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Compactification of the symplectic group via generalized symplectic isomorphisms

By

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§ 1. Introduction

Let G be a connected reductive algebraic group over an algebraically closed field k of characteristic zero. We have a left $(G\times G)$-action on G defined as $(g_1, g_2)\cdot x := g_1 x g_2^{-1}$.

A $(G \times G)$-equivariant embedding $G \hookrightarrow X$ is said to be regular (cf. [BDP], [Br, §1.4]) if the following conditions are satisfied:

(i) X is smooth and the complement $X \setminus G$ is a normal crossing divisor $D_1 \cup \cdots \cup D_n$.

(ii) Each D_i is smooth.

(iii) Every $(G \times G)$-orbit closure in X is a certain intersection of $D_1, \ldots, D_n$.

(iv) For every point $x \in X$, the normal space $T_x X/T_x(Gx)$ contains a dense orbit of the isotropy group G_x.

If $G \hookrightarrow X$ is a $(G\times G)$-equivariant regular compactification of G, then a sum $\sum a_i D_i$ of the boundary divisors is $(G \times G)$-stable. Let $\widetilde{G} \to G$ be a finite covering. If the line bundle $\mathcal{O}(\sum a_i D_i)$ has a $(\widetilde{G}\times\widetilde{G})$-linearization, then the vector space $\mathrm{H}^0(X, \mathcal{O}(\sum a_i D_i))$ of global sections of $\mathcal{O}(\sum a_i D_i)$ becomes a $(\widetilde{G} \times \widetilde{G})$-module. Kato [Ka] and Tchoudjem [T] described the decomposition of this $(\widetilde{G}\times\widetilde{G})$-module into irreducible $(\widetilde{G}\times\widetilde{G})$-modules.

Kausz constructed a regular compactification KGL_n of the general linear group GL_n in [Kausz1]. In [Kausz2] he described the structure of the $(GL_n \times GL_n)$-modules of global sections of line bundles associated to boundary divisors. Although he dealt with only the very special regular compactification KGL_n, a good thing is that his description of the $(GL_n \times GL_n)$-modules is canonical. More precisely, he constructed

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a canonical isomorphism between the $(GL_n \times GL_n)$-modules of global sections of line bundles associated to boundary divisors on KGL_n and the $(GL_n \times GL_n)$-modules of global sections of line bundles on a product of flag varieties. The fact that the decomposition is canonical is important when we apply the compactification of G to the study of the moduli of G-bundles. In fact, Kausz used the canonical decomposition of the above $(GL_n \times GL_n)$-modules, and proved the factorization theorem ([Kausz3]) of generalized theta functions on the moduli stack of vector bundles on a curve. (The factorization theorem has also been obtained by Narasimhan-Ramadas [N-Rd] and Sun [S1], [S2].)

The purpose of this paper is to establish an analogue of the Kausz's results to the symplectic group.

If V is a finite dimensional vector space, the general linear group $GL(V)$ is regarded as a moduli space of isomorphisms $V \to V$. In [Kausz1], Kausz introduced a *generalized isomorphism*. The compactification $KGL(V)$ of $GL(V)$ is the moduli space of generalized isomorphisms from V to V.

Now suppose that V is endowed with a non-degenerate alternate bilinear form. The symplectic group $Sp(V)$ is regarded as a moduli space of symplectic isomorphisms $V \to V$. As a symplectic analogue, we introduce a *generalized symplectic isomorphism* (Definition 3.1). The regular compactification $KSp(V)$ of $Sp(V)$ is defined to be the moduli space of generalized symplectic isomorphisms from V to V. At first glance, it is not clear whether or not $KSp(V)$ is a closed subvariety of $KGL(V)$, but a posteriori we know that it is (Corollary 3.16).

If $\dim V = 2r$, then the complement $KSp(V) \setminus Sp(V)$ is a union of smooth divisors $D_0, \dots, D_{r-1}$ intersecting transversely.

In Section 5 we describe the strata $\cap_{i \in I} D_i$ for $I \subset \{0, \dots, r-1\}$. In particular, we shall obtain a natural isomorphism

$$D_0 \cap \cdots \cap D_{r-1} \simeq \mathrm{SpFl} \times \mathrm{SpFl},$$

where SpFl is a symplectic flag variety parametrizing filtrations $V \supset \mathbb{F}_1(V) \supset \cdots \supset \mathbb{F}_r(V) \supset \mathbb{F}_{r+1}(V) = 0$ such that $\mathbb{F}_i(V)$ is isotropic of dimension $r+1-i$.

In Section 6 we study $Sp(V) \times Sp(V)$-modules $\mathrm{H}^0(KSp(V), \mathcal{O}(\sum a_i D_i))$. The argument here is the same as [Kausz2]. We shall prove, for example, that there is a natural isomorphism

$$\begin{aligned} &\mathrm{H}^0\left(KSp(V), \mathcal{O}(\sum_{i=0}^{r-1} n(r-i)D_i)\right) \\ &\simeq \bigoplus_{n \geq q_1 \geq \cdots \geq q_r \geq 0} \mathrm{H}^0\left(\mathrm{SpFl}, \otimes_{i=1}^r \left(\frac{\mathcal{F}^\perp_{r+2-i}}{\mathcal{F}^\perp_{r+1-i}}\right)^{\otimes q_i}\right) \otimes \mathrm{H}^0\left(\mathrm{SpFl}, \otimes_{i=1}^r \left(\frac{\mathcal{F}^\perp_{r+2-i}}{\mathcal{F}^\perp_{r+1-i}}\right)^{\otimes q_i}\right), \end{aligned}$$

where $V \otimes \mathcal{O}_{\mathrm{SpFl}} \supset \mathcal{F}_1 \supset \cdots \supset \mathcal{F}_r \supset \mathcal{F}_{r+1} = 0$ is the universal filtration.

In Section 7 we shall apply the results about $KSp(V)$ to the study of symplectic bundles on a curve. We shall prove the factorization theorem (Theorem 7.3) of generalized theta functions on the moduli stack of symplectic bundles.

The reason why we develop a symplectic analogue of the Kausz's results is that it has an application to the study of the strange duality for symplectic bundles. In Section 8 we prove a proposition which will be used in a forthcoming paper [A].

Notation and Convention. • We denote by J_2 the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

• For a $2r \times 2r$ matrix $A = (a_{ij})_{1 \leq i,j \leq 2r}$, we denote by $A_{[l,m]}$ the 2×2 minor

$$\begin{pmatrix} a_{2l-1,2m-1} & a_{2l-1,2m} \\ a_{2l,2m-1} & a_{2l,2m} \end{pmatrix}.$$

• The $2r \times 2r$ matrix J_{2r} is defined by

$$(J_{2r})_{[l,m]} = \begin{cases} J_2 & \text{if } l = m \\ \mathbf{O} & \text{if } l \neq m. \end{cases}$$

• For a commutative ring R we denote by $\mathrm{Sp}_{2r}(R)$ the subgroup

$$\left\{ X \in \mathrm{Mat}_{2r \times 2r}(R) \,\middle|\, {}^tX J_{2r} X = J_{2r} \right\}$$

of the group $\mathrm{Mat}_{2r \times 2r}(R)$ of $2r \times 2r$ matrices with entries in R.

• The subgroup $\mathrm{U}^+_{2r}(R)$ of $\mathrm{Sp}_{2r}(R)$ consists of such $X \in \mathrm{Sp}_{2r}(R)$ that $X_{[l,m]}$ is of the form $\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}$ if $l < m$, $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ if $l = m$, and $\begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}$ if $l > m$. The subgroup $\mathrm{U}^-_{2r}(R)$ of $\mathrm{Sp}_{2r}(R)$ is defined as $X \in \mathrm{Sp}_{2r}(R)$ is in $\mathrm{U}^-_{2r}(R)$ iff ${}^tX \in \mathrm{U}^+_{2r}(R)$.

• Let S be a scheme and $*$ be an object (such as a sheaf, a scheme, a morphism etc.) over S. For an S-scheme T, we denote by $(*)_T$ or $*_T$ the base-change of $*$ by $T \to S$.

• Let $f : \mathcal{E} \to \mathcal{F}$ be a morphism of sheaves on a scheme. If $\mathcal{L}$ is a line bundle, the morphism $\mathrm{id} \otimes f : \mathcal{L} \otimes \mathcal{E} \to \mathcal{L} \otimes \mathcal{F}$ is often denoted by f in this paper. When we make use of this abuse of notation, we shall make clear the source and the target of the morphism so that no confusion arises.

• For a product $X \times Y \times Z \times \ldots$, pr_X denotes the projection to X.

§ 2. Review on Kausz's generalized isomorphisms

Here we recall Kausz's result [Kausz1] on the compactification of the general linear group. Most part of this section is copied from [Kausz1].

Definition 2.1. Let $\mathcal{E}$ and $\mathcal{F}$ be locally free sheaves on a scheme S. A *bf-morphism from $\mathcal{E}$ to $\mathcal{F}$* is a tuple

$$g = \left(\mathcal{M}, \mu, \mathcal{E} \xrightarrow{g^\sharp} \mathcal{F}, \mathcal{M} \otimes \mathcal{E} \xleftarrow{g^\flat} \mathcal{F}, r\right),$$

where $\mathcal{M}$ is a line bundle on S, and μ is a global section of $\mathcal{M}$ such that the following holds:

1. The composed morphism $g^\sharp \circ g^\flat$ and $g^\flat \circ g^\sharp$ are both induced by the morphism $\mu : \mathcal{O}_S \to \mathcal{M}$.
2. For every point $x \in S$ with $\mu(x) = 0$, the complex

$$\mathcal{E}|_x \to \mathcal{F}|_x \to (\mathcal{M} \otimes \mathcal{E})|_x \to (\mathcal{M} \otimes \mathcal{F})|_x$$

is exact and the rank of the morphism $\mathcal{E}|_x \to \mathcal{F}|_x$ is r.

Definition 2.2. Let $\mathcal{E}$ and $\mathcal{F}$ be locally free sheaves of rank n on a scheme S. A *generalized isomorphism from $\mathcal{E}$ to $\mathcal{F}$* is a tuple

$$\begin{aligned}\Phi =&(\mathcal{L}_i, \lambda_i, \mathcal{M}_i, \mu_i, \mathcal{E}_i \to \mathcal{M}_i \otimes \mathcal{E}_{i+1}, \mathcal{E}_i \leftarrow \mathcal{E}_{i+1},\\ &\mathcal{F}_{i+1} \to \mathcal{F}_i, \mathcal{L}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \quad (0 \le i \le n-1), h : \mathcal{E}_n \xrightarrow{\sim} \mathcal{F}_n),\end{aligned}$$

where $\mathcal{E} = \mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_n, \mathcal{F}_n, \dots, \mathcal{F}_1, \mathcal{F}_0 = \mathcal{F}$ are locally free sheaves of rank n, and the tuples

$$(\mathcal{M}_i, \mu_i, \mathcal{E}_{i+1} \to \mathcal{E}_i, \mathcal{M}_i \otimes \mathcal{E}_{i+1} \leftarrow \mathcal{E}_i, i)$$
$$(\mathcal{L}_i, \lambda_i, \mathcal{F}_{i+1} \to \mathcal{F}_i, \mathcal{L}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i, i)$$

are bf-morphisms of rank i for $0 \le i \le n-1$, such that for each $x \in S$ the following holds:

1. If $\mu_i(x) = 0$ and (f, g) is one of the following two pairs of morphisms:

$$\mathcal{E}|_x \xrightarrow{f} \left(\left(\otimes_{j=0}^{i-1} \mathcal{M}_j\right) \otimes \mathcal{E}_i\right)|_x \xrightarrow{g} \left(\left(\otimes_{j=0}^{i} \mathcal{M}_j\right) \otimes \mathcal{E}_{i+1}\right)|_x,$$
$$\mathcal{E}_i|_x \xleftarrow{g} \mathcal{E}_{i+1}|_x \xleftarrow{f} \mathcal{E}_n|_x,$$

then $\mathrm{Im}(g \circ f) = \mathrm{Im} g$. Likewise, if $\lambda_i(x) = 0$ and (f, g) is one of the following two pairs of morphisms:

$$\mathcal{F}_n|_x \xrightarrow{f} \mathcal{F}_{i+1}|_x \xrightarrow{g} \mathcal{F}_i|_x,$$
$$\left(\left(\otimes_{j=0}^{i} \mathcal{L}_j\right) \otimes \mathcal{F}_{i+1}\right)|_x \xleftarrow{g} \left(\left(\otimes_{j=0}^{i-1} \mathcal{L}_j\right) \otimes \mathcal{F}_i\right)|_x \xleftarrow{f} \mathcal{F}|_x,$$

then $\mathrm{Im}(g \circ f) = \mathrm{Im} g$.

