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THE SECONDARY SPHERICAL FUNCTIONS AND GREEN CURRENTS ASSOCIATED WITH CERTAIN SYMMETRIC PAIRS

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ABSTRACT. We construct a Poincaré dual form and a Green current for a modular cycle of higher codimension on an arithmetic quotient of a certain Hermitian symmetric domain, generalizing the classical construction of the automorphic Green function for the modular curves.

1. INTRODUCTION

Arithmetic quotients of hermitian symmetric domains are important objects to investigate. For example, the moduli spaces of abelian varieties with certain endomorphisms and polarization types, and the moduli spaces of K3 surfaces are realized as such. To understand the cohomology groups and the cycle geometry of these quotients is very interesting arithmetic problem. There is a history to investigate this theme around the time of establishment of the Matsushima isomorphism. The construction method of cycles by means of equivariant embeddings of locally symmetric spaces are called 'generalized modular symbols '. (*cf.* [4]).

If both the embedded and the ambient spaces are of hermitian type, there is an extensive study by Satake [8] for possible embeddings. Sometimes they have been called modular embeddings. Let

$$j : \Delta\backslash H/H\cap K \to \Gamma\backslash G/K$$

be a modular embedding with G a semisimple Lie group, K a maximal compact subgroup of G, H a symmetric subgroup such that $H\cap K$ is maximally compact in H and Γ, Δ are compatible arithmetic subgroups of G, H respectively. Then j yields the restriction map of cohomology

$$j^*_\heartsuit : \mathrm{H}^*_\heartsuit(\Gamma\backslash G/K, \mathbb{C}) \to \mathrm{H}^*_\heartsuit(\Delta\backslash H/H\cap K, \mathbb{C})$$

($\heartsuit \in \{\mathrm{empty}, \mathrm{c}, !\}$ a support condition of cohomology theories). Then we have the Poincaré dual map

$$(j_\heartsuit)_* : \mathrm{H}^{2m-q}_\spadesuit(\Delta\backslash H/H\cap K, \mathbb{C}) \to \mathrm{H}^{2n-q}_\spadesuit(\Gamma\backslash G/K, \mathbb{C})$$

with $\spadesuit$ the support condition dual to $\heartsuit$. We propose here a
Problem: Construct the Poincaré dual map $(j_\heartsuit)_*$ explicitly.

This problem seems to be quite difficult to answer generally. But at least for special case, we have a tractable method: to use Poincaré series and derived Green currents.

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In a previous paper [5], we discuss for the case when the complex codimension of $\Delta\backslash H/H\cap K$ in $\Gamma\backslash G/K$ is one. We can extend the similar construction for higher codimensional case associated with the symmetric pair $\mathrm{U}(p,q)$, $\mathrm{U}(p-1,q)\times\mathrm{U}(1)$ in this article. We note that its dual symmetric pair $\mathrm{U}(p+q-1,1)$, $\mathrm{U}(p-1,1)\times\mathrm{U}(q)$, which yields a class of higher codimensional cycles in a discrete quotient of a complex hyperball, is already treated in [9] by a similar method.

Because this article is a short summary of the forthcoming full paper, no proof is included.

Notation:

The number 0 is included in the set of natural numbers: $\mathbb{N}=\{0,1,2,\dots\}$.

For any matrix $B=(b_{ij})$ with coefficients in $\mathbb{C}$, $B^*=(\bar{b}_{ji})$ denotes its conjugate-transpose matrix.

We follow the usual convention that the Lie algebra of a real Lie group G is denoted by the corresponding German letter $\mathfrak{g}$.

2. PRELIMINARY

2.1. Unitary group and its symmetric space. Let $G=\{g\in\mathrm{GL}_{p+q}(\mathbb{C})\,|\,g^*I_{p,q}g=I_{p,q}\}$ be the unitary group of the Hermitian form $I_{p,q}=\mathrm{diag}(1_p,-1_q)$ with signature $(p+,q-)$. We assume $p\geqslant q\geqslant 2$ from now on.

The inner automorphism $\theta: g\mapsto I_{p,q}\,g\,I_{p,q}$ is a Cartan involution of G and its fixed point set

$$K=\{\mathrm{diag}(k_1,k_2)\,|\,k_1\in\mathrm{U}(p),\ k_2\in\mathrm{U}(q)\}$$

yields a maximal compact subgroup of G. The (-1)-eigenspace of $d\theta:\mathfrak{g}\to\mathfrak{g}$ denoted by $\mathfrak{p}$ is identified with the tangent space of the G-homogeneous manifold G/K at its origin $o=K$. The adjoint action $J=\mathrm{Ad}(z_o)|\mathfrak{p}$ by the element $z_o=\mathrm{diag}(\sqrt{-1}\,1_p,1_q)$ in the center of K yields a K-invariant complex structure on $\mathfrak{p}\cong T_o(G/K)$, which propagates a G-invariant complex structure on G/K. The complexification $\mathfrak{p}_{\mathbb{C}}$ is decomposed to its holomorphic and anti-holomorphic subspaces: $\mathfrak{p}_{\mathbb{C}}=\mathfrak{p}_+\oplus\mathfrak{p}_-$ with $\mathfrak{p}_\pm=\{X\in\mathfrak{p}_{\mathbb{C}}\,|\,J(X)=\pm\sqrt{-1}X\}$. If we identify $\mathfrak{g}_{\mathbb{C}}=\mathfrak{gl}_{p+q}(\mathbb{C})$ naturally, we have

$$\mathfrak{p}_+=\{p_+(x')=\left[\begin{smallmatrix}0&x'\\0&0\end{smallmatrix}\right]\in\mathfrak{gl}_{p+q}(\mathbb{C})\,|\,x'\in\mathrm{M}_{p,q}(\mathbb{C})\},$$
$$\mathfrak{p}_-=\{p_-(x'')=\left[\begin{smallmatrix}0&0\\x''&0\end{smallmatrix}\right]\in\mathfrak{gl}_{p+q}(\mathbb{C})\,|\,x''\in\mathrm{M}_{p,q}(\mathbb{C})\}.$$

Let $X\mapsto\bar{X}$ be the complex conjugate in $\mathfrak{gl}_{p+q}(\mathbb{C})$ with respect to its real form $\mathfrak{g}$. Then $\bar{X}=-I_{p,q}X^*I_{p,q}\ (\forall X\in\mathfrak{g}_{\mathbb{C}})$ and $\overline{p_\pm(x)}=p_\mp(x^*)\ (\forall x\in\mathrm{M}_{p,q}(\mathbb{C}))$.

The non-degenerate $\mathbb{R}$-bilinear form $B_{\mathfrak{g}}(X,Y)=2^{-1}\mathrm{tr}(XY)$ on $\mathfrak{g}$ entails a positive definite K-invariant inner product $B_{\mathfrak{p}}$ on $\mathfrak{p}$, which propagates a G-invariant metric on G/K. The mertic on G/K is Kaehler and the associated 2-form form is given by

$$\omega_{\mathfrak{p}}(X,Y)=B_{\mathfrak{p}}(X,JY),\quad X,Y\in\mathfrak{p}\tag{2.1}$$

on $\mathfrak{p}\cong T_o(G/K)$.

Let $B_{\mathfrak{p}^*_{\mathbb{C}}}$ be the complex bilinear extension of the inner product $B_{\mathfrak{p}^*}$ on $\mathfrak{p}^*$ dual to $B_{\mathfrak{p}}$. Then $\mathfrak{p}^*_{\mathbb{C}}$ is equipped with the hermitian inner product $(\xi|\xi')=B_{\mathfrak{p}^*_{\mathbb{C}}}(\xi,\bar{\xi}')$, which is extended to the exterior algebra $\bigwedge \mathfrak{p}_{\mathbb{C}}^*$ canonically. Note the natural decomposition of $\bigwedge \mathfrak{p}_{\mathbb{C}}^*$ to its bidegree (a, b) part

$$\overset{a,b}{\bigwedge} \mathfrak{p}_{\mathbb{C}}^* = \overset{a}{\bigwedge} \mathfrak{p}_+^* \otimes \overset{b}{\bigwedge} \mathfrak{p}_-^*$$

is orthogonal. It is sometimes convenient to note that the inner product of typical elements $\xi = p_\epsilon(x) \in \mathfrak{p}_\epsilon$, $(x \in \mathrm{M}_{p,q}(\mathbb{C}), \epsilon = \pm)$ and $\eta = p_{\epsilon'}(y) \in \mathfrak{p}_{\epsilon'}$, $(y \in \mathrm{M}_{p,q}(\mathbb{C}), \epsilon' = \pm)$ is computed as $(\xi|\eta) = \frac{1}{2}\delta_{\epsilon,\epsilon'}\mathrm{tr}(xy^*)$.

