = H(D) \otimes H(2)$ is of the form $I \hat{\otimes} D[2]$. q.e.d.

Analogous result is proved.

Lemma 2-2 For two topological groups H_1, H_2 , let $\omega \equiv \{H(\omega), Tg\}$ be an irreducible unitary representation of $H \equiv H_1 \times H_2$, then the restriction $\omega\,|H_1$ of ω to H_1 is a factor representation of H_1.

Moreover if H_1 is type I group, then ω is the outer tensor product of irreducible unitary representations ω_j of H_j ($j = 1, 2$), that is, $\omega = \omega_1 \hat{\otimes}\, \omega_2$.

Proof. If $\omega\,|H_1$ is not a factor representation, there exists a non-trivial projection P belonging to the double commmutant $(\omega\,|H_1)''$. $PH(\omega)$ and $(I-P)H(\omega)$ are both non-trivial $H_1 \times H_2$ invariant subspaces. This contradicts the assumption of irreducibility.

Next, if H_1 is of Type 1, there exists an irreducible representation ω_1 of H_1 and $\omega\,|\,H_1$ is a multiple of ω_1 and the space is written as $H(\omega) = H(\omega_1) \otimes H(\omega_2)$, the tensor product of the space of ω_1 with some space $H(\omega_2)$ on which operators in $(\omega\,|H_1)'$ act, surely some representation ω_2 of H_2 . Again the irreducibility assumption of $\omega = \omega_1 \hat{\otimes}\, \omega_2$ leads us to the irreducibility of ω_2 . q.e.d.

Corollary In our group G, if all Gk are type 1 groups, then any irreducible unitary representation ω of G is of quasi-direct product type.

Proof. For $G(j) \equiv \Pi_{k \leqq j} G_k$, we use Lemma 2-2 repeatedly, And we conclude that every irreducible representation of G(j) is of the form $\omega(j) = \hat{\otimes}\, \omega_k$ $(1 \leqq k \leqq j)$, where $\omega_k \equiv \{ H_k, T^k g\}$ is an irreducible representation of Gk.

Again we apply Lemma 2-2 to the case of $H = G(j) \times G[j]^{\vee}$, where $H_1 = G(j)$, $H_2 = G[j]^{\vee}$. We get that any irreducible representation of G is of the form $\omega(j) \hat{\otimes}\, \omega[j]$, where $\omega[j]$ is an irreducible representation of $G[j]^{\vee}$. In other words, for arbitrary given irreducible representation $\omega \equiv \{H, Tg\}$ of G, there exist irreducible representations ω_k of Gk determined for any $k \leqq j$, and ω is written in the form

$$\omega = (\hat{\otimes}_{k \leqq j}\, \omega_k) \hat{\otimes}\, \omega[j] .$$
q.e.d.

[Remark] If the assumption, " all Gk are Type I" , is omitted, then we have the following example for which the assertion of Colloraly 1 fails.

Example 2-2 Consider H the free group with two generators (Yoshizawa Group). On $L^2(H)$, we have two groups of operators, $K_L = \{L_h;\ h \in H\}$ (left translations) and $K_R = \{R_h;\ h \in H\}$ (right translations). It is well known that both of the regular representations $\{L^2(H), L_h\}$ and $\{L^2(H), R_h\}$ of H are type II factors and so H is not type I group.

We take in our Corollary, Gk = H (k=1,2,3,. • • • ..) and consider the representation ω of G on the space $H = \hat{\otimes}\, H_k$ (Here $\forall$ k, $H_k = L^2(H)$) with any reference vector

$f = \otimes_k f_k$ ($\forall$ k, $f_k \in \mathcal{H}_k$, $\| f_k \| = 1$) and the representation operators are

$G \ni g = (g_1, g_2, g_3, \ldots) \rightarrow Tg = Lg_1 Rg_2 \otimes Lg_2 Rg_3 \otimes Lg_3 Rg_4 \otimes \cdots$.

But the representation $\omega(1,2)$ of $H_1 \times H_2$ $(H_1 = H_2 = H)$ on $L^2(H)$ given by $H_1 \times H_2 \ni (h_1, h_2) \rightarrow L_{h_1} R_{h_2}$ is irreducible . Apply this to the case of $G_j = H_1$ and $G_{j+1} = H_2$, then we can assert that the representation

$\omega(j, j+1): G_j \times G_{j+1} \ni (g_j, g_{j+1}) \rightarrow Lg_j Rg_{j+1}$ is irreducible.

Extend this representation to G, as $\omega(j)^{\vee} \equiv \hat{\otimes} I . \hat{\otimes} \omega(j, j+1) \hat{\otimes} (\hat{\otimes} I)$, then this representation is irreducible.