2. We have $(h|_x)\left(\mathrm{Ker}(\mathcal{E}_n|_x \to \mathcal{E}_0|_x)\right) \cap \mathrm{Ker}(\mathcal{F}_n|_x \to \mathcal{F}_0|_x) = \{0\}$.

Definition 2.3. A *quasi-equivalence* between two generalized isomorphisms

$$\begin{aligned}\Phi =&(\mathcal{L}_i, \lambda_i, \mathcal{M}_i, \mu_i, \mathcal{E}_i \to \mathcal{M}_i \otimes \mathcal{E}_{i+1}, \mathcal{E}_i \leftarrow \mathcal{E}_{i+1},\\ &\mathcal{F}_{i+1} \to \mathcal{F}_i, \mathcal{L}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \quad (0 \le i \le n-1), h : \mathcal{E}_n \xrightarrow{\sim} \mathcal{F}_n),\\ \Phi' =&(\mathcal{L}'_i, \lambda'_i, \mathcal{M}'_i, \mu'_i, \mathcal{E}'_i \to \mathcal{M}'_i \otimes \mathcal{E}'_{i+1}, \mathcal{E}'_i \leftarrow \mathcal{E}'_{i+1},\\ &\mathcal{F}'_{i+1} \to \mathcal{F}'_i, \mathcal{L}'_i \otimes \mathcal{F}'_{i+1} \leftarrow \mathcal{F}'_i \quad (0 \le i \le n-1), h' : \mathcal{E}'_n \xrightarrow{\sim} \mathcal{F}'_n)\end{aligned}$$

from $\mathcal{E}$ to $\mathcal{F}$ consists of isomorphisms $\mathcal{L}_i \simeq \mathcal{L}'_i$ and $\mathcal{M}_i \simeq \mathcal{M}'_i$ for $0 \le i \le n-1$, and isomorphisms $\mathcal{E}_i \simeq \mathcal{E}'_i$ and $\mathcal{F}_i \simeq \mathcal{F}'_i$ for $0 \le i \le n$, such that all the obvious diagrams are commutative. A quasi-equivalence between Φ and Φ' is called an equivalence if the isomorphisms $\mathcal{E}_0 \simeq \mathcal{E}'_0$ and $\mathcal{F}_0 \simeq \mathcal{F}'_0$ are in fact the identity on $\mathcal{E}$ and $\mathcal{F}$ respectively.

Remark 2.4. In [Kausz1, Page 579], Kausz proved that there is at most one equivalence between Φ an Φ'.

Let S be a scheme, $\mathcal{E}$ and $\mathcal{F}$ locally free sheaves on S. We denote by $\mathcal{KGL}(\mathcal{E}, \mathcal{F})$ the functor from the category of S-schemes to the category of sets that associates to an S-scheme T the set of equivalence classes of generalized isomorphisms from $\mathcal{E}_T$ to $\mathcal{F}_T$. Then [Kausz1, Theorem 5.5] says:

Theorem 2.5. *The functor $\mathcal{KGL}(\mathcal{E}, \mathcal{F})$ is represented by a scheme $KGL(\mathcal{E}, \mathcal{F})$ which is smooth and projective over S.*

Kausz also considered a compactification of PGL_n.

Definition 2.6. Let S be a scheme and $\mathcal{E}$, $\mathcal{F}$ locally free $\mathcal{O}_S$-modules of rank n. A *complete collineation* from $\mathcal{E}$ to $\mathcal{F}$ is a tuple

$$\Psi = (\mathcal{L}_i, \lambda_i; \mathcal{F}_{i+1} \to \mathcal{F}_i, \mathcal{L}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \quad (0 \le i \le n-1),)$$

where $\mathcal{E} = \mathcal{F}_n, \mathcal{F}_{n-1}, \dots, \mathcal{F}_1, \mathcal{F}_0 = \mathcal{F}$ are locally free $\mathcal{O}_S$-modules of rank n, the tuples

$$(\mathcal{L}_i, \lambda_i, \mathcal{F}_{i+1} \to \mathcal{F}_i, \mathcal{L}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i, i)$$

are bf-morphisms of rank i for $0 \le i \le n-1$ and $\lambda_0 = 0$, such that for each point $x \in S$ and index $i \in \{0, \dots, n-1\}$ with the property that $\lambda_i(x) = 0$, the following holds:
If (f, g) is one of the following two pairs of morphisms:

$$\begin{aligned}\mathcal{F}_n|_x \xrightarrow{f} \mathcal{F}_{i+1}|_x \xrightarrow{g} \mathcal{F}_i|_x,\\ \left(\left(\otimes_{j=0}^{i}\mathcal{L}_i\right) \otimes \mathcal{F}_{i+1}\right)|_x \xleftarrow{g} \left(\left(\otimes_{j=0}^{i-1}\mathcal{L}_i\right) \otimes \mathcal{F}_i\right)|_x \xleftarrow{f} \mathcal{F}_0|_x,\end{aligned}$$

then $\mathrm{Im}(g \circ f) = \mathrm{Im}(g)$.

Two complete collineations Ψ and Φ' from $\mathcal{E}$ to $\mathcal{F}$ are called *equivalent* if there are isomorphisms $\mathcal{L}_i \simeq \mathcal{L}'_i$, $\mathcal{F}_i \simeq \mathcal{F}'_i$ such that all the obvious diagrams commute and such that $\mathcal{F}_n \simeq \mathcal{F}'_n$ and $\mathcal{F}_0 \simeq \mathcal{F}'_0$ are the identity on $\mathcal{E}$ and $\mathcal{F}$ respectively.

Let S be a scheme, and $\mathcal{E}$, $\mathcal{F}$ locally free $\mathcal{O}_S$-modules of rank n. We denote by $\overline{\mathcal{PG}l}(\mathcal{E},\mathcal{F})$ the functor from the category of S-schemes to the category of sets that associates to an S-scheme T the set of equivalence classes of complete collineations from $\mathcal{E}_T$ to $\mathcal{F}_T$. Then [Kausz1, Corollary 8.2] says:

Theorem 2.7. *The functor $\overline{\mathcal{PG}l}(\mathcal{E},\mathcal{F})$ is represented by a scheme $\overline{PGl}(\mathcal{E},\mathcal{F})$ which is smooth and projective over S.*

In fact, $\overline{PGl}(\mathcal{E},\mathcal{F})$ is a closed subscheme of $KGL(\mathcal{E},\mathcal{F})$.

The following lemma is an easy consequence of [Kausz1, Lemma 6.1 and Proposition 6.2].

Lemma 2.8. *Let $\mathcal{A}$, $\mathcal{B}$ be vector bundles of rank m, and let*

$$(\mathcal{L},\lambda,\mathcal{A}\xrightarrow{g^\sharp}\mathcal{B},\mathcal{L}\otimes\mathcal{A}\xleftarrow{g^\flat}\mathcal{B},i)$$

be a bf-morphism of rank i.

(1) *There is a natural isomorphism*

$$\mathcal{L}^{\otimes(m-i)}\otimes\det\mathcal{A}\simeq\det\mathcal{B}.$$

(2) *If $\lambda=0$, then $\mathrm{Im}(\mathcal{A}\to\mathcal{B})=\mathrm{Ker}(\mathcal{B}\to\mathcal{L}\otimes\mathcal{A})$ and $\mathrm{Ker}(\mathcal{A}\to\mathcal{B})=\mathrm{Im}(\mathcal{L}^\vee\otimes\mathcal{B}\to\mathcal{A})$, and they are subbundles of rank i and of rank $m-i$ of $\mathcal{B}$ and $\mathcal{A}$ respectively.*

§ 3. generalized symplectic isomorphism

As a symplectic analogue of generalized isomorphisms, we first introduce generalized symplectic isomorphisms (Definition 3.1). Then we shall prove that the moduli space of generalized symplectic isomorphisms gives a compactification of the symplectic group.

Definition 3.1. Let S be a scheme, $\mathcal{E}$ and $\mathcal{F}$ locally free $\mathcal{O}_S$-modules of rank $2r$, $\mathcal{P}$ a line bundle on S, and $\pi_{\mathcal{E}}:\mathcal{E}\otimes\mathcal{E}\to\mathcal{P}$ and $\pi_{\mathcal{F}}:\mathcal{F}\otimes\mathcal{F}\to\mathcal{P}$ non-degenerate alternate bilinear forms.

A *generalized symplectic isomorphism from $\mathcal{E}$ to $\mathcal{F}$* is a tuple

$$\begin{aligned}\Phi=&(\mathcal{M}_i,\mu_i,\mathcal{E}_i\to\mathcal{M}_i\otimes\mathcal{E}_{i+1},\mathcal{E}_i\leftarrow\mathcal{E}_{i+1},\\ &\mathcal{F}_{i+1}\to\mathcal{F}_i,\mathcal{M}_i\otimes\mathcal{F}_{i+1}\leftarrow\mathcal{F}_i\quad(0\le i\le r-1),h:\mathcal{E}_r\xrightarrow{\sim}\mathcal{F}_r),\end{aligned}\tag{3.1}$$

where $\mathcal{E}=\mathcal{E}_0,\mathcal{E}_1,\dots,\mathcal{E}_r,\mathcal{F}_r,\dots,\mathcal{F}_1,\mathcal{F}_0=\mathcal{F}$ are locally free $\mathcal{O}_S$-modules of rank $2r$ and the tuples

$$(\mathcal{M}_i,\mu_i,\mathcal{E}_{i+1}\xrightarrow{e_i^\sharp}\mathcal{E}_i,\mathcal{M}_i\otimes\mathcal{E}_{i+1}\xleftarrow{e_i^\flat}\mathcal{E}_i,r+i)$$

$$\text{and }(\mathcal{M}_i,\mu_i,\mathcal{F}_{i+1}\xrightarrow{f_i^\sharp}\mathcal{F}_i,\mathcal{M}_i\otimes\mathcal{F}_{i+1}\xleftarrow{f_i^\flat}\mathcal{F}_i,r+i)$$

are bf-morphisms of rank $r+i$ for $0 \le i \le r-1$ such that for each $x \in S$ the following holds:

1. If $\mu_i(x) = 0$ and (f,g) is one of the following pairs of morphisms

$$\begin{aligned}
&\mathcal{E}_r|_x \xrightarrow{f} \mathcal{E}_{i+1}|_x \xrightarrow{g} \mathcal{E}_i|_x,\\
&\mathcal{E}|_x \xrightarrow{f} \left(\left(\otimes_{j=0}^{i-1}\mathcal{M}_j\right)\otimes\mathcal{E}_i\right)|_x, \xrightarrow{g} \left(\left(\otimes_{j=0}^{i}\mathcal{M}_j\right)\otimes\mathcal{E}_{i+1}\right)|_x,\\
&\mathcal{F}_r|_x \xrightarrow{f} \mathcal{F}_{i+1}|_x \xrightarrow{g} \mathcal{F}_i|_x,\\
&\mathcal{F}|_x \xrightarrow{f} \left(\left(\otimes_{j=0}^{i-1}\mathcal{M}_j\right)\otimes\mathcal{F}_i\right)|_x \xrightarrow{g} \left(\left(\otimes_{j=0}^{i}\mathcal{M}_j\right)\otimes\mathcal{F}_{i+1}\right)|_x,
\end{aligned}$$

then $\mathrm{Im}(g\circ f) = \mathrm{Im}(g)$.