The Hodge star operator $*$ is defined to be the $\mathbb{C}$-linear endomorphism of $\bigwedge \mathfrak{p}_{\mathbb{C}}^*$ such that $*\bar{\alpha} = \overline{*\alpha}$ and such that $(\alpha|\beta)\mathrm{vol}_{\mathfrak{p}} = \alpha \wedge *\bar{\beta}$. Here $\mathrm{vol}_{\mathfrak{p}} = \frac{1}{(pq)!}\omega_{\mathfrak{p}}^{pq}$ is the Kähler volume form. For $\alpha \in \bigwedge \mathfrak{p}_{\mathbb{C}}^*$, let us define $e(\alpha) : \bigwedge \mathfrak{p}_{\mathbb{C}}^* \to \bigwedge \mathfrak{p}_{\mathbb{C}}^*$ by $e(\alpha)\beta = \alpha \wedge \beta$. The operator $L = e(\omega_{\mathfrak{p}})$ is commonly called Lefshetz operator. The adjoint of $e(\alpha)$ with respect to the hermitian inner product of $\bigwedge \mathfrak{p}_{\mathbb{C}}^*$ is denoted by $e^*(\alpha)$. In particular, the operator $e^*(\omega_{\mathfrak{p}})$, the adjoint of the Lefshetz operator, is denoted by Λ

2.2. **A symmetric subgroup.** Let us consider the involution σ of G defined by

$$\sigma(g) = \mathrm{diag}(1_{p-1}, -1, 1_q)\, g\, \mathrm{diag}(1_{p-1}, -1, 1_q).$$

Let $H = G^\sigma$ be the σ-fixed point subgroup of G. Since θ is commutative with σ, the restriction $\theta|H$ provides H with a Cartan involution. The θ-fixed points

$$H^\theta = H \cap K = \{\mathrm{diag}(h_1, u, h_2)|\, h_1 \in \mathrm{U}(p-1),\, u \in \mathrm{U}(1),\, h_2 \in \mathrm{U}(q)\}$$

is a maximal compact subgroup of H. The Cartan decomposition of the Lie algebra $\mathfrak{h}$ of H is $\mathfrak{h} = (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{h})$. Since the element z_o defining the complex structure J of $\mathfrak{p}$ belongs to the center of $H \cap K$, J yields a $H \cap K$-invariant complex structure of the real vector space $\mathfrak{h} \cap \mathfrak{p} \cong T_o(H/H \cap K)$, which propagates an H-invariant complex structure of H-homogeneous manifold $H/H \cap K$. We put $H/H \cap K$ the H-invariant metric coming from the restriction of $B_{\mathfrak{p}}$ to $\mathfrak{h} \cap \mathfrak{p}$. The metric is Kähler and the associated 2-form on $\mathfrak{p} \cap \mathfrak{h} \cong T_o(H/H \cap K)$ is $\omega_{\mathfrak{p}\cap\mathfrak{h}} = \omega_{\mathfrak{p}}|(\mathfrak{p} \cap \mathfrak{h}) \times (\mathfrak{p} \cap \mathfrak{h})$.

As a consequence of the constructions so far, the inclusion $H/H \cap K \hookrightarrow G/K$ is a holomorphic map between Kähler manifolds and $\mathrm{codim}_{\mathbb{C}}(G/K; H/H \cap K) = q$.

2.3. **Root vectors.** For $1 \leqslant i, j \leqslant p + q$, let $E_{ij} = (\delta_{i\alpha}\delta_{j\beta})$ denotes the matrix unit in $\mathrm{M}_{p+q}(\mathbb{C})$. The matrices $E_{i,j}$ comprise a $\mathbb{C}$-basis of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}_{p+q}(\mathbb{C})$.

Let $\mathfrak{q}$ be the (-1)-eigenspace of $d\sigma : \mathfrak{g} \to \mathfrak{g}$. Since θ and σ are mutually commutative involutions, $\mathfrak{g}$ is decomposed to their joint eigenspaces: $\mathfrak{g} = (\mathfrak{k}\cap\mathfrak{h})\oplus(\mathfrak{p}\cap\mathfrak{q})\oplus(\mathfrak{p}\cap\mathfrak{h})\oplus(\mathfrak{k}\cap\mathfrak{q})$. The pair $(\mathfrak{g}, \mathfrak{h})$ is a symmetric pair of split rank one, and $\mathfrak{a} = \mathbb{R}Y_0$ with $Y_0 = E_{p,p+1} + E_{p+1,p}$ is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$. The set of $\mathfrak{a}$-roots in $\mathfrak{g}$ is $\Sigma(\mathfrak{a}) = \{\pm\lambda, \pm 2\lambda\}$. Here $\lambda \in \mathfrak{a}^*$ is the unique simple root such that $\lambda(Y_0) = 1$. The signature ([7]) of each root is computed as $\begin{pmatrix} m^+(\lambda) & m^+(2\lambda) \\ m^-(\lambda) & m^-(2\lambda) \end{pmatrix} = \begin{pmatrix} 2(q-1) & 1 \\ 2(p-1) & 0 \end{pmatrix}$.

Set $M = Z_{H\cap K}(\mathfrak{a})$. Then

$$M = \{\mathrm{diag}(x_1, u, u, x_2)|\, x_1 \in \mathrm{U}(p-1),\, u \in \mathrm{U}(1),\, x_2 \in \mathrm{U}(q-1)\}$$

coincides with $Z_K(\mathfrak{a})$.

For $1 \leqslant i \leqslant p-1$ and $1 \leqslant j \leqslant q-1$, set

$$\begin{aligned}
X_0^{\mathfrak{q}} &= E_{p,p+1}, & \bar{X}_0^{\mathfrak{q}} &= E_{p+1,p}, & Z_0^{\mathfrak{h}} &= \sqrt{-1}(E_{p,p} - E_{p+1,p+1}),\\
X_j^{\mathfrak{q}} &= E_{p,p+j+1}, & \bar{X}_j^{\mathfrak{q}} &= E_{p+j+1,p},\\
Z_j^{\mathfrak{h}} &= -E_{p+1,p+j+1}, & \bar{Z}_j^{\mathfrak{h}} &= E_{p+j+1,p+1},\\
X_i^{\mathfrak{h}} &= E_{i,p+1}, & \bar{X}_i^{\mathfrak{h}} &= E_{p+1,i},\\
Z_i^{\mathfrak{q}} &= E_{i,p}, & \bar{Z}_i^{\mathfrak{q}} &= -E_{p,i},\\
X_{ij}^{\mathfrak{h}} &= E_{i,p+j+1}, & \bar{X}_{ij}^{\mathfrak{h}} &= E_{p+j+1,i}.
\end{aligned}$$

This notation is consistent with the complex conjugation.