Finally as the inner tensor product of representations, $\omega = \{ \otimes_j \omega(j)^{\vee}, f \}$ of G is irreducible. And this irreducible representation is not of the above form.

Proposition 2-1 For our group G, any positive definite function associated to an irreducible unitary representation is a limit of a sequence of ones associated to elements in Ω_D with Fell-topology.

Proof. By Lemma 1-1, any compact set C is contained in some Gj.

In other hand, by Lemma 2-2 and Corollary, any irreducible representation ω of G is quasi -direct product type as $\forall m$, $\omega = (\hat{\otimes}_{k \leq m} \omega_k) \hat{\otimes} \omega[m]$. So, for a matrix element $f(g) \equiv \langle Tgv, v \rangle$ ($v = \otimes_{k \leq m} v_k \otimes v(m)$) associated to ω, consider the DRP $(\hat{\otimes}_k^{\infty} \omega_k, v_0)$, where $v_0 = \otimes_k v_k$. Then

$\forall g \in C$, $f(g) \equiv \langle Tgv, v \rangle = \prod_{k \leq j} \langle Tg_k v_k, v_k \rangle \times 1 = \langle Tgv_0, v_0 \rangle$

This shows that f coincides with a matrix element $f_0 = \langle Tgv_0, v_0 \rangle$ on C. q.e.d.

§ 3 Duality theorem

In this section, we treat our group G, that is, an inductive limit group of countable direct product groups for which all component groups are type I locally compact groups.

We show a duality theorem for G.

As in § 1, we put $\Omega \equiv \{\omega\}$ all unitary representations of G, and $U \equiv \{U(\omega)\}$ a given birepresentation on Ω.

By definition, each $U(\omega)$ is a bounded operator on the representation space of ω, and $\{U(\omega)\}$ satisfies the following

(1) $\omega_1, \omega_2 \in \Omega$, if $\omega_1 \sim_A \omega_2$, then $U(\omega_1) = A^{-1} U(\omega_2) A$,

(2) $\forall\ \omega_1, \omega_2 \in \Omega$, $U(\omega_1 \oplus \omega_2) = U(\omega_1) \oplus U(\omega_2)$,

(3) $\forall\ \omega_1, \omega_2 \in \Omega$, $U(\omega_1 \otimes \omega_2) = U(\omega_1) \otimes U(\omega_2)$,

(4) $\forall\ \omega \in \Omega$, $U(\omega) \neq 0$.

(5) $U(\omega)$ is weakly continuous with respect to Fell-topology.

Lemma 3-1 For a given birepresentation $U \equiv \{U(\omega)\}$, there exist a unique element $g_U \in G$ such that $U(\omega) = Tg_U(\omega)$ for any DPR ω.

Proof. Step 1. At first, for any k, we consider the set $\Omega(k)$ of index k single type representations. As we remarked, $\Omega(k)$ is identified with the weak dual $\Omega(G_k)$ of G_k.

By restricting our birepresentation $\{U(\omega)\}$ to $\Omega(k)$, we obtain a birepresentation on the weak dual $\Omega(G_k)$ of locally compact group G_k.

We can use the duality theorem for this restriction, and get unique element $g_k \in G_k$, such that for any ω_k in $\Omega(k)$, $U(\omega_k) = Tg_k(\omega_k)$.

Step 2. Next we treat FT-representation $\omega[j]$.

Let $\omega[j] = (\hat{\otimes}_{k \leqq j}\, \omega_k) \hat{\otimes}\, I(G[j]^{\vee}) = \otimes_{k \leqq j}\, \omega(k) \otimes I(G)$, where $I(G) \equiv \hat{\otimes}\, I_k$ shows the trivial representation of G.

From (3) of the definition of birepresentation,. $U(\omega[j]) = \otimes_{k \leqq j}\, Tg_k(\omega_k) = \otimes_{k \leqq j}\, Tg(\omega[j])$ ($g = (g_1, g_2, \cdots g_j, e, e, e, \cdots)$)

It is remarkable that the above g_k depend only on the given birepresentation U and not on j.

Step 3. In the case where the representation $\omega \equiv \omega(v) = \{\hat{\otimes}_k\, \omega_k, v \equiv \otimes_k v_k\}$ is DPR, for any j, we can write $\omega(v) = \omega[j] \otimes (\omega[j])^\wedge(v[j])$, where

$$(\omega[j])^\wedge(v[j]) = I(G(j)) \hat{\otimes} (\hat{\otimes}_{k>j}\, \omega_k) \text{ and } v[j] = (\otimes 1) \otimes (\otimes_{k>j}\, v_k).$$

Thus $U(\omega(v)) = U(\omega[j]) \hat{\otimes}\, U((\omega[j])^\wedge(v[j]))$

$= (\otimes_{k \leqq j}\, Tg_k(\omega_k)) \hat{\otimes}\, U((\omega[j])^\wedge(v[j]))$

This means that birepresentation operator $U(\omega(v))$ operates on the reference vector $v \equiv \otimes_k v_k$ as follows.