2. $(h|_x)\left(\mathrm{Ker}\left(\mathcal{E}_r|_x \to \mathcal{E}_0|_x\right)\right) \cap \mathrm{Ker}\left(\mathcal{F}_r|_x \to \mathcal{F}_0|_x\right) = \{0\}$.
3. The following diagram is commutative:

$$\begin{array}{ccc}
 & \left\{\left(\otimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{E}_0\right)\times_{\mathcal{E}_k}\mathcal{E}_r\right\}\otimes\left\{\left(\otimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{F}_0\right)\times_{\mathcal{F}_k}\mathcal{F}_r\right\} & \\
\alpha\swarrow & & \searrow\beta \\
\left(\otimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{E}_0\right)\otimes\mathcal{E}_0 & & \mathcal{F}_0\otimes\left(\otimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{F}_0\right) \\
\gamma\searrow & & \swarrow\delta \\
 & \left(\otimes_{j=0}^{k-1}\mathcal{M}_j^\vee\right)\otimes\mathcal{P}, &
\end{array} \tag{3.2}$$

where γ and δ are induced by $\pi_\mathcal{E}$ and $\pi_\mathcal{F}$ respectively, and

$$\begin{aligned}
\alpha &= q_k^\mathcal{E}\otimes\left(e_0^\sharp\circ\cdots\circ e_{r-1}^\sharp\circ h^{-1}\circ p_k^\mathcal{F}\right)\\
\beta &= \left(f_0^\sharp\circ\cdots\circ f_{r-1}^\sharp\circ h\circ p_k^\mathcal{E}\right)\otimes q_k^\mathcal{F},
\end{aligned}$$

where $p_k^\mathcal{E}$, $q_k^\mathcal{E}$, $p_k^\mathcal{F}$ and $q_k^\mathcal{F}$ are defined by

$$\begin{array}{ccc}
\left(\bigotimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{E}_0\right)\times_{\mathcal{E}_k}\mathcal{E}_r & \xrightarrow{p_k^\mathcal{E}} & \mathcal{E}_r \\
q_k^\mathcal{E}\downarrow & \square & \downarrow e_k^\sharp\circ\cdots\circ e_{r-1}^\sharp \\
\bigotimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{E}_0 & \xrightarrow[e_{k-1}^\flat\circ\cdots\circ e_0^\flat]{} & \mathcal{E}_k
\end{array} \tag{3.3}$$

and

$$\begin{array}{ccc}
\left(\bigotimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{F}_0\right)\times_{\mathcal{F}_k}\mathcal{F}_r & \xrightarrow{p_k^\mathcal{F}} & \mathcal{F}_r \\
q_k^\mathcal{F}\downarrow & \square & \downarrow f_k^\sharp\circ\cdots\circ f_{r-1}^\sharp \\
\bigotimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{F}_0 & \xrightarrow[f_{k-1}^\flat\circ\cdots\circ f_0^\flat]{} & \mathcal{F}_k.
\end{array} \tag{3.4}$$

3.2. We can consider the composition of a generalized symplectic morphism with symplectic isomorphisms as follows. Let $\alpha:\mathcal{E}\to\mathcal{E}$ and $\beta:\mathcal{F}\to\mathcal{F}$ be symplectic isomorphisms. Replacing the morphisms $\mathcal{E}_1 \xrightarrow{e_0^\sharp} \mathcal{E}_0$, $\mathcal{M}_0 \otimes \mathcal{E}_1 \xleftarrow{e_0^\flat} \mathcal{E}_0$, $\mathcal{M}_0 \otimes \mathcal{F}_1 \xleftarrow{f_0^\flat} \mathcal{F}_0$ and $\mathcal{F}_1 \xrightarrow{f_0^\sharp} \mathcal{F}_0$ with $\mathcal{E}_1 \xrightarrow{\alpha \circ e_0^\sharp} \mathcal{E}_0$, $\mathcal{M}_0 \otimes \mathcal{E}_1 \xleftarrow{e_0^\flat \circ \alpha^{-1}} \mathcal{E}_0$, $\mathcal{M}_0 \otimes \mathcal{F}_1 \xleftarrow{f_0^\flat \circ \beta^{-1}} \mathcal{F}_0$ and $\mathcal{F}_1 \xrightarrow{\beta \circ f_0^\sharp} \mathcal{F}_0$ respectively, we obtain another generalized symplectic isomorphism from $\mathcal{E}$ to $\mathcal{F}$, which we denote by $\beta \circ \Phi \circ \alpha^{-1}$.

Definition 3.3. Let S be a scheme, $\mathcal{E}$ and $\mathcal{F}$ rank $2r$ locally free $\mathcal{O}_S$-modules, $\mathcal{P}$ a line bundle on S, $\pi_{\mathcal{E}} : \mathcal{E} \otimes \mathcal{E} \to \mathcal{P}$ and $\pi_{\mathcal{F}} : \mathcal{F} \otimes \mathcal{F} \to \mathcal{P}$ non-degenerate alternate bilinear forms.

A *quasi-equivalence* between two generalized symplectic isomorphisms

$$\begin{aligned} \Phi =&(\mathcal{M}_i, \mu_i, \mathcal{E}_i \to \mathcal{M}_i \otimes \mathcal{E}_{i+1}, \mathcal{E}_i \leftarrow \mathcal{E}_{i+1}, \\ &\mathcal{F}_{i+1} \to \mathcal{F}_i, \mathcal{M}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \quad (0 \le i \le r-1), h : \mathcal{E}_r \to \mathcal{F}_r) \\ \Phi' =&(\mathcal{M}'_i, \mu'_i, \mathcal{E}'_i \to \mathcal{M}'_i \otimes \mathcal{E}'_{i+1}, \mathcal{E}'_i \leftarrow \mathcal{E}'_{i+1}, \\ &\mathcal{F}'_{i+1} \to \mathcal{F}'_i, \mathcal{M}'_i \otimes \mathcal{F}'_{i+1} \leftarrow \mathcal{F}'_i \quad (0 \le i \le r-1), h' : \mathcal{E}'_r \to \mathcal{F}'_r) \end{aligned}$$

from $\mathcal{E}$ to $\mathcal{F}$ consists of isomorphisms $\mathcal{M}_i \simeq \mathcal{M}'_i$ $(0 \le i \le r-1)$ by which μ_i maps to μ'_i, and isomorphisms $\mathcal{E}_i \simeq \mathcal{E}'_i$ and $\mathcal{F}_i \simeq \mathcal{F}'_i$ $(0 \le i \le r)$ such that $\mathcal{E}_0 \simeq \mathcal{E}'_0$ and $\mathcal{F}_0 \simeq \mathcal{F}'_0$ are symplectic and the obvious diagrams are commutative.

A quasi-equivalence between Φ and Φ' is called an equivalence if the isomorphisms $\mathcal{E}_0 \simeq \mathcal{E}'_0$ and $\mathcal{F}_0 \simeq \mathcal{F}'_0$ are in fact the identity on $\mathcal{E}$ and $\mathcal{F}$ respectively.

Definition 3.4. Let S be a scheme. Let $\mathcal{E} = \mathcal{F} = \mathcal{O}_S^{\oplus 2r}$ be given the non-degenerate alternate bilinear form by the matrix J_{2r}. To a tuple $(m_0, \dots, m_{r-1})$ of regular functions on S, we associate the following generalized symplectic isomorphisms from $\mathcal{E}$ to $\mathcal{F}$:

$$\begin{aligned} \Phi(m_0, \dots, m_{r-1}) :=&(\mathcal{M}_i, \mu_i, \mathcal{E}_i \to \mathcal{M}_i \otimes \mathcal{E}_{i+1}, \mathcal{E}_i \leftarrow \mathcal{E}_{i+1}, \\ &\mathcal{F}_{i+1} \to \mathcal{F}_i, \mathcal{M}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i, h : \mathcal{E}_r \xrightarrow{\sim} \mathcal{F}_r), \end{aligned} \tag{3.5}$$

where $\mathcal{M}_i = \mathcal{O}_S$, $\mu_i = m_i$ for $0 \le i \le r-1$, and $\mathcal{E}_i = \mathcal{F}_i = \mathcal{O}_S^{\oplus 2r}$ for $0 \le i \le r$; the morphisms $\mathcal{E}_i \to \mathcal{M}_i \otimes \mathcal{E}_{i+1}$ and $\mathcal{E}_i \leftarrow \mathcal{E}_{i+1}$ (both are from $\mathcal{O}_S^{\oplus 2r}$ to $\mathcal{O}_S^{\oplus 2r}$) are described by the $2r \times 2r$ diagonal matrices

$$\mathrm{diag}(1, m_i, 1, m_i, \dots, 1, m_i, \overbrace{m_i, \dots, m_i}^{2i \text{ times}}) \tag{3.6}$$

and

$$\mathrm{diag}(m_i, 1, m_i, 1, \dots, m_i, 1, \overbrace{1, \dots, 1}^{2i \text{ times}}) \tag{3.7}$$

respectively; the morphisms $\mathcal{F}_{i+1} \to \mathcal{F}_i$ and $\mathcal{M}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i$ by the matrices

$$\mathrm{diag}(1, m_i, 1, m_i, \dots, 1, m_i, \overbrace{1, \dots, 1}^{2i \text{ times}}) \tag{3.8}$$

and

$$\mathrm{diag}(m_i, 1, m_i, 1, \dots, m_i, 1, \overbrace{m_i, \dots, m_i}^{2i \text{ times}}) \tag{3.9}$$

respectively; and the isomorphism $h : \mathcal{E}_r \to \mathcal{F}_r$ is the identity.