Then we have

$$\begin{aligned}
\mathfrak{p}_+ \cap \mathfrak{q}_{\mathbb{C}} &= \langle X_j^{\mathfrak{q}}\,(0 \leqslant j \leqslant q-1)\rangle_{\mathbb{C}},\\
\mathfrak{p}_- \cap \mathfrak{q}_{\mathbb{C}} &= \langle \bar{X}_j^{\mathfrak{q}}\,(0 \leqslant j \leqslant q-1)\rangle_{\mathbb{C}},\\
\mathfrak{p}_+ \cap \mathfrak{h}_{\mathbb{C}} &= \langle X_i^{\mathfrak{h}}\,(1 \leqslant i \leqslant p-1)\rangle_{\mathbb{C}} \oplus \langle X_{ij}^{\mathfrak{h}}\,(1 \leqslant i \leqslant p-1, 1 \leqslant j \leqslant q-1)\rangle_{\mathbb{C}},\\
\mathfrak{p}_- \cap \mathfrak{h}_{\mathbb{C}} &= \langle \bar{X}_i^{\mathfrak{h}}\,(1 \leqslant i \leqslant p-1)\rangle_{\mathbb{C}} \oplus \langle \bar{X}_{ij}^{\mathfrak{h}}\,(1 \leqslant i \leqslant p-1, 1 \leqslant j \leqslant q-1)\rangle_{\mathbb{C}},\\
(\mathfrak{k} \cap \mathfrak{h})_{\mathbb{C}} &= \langle Z_0^{\mathfrak{h}}, Z_j^{\mathfrak{h}}, \bar{Z}_j^{\mathfrak{h}}\,(1 \leqslant j \leqslant q-1)\rangle_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}},\\
(\mathfrak{k} \cap \mathfrak{q})_{\mathbb{C}} &= \langle Z_i^{\mathfrak{q}}, \bar{Z}_i^{\mathfrak{q}}\,(1 \leqslant i \leqslant p-1)\rangle_{\mathbb{C}}.
\end{aligned}$$

Consider the one parameter subgroup

$$a_t = \exp(tY_0) = \operatorname{diag}\left(1_{p-1}, \left[\begin{smallmatrix}\cosh t & \sinh t\\ \sinh t & \cosh t\end{smallmatrix}\right], 1_{q-1}\right), \quad (t \in \mathbb{R})$$

of G. Then by general theory, the group G is a disjoint union of double cosets Ha_tK $(t \geqslant 0)$ and the Lie algebra $\mathfrak{g} = \mathrm{Ad}(a_t)^{-1}\mathfrak{h} + \mathfrak{a} + \mathfrak{k}$ if $t > 0$. We have

$$\begin{aligned}
X_0^{\mathfrak{q}} &= \tfrac{1}{2}Y_0 - \tfrac{1}{2}\tfrac{\sqrt{-1}}{\sinh(2t)}\,\mathrm{Ad}(a_t)^{-1}Z_0^{\mathfrak{h}} + \tfrac{\sqrt{-1}}{2}\tfrac{\cosh(2t)}{\sinh(2t)}Z_0^{\mathfrak{h}},\\
X_j^{\mathfrak{q}} &= \tfrac{1}{\sinh t}\mathrm{Ad}(a_t)^{-1}Z_j^{\mathfrak{h}} - \tfrac{\cosh t}{\sinh t}Z_j^{\mathfrak{h}}, \quad (1 \leqslant j \leqslant q-1),\\
X_i^{\mathfrak{h}} &= \tfrac{1}{\cosh t}\mathrm{Ad}(a_t)^{-1}X_i^{\mathfrak{h}} - \tfrac{\sinh t}{\cosh t}Z_i^{\mathfrak{q}}, \quad (1 \leqslant i \leqslant p-1).
\end{aligned}$$

2.4. **Invariant measures.** Let dk and dk_0 be the Haar measures of the compact groups K and $H \cap K$ with total volume 1 respectively. Then we can take a unique Haar measure dg (resp. dh) of G (resp. H) such that the quotient measure $\frac{dg}{dk}$ (resp. $\frac{dh}{dk_0}$) coincides with the invariant measure on the symmetric space G/K (resp. H/K_H) determined by the Kähler volume form.

Lemma 1. *For any integrable function f on G, we have*

$$\int_G f(g)\,dg = \int_H dh \int_K dk \int_0^{+\infty} f(ha_tk)\,\varrho(t)\,dt \tag{2.2}$$

with dt the Lebesgue measure of $\mathbb{R}$ and

$$\varrho(t) = \tfrac{2\pi^q}{\Gamma(q)}(\sinh t)^{2q-1}(\cosh t)^{2p-1}.$$

3. CERTAIN INVARIANT TENSORS

For a C^∞-manifold U, let $A(U)$ denote the space of C^∞-differential forms on U and $A_c(U)$ the subspace of those forms with compact support; when necessary we topologize these spaces in the usual way.

When U has a complex structure, $A^{a,b}(U)$ denotes the space of C^∞-differential forms of bidegree (a, b).

3.1. Current defined by the symmetric subgroup. Let $j : H/H \cap K \hookrightarrow G/K$ be the natural inclusion. Then a (q, q)-current $\delta_{H/H\cap K}$ on G/K is defined by the integration

$$\langle \delta_{H/H\cap K}, \alpha \rangle = \int_{H/H\cap K} j^*\alpha, \quad \alpha \in A_c(G/K).$$

Lemma 2. *For $\alpha \in A_c(G/K)$, we have*

$$\langle \Lambda^{q-d}\delta_{H/H\cap K}, *\bar{\alpha} \rangle = \int_H (\Lambda^{q-d}(*\mathrm{vol}_{\mathfrak{p}\cap\mathfrak{h}})|\alpha(h))\, dh.$$

Here

$$\mathrm{vol}_{\mathfrak{p}\cap\mathfrak{h}} = \tfrac{1}{(q(p-1))!}\omega_{\mathfrak{p}\cap\mathfrak{h}}^{q(p-1)} \quad \in \quad \bigwedge^{(p-1)q,(p-1)q} \mathfrak{p}_{\mathbb{C}}^*$$

is the $K \cap H$-invariant tensor corresponding to the Kähler volume form of $H/H \cap K$.

For our purpose, it is important to understand the nature of the tensor $\Lambda^{q-d}(*\mathrm{vol}_{\mathfrak{p}\cap\mathfrak{h}})$ in some detail. First we have

Lemma 3. *For $0 \leqslant d \leqslant q$,*

$$\Lambda^{q-d}(*\mathrm{vol}_{\mathfrak{p}\cap\mathfrak{h}}) = \tfrac{(q-d)!}{d!}\gamma_{pp}^d.$$

3.2. K-spectrum of certain cyclic K-module. The aim of this subsection is to obtain an $\Omega_{\mathfrak{k}}$-eigendecomposition of the tensor $\Lambda^{q-d}(*\mathrm{vol}_{\mathfrak{p}\cap\mathfrak{h}})$. Here $\Omega_{\mathfrak{k}}$ is the Casimir element of K corresponding to the invariant form $B_{\mathfrak{g}}$.

The coadjoint representation of K on $\mathfrak{p}_{\mathbb{C}}^*$ is naturally extended to a unitary representation $\tau : K \to \mathrm{GL}(\bigwedge \mathfrak{p}_{\mathbb{C}}^*)$ in such a way that $\tau(k)(\alpha \wedge \beta) = \tau(k)\alpha \wedge \tau(k)\beta$ holds for $\alpha, \beta \in \bigwedge \mathfrak{p}_{\mathbb{C}}^*$ and $k \in K$. For $(a, b) \in \mathbb{N}^2$, $\tau^{a,b}$ denotes the subrepresentation of τ on $\bigwedge^{a,b} \mathfrak{p}_{\mathbb{C}}^*$.

For $1 \leqslant i \leqslant p$, $q \leqslant j \leqslant q$, let us define $\omega_{ij} \in \mathfrak{p}_{\mathbb{C}}^*$ by $\omega_{ij}(E_{\alpha,p+\beta}) = \delta_{i\alpha}\delta_{j\beta}$ $(1 \leqslant \alpha \leqslant p, 1 \leqslant \beta \leqslant q)$, $\omega_{ij}|\mathfrak{p}_- = 0$. Then ω_{ij}'s and their complex conjugates $\bar{\omega}_{ij}$ comprise a $\mathbb{C}$-basis of $\mathfrak{p}_{\mathbb{C}}^*$ dual of the basis of matrix units in $\mathfrak{p}_{\mathbb{C}}$.

For $\gamma \in \bigwedge^{1,1} \mathfrak{p}_{\mathbb{C}}^*$, the r-fold wedge product $\gamma \wedge \gamma \wedge \cdots \wedge \gamma$ is denoted by γ^r. In order to have a decomposition of γ_{pp}^d into eigenvectors of the Casimir operator $\Omega_{\mathfrak{k}}$, we first analyze the K-spectrum of $U(\mathfrak{k}_{\mathbb{C}})\gamma_{pp}^d$, the cyclic $U(\mathfrak{k}_{\mathbb{C}})$-submodule of $\bigwedge^{d,d} \mathfrak{p}_{\mathbb{C}}^*$ generated by γ_{pp}^d.