(*) The k-th component vector v_k changes to $Tg_k(\omega_k)v_k$.

Step 4. We consider a full regular representation $\Re(f) \equiv \{\hat{\otimes}_k\, \Re_k, f \equiv \otimes_k f_k\}$. The above result means $U(\Re(f))$ must transfer the reference vector $f \equiv \otimes_k f_k$ to $\otimes_k Rg_k f_k$.

If for any j, $U(\Re_j(f_j)) \neq 0$, then $U(\Re(f))f = \otimes_k Rg_k f_k$.

Now we assume that there exists an infinite set $K = \{k\}$ such that for $\forall k \in K$, $g_k \neq e$. In regular representation $\Re_k$ of locally compact group G_k, for non-unit element g_k, there exists a normalized L^2-function f_k such that $[f_k] \cap [Rg_k f_k] = \varnothing$, that is, $\| f_k - Rg_k f_k \| = 2$. Therefore the vector $U(\Re(f))f = \otimes_k Rg_k f_k$ can not belong to the space of $\Re(f)$.

This contradicts the assumption that for birepresentation, its component $U(\omega)$ for any ω is a bounded operator on the representation space of ω.

Step 5. After the result in Step 4, for any given birepresentation U, the element of corresponding sequence $\{g_k\}$ ($g_k \in G_k$) must be unit e except only a finite number of k, in other words, $\{g_k\}$ is of the form $(g_1, g_2, g_3, \ldots, g_j, e, e, \cdots)$. Therefore there exists an element $g_U = g_1 \times g_2 \times g_3 \times \ldots \times g_j \times e \times e \times \cdots$ in G and

$U(\omega) = \hat{\otimes}_{k \leqq j}\, Tg_k(\omega_k) \hat{\otimes} (\hat{\otimes}\, I) = Tg(\omega(j) \hat{\otimes} (\hat{\otimes}\, I)) = Tg_U(\omega)$. q.e.d.

Corollary For our group G, $U(\omega) = Tg_U(\omega)$ for any irreducible unitary representation ω.

Proof. By Proposition 2-1, any positive definite function associated to an irreducible unitary representation of G is a limit of ones associated to an element in Ω_D. And by definition, birepresentation $\{U(\omega)\}$ is w-continuous with respect to Fell-topology.

From Lemma 3-1, on Ω_D, $< Tg(\omega)U(\omega)v, v > = < Tg(\omega)Tg_U(\omega)v, v >$, $(\forall v \in H(\omega), \forall g \in G)$. Take the limit, and we get this for any irreducible ω too. That is, $U(\omega) = Tg_U(\omega)$, for any irreducible representation $\omega \in \Omega$. q.e.d.

Theorem Consider the inductive limit group G of countable direct product type of type I locally compact groups Gk, $k = 1,2,3, \cdots$.

Then any birepresentation $U \equiv \{U(\omega)\}$ coincides with $Tg = \{Tg(\omega)\}$ for some $g \in G$.

That is, the set of all birepresentations corresponds to G one to one way as a group.

Proof Use the notations in Lemma 3-1.

By the results of and Corollary of Lemma 3-1, $U(\omega) = Tg_U(\omega)$ for any irreducible ω.

Now we show that $U(\omega) = Tg_U(\omega)$ $(\forall \omega \in \Omega)$,then the proof is completed.

For any normalized positive definite function $\varphi(g) \equiv < Tg(\omega)v(\omega), v(\omega) >$ associated to ω, take the function $U(\varphi)(g) \equiv < Tg(\omega)U(\omega)v(\omega), v(\omega) >$.

If ω is irreducible, $\forall g \in G$, $U(\varphi)(g) = Tg_U(\varphi)(g)$. Since the function $\Omega p \ni \varphi \rightarrow U(\varphi)(g)$ $(\forall g \in G)$ is continuous, using the result of Proposition 1-2, we obtain $\forall \varphi \in \Omega p$, $\forall g \in G$, $U(\varphi)(g) = Tg_U(\varphi)(g)$. That is,

$\forall \omega \in \Omega, \forall v \in H(\omega), \forall g \in G, < Tg(\omega)U(\omega)v, v > = < Tg(\omega)Tg_U(\omega)v, v >$.

So $U(\omega)v = Tg_U(\omega)v$ $(\forall v \in H(\omega))$, i.e. $U(\omega) = Tg_U(\omega)$ $(\forall \omega \in \Omega)$. q.e.d.

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