Notation 3.5. We define the subgroup W_{2r} of $\mathrm{Mat}_{2r\times 2r}$ as follows. A matrix $A \in \mathrm{Mat}_{2r\times 2r}$ is in W_{2r} iff there exists a $\sigma \in \mathfrak{S}_r$ such that $A_{[i,j]} = \mathbf{O}$ if $i \neq \sigma(j)$, and

$$A_{[\sigma(j),j]} \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Definition 3.6. Let S, $\mathcal{E}$ and $\mathcal{F}$ as in Definition 3.4. Let

$$\begin{aligned} \Phi =& (\mathcal{M}_i, \mu_i, \mathcal{E}_i \to \mathcal{M}_i \otimes \mathcal{E}_{i+1}, \mathcal{E}_i \leftarrow \mathcal{E}_{i+1}, \\ & \mathcal{F}_{i+1} \to \mathcal{F}_i, \mathcal{M}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \quad (0 \le i \le r-1), h : \mathcal{E}_r \xrightarrow{\sim} \mathcal{F}_r) \end{aligned}$$

be a generalized symplectic isomorphism from $\mathcal{E}$ to $\mathcal{F}$. A *diagonalization of* Φ *with respect to* $(\alpha, \beta) \in W_{2r} \times W_{2r}$ is a tuple $(u_i, v_i\,(0 \le i \le r); \psi_i\,(0 \le i \le r-1))$ of isomorphisms, where $u_i : \mathcal{O}_S^{\oplus 2r} \xrightarrow{\sim} \mathcal{E}_i$, $v_i : \mathcal{O}_S^{\oplus 2r} \xrightarrow{\sim} \mathcal{F}_i$ and $\psi_i : \mathcal{O}_S \xrightarrow{\sim} \mathcal{M}_i$ such that $(u_i, v_i\,(0 \le i \le r); \psi_i\,(0 \le i \le r-1))$ establishes a quasi-equivalence between $\Phi(\psi_0^{-1}(\mu_0), \dots, \psi_{r-1}^{-1}(\mu_{r-1}))$ and Φ such that $\alpha^{-1} \circ u_0 : \mathcal{O}^{\oplus 2r} \to \mathcal{O}^{\oplus 2r} = \mathcal{E}$ is in $\mathrm{U}_{2r}^{+}(\mathcal{O}_S)$ and $\beta^{-1} \circ v_0 : \mathcal{O}^{\oplus 2r} \to \mathcal{O}^{\oplus 2r} = \mathcal{F}$ is in $\mathrm{U}_{2r}^{-}(\mathcal{O}_S)$.

Remark 3.7. Clearly Φ has a diagonalization with respect to $(\alpha, \beta) \in W_{2r} \times W_{2r}$ if and only if $\beta^{-1} \circ \Phi \circ \alpha$ has a diagonalization with respect to $(\mathrm{id}, \mathrm{id}) \in W_{2r} \times W_{2r}$.

Proposition 3.8. *Let S be a scheme and let $\mathcal{E} = \mathcal{F} = \mathcal{O}_S^{\oplus 2r}$ be given the non-degenerate alternate bilinear forms by the matrix J_{2r}. Let*

$$\begin{aligned} \Phi =& (\mathcal{M}_i, \mu_i, \mathcal{E}_i \to \mathcal{M}_i \otimes \mathcal{E}_{i+1}, \mathcal{E}_i \leftarrow \mathcal{E}_{i+1}, \\ & \mathcal{F}_{i+1} \to \mathcal{F}_i, \mathcal{M}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \quad (0 \le i \le r-1), h : \mathcal{E}_r \xrightarrow{\sim} \mathcal{F}_r), \end{aligned} \tag{3.10}$$

be a generalized symplectic isomorphism from $\mathcal{E}$ to $\mathcal{F}$.

(1) *For every point $s \in S$, there exists an open neighborhood U of s such that $\Phi|_U$ has a diagonalization with respect to some $(\alpha, \beta) \in W_{2r} \times W_{2r}$.*

(2) *Assume moreover that $S = \mathrm{Spec} K$ with K the quotient field of a valuation ring R. Then the above diagonalization is chosen such that $\alpha^{-1} \circ u_0 \in U_{2r}^{+}(R)$, $\beta^{-1} \circ v_0 \in U_{2r}^{-}(R)$ and $\psi_i^{-1}(\mu_i) \in R$.*

Proof. (1) We proceed by induction on r. Let $\mathbf{e}_1, \ldots, \mathbf{e}_{2r}$ be the standard basis of $\mathcal{E} = \mathcal{O}_S^{\oplus 2r}$, and $\mathbf{f}_1, \ldots, \mathbf{f}_{2r}$ that of $\mathcal{F} = \mathcal{O}_S^{\oplus 2r}$.

By the conditions 1 and 2 of Definition 3.1,

$$g := f_0^\sharp \circ \cdots \circ f_{r-1}^\sharp \circ h \circ e_{r-1}^\flat \circ \cdots \circ e_0^\flat : \mathcal{E}_0 \to \left(\otimes_{j=0}^{r-1} \mathcal{M}_j\right) \otimes \mathcal{F}_0$$

is nonzero at every point of S. We can find $(\alpha', \beta') \in W_{2r} \times W_{2r}$ such that

$$\sigma := (\beta'^{-1} \circ g \circ \alpha'(\mathbf{e}_1), \mathbf{f}_2) \in \otimes_{j=0}^{r-1} \mathcal{M}_j \tag{3.11}$$

is nowhere vanishing in a neighborhood of s. Replacing S by this neighborhood, we may assume that σ is nowhere vanishing on S. Then the composite of morphisms

$$\mathcal{O}\mathbf{e}_1 \subset \mathcal{O}^{\oplus 2r} \xrightarrow{\alpha'} \mathcal{O}^{\oplus 2r} = \mathcal{E} \xrightarrow{e_{l-1}^\flat \circ \cdots \circ e_0^\flat} \left(\otimes_{j=0}^{l-1} \mathcal{M}_j\right) \otimes \mathcal{E}_l$$

induces a line subbundle $\otimes_{j=0}^{l-1} \mathcal{M}_j^\vee \hookrightarrow \mathcal{E}_l$. By the condition 3 of Definition 3.1, we have $(\mathbf{e}_1, \alpha'^{-1} \circ g' \circ \beta'(\mathbf{f}_2)) = \sigma$, where

$$g' := e_0^\sharp \circ \cdots \circ e_{r-1}^\sharp \circ h^{-1} \circ f_{r-1}^\flat \circ \cdots \circ f_0^\flat : \mathcal{F}_0 \to \left(\otimes_{j=0}^{r-1} \mathcal{M}_j\right) \otimes \mathcal{E}_0.$$

Thus the composite of morphisms

$$\mathcal{O}\mathbf{f}_2 \subset \mathcal{O}^{\oplus 2r} \xrightarrow{\beta'} \mathcal{O}^{\oplus 2r} = \mathcal{F} \xrightarrow{f_{l-1}^\flat \circ \cdots \circ f_0^\flat} \left(\otimes_{j=0}^{l-1} \mathcal{M}_j\right) \otimes \mathcal{F}_l$$

also induces a line subbundle $\otimes_{j=0}^{l-1} \mathcal{M}_j^\vee \hookrightarrow \mathcal{F}_l$

For $0 \leq l \leq r$, we put

$$\begin{aligned} \mathcal{F}_l \ni \mathbf{f}_{1,l} &:= \frac{1}{\sigma} f_l^\sharp \circ \cdots \circ f_{r-1}^\sharp \circ h \circ e_{r-1}^\flat \circ \cdots \circ e_0^\flat \circ \alpha'(\mathbf{e}_1) \\ \mathcal{E}_l \ni \mathbf{e}_{2,l} &:= \frac{1}{\sigma} e_l^\sharp \circ \cdots \circ e_{r-1}^\sharp \circ h^{-1} \circ f_{r-1}^\flat \circ \cdots \circ f_0^\flat \circ \beta'(\mathbf{f}_2). \end{aligned} \tag{3.12}$$

Then you can check that $\mathcal{E}_l \supset \left(\otimes_{j=0}^{l-1} \mathcal{M}_j^\vee\right) \oplus \mathcal{O}\mathbf{e}_{2,l}$ and $\mathcal{F}_l \supset \mathcal{O}\mathbf{f}_{1,l} \oplus \left(\otimes_{j=0}^{l-1} \mathcal{M}_j^\vee\right)$ are subbundles.

Let $\gamma : \mathcal{F} = \mathcal{O}^{\oplus 2r} \to \mathcal{O}$ be given by $x \mapsto (x, \beta'(\mathbf{f}_2))$, and $\delta : \mathcal{E} = \mathcal{O}^{\oplus 2r} \to \mathcal{O}$ by $y \mapsto (\alpha'(\mathbf{e}_1), y)$. Put

(3.13)

$$\begin{aligned} \mathcal{E}_l \supset \overline{\mathcal{E}_l} &:= \mathrm{Ker}(\gamma \circ f_0^\sharp \circ \cdots \circ f_{r-1}^\sharp \circ h \circ e_{r-1}^\flat \circ \cdots \circ e_l^\flat) \cap \mathrm{Ker}(\delta \circ e_0^\sharp \circ \cdots \circ e_{l-1}^\sharp) \\ \mathcal{F}_l \supset \overline{\mathcal{F}_l} &:= \mathrm{Ker}(\gamma \circ e_0^\sharp \circ \cdots \circ e_{r-1}^\sharp \circ h^{-1} \circ f_{r-1}^\flat \circ \cdots \circ f_l^\flat) \cap \mathrm{Ker}(\gamma \circ f_0^\sharp \circ \cdots \circ f_{l-1}^\sharp). \end{aligned}$$

Then $\overline{\mathcal{E}_l}$ and $\overline{\mathcal{F}_l}$ are vector subbundles of $\mathcal{E}_l$ and $\mathcal{F}_l$ respectively, and we have the direct sum decompositions

$$\mathcal{E}_l = \left(\otimes_{j=0}^{l-1} \mathcal{M}_j^\vee\right) \oplus \mathcal{O}\mathbf{e}_{2,l} \oplus \overline{\mathcal{E}_l}, \quad \mathcal{F}_l = \mathcal{O}\mathbf{f}_{1,l} \oplus \left(\otimes_{j=0}^{l-1} \mathcal{M}_j^\vee\right) \oplus \overline{\mathcal{F}_l} \tag{3.14}$$

for $0 \leq l \leq r$. Moreover the rank $r+l$ bf-morphism

$$(\mathcal{M}_l, \mu_l, \mathcal{E}_{l+1} \to \mathcal{E}_l, \mathcal{M}_l \otimes \mathcal{E}_{l+1} \leftarrow \mathcal{E}_l, r+l)$$

is a direct sum of the bf-morphisms

$$\Big(\mathcal{M}_l, \mu_l, (\otimes_{j=0}^{l} \mathcal{M}_j^\vee) \oplus \mathcal{O}\mathbf{e}_{2,l+1} \to (\otimes_{j=0}^{l-1} \mathcal{M}_j^\vee) \oplus \mathcal{O}\mathbf{e}_{2,l}),$$
$$(\otimes_{j=0}^{l-1} \mathcal{M}_j^\vee) \oplus \mathcal{M}_l\mathbf{e}_{2,l+1} \leftarrow (\otimes_{j=0}^{l-1} \mathcal{M}_j^\vee) \oplus \mathcal{O}\mathbf{e}_{2,l}, 1\Big)$$

and

$$\big(\mathcal{M}_l, \mu_l, \overline{\mathcal{E}}_{l+1} \to \overline{\mathcal{E}_l}, \mathcal{M}_l \otimes \overline{\mathcal{E}}_{l+1} \leftarrow \overline{\mathcal{E}_l}, r+l-1\big)\,.$$