Since $\mathfrak{k}_{\mathbb{C}} \cong \mathfrak{gl}_p(\mathbb{C}) \oplus \mathfrak{gl}_q(\mathbb{C})$, the highest weight of an irreducible representation of $\mathfrak{k}$ is supposed to take the form

$$\lambda = [l_1, l_2, \ldots, l_p] \oplus [m_1, m_2, \ldots, m_q]$$

with $l_i, m_j \in \mathbb{Z}$ such that $l_1 \geqslant l_2 \geqslant \cdots \geqslant l_p$, $m_1 \geqslant m_2 \geqslant \cdots m_q$.

Lemma 4. *Let $0 \leqslant d \leqslant q$ and V an irreducible submodule of $U(\mathfrak{k}_{\mathbb{C}})\gamma_{pp}^d$. Then the highest weight of V is of the form $[\kappa, 0, \dots, 0, -\kappa] \oplus [0, \dots, 0]$ with an integer $0 \leqslant \kappa \leqslant d$.*

For $0 \leqslant \kappa \leqslant d$, let $V_\kappa^{(d)}$ be the $[\kappa, 0, \dots, 0, -\kappa] \oplus [0, \dots, 0]$-isotypic part of $U(\mathfrak{k}_{\mathbb{C}})\gamma_{pp}^d$. Then Lemma 4 implies

$$U(\mathfrak{k}_{\mathbb{C}})\gamma_{pp}^d = \bigoplus_{\kappa=0}^{d} V_\kappa^{(d)}. \tag{3.1}$$

Note that $V_0^{(d)}$ is a trivial representation of K.

Corollary 5. *Let $0 \leqslant d \leqslant q$. Then the operator $\prod_{\kappa=0}^d(-4^{-1}\Omega_{\mathfrak{k}} + \kappa(\kappa+p-1))$ annihilates the tensor γ_{pp}^d:*

$$\prod_{\kappa=0}^{d}(-4^{-1}\Omega_{\mathfrak{k}} + \kappa(\kappa+p-1))\,\gamma_{pp}^d = 0.$$

For $0 \leqslant d \leqslant q$, $0 \leqslant \kappa \leqslant d$, set

$$\theta_\kappa^{(d)} = \frac{(q-d)!}{d!} \prod_{\substack{0 \leqslant \alpha \leqslant d \\ \alpha \neq \kappa}} \frac{-4^{-1}\Omega_{\mathfrak{k}} + \alpha(\alpha+p-1)}{(\alpha-\kappa)(\kappa+\alpha+p-1)} \gamma_{pp}^d \quad \in \bigwedge^{d,d} \mathfrak{p}_{\mathbb{C}}^*.$$

Proposition 6. *Let $0 \leqslant d \leqslant q$.*

(1) *For each $0 \leqslant \kappa \leqslant d$, the tensor $\theta_\kappa^{(d)}$ is a nonzero eigenvector of $\Omega_{\mathfrak{k}}$ with the eigenvalue $4\kappa(\kappa+p-1)$, i.e.,*

$$\Omega_{\mathfrak{k}}\theta_\kappa^{(d)} = 4\kappa(\kappa+p-1)\theta_\kappa^{(d)}, \quad \theta_\kappa^{(d)} \neq 0.$$

The tensor $\theta_\kappa^{(d)}$ is $H \cap K$-invariant and is a $U(\mathfrak{k}_{\mathbb{C}})$-cyclic vector of $V_\kappa^{(d)}$. The representation $V_d^{(d)}$ is irreducible.

(2) *We have*

$$\Lambda^{q-d}(*\mathrm{vol}_{\mathfrak{p}\cap\mathfrak{h}}) = \sum_{\kappa=0}^{d} \theta_\kappa^{(d)}.$$

Moreover, the tensors $\theta_d^{(d)}$ $(0 \leqslant d \leqslant q)$ are primitive, i.e., $\Lambda\theta_d^{(d)} = 0$; we have $\Lambda\theta_\kappa^{(d)} = \theta_\kappa^{(d-1)}$ $(0 \leqslant \kappa < d)$.

REMARK: The $\mathfrak{k}$-module $V_\kappa^{(d)}$ with $0 \leqslant \kappa < d$ is not necessarily irreducible. For example, when $p = q = 2$, $V_0^{(2)} = \mathbb{C} \oplus \mathbb{C}$ is two dimensional and contains a non trivial K-invariant tensor orthogonal to $\omega_{\mathfrak{p}}^2$.

4. POLAR DECOMPOSITION OF SEVERAL DIFFERENTIAL OPERATORS

In this subsection we have an expression of several differential operators acting on the space of H-invariant forms $A((G - HK)/K)^H$.

4.1. **Differential forms.** For any right K-stable open subset S of G and a unitary representation (ρ, W) of K, let $C^\infty(S/K;\rho)$ denotes the space of C^∞-function $\varphi : S \to W$ such that

$$\varphi(gk) = \rho(k)^{-1}\varphi(g)\,(\forall g \in S,\, \forall k \in K).$$

For $g \in G$, let $L_g : xK \mapsto gxK$ be the left translation on G/K by g. Its tangent map $T_o(L_g)$ at the origin $o = K$ is regarded as a linear map $\mathfrak{p} \to T_{gK}(G/K)$. Given $\alpha \in A(S/K)$, a function $\tilde{\alpha} \in C^\infty(S/K;\tau)$ is defined by the formula

$$\langle\tilde{\alpha}(g), \xi\rangle = \langle\alpha(gK), (\textstyle\bigwedge T_o(L_g))\xi\rangle, \quad (\forall g \in S,\, \forall \xi \in \textstyle\bigwedge \mathfrak{p}).$$

The map $\alpha \mapsto \tilde{\alpha}$ yields a linear bijection from the space of forms $A^{a,b}(S/K)$ onto the space of functions $C^\infty(S/K;\tau^{a,b})$; we identify these two spaces by this isomorphism.

Since $G - HK$ is left H-stable and right K-stable open subset of G, both $A^{a,b}((G-HK)/K)$ and $C^\infty((G-HK)/K;\tau^{a,b})$ have natural left actions by H, and the isomorphism $A^{a,b}((G-HK)/K) \cong C^\infty((G-HK)/K;\tau^{a,b})$ preserves the H-actions.

Lemma 7. *Let $\varphi \in C^\infty((G-HK)/K;\tau)^H$. Then for each $t > 0$, the value $\varphi(a_t)$ belongs to the M-invariant part $(\bigwedge \mathfrak{p}^*_{\mathbb{C}})^M$. Conversely, given a C^∞-function $\phi : (0,+\infty) \to (\bigwedge \mathfrak{p}^*_{\mathbb{C}})^M$, there exists a unique function $\varphi \in C^\infty((G-HK)/K;\tau)$ such that $\varphi(a_t) = \phi(t)\,(\forall t > 0)$.*

4.2. **Laplacian.** Let $\Omega_{\mathfrak{m}}$, $\Omega_{\mathfrak{k}}$, $\Omega_{\mathfrak{h}\cap\mathfrak{k}}$ and $\Omega_{\mathfrak{g}}$ be Casimir element of M, K, $H \cap K$ and G respectively, corresponding to the invariant form $B_{\mathfrak{g}}$. Then

$$\Omega_{\mathfrak{k}\cap\mathfrak{h}} = \Omega_{\mathfrak{m}} - (Z_0^{\mathfrak{h}})^2 - 2\sum_{j=1}^{q-1}(Z_j^{\mathfrak{h}}\bar{Z}_j^{\mathfrak{h}} + \bar{Z}_j^{\mathfrak{h}}Z_j^{\mathfrak{h}}),$$

$$\Omega_{\mathfrak{k}} = \Omega_{\mathfrak{m}} - (Z_0^{\mathfrak{h}})^2 - 2\sum_{j=1}^{q-1}(Z_j^{\mathfrak{h}}\bar{Z}_j^{\mathfrak{h}} + \bar{Z}_j^{\mathfrak{h}}Z_j^{\mathfrak{h}}) - 2\sum_{i=1}^{p-1}(Z_i^{\mathfrak{q}}\bar{Z}_i^{\mathfrak{q}} + \bar{Z}_i^{\mathfrak{q}}Z_i^{\mathfrak{q}}),$$

$$\Omega_{\mathfrak{g}} = \Omega_{\mathfrak{k}} + 2\sum_{j=0}^{q-1}(X_j^{\mathfrak{q}}\bar{X}_j^{\mathfrak{q}} + \bar{X}_j^{\mathfrak{q}}X_j^{\mathfrak{q}}) + 2\sum_{i=1}^{p-1}(X_i^{\mathfrak{h}}\bar{X}_i^{\mathfrak{h}} + \bar{X}_i^{\mathfrak{h}}X_i^{\mathfrak{h}}) + 2\sum_{i=1}^{p-1}\sum_{j=1}^{q-1}(X_{ij}^{\mathfrak{h}}\bar{X}_{ij}^{\mathfrak{h}} + \bar{X}_{ij}^{\mathfrak{h}}X_{ij}^{\mathfrak{h}}).$$