Likewise $(\mathcal{M}_l, \mu_l, \mathcal{F}_{l+1} \to \mathcal{F}_l, \mathcal{M}_l \otimes \mathcal{F}_{l+1} \leftarrow \mathcal{F}_l, r+l)$ is a direct sum of the bf-morphisms

$$\Big(\mathcal{M}_l, \mu_l, \mathcal{O}\mathbf{f}_{1,l+1} \oplus (\otimes_{j=0}^{k} \mathcal{M}_j^\vee) \to \mathcal{O}\mathbf{f}_{1,l} \oplus (\otimes_{j=0}^{l-1} \mathcal{M}_j^\vee),$$
$$\mathcal{M}_l\mathbf{f}_{1,l+1} \oplus (\otimes_{j=0}^{l-1} \mathcal{M}_j^\vee) \leftarrow \mathcal{O}\mathbf{f}_{1,l} \oplus (\otimes_{j=0}^{l-1} \mathcal{M}_j^\vee), 1\Big)$$

and

$$\big(\mathcal{M}_l, \mu_l, \overline{\mathcal{F}}_{l+1} \to \overline{\mathcal{F}_l}, \mathcal{M}_l \otimes \overline{\mathcal{F}}_{l+1} \leftarrow \overline{\mathcal{F}_l}, r+l-1\big)\,.$$

Note that $\overline{\mathcal{E}_r} \xrightarrow{e^\sharp_{r-1}} \overline{\mathcal{E}}_{r-1}$ and $\overline{\mathcal{F}_r} \xrightarrow{f^\sharp_{r-1}} \overline{\mathcal{F}}_{r-1}$ are isomorphisms. Let $\overline{h}$ be the composed isomorphism $f^\sharp_{r-1} \circ h \circ e^{\sharp-1}_{r-1} : \overline{\mathcal{E}}_{r-1} \to \overline{\mathcal{F}}_{r-1}$. Then the bf-morphisms

$$\begin{aligned} &\big(\mathcal{M}_i, \mu_i, \overline{\mathcal{E}}_{i+1} \to \overline{\mathcal{E}}_i, \overline{\mathcal{E}}_i \to \mathcal{M}_i \otimes \overline{\mathcal{E}}_{i+1}, r-1-i\big) \\ &\big(\mathcal{M}_i, \mu_i, \overline{\mathcal{F}}_{i+1} \to \overline{\mathcal{F}}_i, \overline{\mathcal{F}}_i \to \mathcal{M}_i \otimes \overline{\mathcal{F}}_{i+1}, r-1-i\big) \end{aligned} \tag{3.15}$$

$(0 \leq i \leq r-2)$, and the isomorphism $\overline{h} : \overline{\mathcal{E}}_{r-1} \to \overline{\mathcal{F}}_{r-1}$ give an generalized symplectic isomorphism $\overline{\Phi}$ from $\overline{\mathcal{E}}_0$ to $\overline{\mathcal{F}}_0$.

Since $(\beta'^{-1}(\mathbf{f}_{1,0}), \mathbf{f}_2) = 1$, we have $\beta'^{-1}(\mathbf{f}_{1,0}) =^t (1, c_2, \ldots, c_{2r-1}, c_{2r})$. Similarly we have $\alpha'^{-1}(\mathbf{e}_{2,0}) =^t (d_1, 1, d_3, \ldots, d_{2r})$.

Let $\theta'_{\mathcal{E}}$ and $\theta'_{\mathcal{F}}$ be the isomorphisms $\mathcal{O}_S^{\oplus 2r} \to \mathcal{O}_S^{\oplus 2r}$ defined by the matrices

$$\left(\begin{array}{cc|cc|c|cc} 1 & d_1 & -d_4 & d_3 & \ldots & -d_{2r} & d_{2r-1} \\ & 1 & & & & & \\ \hline & d_3 & 1 & & & & \\ & d_4 & & 1 & & & \\ \hline & \vdots & & & \ddots & & \\ \hline & d_{2r-1} & & & & 1 & \\ & d_{2r} & & & & & 1 \end{array}\right) \tag{3.16}$$

and

$$
\left(\begin{array}{cc|cc|c|cc}
1 & & & & & & \\
c_2 & 1 & c_4 & -c_3 & \dots & c_{2r} & -c_{2r-1} \\
\hline
c_3 & & 1 & & & & \\
c_4 & & & 1 & & & \\
\hline
\vdots & & & & \ddots & & \\
\hline
c_{2r-1} & & & & & 1 & \\
c_{2r} & & & & & & 1
\end{array}\right) \tag{3.17}
$$

respectively, where no entry is understood to be zero.

Restricting the symplectic isomorphisms $\alpha' \circ \theta'_{\mathcal{E}} : \mathcal{O}^{\oplus 2r} \to \mathcal{O}^{\oplus 2r} = \mathcal{E}_0$ and $\beta' \circ \theta'_{\mathcal{F}} : \mathcal{O}^{\oplus 2r} \to \mathcal{O}^{\oplus 2r} = \mathcal{F}_0$ to the last $(2r-2)$ direct summands $\mathcal{O}^{\oplus 2r-2} \subset \mathcal{O}^{\oplus 2r}$, we have symplectic isomorphisms $\mathcal{O}^{\oplus 2r-2} \simeq \overline{\mathcal{E}}_0$ and $\mathcal{O}^{\oplus 2r-2} \simeq \overline{\mathcal{F}}_0$. We regard $\overline{\mathcal{E}}_0$ and $\overline{\mathcal{F}}_0$ as equal to $\mathcal{O}^{\oplus 2r-2}$ by these isomorphisms. By induction hypothesis, the generalized isomorphism $\overline{\Phi}$ has a diagonalization with respect to $(\overline{\alpha}, \overline{\beta}) \in W_{2r-2} \times W_{2r-2}$ in a neighborhood of s. Replacing S by this neighborhood, we may assume that $\overline{\Phi}$ has a diagonalization with respect to $(\overline{\alpha}, \overline{\beta}) \in W_{2r-2} \times W_{2r-2}$ on S. So we have isomorphisms

$$
\begin{aligned}
\psi_i : \mathcal{O}_S \xrightarrow{\sim} \mathcal{M}_i \qquad (0 \le i \le r-2),& \\
\overline{u}_0 : \mathcal{O}^{\oplus 2r-2} \to \mathcal{O}^{\oplus 2r-2} \simeq \overline{\mathcal{E}}_0, \qquad \overline{v}_0 : \mathcal{O}^{\oplus 2r-2} \to \mathcal{O}^{\oplus 2r-2} \simeq \overline{\mathcal{F}}_0,& \\
\overline{u}_l : \mathcal{O}^{\oplus 2r-2} \to \overline{\mathcal{E}}_l, \qquad \overline{v}_l : \mathcal{O}^{\oplus 2r-2} \to \overline{\mathcal{F}}_l \quad (1 \le l \le r-1)&
\end{aligned} \tag{3.18}
$$

such that $\overline{\alpha}^{-1} \circ \overline{u}_0 \in \mathrm{U}^+_{2r-2}(\mathcal{O}_S)$ and $\overline{\beta}^{-1} \circ \overline{v}_0 \in \mathrm{U}^-_{2r-2}(\mathcal{O}_S)$. Since $\sigma \in \otimes_{j=0}^{r-1} \mathcal{M}_j$ is nowhere vanishing, there is a unique isomorphism $\psi_{r-1} : \mathcal{O}_S \to \mathcal{M}_{r-1}$ such that $\left(\otimes_{j=0}^{r-1} \psi_j\right)(1) = \sigma$.

For $1 \le l \le r-1$, let

$$
\begin{aligned}
u_l &:= \left(\left(\otimes_{j=0}^{l-1} \psi_j^\vee\right) \oplus \mathrm{id}\right) \oplus \overline{u}_l : \mathcal{O}^{\oplus 2} \oplus \mathcal{O}^{\oplus 2r-2} \to \left(\otimes_{j=0}^{l-1} \mathcal{M}_j^\vee \oplus \mathcal{O}\right) \oplus \overline{\mathcal{E}}_l = \mathcal{E}_l \\
v_l &:= \left(\mathrm{id} \oplus \left(\otimes_{j=0}^{l-1} \psi_j^\vee\right)\right) \oplus \overline{v}_l : \mathcal{O}^{\oplus 2} \oplus \mathcal{O}^{\oplus 2r-2} \to \left(\mathcal{O} \oplus \otimes_{j=0}^{l-1} \mathcal{M}_j^\vee\right) \oplus \overline{\mathcal{F}}_l = \mathcal{F}_l
\end{aligned} \tag{3.19}
$$

and let

$$
\begin{aligned}
u_r &:= \left(\left(\otimes_{j=0}^{r-1} \psi_j^\vee\right) \oplus \mathrm{id}\right) \oplus \left((e^\sharp_{r-1})^{-1} \circ \overline{u}_{r-1}\right) : \mathcal{O}^{\oplus 2} \oplus \mathcal{O}^{\oplus 2r-2} \\
&\qquad\qquad \to \left(\otimes_{j=0}^{r-1} \mathcal{M}_j^\vee \oplus \mathcal{O}\right) \oplus \overline{\mathcal{E}}_r = \mathcal{E}_r \\
v_r &:= \left(\mathrm{id} \oplus \left(\otimes_{j=0}^{r-1} \psi_j^\vee\right)\right) \oplus \left((f^\sharp_{r-1})^{-1} \circ \overline{v}_{r-1}\right) : \mathcal{O}^{\oplus 2} \oplus \mathcal{O}^{\oplus 2r-2} \\
&\qquad\qquad \to \left(\mathcal{O} \oplus \otimes_{j=0}^{r-1} \mathcal{M}_j^\vee\right) \oplus \overline{\mathcal{F}}_r = \mathcal{F}_r.
\end{aligned} \tag{3.20}
$$

Let $u_0 : \mathcal{O}^{\oplus 2} \oplus \mathcal{O}^{\oplus 2r-2} = \mathcal{O}^{\oplus 2r} \to \mathcal{O}^{\oplus 2r} = \mathcal{E}_0$ be the morphism $\alpha' \circ \theta'_{\mathcal{E}} \circ (\mathrm{id} \oplus \overline{u}_0)$ and let $v_0 : \mathcal{O}^{\oplus 2} \oplus \mathcal{O}^{\oplus 2r-2} = \mathcal{O}^{\oplus 2r} \to \mathcal{O}^{\oplus 2r} = \mathcal{F}_0$ be the morphism $\beta' \circ \theta'_{\mathcal{F}} \circ (\mathrm{id} \oplus \overline{v}_0)$. We have

$$\alpha' \circ \theta'_{\mathcal{E}} \circ (\mathrm{id} \oplus \overline{u}_0) = \alpha' \circ (\mathrm{id} \oplus \overline{\alpha}) \circ \left\{ (\mathrm{id} \oplus \overline{\alpha})^{-1} \circ \theta'_{\mathcal{E}} \circ (\mathrm{id} \oplus \overline{\alpha}) \right\} \circ (\mathrm{id} \oplus (\overline{\alpha}^{-1} \circ \overline{u}_0)),$$

and we have $\alpha := \alpha' \circ (\mathrm{id} \oplus \overline{\alpha}) \in W_{2r}$ and $\left\{ (\mathrm{id} \oplus \overline{\alpha})^{-1} \circ \theta'_{\mathcal{E}} \circ (\mathrm{id} \oplus \overline{\alpha}) \right\} \circ (\mathrm{id} \oplus (\overline{\alpha}^{-1} \circ \overline{u}_0)) \in \mathrm{U}^+_{2r}(\mathcal{O}_S)$. Similarly, if we put $\beta := \beta' \circ (\mathrm{id} \oplus \overline{\beta})$, then $\beta^{-1} \circ v_0 \in \mathrm{U}^-_{2r}(\mathcal{O}_S)$. Therefore these data give a diagonalization of Φ with respect to $(\alpha, \beta) \in W_{2r} \times W_{2r}$.

(2) Again we proceed by induction on r. We follow closely the argument in (1) and use the same notation. Let $v : K \setminus \{0\} \to \Gamma$ be the valuation, where Γ is the valuation group of R. (By convention $v(0) = +\infty$.) When V is a one-dimensional K-vector space, we denote $v(x) \leq v(y)$ for $x, y \in V$ if for one (and all) K -linear isomorphism $\iota : V \to K$, we have $v(\iota(x)) \leq v(\iota(y))$.

In the proof of (1), we can choose $(\alpha', \beta') \in W_{2r} \times W_{2r}$ such that

$$v\left((\beta'^{-1} \circ g \circ \alpha'(\mathbf{e}_1), \mathbf{f}_2) \right) \leq v\left((\beta'^{-1} \circ g \circ \alpha'(\mathbf{e}_i), \mathbf{f}_j) \right) \tag{3.21}$$

for $1 \leq i, j \leq 2r$. Then for any $x \in R^{2r} \subset K^{2r} = \mathcal{E}$ and $y \in R^{2r} \subset K^{2r} = \mathcal{F}$, we have $(g(x), y)/\sigma \in R$. Therefore we have $d_i, c_j \in R$ in (3.16) and (3.17). By induction hypothesis, we can choose the diagonalization (3.18) of $\overline{\Phi}$ in (1) such that $\psi_i^{-1}(\mu_i) \in R$ $(0 \leq i \leq r-2)$ and $\overline{\alpha}^{-1} \circ \overline{u}_0 \in \mathrm{U}^+_{2r-2}(R)$ and $\overline{\beta}^{-1} \circ \overline{v}_0 \in \mathrm{U}^-_{2r-2}(R)$.

Therefore arguing as in (1), we obtain a diagonalization of Φ with respect to $(\alpha, \beta) \in W_{2r} \times W_{2r}$ such that $\alpha^{-1} \circ u_0 \in \mathrm{U}^+_{2r}(R)$, $\beta^{-1} \circ v_0 \in \mathrm{U}^-_{2r}(R)$, $\psi_i^{-1}(\mu_i) \in R$ $(0 \leq i \leq r-2)$, and that

$$\xi(g(x)) \in R^{2r} \subset K^{2r} = \mathcal{F}_0 \tag{3.22}$$

for any $x \in R^{2r} \subset K^{2r} = \mathcal{E}_0$, where ξ is the inverse of the morphism $\left(\otimes_{j=0}^{r-1} \psi_j \right) \otimes \mathrm{id}_{\mathcal{F}_0}$: $\mathcal{F}_0 \to \left(\otimes_{j=0}^{r-1} \mathcal{M}_j \right) \otimes \mathcal{F}_0$.

It remains to show that $\psi_{r-1}^{-1}(\mu_{r-1}) \in R$. If $r = 1$, then $(\xi \circ g \circ u_0)({}^t(0,1)) = v_0\left({}^t(0, \psi_0^{-1}(\mu_0)^2) \right)$. Hence we have $\psi_0^{-1}(\mu_0) \in R$ by (3.22). If $r \geq 2$, then considering (3.22) for $x = u_0({}^t(0,0,1,0,\ldots,0))$, we know that $\psi_{r-1}^{-1}(\mu_{r-1}) \in R$. ☐

Proposition 3.9. *Let S, $\mathcal{E}$, $\mathcal{F}$ and Φ as in Proposition 3.8. For a given pair $(\alpha, \beta) \in W_{2r} \times W_{2r}$, there exists at most one diagonalization of Φ with respect to (α, β).*

Proof. This proposition follows from the fact that the construction of the diagonalization of Φ given in the proof of Proposition 3.8 is the unique way. A rigorous proof is as follows.

Let $\mathbf{e}_1, \ldots, \mathbf{e}_{2r}$ be the standard basis of $\mathcal{E} = \mathcal{O}_S^{\oplus 2r}$, and $\mathbf{f}_1, \ldots, \mathbf{f}_{2r}$ that of $\mathcal{F} = \mathcal{O}_S^{\oplus 2r}$. By Remark 3.7, we may assume that $(\alpha, \beta) = (\mathrm{id}, \mathrm{id})$. Let us be given two diagonalization of Φ with respect to $(\mathrm{id},\mathrm{id})$:

$$u_i^{(m)} : \mathcal{O}^{\oplus 2r} \to \mathcal{E}_i, \quad v_i^{(m)} : \mathcal{O}^{\oplus 2r} \to \mathcal{F}_i \quad (0 \le i \le r),$$
$$\psi_i^{(m)} : \mathcal{O}_S \to \mathcal{M}_i \quad (0 \le i \le r-1)$$

with the entries of $u_0^{(m)}$ and $v_0^{(m)}$

$$\left(u_0^{(m)}\right)_{[a,b]} = \begin{cases} \begin{pmatrix} x_{ab}^{(m)} & y_{ab}^{(m)} \\ 0 & 0 \end{pmatrix} & \text{if } a < b \\ \begin{pmatrix} 1 & y_{ab}^{(m)} \\ 0 & 1 \end{pmatrix} & \text{if } a = b \\ \begin{pmatrix} 0 & y_{ab}^{(m)} \\ 0 & w_{ab}^{(m)} \end{pmatrix} & \text{if } a > b \end{cases}$$

$$\left(v_0^{(m)}\right)_{[a,b]} = \begin{cases} \begin{pmatrix} 0 & 0 \\ z_{ab}^{(m)} & w_{ab}^{(m)} \end{pmatrix} & \text{if } a < b \\ \begin{pmatrix} 1 & 0 \\ z_{ab}^{(m)} & 1 \end{pmatrix} & \text{if } a = b \\ \begin{pmatrix} x_{ab}^{(m)} & 0 \\ z_{ab}^{(m)} & 0 \end{pmatrix} & \text{if } a > b, \end{cases}$$

$(m = 1, 2)$.

Both $\otimes_{j=0}^{r-1}\psi_j^{(1)} : \mathcal{O}_S \to \otimes_{j=0}^{r-1}\mathcal{M}_j$ and $\otimes_{j=0}^{r-1}\psi_j^{(2)} : \mathcal{O}_S \to \otimes_{j=0}^{r-1}\mathcal{M}_j$ are induced by

$$\begin{aligned} \mathcal{O}\mathbf{e}_1 \subset \oplus_{i=1}^{2r}\mathcal{O}\mathbf{e}_i = \mathcal{E} &\xrightarrow{f_0^\sharp\circ\cdots\circ f_{r-1}^\sharp\circ h\circ e_{r-1}^\flat\circ\cdots\circ e_0^\flat} \left(\otimes_{j=0}^{r-1}\mathcal{M}_j\right) \otimes \mathcal{F} \\ &= \oplus_{i=1}^{2r}\left(\otimes_{j=0}^{r-1}\mathcal{M}_j\right)\mathbf{f}_i \to \left(\otimes_{j=0}^{r-1}\mathcal{M}_j\right)\mathbf{f}_1, \end{aligned}$$

hence we have $\otimes_{j=0}^{r-1}\psi_j^{(1)} = \otimes_{j=0}^{r-1}\psi_j^{(2)}$.

${}^t(y_{11}^{(m)}, 1, y_{21}^{(m)}, w_{21}^{(m)}, \ldots, y_{r1}^{(m)}, w_{r1}^{(m)})$ (resp. ${}^t(1, z_{11}^{(m)}, x_{21}^{(m)}, z_{21}^{(m)}, \ldots, x_{r1}^{(m)}, z_{r1}^{(m)})$) corresponds to the morphism

$$\begin{aligned} \mathcal{O}\mathbf{f}_2 \subset \oplus_{i=1}^{2r}\mathcal{O}\mathbf{f}_i = \mathcal{F} &\xrightarrow{e_0^\sharp\circ\cdots\circ e_{r-1}^\sharp\circ h^{-1}\circ f_{r-1}^\flat\circ\cdots\circ f_0^\flat} \left(\otimes_{j=0}^{r-1}\mathcal{M}_j\right) \otimes \mathcal{E} \\ &\xrightarrow{(\varphi_{r-1}^{(m)})^{-1}} \mathcal{E} = \mathcal{O}^{\oplus 2r} \end{aligned}$$

(resp.

$$\begin{aligned} \mathcal{O}\mathbf{e}_1 \subset \oplus_{i=1}^{2r}\mathcal{O}\mathbf{e}_i = \mathcal{E} &\xrightarrow{f_0^\sharp\circ\cdots\circ f_{r-1}^\sharp\circ h\circ e_{r-1}^\flat\circ\cdots\circ e_0^\flat} \left(\otimes_{j=0}^{r-1}\mathcal{M}_j\right) \otimes \mathcal{F} \\ &\xrightarrow{(\varphi_{r-1}^{(m)})^{-1}} \mathcal{F} = \mathcal{O}^{\oplus 2r}), \end{aligned}$$

therefore $x_{a1}^{(1)} = x_{a1}^{(2)}$, $y_{a1}^{(1)} = y_{a1}^{(2)}$, $z_{a1}^{(1)} = z_{a1}^{(2)}$, $w_{a1}^{(1)} = w_{a1}^{(2)}$. From this we know that the restrictions of $u_i^{(1)}$ and $u_i^{(2)}$ (resp. $v_i^{(1)}$ and $v_i^{(2)}$) to the first two factors $\mathcal{O}^{\oplus 2} \subset \mathcal{O}^{\oplus 2r}$ are equal for $0 \le i \le r$. Let $\gamma : \mathcal{F} = \mathcal{O}^{\oplus 2r} \to \mathcal{O}$ and $\delta : \mathcal{E} = \mathcal{O}^{\oplus 2r} \to \mathcal{O}$ be given by $x \mapsto (x, \mathbf{f}_2)$ and $y \mapsto (\mathbf{e}_1, y)$ respectively.

Let $\overline{\mathcal{E}}_i$ and $\overline{\mathcal{F}}_i$ $(0 \le i \le r)$ be as in (3.13). In particular we have

$$\overline{\mathcal{E}}_0 = \langle {}^t(1, 0, \ldots, 0), {}^t(y_{11}^{(m)}, 1, \ldots, y_{r1}^{(m)}, w_{r1}^{(m)}) \rangle^{\perp},$$
$$\overline{\mathcal{F}}_0 = \langle {}^t(1, z_{11}^{(m)}, \ldots, x_{r1}^{(m)}, z_{r1}^{(m)}), {}^t(0, 1, 0, \ldots, 0) \rangle^{\perp}.$$

As in the proof of Proposition 3.8, Φ induces a generalized symplectic isomorphism $\overline{\Phi}$ from $\overline{\mathcal{E}}_0$ to $\overline{\mathcal{F}}_0$.