Let us introduce the operators

$$\mathcal{S}_{\mathfrak{k},\mathfrak{q}} = \tfrac{1}{2}\sum_{i=1}^{p-1}\tau(Z_i^{\mathfrak{q}}\bar{Z}_i^{\mathfrak{q}} + \bar{Z}_i^{\mathfrak{q}}Z_i^{\mathfrak{q}}) = \tfrac{-1}{4}\{\tau(\Omega_{\mathfrak{k}}) - \tau(\Omega_{\mathfrak{k}\cap\mathfrak{h}})\},$$

$$\mathcal{S}_{\mathfrak{k},\mathfrak{h}} = \tfrac{1}{2}\sum_{j=1}^{q-1}\tau(Z_j^{\mathfrak{h}}\bar{Z}_j^{\mathfrak{h}} + \bar{Z}_j^{\mathfrak{h}}Z_j^{\mathfrak{h}}) = \tfrac{-1}{4}\{\tau(\Omega_{\mathfrak{k}\cap\mathfrak{h}}) - \tau(\Omega_{\mathfrak{m}}) + \tau(Z_0^{\mathfrak{h}})^2\}$$

acting on $\bigwedge \mathfrak{p}^*_{\mathbb{C}}$. Let $\triangle$ be the Hodge laplacian acting on $A((G-HK)/K)$.

Proposition 8. *Let $\varphi \in C^\infty((G-HK)/K;\tau)^H$ and set $\phi(t) = \varphi(a_t)\,(t > 0)$. Then $\triangle(\varphi)(a_t) = -\mathfrak{D}_t\phi(t)\,(t > 0)$ with $\mathfrak{D}_t$ the $(\bigwedge \mathfrak{p}^*_{\mathbb{C}})^M$-valued differential operator*

$$\mathfrak{D}_t = \tfrac{d^2}{dt^2} + ((2p-1)\tanh t + (2q-1)\coth t)\,\tfrac{d}{dt} + \tfrac{4\mathcal{S}_{\mathfrak{k},\mathfrak{h}}}{\sinh^2 t} + \tfrac{-4\mathcal{S}_{\mathfrak{k},\mathfrak{q}}}{\cosh^2 t} + \tfrac{1}{4}(\coth t - \tanh t)^2\tau(Z_0^{\mathfrak{h}})^2 + \tau(\Omega_{\mathfrak{m}}).$$

4.3. $\partial\bar{\partial}$-operator. Since $(G-HK)/K$ is an open subset of the complex manifold G/K, we have usual operators ∂ and $\bar{\partial}$ acting on $A((G-HK)/K)$. The aim here is to obtain an expression of the composite operator $\partial\bar{\partial}$ on the H-invariant forms.

Let us introduce operators acting on $\bigwedge \mathfrak{p}_{\mathbb{C}}^*$:

$$\mathcal{P}_+ = \sum_{i=1}^{p-1} e(\bar{\omega}_{i1})\tau(\bar{Z}_i^{\mathfrak{q}}), \quad \mathcal{P}_- = \sum_{i=1}^{p-1} e(\omega_{i1})\tau(Z_i^{\mathfrak{q}}), \quad e(\eta_{\mathfrak{h}}) = \tfrac{\sqrt{-1}}{2}\sum_{i=1}^{p-1} e(\omega_{i1}\wedge\bar{\omega}_{i1}),$$

$$\mathcal{R}_+ = \sum_{j=1}^{q-1} e(\bar{\omega}_{p,j+1})\tau(\bar{Z}_j^{\mathfrak{h}}), \quad \mathcal{R}_- = \sum_{j=1}^{q-1} e(\omega_{p,j+1})\tau(Z_j^{\mathfrak{h}}), \quad e(\eta_{\mathfrak{q}}) = \tfrac{\sqrt{-1}}{2}\sum_{j=1}^{q-1} e(\omega_{p,j+1}\wedge\bar{\omega}_{p,j+1}),$$

and

$$\begin{aligned}
\mathcal{A} &= \tfrac{2}{\sqrt{-1}}e(\eta_{\mathfrak{h}}) + \tfrac{1}{2}e(\omega_0\wedge\bar{\omega}_0) - e(\bar{\omega}_0)\mathcal{P}_- + e(\omega_0)\mathcal{P}_+,\\
\mathcal{B} &= \tfrac{2}{\sqrt{-1}}e(\eta_{\mathfrak{q}}) + \tfrac{1}{2}e(\omega_0\wedge\bar{\omega}_0) - e(\bar{\omega}_0)\mathcal{R}_- + e(\omega_0)\mathcal{R}_+,\\
\mathcal{C} &= e(\omega_0)(\mathcal{P}_+ + \mathcal{R}_+) + \mathcal{P}_-\mathcal{R}_+ + \mathcal{R}_-\mathcal{P}_+.
\end{aligned}$$

Here we set $\omega_0 = \omega_{p,1}$.

Proposition 9. *Let $\varphi \in C^\infty((G-HK)/K;\tau)^H$ ad set $\phi(t) = \varphi(a_t)\,(t>0)$. Then $(\partial\bar{\partial}\varphi)(a_t) = \mathcal{E}_t\phi(t)\,(t>0)$ with $\mathcal{E}_t$ the $(\bigwedge\mathfrak{p}_{\mathbb{C}}^*)^M$-valued differential operator*

$$\begin{aligned}
\mathcal{E}_t = {}& \tfrac{1}{4}e(\omega_0\wedge\bar{\omega}_0)\tfrac{d^2}{dt^2} + \tfrac{1}{2}(\tanh t\,\mathcal{A} + \coth t\,\mathcal{B})\tfrac{d}{dt}\\
&+ \tanh^2 t\,\mathcal{P}_-\mathcal{P}_+ + \coth^2 t\,\mathcal{R}_-\mathcal{R}_+ + \mathcal{C}\\
&+ \tfrac{\sqrt{-1}}{4}(1+\tanh^2 t)\left(\tfrac{2}{\sqrt{-1}}e(\eta_{\mathfrak{h}}) - e(\bar{\omega}_0)\mathcal{P}_- - e(\omega_0)\mathcal{P}_+\right)\tau(Z_0^{\mathfrak{h}})\\
&+ \tfrac{\sqrt{-1}}{4}(1+\coth^2 t)\left(\tfrac{2}{\sqrt{-1}}e(\eta_{\mathfrak{q}}) - e(\bar{\omega}_0)\mathcal{R}_- - e(\omega_0)\mathcal{R}_+\right)\tau(Z_0^{\mathfrak{h}})\\
&+ \tfrac{\sqrt{-1}}{2}e(\omega_0\wedge\bar{\omega}_0)\tau(Z_0^{\mathfrak{h}}) + \tfrac{1}{16}(\tanh t + \coth t)^2\tau(Z_0^{\mathfrak{h}})^2.
\end{aligned}$$

5. THE SECONDARY SPHERICAL FUNCTIONS

In this section, we fix an integer $0 \leqslant d \leqslant q$ and set

$$D^{(d)} = \mathbb{C} - \{2d+p-q-1-2n \mid n\in\mathbb{N}\}.$$

Here is the main theorem of this section.