Choose

$$\begin{pmatrix} -w_{21}^{(m)} \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} y_{21}^{(m)} \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \cdots \begin{pmatrix} -w_{r1}^{(m)} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} y_{r1}^{(m)} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 \\ z_{21}^{(m)} \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -x_{21}^{(m)} \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ z_{r1}^{(m)} \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -x_{r1}^{(m)} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

as bases of $\overline{\mathcal{E}}_0$ and $\overline{\mathcal{F}}_0$ respectively. Then with respect to these bases,

$$\overline{u}_i^{(m)} : \mathcal{O}_S^{\oplus 2r-2} \to \overline{\mathcal{E}_i}, \quad \overline{v}_i^{(m)} : \mathcal{O}_S^{\oplus 2r-2} \to \overline{\mathcal{F}_i} \quad (0 \le i \le r-1),$$
$$\psi_i^{(m)} : \mathcal{O}_S \to \mathcal{M}_i \quad (0 \le i \le r-2)$$

give diagonalizations of $\overline{\Phi}$ (with respect to $(\mathrm{id}, \mathrm{id})$), where $\overline{u}_i^{(m)}$ and $\overline{v}_i^{(m)}$ are the restrictions of $u_i^{(m)}$ and $v_i^{(m)}$ to the last $(2r-2)$ factors. By induction hypothesis we have

$$\psi_i^{(1)} = \psi_i^{(2)} \quad (0 \le i \le r-2),$$
$$\overline{u}_i^{(1)} = \overline{u}_i^{(2)} \text{ and } \overline{v}_i^{(1)} = \overline{v}_i^{(2)} \quad (0 \le i \le r-1).$$

Since the restrictions of $e^{\sharp}_{r-1}$ and $f^{\sharp}_{r-1}$ respectively to $\overline{\mathcal{E}}_r$ and $\overline{\mathcal{F}}_r$ induce isomorphisms $\overline{\mathcal{E}}_r \xrightarrow{\sim} \overline{\mathcal{E}}_{r-1}$ and $\overline{\mathcal{F}}_r \xrightarrow{\sim} \overline{\mathcal{F}}_{r-1}$, the equality $\overline{u}^{(1)}_{r-1} = \overline{u}^{(2)}_{r-1}$and $\overline{v}^{(1)}_{r-1} = \overline{v}^{(2)}_{r-1}$ implies that $\overline{u}^{(1)}_r = \overline{u}^{(2)}_r$and $\overline{v}^{(1)}_r = \overline{v}^{(2)}_r$. All together we have $\varphi^{(1)}_i = \varphi^{(2)}_i$ $(0 \leq i \leq r-1)$, $u^{(1)}_i = u^{(2)}_i$ and $v^{(1)}_i = v^{(2)}_i$ $(0 \leq i \leq r)$. □

Remark 3.10. By Proposition 3.9 we know that given two generalized symplectic isomorphisms Φ_1 and Φ_2 from $\mathcal{E}$ to $\mathcal{F}$, there exists at most one equivalence between Φ_1 and Φ_2. (cf. [Kausz1, the proof of Theorem 5.5 in page 579].)

Proposition 3.11. *Let Φ be as in Proposition 3.8. For a point $s \in S$, if $\Phi \otimes_S \overline{k(s)}$, the pull-back of Φ to* $\mathrm{Spec}\overline{k(s)}$, *has a diagonalization with respect to $(\alpha, \beta) \in W_{2r} \times W_{2r}$, then Φ has a diagonalization in a neighborhood of $s \in S$.*

Proof. We may assume that $(\alpha, \beta) = (\mathrm{id}, \mathrm{id})$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_{2r}$ be the standard basis of $\mathcal{E} = \mathcal{O}_S^{\oplus 2r}$, and $\mathbf{f}_1, \ldots, \mathbf{f}_{2r}$ that of $\mathcal{F} = \mathcal{O}_S^{\oplus 2r}$. Since $\Phi \otimes_S \overline{k(s)}$ has a diagonalization with respect to $(\mathrm{id}, \mathrm{id})$, the morphism

$$\begin{aligned} \mathcal{O}\mathbf{e}_1 \subset \mathcal{O}^{\oplus 2r} = \mathcal{E} &\xrightarrow{f^{\sharp}_0 \circ \cdots \circ f^{\sharp}_{r-1} \circ h \circ e^{\flat}_{r-1} \circ \cdots \circ e^{\flat}_0} \left(\otimes_{j=0}^{r-1} \mathcal{M}_j\right) \otimes \mathcal{F} \\ &\to \left(\otimes_{j=0}^{r-1} \mathcal{M}_j\right) \mathbf{f}_1 \end{aligned}$$

is nonzero at s, hence nonzero in a neighborhood of s. If we define subbundles $\overline{\mathcal{E}}_l \subset \mathcal{E}_l$ and $\overline{\mathcal{F}}_l \subset \mathcal{F}_l$ as in the proof of Proposition 3.8, we obtain a generalized symplectic isomorphism $\overline{\Phi}$ from $\overline{\mathcal{E}}_0$ to $\overline{\mathcal{F}}_0$ that has a diagonalization with respect to $(\mathrm{id}, \mathrm{id})$ at $\mathrm{Spec}\overline{k(s)}$. By induction hypothesis, it has a diagonalization with respect to $(\mathrm{id}, \mathrm{id})$ in a neighborhood of $s \in S$. So Φ has a diagonalization with respect to $(\mathrm{id}, \mathrm{id})$. □

Definition 3.12. Let S be a scheme, P a line bundle on S, $\mathcal{E}$ and $\mathcal{F}$ locally free $\mathcal{O}_S$-modules of rank $2r$, $\mathcal{E} \otimes \mathcal{E} \to \mathcal{P}$ and $\mathcal{F} \otimes \mathcal{F} \to \mathcal{P}$ non-degenerate alternate bilinear forms.

The functor $\mathcal{KS}p(\mathcal{E}, \mathcal{F})$ from the category of S-schemes to the category of sets is defined to associate to an S-scheme T the set of equivalence classes of generalized symplectic isomorphisms from $\mathcal{E}_T$ to $\mathcal{F}_T$.

Proposition 3.13. *The functor $\mathcal{KS}p(\mathcal{E}, \mathcal{F})$ is represented by a scheme which is smooth and of finite presentation over S.*

Proof. If we prove the representability locally on S, then by Remark 3.10 we can glue together locally-constructed universal families. So we may assume that $\mathcal{E} = \mathcal{F} = \mathcal{O}_S^{\oplus 2r}$ and the symplectic bilinear forms are given by the matrix J_{2r}.

For a pair $(\alpha, \beta) \in W_{2r} \times W_{2r}$, we define the subfunctor $\mathcal{KS}p(\mathcal{E}, \mathcal{F})^{(\alpha, \beta)} \subset \mathcal{KS}p(\mathcal{E}, \mathcal{F})$ to associate to an S-scheme T the set of equivalence classes of generalized symplectic isomorphisms from $\mathcal{E}_T$ to $\mathcal{F}_T$ that have a diagonalization with respect to (α, β).

By Proposition 3.11, $\mathcal{KS}p(\mathcal{E},\mathcal{F})^{(\alpha,\beta)}$ is an open subfunctor of $\mathcal{KS}p(\mathcal{E},\mathcal{F})$. Since Remark 3.10 guarantees that the universal families glue together, it suffices to prove that $\mathcal{KS}p(\mathcal{E},\mathcal{F})^{(\alpha,\beta)}$ is represented by a smooth scheme of finite presentation over S.

For an S-scheme T, let us given a generalized symplectic isomorphism

$$\begin{aligned}\Phi =&(\mathcal{M}_i,\mu_i,\mathcal{E}_i\to\mathcal{M}_i\otimes\mathcal{E}_{i+1},\mathcal{E}_i\leftarrow\mathcal{E}_{i+1},\\ &\mathcal{F}_{i+1}\to\mathcal{F}_i,\mathcal{M}_i\otimes\mathcal{F}_{i+1}\leftarrow\mathcal{F}_i\quad(0\le i\le r-1),h:\mathcal{E}_r\to\mathcal{F}_r),\end{aligned}$$

from $\mathcal{E}_T$ to $\mathcal{F}_T$ with its unique diagonalization with respect to (α,β)

$$\begin{aligned}u_i:\mathcal{O}_T^{\oplus 2r}\to\mathcal{E}_i,\qquad v_i:\mathcal{O}_T^{\oplus 2r}\to\mathcal{F}_i\quad(0\le i\le r)\\ \psi_i:\mathcal{O}_T\to\mathcal{M}_i\quad(0\le i\le r-1)\end{aligned}$$

with $\alpha^{-1}\circ u_0\in \mathrm{U}_{2r}^{+}(\mathcal{O}_T)$ and $\beta^{-1}\circ v_0\in \mathrm{U}_{2r}^{-}(\mathcal{O}_T)$.

The global sections $\psi_i^{-1}(\mu_i)$ $(0\le i\le r-1)$ give rise to a morphism $g_1:T\to\mathbb{A}_S^r$. The matrices $\alpha^{-1}\circ u_0\in \mathrm{U}_{2r}^{+}(\mathcal{O}_T)$ and $\beta^{-1}\circ v_0\in \mathrm{U}_{2r}^{-}(\mathcal{O}_T)$ give rise to morphisms $g_2:T\to \mathrm{U}_{2r}^{+}(\mathcal{O}_S)$ and $g_3:T\to \mathrm{U}_{2r}^{-}(\mathcal{O}_S)$. Conversely, given $g_1:T\to\mathbb{A}_S^r$, $g_2:T\to \mathrm{U}_{2r}^{+}(\mathcal{O}_S)$ and $g_3:T\to \mathrm{U}_{2r}^{-}(\mathcal{O}_S)$, we can recover an object of $\mathcal{KS}p(\mathcal{E},\mathcal{F})^{(\alpha,\beta)}$. Therefore the functor $\mathcal{KS}p(\mathcal{E},\mathcal{F})^{(\alpha,\beta)}$ is representable by a scheme $KSp(\mathcal{E},\mathcal{F})^{(\alpha,\beta)}$, and we have an isomorphism

$$KSp(\mathcal{E},\mathcal{F})^{(\alpha,\beta)}\simeq \mathrm{U}_{2r}^{+}(\mathcal{O}_S)\times_S\mathbb{A}_S^r\times_S \mathrm{U}_{2r}^{-}(\mathcal{O}_S). \tag{3.23}$$

□

Definition 3.14. We denote by $KSp(\mathcal{E},\mathcal{F})$ the S-scheme that represents the functor $\mathcal{KS}p(\mathcal{E},\mathcal{F})$.

In order to prove the projectivity of $KSp(\mathcal{E},\mathcal{F})$, we shall construct a closed immersion of $KSp(\mathcal{E},\mathcal{F})$ to $KGL(\mathcal{E},\mathcal{F})$.

Let S be a scheme, $\mathcal{P}$ a line bundle on S, $\mathcal{E}$ and $\mathcal{F}$ rank $2r$ locally free $\mathcal{O}_S$-modules, $\mathcal{E}\otimes\mathcal{E}\to\mathcal{P}$ and $\mathcal{F}\otimes\mathcal{F}\to\mathcal{P}$ non-degenerate alternate bilinear forms.

We compare the scheme $KSp(\mathcal{E},\mathcal{F})$ and $KGL(\mathcal{E},\mathcal{F})$.