Theorem 10. (1) *There exists a unique family $\varphi_s^{(d)}$ $(s\in D^{(d)})$ of functions with the properties:*

(i) *For $s\in D^{(d)}$, $\varphi_s^{(d)} \in C^\infty((G-HK)/K;\tau^{d,d})^H$.*

(ii) *For each $g\in G-HK$, the value $\varphi_s^{(d)}(g)$ depends on $s\in D^{(d)}$ holomorphically.*

(iii) *For each $s\in D^{(d)}$,*

$$\Omega_{\mathfrak{g}}\varphi_s^{(d)}(g) = (s^2-(p+q-1)^2)\varphi_s^{(d)}(g), \quad (g\in G-HK).$$

(iv) *It has the 'small-time behavior'*

$$\lim_{t\to 0+0} t^{2(q-1)}\varphi_s^{(d)}(a_t) = \Lambda^{q-d}(*\mathrm{vol}_{\mathfrak{p}\cap\mathfrak{h}}).$$

(v) *It has the 'large-time behavior'*

$$\varphi_s^{(d)}(a_t) = O(e^{-(\mathrm{Re}(s)+p+q-1)t}), \quad (t \to +\infty).$$

(2) *The radial value* $\varphi_s^{(d)}(a_t)$ *is given by the explicit formula*

$$\varphi_s^{(d)}(a_t) = \sum_{\kappa=0}^{d} F_\kappa(s;t)\,\theta_\kappa^{(d)}, \quad (t > 0).$$

Here for each $\kappa \in \mathbb{N}$, $s \in \mathbb{C}$ *and* $t > 0$, *we set*

$$\begin{aligned} F_\kappa(s;t) = {} & \frac{\Gamma(\frac{s+p+q-1}{2}+\kappa)\Gamma(\frac{s-p+q+1}{2}-\kappa)}{\Gamma(s+1)\Gamma(q-1)} \\ & \times (\cosh t)^{-(s+p+q-1)}\,{}_2F_1\left(\tfrac{s+p+q-1}{2}+\kappa, \tfrac{s-p+q+1}{2}-\kappa; s+1; \tfrac{1}{\cosh^2 t}\right). \end{aligned}$$

The next corollary says that only the function $\varphi_s^{(q)}$ is essential, from which others $\varphi_s^{(d)}$ with smaller bidegree (d,d) are obtained by successive application of Λ.

Corollary 11. *We have* $\Lambda\varphi_s^{(d)} = \varphi_s^{(d-1)}$ *whenever* $1 \leqslant d \leqslant q$, $s \in D^{(d)}$.

5.1. **Some properties of the secondary spherical functions.** In this subsection, we fix a family of functions $\varphi_s^{(d)} (s \in D^{(d)})$ satisfying the conditions (i),(ii),(iii),(iv) and (v) in Theorem 10. Starting with these five properties, we deduce several substantial results which will be used not only to prove Theorem 10 but also to study Poincaré series in the next section .

Proposition 12. *There exists a* $(\bigwedge^{d,d}\mathfrak{p}_{\mathbb{C}}^*)^M$*-valued holomorphic function* $\mathsf{R}(s,z)$ *on* $D^{(d)} \times \{|z|<1\}$ *such that*

$$\varphi_s^{(d)}(a_t) = (1-z)^{(s+p+q-1)/2}\,\mathsf{R}(s,1-z), \qquad (s \in D^{(d)},\ z = \tanh^2 t \in (0,1)).$$

There exists a unique family $\mathsf{c}_\alpha(s)\,(0 \leqslant \alpha \leqslant q-2)$ *of tensors in* $(\bigwedge^{d,d}\mathfrak{p}_{\mathbb{C}}^*)^{H\cap K}$ *such that the following properties hold.*

(1) *There exist* $N \in \mathbb{N}$ *and* $(\bigwedge^{d,d}\mathfrak{p}_{\mathbb{C}}^*)^M$*-valued holomorphic functions* $\mathsf{P}_h(s,z)\,(0 \leqslant h \leqslant N)$ *on* $D^{(d)} \times \{|z|<1\}$ *such that*

$$\varphi_s^{(d)}(a_t) = \sum_{\alpha=0}^{q-2} \frac{\mathsf{c}_\alpha(s)}{z^{q-\alpha-1}} + \sum_{h=0}^{N} (\log z)^i\,\mathsf{P}_h(s,z), \quad (s \in D^{(d)},\ z = \tanh^2 t \in (0,1)).$$

(2) *We have* $\mathsf{c}_0(s) = \Lambda^{q-d}(*\mathrm{vol}_{\mathfrak{p}\cap\mathfrak{h}})$, *and* $\mathsf{c}_\alpha(s)$ *satisfies the recurrence relation:*

$$\begin{aligned} 4\alpha(q-\alpha-1)\,\mathsf{c}_\alpha(s) = \sum_{\kappa=0}^{\alpha-1} \{ & \tau(\Omega_{\mathfrak{k}}) + (\alpha-\kappa)((p+q-1)^2 - s^2) \\ & - 4(p+q-2)(q-\kappa-1)\}\,\mathsf{c}_\kappa(s), \quad (0 < \alpha \leqslant q-2). \end{aligned}$$

(3) *For* $0 \leqslant \alpha \leqslant q-2$, $\mathsf{c}_\alpha(s)$ *is a polynomial function in* $s \in \mathbb{C}$ *such that* $\mathsf{c}_\alpha(s) = \mathsf{c}_\alpha(-s)$ *and* $\deg \mathsf{c}_\alpha(s) = 2\alpha$.

Since HK is a zero set of G with respect to the Haar measure, the form $\varphi_s^{(d)}$ is regarded as a measurable form on G/K.

Lemma 13. *The measurable form $\varphi_s^{(d)}$ on G/K is locally integrable.*

Proposition 14. *The (d,d)-current $\varphi_s^{(d)}$ satisfies the differential equation:*

$$\{\Delta + s^2 - (p+q-1)^2\}\varphi_s^{(d)} = \tfrac{4\pi^q}{\Gamma(q-1)}\Lambda^{q-d}\delta_{H/H\cap K}.$$

5.2. **A differential relation.** The functions $\phi_s = \varphi_s^{(q-1)} = \Lambda\varphi_s^{(q)}$ and $\psi_s = \varphi_s^{(q)}$ are of particular importance in our investigation of the modular cycles arising from H. They are related by the simple formula:

Theorem 15. *Let $s \in D^{(q)}$. Then we have*

$$\partial\bar{\partial}\phi_s(g) = \frac{-\sqrt{-1}\{s^2 - (p+q-1)^2\}}{2}\psi_s(g), \quad g \in G - HK.$$

6. POINCARÉ SERIES

Let Γ be a discrete torsion free subgroup of G such that the quotient spaces $\Gamma\backslash G$ and $\Gamma \cap H\backslash H$ have finite invariant volumes. For simplicity we set $\Gamma_H = \Gamma \cap H$ and $K_H = H \cap K$. Since Γ is torsion free, the Kähler manifold structures on the discrete quotients $\Gamma_H\backslash H/K_H$ and $\Gamma\backslash G/K$ are entailed from those on their universal coverings H/K_H and G/K. Moreover, ΓH is a closed subset of G and the inclusion $\Gamma\cap H\backslash H \hookrightarrow \Gamma\backslash G$ has the closed image.

6.1. **Currents defined by Poincaré series.** Let $\varphi_s^{(d)}$ ($s \in D^{(d)}$) be the secondary spherical function of bidegree (d,d) constructed in Theorem 10. For $r \in \mathbb{N}$, we define an auxiliary function $\varphi_{s,r}^{(d)}$ by

$$\varphi_{s,r}^{(d)}(g) = \frac{1}{r!}\left(\frac{-1}{2s}\frac{d}{ds}\right)^r \varphi_s^{(d)}(g), \quad (s \in D^{(d)},\ g \in G - HK).$$

Let us consider the Poincaré series

$$\Phi_{s,r}^{(d)}(g) = \tfrac{\Gamma(q-1)}{\pi^q} \sum_{\gamma \in \Gamma_H\backslash\Gamma} \varphi_{s,r}^{(d)}(\gamma g) \tag{6.1}$$

for (s,g) belonging to the set $\{s \in \mathbb{C} | \operatorname{Re}(s) > p+q-1\} \times (G - \Gamma HK)$, where the series is convergent as the next theorem shows. Note $\operatorname{Re}(s) > p+q-1$ is contained in the domain $D^{(d)}$.

Proposition 16. *Let U be a compact subset of $G - \Gamma HK$ and ϵ a positive real number. Then the series (6.1) converges absolutely and uniformly on $U \times \{s \in \mathbb{C} | \operatorname{Re}(s) \geqslant p+q-1+\epsilon\}$.*

Proposition 17. *We have*

$$\int_{\Gamma\backslash G} \sum_{\gamma \in \Gamma_H\backslash\Gamma} \|\varphi_{s,r}^{(d)}(g)\|\, dg < +\infty$$

In particular, the measurable function $\Phi_{s,r}^{(d)}(g)$ on $\Gamma\backslash G/K$ is integrable.

Therefore the measurable (d,d)-form $\Phi_{s,r}^{(d)}$ on $\Gamma\backslash G/K$ yields a current, denoted by the same notation $\Phi_{s,r}^{(d)}$, by the integration:

$$\langle \Phi_{s,r}^{(d)}, \alpha \rangle = \int_{\Gamma\backslash G/K} \Phi_{s,r}^{(d)} \wedge \alpha, \quad (\forall \alpha \in A_c(\Gamma\backslash G/K)).$$

6.2. **Poisson equation.** Let $C_H^\Gamma : \Gamma_H\backslash H/K_H \to \Gamma\backslash G/K$ be the holomorphic map obtained from the inclusion $H/K_H \hookrightarrow G/K$ by passing to the discrete quotients. The image of C_H^Γ is a closed complex analytic subset of $\Gamma\backslash G/K$. Sometimes we use the simpler notation C for C_H^Γ. Our currents $\Phi_{s,r}^{(d)}$ satisfy the generalized Poisson equation:

Proposition 18.

$$\{\Delta + s^2 - (p+q-1)^2\}^{r+1}\Phi_{s,r}^{(d)} = 4\Lambda^{q-d}\delta_C$$

for $\mathrm{Re}(s) > p+q-1$, $r \in \mathbb{N}$.

6.3. **Spectral expansion of Poincaré series.** In order to obtain meromorphic continuation of the function $s \mapsto \Phi_{s,0}^{(d)}$ beyond the convergence region $\mathrm{Re}(s) > p+q-1$, we want to use L^2-theory, i.e., spectral decomposition of the Laplace-Beltrami operator acting on the Hilbert space of square integrable (d,d)-forms. Unfortunately, the form $\Phi_{s,0}^{(d)}$ is *not* square-integrable, even when $\Gamma\backslash G$ is compact. This difficulty is circumvented by considering $\Phi_{s,r}^{(d)}$ with large r.

Proposition 19. *Let* $r \geqslant q-1$. *Suppose one of the conditions (a) and (b) is satisfied:*

(a) $\Gamma\backslash G$ *is compact, and* $\mathrm{Re}(s) > p+q-1$.

(b) G *has a* $\mathbb{Q}$*-structure with respect to which* H *is* $\mathbb{Q}$*-rational and* Γ *is arithmetic, and* $\mathrm{Re}(s) > (p+q-1)(3-2p^{-1})$.

Then the measurable (d,d)*-form* $\Phi_{s,r}^{(d)}$ *on* $\Gamma\backslash G/K$ *is* $L^{2+\epsilon}$ *for some* $\epsilon > 0$.

REMARK: Since $\mathrm{vol}(\Gamma\backslash G) < +\infty$, $L^{2+\epsilon}$ implies L^2 for a function on $\Gamma\backslash G$ by Hölder's inequality.

Let $A_{(2)}^{d,d}(\Gamma\backslash G/K)$ be the completion of the space $A_c^{d,d}(\Gamma\backslash G/K)$ by the inner product

$$\langle \alpha | \beta \rangle = \int_{\Gamma\backslash G/K} \alpha \wedge *\bar{\beta}.$$

From now on we further assume that Γ *is a uniform lattice, i.e., the manifold* $\Gamma\backslash G/K$ *is compact.* Then the Laplace-Beltrami operator Δ with the domain $A^{d,d}(\Gamma\backslash G/K)$ is essentially self-adjoint operator on the Hilbert space $A_{(2)}^{d,d}(\Gamma\backslash G/K)$. The domain of $\bar{\Delta}$, the minimal closed extension of Δ, consists of all $\alpha \in A_{(2)}^{d,d}(\Gamma\backslash G/K)$ such that the distribution $\Delta\alpha$ belongs to $A_{(2)}^{d,d}(\Gamma\backslash G/K)$. There exists an orthonormal basis $\{\alpha_n\}_{n\in\mathbb{N}}$ of $A_{(2)}^{d,d}(\Gamma\backslash G/K)$ consisting of eigenvectors of $\bar{\Delta}$; let $\{\lambda_n\}$ be the corresponding system of eigenvalues: $\bar{\Delta}\alpha_n = \lambda_n\alpha_n$. Note α_n's are C^∞-forms and λ_n's are non-negative real numbers because the differential operator Δ is positive, formally self-adjoint and elliptic.

Theorem 20. *Let $r \geqslant q-1$ and $\mathrm{Re}(s) > p+q-1$. Then*

$$\Phi^{(d)}_{s,r} = \sum_{n=0}^{\infty} \frac{4\langle \Lambda^{q-d}\delta_C, *\bar{\alpha}_n\rangle}{\{\lambda_n + s^2 - (p+q-1)^2\}^{r+1}}\alpha_n$$

is the spectral expansion of $\Phi^{(d)}_{s,r} \in A^{d,d}_{(2)}(\Gamma\backslash G/K)$.

The spectral expansion yields a meromorphic continuation of $\Phi^{(d)}_{s,r}$.

Theorem 21. *Let $r \in \mathbb{N}$ and $\beta \in A(\Gamma\backslash G/K)$. The function $s \mapsto \langle \Phi^{(d)}_{s,r}|\beta\rangle$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$. A point $s_0 \in \mathbb{C}$ with $\mathrm{Re}(s_0) \geqslant 0$ is a pole of $\langle\Phi^{(d)}_{s,r}|\beta\rangle$ if and only if there exists an index $n \in \mathbb{N}$ such that $\langle \Lambda^{q-d}\delta_C, *\bar{\alpha}_n\rangle \neq 0$, $\langle \alpha_n|\beta\rangle \neq 0$ and $s_0^2 - (p+q-1)^2 = -\lambda_n$. The function*

$$\langle\Phi^{(d)}_{s,r}|\beta\rangle + \sum_{\substack{n\in\mathbb{N}\\ \lambda_n=(p+q-1)^2-s_0^2}} \frac{-4\langle \Lambda^{q-d}\delta_C, *\bar{\alpha}_n\rangle}{\{\lambda_n + s^2 - (p+q-1)^2\}^{r+1}}\langle\alpha_n|\beta\rangle$$

is holomorphic at $s = s_0$. We have the functional equation $\Phi^{(d)}_{s,r} = \Phi^{(d)}_{-s,r}$.

7. AUTOMORPHIC GREEN CURRENT

Set $G^\Gamma_H(s) = \Phi^{(q-1)}_{s,0}$ and $\Psi^\Gamma_H(s) = \Phi^{(q)}_{s,0}$.

Theorem 22. *The equations*

$$\{\triangle + s^2 - (p+q-1)^2\}\, G^\Gamma_H(s) = 4\Lambda\delta_{C^\Gamma_H},$$
$$\{\triangle + s^2 - (p+q-1)^2\}\, \Psi^\Gamma_H(s) = 4\delta_{C^\Gamma_H},$$
$$\mathrm{d_c d}\, G^\Gamma_H(s) + \{s^2 - (p+q-1)^2\}\Psi^\Gamma_H(s) = 4\delta_{C^\Gamma_H}$$

hold.

Since $G^\Gamma_H(s)$ and $\Psi^\Gamma_H(s)$ are meromorphic on $\mathbb{C}$ with at most simple poles at $s = p+q-1$, we can consider the constant term and the residue of their Laurent expansion:

$$\mathcal{G}^\Gamma_H = \tfrac{1}{4}\mathrm{CT}_{s=p+q-1}G^\Gamma_H(s), \quad \Psi^\Gamma_H = \tfrac{p+q-1}{2}\mathrm{Res}_{s=p+q-1}\Psi^\Gamma_H(s).$$

Theorem 23. *We have*

$$\triangle\,\mathcal{G}^\Gamma_H = \Lambda\delta_{C^\Gamma_H}, \quad \triangle\,\Psi^\Gamma_H = 0, \quad \mathrm{d_c d}\,\mathcal{G}^\Gamma_H + \Psi^\Gamma_H = \delta_{C^\Gamma_H}.$$

By the Hodge theory for compact Kähler manifolds, the fundamental class of the cycle C^Γ_H has a unique harmonic representative in $A^{q,q}(\Gamma\backslash G/K)$ called *the Poincaré dual form* of C^Γ_H. Our result tells an explicit way how to construct that harmonic form. Indeed, the second equation in Theorem 23 shows the (q,q)-form Ψ^Γ_H is harmonic and the third one means Ψ^Γ_H is cohomologus to the current $\delta_{C^\Gamma_H}$. Therefore, Ψ^Γ_H meets the requirements of the Poincaré dual form.

Theorem 23 also tells that $(q-1,q-1)$-current $\mathcal{G}^\Gamma_H$ is a *Green current* for the cycle C^Γ_H (*cf.* Gillet-Soulé [2]). Though there are many Green currents for C^Γ_H, our construction fixes a choice, whose dependence on Γ is tractable. Note the singularity of our Green current is different from the one considered by Gillet-Soulé.

8. SOME GLOBAL CONSEQUENCE

Along the K-module decomposition (3.1), the current $\Psi_H^\Gamma(s)$ is decomposed as

$$\Psi_H^\Gamma(s) = \sum_{\kappa=0}^{q} \Psi_{H,\kappa}^\Gamma(s), \quad \Psi_{H,\kappa}^\Gamma(s) \in C^\infty((G-\Gamma HK)/K; V_\kappa^{(d)})^\Gamma.$$

Each component function $\Psi_{H,\kappa}^\Gamma(s)$ is also meromorphic in $s \in \mathbb{C}$ and the Poincaré dual form Ψ_H^Γ is a sum of forms $\Psi_{H,\kappa}^\Gamma = \frac{\rho_0}{2}\mathrm{Res}_{s=\rho_0}\Psi_{H,\kappa}^\Gamma(s)$ $(0 \leqslant \kappa \leqslant q)$, each of which is also harmonic. Moreover $\Psi_{H,q}^\Gamma$ is primitive, i.e., $\Lambda\Psi_{H,q}^\Gamma = 0$.

Proposition 24. *We have*

$$\Psi_{H,0}^\Gamma = \frac{\mathrm{vol}(\Gamma_H\backslash H/K_H)}{\mathrm{vol}(\Gamma\backslash G/K)}\,\theta_0^{(q)}.$$

In particular, $\Psi_{H,0}^\Gamma \neq 0$.

About the intermediate forms $\Psi_{H,\kappa}^\Gamma$ $(1 \leqslant \kappa \leqslant q-1)$, we have the vanishing theorem.

Proposition 25. *For* $0 < \kappa < q$*, we have* $\Psi_{H,\kappa}^\Gamma = 0$.

The remaining is the primitive form $\Psi_{H,q}^\Gamma$, which can be regarded as the essential ingredient of the Poincaré dual form.

The secondary spherical function $\psi_s = \varphi_s^{(q)}$ has a simple pole at $s = \rho_0 := p+q-1$ with $\psi_H = \mathrm{Res}_{s=\rho_0}\psi_s$ such that

$$\psi_H(a_t) = 2\frac{\Gamma(\rho_0+q)}{\Gamma(\rho_0+1)\Gamma(q-1)}(\cosh t)^{-2\rho_0}\,\theta_q^{(q)}, \quad (\forall t > 0). \tag{8.1}$$

The (q,q)-current ψ_H is a harmonic form belonging to the space $A^{q,q}(G/K)^H$. The coefficient functions $g \mapsto (\psi_H(g)|v)$ $(v \in V_q^{(q)})$ belong to $L^2(H\backslash G)$ and together with their right $U(\mathfrak{g}_\mathbb{C})$ translates span an irreducible $(\mathfrak{g}_\mathbb{C}, K)$-submodule π_q of $L^2(H\backslash G)$.

Proposition 26. *Our global construction* $\Psi_{H,q}^\Gamma$*, if non-zero, yields an automorphic realization of* π_q *in the sapce of* L^2*-automorphic forms* $L^2(\Gamma\backslash G)$.

Remark 1: The representation π_q satisfies $\mathrm{H}^{q,q}(\mathfrak{g}_\mathbb{C}, K; \mathbb{C}) \cong \mathbb{C}$.
Remark 2: It is a subtle and difficult arithmetic problem to find whether the primitive form $\Psi_{H,q}^\Gamma$ for a given Γ is zero or not. Analogous non-vanishing statements of the Poincaré series constructed from an ordinary spherical function with regular spectral parameter (for small Γ) are found in Oshima [6] and Tong-Wang [10].

9. REMARKS AND FURTHER OBSERVATIONS

- Though our global results after Proposition 20 in this paper are stated under the assumption that $\Gamma\backslash G$ is compact, the same statements (except a proper modification of the functional equation of $\Phi_{s,r}^{(d)}$) should be true for arithmetic non-uniform lattices Γ. But the situation is technically more sophisticated.
- Finally, we should say a few words about existing works related to the theme of this article.

  When the complex codimension of $H/H\cap K$ in G/K is one, the modular construction of Green current of C_H^Γ is obtained in [5] by the same way as explained here. If G/K and $H/H \cap K$ are type IV symmetric domains and if Γ is a discriminant group of some rational quadratic form, Bruiner [1] constructed a Green function for a 'Heegner divisor' (which is a member of the divisor class group of $\Gamma\backslash G/K$ expressed as a linear combination of $C^{\Gamma}_{H_i}$ for various H_i's defined over $\mathbb{Q}$) by a 'regularized theta lifting'. It turns out the Green function in [1] is built from the one in [5] according to the formation of the relevant Heegner divisor.

Based on a work of Oshima-Matsuki, Tong-Wang [10] provides a fairly general and simple method to construct an automorphic realization of a discrete series of a symmetric space, which yields a modular construction of the Poincaré dual form associated with a cohomology class defined by the symmetric subgroup in a cohomology group with coefficients in a local system. For analytical reasons, they need to assume that the coefficient system should be sufficiently regular. This requirement is related to the L^1-condition of the discrete series, which is indispensable to guarantee the convergence of the Poincaré series they use. This is a serious technical limitation to obtain the Poincaré dual forms in the cohomology with *constant coefficient.*

To be more concrete, let us pick the representation π_q defined above as an example. It is easy to see that π_q is *not* integrable; so one can not expect the convergence of the Poincaré series '$\sum_{\gamma\in\Gamma_H\backslash\Gamma}\psi_H(\gamma g)$' used in [10]. Though the secondary spherical function ψ_s has a singularity, it is good enough to assure the convergence of the Poincaré series $\Psi_s(g)=\sum_{\gamma\in\Gamma_H\backslash\Gamma}\psi_s(\gamma g)$ for large Re(s). We can recover the object '$\sum_{\gamma\in\Gamma_H\backslash\Gamma}\psi_H(\gamma g)$' properly by taking the residue at $s=p+q-1$ after the meromorphic continuation of the series Ψ_s. This regularization procedure reminds us of the 'Hecke's trick' which is used to obtain an Eisenstein series with low weight in the classical theory of elliptic modular forms ([11]). In this analogy, the construction of the automorphic Green current $\mathcal{G}^{\Gamma}_H$ can be regarded as a kind of the second limit formula of Kronecker ([11]).

- The paper [3] is also related to this paper. There we also considered modular symbols derived from the injection $H \hookrightarrow G$ for a symmetric pair (G,H). But in this case the image of the locally symmetric space associated with the subgroup H is totally real, in contrast that here in this paper we consider the holomorphic embedding. Actually we have some evidence to believe that the modular symbols considered in [3] have the extremal Hodge components (i.e., $(m,0)$-type components), but our cycles are algebraic hence have only (p,p)-type Hodge components. In this sense, the two results in this paper and [3] might be the two edges of some more general phenomena.
- Because of the lack of time and energy we could not handle the cases of symmetric pairs $(G,H)=(SO(2,2m),SU(1,m))$. Since the representations of the maximal compact subgroup $K\cong SO(2m)$ of G are a bit more complicated to describe than the case of $U(p)\times U(q)$, one might be asked more in computation. But we believe that basically the same scenario as in this paper is valid for this case too.

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