Let

$$\begin{aligned}\Phi =&(\mathcal{M}_i,\mu_i,\mathcal{E}_i\to\mathcal{M}_i\otimes\mathcal{E}_{i+1},\mathcal{E}_i\leftarrow\mathcal{E}_{i+1},\\ &\mathcal{F}_{i+1}\to\mathcal{F}_i,\mathcal{M}_i\otimes\mathcal{F}_{i+1}\leftarrow\mathcal{F}_i\quad(0\le i\le r-1),h:\mathcal{E}_r\xrightarrow{\sim}\mathcal{F}_r),\end{aligned}$$

be a generalized symplectic isomorphism from $\mathcal{E}$ to $\mathcal{F}$. If we let

$$\begin{aligned}&\mathcal{E}'_i:=\mathcal{E}_0,\quad \mathcal{F}'_i:=\mathcal{F}_0\qquad(0\le i\le r-1),\\ &\mathcal{E}'_i:=\mathcal{E}_{r-i},\quad \mathcal{F}'_i:=\mathcal{F}_{r-i}\qquad(r\le i\le 2r),\\ &\mathcal{L}'_i=\mathcal{M}'_i:=\mathcal{O}_S,\quad \lambda'_i=\mu'_i:=1\qquad(0\le i\le r-1),\\ &\mathcal{L}'_i=\mathcal{M}'_i:=\mathcal{M}_{i-r},\quad \lambda'_i=\mu'_i:=\mu_{i-r}\qquad(r\le i\le 2r-1),\end{aligned}$$

then

$$\begin{aligned}\Psi =&(\mathcal{L}'_i, \lambda'_i, \mathcal{M}'_i, \mu'_i, \mathcal{E}'_i \to \mathcal{M}'_i \otimes \mathcal{E}'_{i+1}, \mathcal{E}'_i \leftarrow \mathcal{E}'_{i+1}, \\ &\mathcal{F}'_{i+1} \to \mathcal{F}'_i, \mathcal{L}'_i \otimes \mathcal{F}'_{i+1} \leftarrow \mathcal{F}'_i \ (0 \le i \le 2r-1), h : \mathcal{E}'_{2r} \to \mathcal{F}'_{2r})\end{aligned} \tag{3.24}$$

is a generalized isomorphism from $\mathcal{E}$ to $\mathcal{F}$. By this correspondence, we have a natural transformation

$$\tau : \mathcal{KS}p(\mathcal{E}, \mathcal{F}) \to \mathcal{KGL}(\mathcal{E}, \mathcal{F}).$$

Proposition 3.15. *For any S-scheme T, the morphism*

$$\mathcal{KS}p(\mathcal{E}, \mathcal{F})(T) \to \mathcal{KGL}(\mathcal{E}, \mathcal{F})(T)$$

of sets is injective.

Proof. For $l = 1, 2$, let

$$\begin{aligned}\Phi^{(l)} =&(\mathcal{M}_i^{(l)}, \mu_i^{(l)}, \mathcal{E}_i^{(l)} \xrightarrow{e_i^{\flat(l)}} \mathcal{M}_i^{(l)} \otimes \mathcal{E}_{i+1}^{(l)}, \mathcal{E}_i^{(l)} \xleftarrow{e_i^{\sharp(l)}} \mathcal{E}_{i+1}^{(l)}, \\ &\mathcal{F}_{i+1}^{(l)} \xrightarrow{f_i^{\sharp(l)}} \mathcal{F}_i^{(l)}, \mathcal{M}_i^{(l)} \otimes \mathcal{F}_{i+1}^{(l)} \xleftarrow{f_i^{\flat(l)}} \mathcal{F}_i^{(l)} \quad (0 \le i \le r-1), \\ &h^{(l)} : \mathcal{E}_r^{(l)} \to \mathcal{F}_r^{(l)}),\end{aligned} \tag{3.25}$$

be a generalized symplectic isomorphisms from $\mathcal{E}_T$ to $\mathcal{F}_T$. Let $s_{\mathcal{E},i} : \mathcal{M}_i^{(1)} \to \mathcal{M}_i^{(2)}$ and $s_{\mathcal{F},i} : \mathcal{M}_i^{(1)} \to \mathcal{M}_i^{(2)}$ $(0 \le i \le r-1)$ be isomorphisms such that $s_{\mathcal{E},i}(\mu_i^{(1)}) = \mu_i^{(2)}$ and $s_{\mathcal{F},i}(\mu_i^{(1)}) = \mu_i^{(2)}$. Let $t_{\mathcal{E},i} : \mathcal{E}_i^{(1)} \to \mathcal{E}_i^{(2)}$ and $t_{\mathcal{F},i} : \mathcal{F}_i^{(1)} \to \mathcal{F}_i^{(2)}$ be isomorphisms such that $t_{\mathcal{E},0} = \mathrm{id}_{\mathcal{E}}$ and $t_{\mathcal{F},0} = \mathrm{id}_{\mathcal{F}}$, and that

$$\begin{aligned}&t_{\mathcal{E},i} \circ e_i^{\sharp(1)} = e_i^{\sharp(2)} \circ t_{\mathcal{E},i+1},\ (s_{\mathcal{E},i} \otimes t_{\mathcal{E},i+1}) \circ e_i^{\flat(1)} = e_i^{(2)\flat} \circ t_{\mathcal{E},i} \\ &t_{\mathcal{F},i} \circ f_i^{\sharp(1)} = f_i^{\sharp(2)} \circ t_{\mathcal{F},i+1},\ (s_{\mathcal{F},i} \otimes t_{\mathcal{F},i+1}) \circ f_i^{\flat(1)} = f_i^{(2)\flat} \circ t_{\mathcal{F},i} \ (0 \le i \le r-1) \\ &t_{\mathcal{F},r} \circ h^{(1)} = h^{(2)} \circ t_{\mathcal{E},r}.\end{aligned} \tag{3.26}$$

Then $s_{\mathcal{E},i}$, $s_{\mathcal{F},i}$, $t_{\mathcal{E},j}$ and $t_{\mathcal{F},j}$ $(0 \le i \le r-1, 0 \le j \le r)$ give an equivalence between $\Phi^{(1)}$ and $\Phi^{(2)}$ as generalized isomorphisms. If $s_{\mathcal{E},i} = s_{\mathcal{F},i}$ $(0 \le i \le r-1)$, then they give an equivalence between $\Phi^{(1)}$ and $\Phi^{(2)}$ as generalized symplectic isomorphisms. Therefore the proposition follows from the next claim.

Claim. $s_{\mathcal{E},i} = s_{\mathcal{F},i} \quad (0 \le i \le r-1)$.

Proof of Claim. By the commutativity of the diagram (3.2), we have

$$\begin{aligned}&(1 \otimes \pi_{\mathcal{F}}) \circ ((f_0^{\sharp(l)} \circ \cdots \circ f_{r-1}^{\sharp(l)} \circ h^{(l)} \circ p_k^{\mathcal{E}^{(l)}}) \otimes q_k^{\mathcal{F}^{(l)}}) \\ &= (1 \otimes \pi_{\mathcal{E}}) \circ (q_k^{\mathcal{E}^{(l)}} \otimes (e_0^{\sharp(l)} \circ \cdots \circ e_{r-1}^{\sharp(l)} \circ h^{(l)-1} \circ p_k^{\mathcal{F}^{(l)}}))\end{aligned} \tag{3.27}$$

as morphisms from $\{(\bigotimes_{j=0}^{k-1}\mathcal{M}_j^{(l)\vee}\otimes\mathcal{E}_0^{(l)})\times_{\mathcal{E}_k^{(l)}}\mathcal{E}_r^{(l)}\}\otimes\{(\bigotimes_{j=0}^{k-1}\mathcal{M}_j^{(l)\vee}\otimes\mathcal{F}_0^{(l)})\times_{\mathcal{F}_k^{(l)}}\mathcal{F}_r^{(l)}\}$ to $\bigotimes_{j=0}^{k-1}\mathcal{M}_j^{(l)\vee}\otimes\mathcal{P}$ for $1\le k\le r$ and $l=1,2$. Using the equalities in (3.26) we know that

$$\begin{aligned}
&\left(\otimes_{j=0}^{k-1}s_{\mathcal{F},j}\otimes 1\right)\circ(1\otimes\pi_{\mathcal{F}})\circ\left((f_0^{\sharp(1)}\circ\cdots\circ f_{r-1}^{\sharp(1)}\circ h^{(1)}\circ p_k^{\mathcal{E}^{(1)}})\otimes q_k^{\mathcal{F}^{(1)}}\right)\\
&=(1\otimes\pi_{\mathcal{F}})\circ\left((f_0^{\sharp(2)}\circ\cdots\circ f_{r-1}^{\sharp(2)}\circ h^{(2)}\circ p_k^{\mathcal{E}^{(2)}})\otimes q_k^{\mathcal{F}^{(2)}}\right)\\
&\circ\left(\left((\otimes_{j=0}^{k-1}s_{\mathcal{E},j}\otimes t_{\mathcal{E},0})\times t_{\mathcal{E},r}\right)\otimes\left((\otimes_{j=0}^{k-1}s_{\mathcal{F},j}\otimes t_{\mathcal{F},0})\times t_{\mathcal{F},r}\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\otimes_{j=0}^{k-1}s_{\mathcal{E},j}\otimes 1\right)\circ(1\otimes\pi_{\mathcal{E}})\circ\left(q_k^{\mathcal{E}^{(1)}}\otimes(e_0^{\sharp(2)}\circ\cdots\circ e_{r-1}^{\sharp(2)}\circ h^{(1)-1}\circ p_k^{\mathcal{F}^{(1)}})\right)\\
&=(1\otimes\pi_{\mathcal{E}})\circ\left(q_k^{\mathcal{E}^{(2)}}\otimes(e_0^{\sharp(2)}\circ\cdots\circ e_{r-1}^{\sharp(2)}\circ h^{(2)-1}\circ p_k^{\mathcal{F}^{(2)}})\right)\\
&\circ\left(\left((\otimes_{j=0}^{k-1}s_{\mathcal{E},j}\otimes t_{\mathcal{E},0})\times t_{\mathcal{E},r}\right)\otimes\left((\otimes_{j=0}^{k-1}s_{\mathcal{F},j}\otimes t_{\mathcal{F},0})\times t_{\mathcal{F},r}\right)\right)
\end{aligned}$$

as morphisms from $\{(\bigotimes_{j=0}^{k-1}\mathcal{M}_j^{(1)\vee}\otimes\mathcal{E}_0^{(1)})\times_{\mathcal{E}_k^{(1)}}\mathcal{E}_r^{(1)}\}\otimes\{(\bigotimes_{j=0}^{k-1}\mathcal{M}_j^{(1)\vee}\otimes\mathcal{F}_0^{(1)})\times_{\mathcal{F}_k^{(1)}}\mathcal{F}_r^{(1)}\}$ to $\bigotimes_{j=0}^{k-1}\mathcal{M}_j^{(2)\vee}\otimes\mathcal{P}$. From these equalities, we know that if we denote the morphism in (3.27) by b_l ($l=1,2$), then we have

$$\left(\otimes_{j=0}^{k-1}s_{\mathcal{E},j}\otimes 1\right)\circ b_1=\left(\otimes_{j=0}^{k-1}s_{\mathcal{F},j}\otimes 1\right)\circ b_1.$$

Using diagonalization locally, you can check that b_1 is surjective. So we have $\bigotimes_{j=0}^{k-1}s_{\mathcal{E},j}=\bigotimes_{j=0}^{k-1}s_{\mathcal{F},j}$ ($1\le k\le r$). Hence $s_{\mathcal{E},j}=s_{\mathcal{F},j}$ ($0\le j\le r-1$). This completes the proof of the claim. □

This is the end of the proof of Proposition 3.15. □

The natural transformation $\tau:\mathcal{KS}p(\mathcal{E},\mathcal{F})\to\mathcal{KGL}(\mathcal{E},\mathcal{F})$ induces a morphism $\iota: KSp(\mathcal{E},\mathcal{F})\to KGL(\mathcal{E},\mathcal{F})$ of S-schemes.

Corollary 3.16. *The morphism ι is a closed immersion.*

Proof. We can check this locally on S, so we may assume that S is an affine scheme, and that $\mathcal{P}=\mathcal{O}_S$, $\mathcal{E}=\mathcal{F}=\mathcal{O}_S^{\oplus 2r}$, and that $\mathcal{E}\otimes\mathcal{E}\to\mathcal{P}$ and $\mathcal{F}\otimes\mathcal{F}\to\mathcal{P}$ are given by the matrix J_{2r}.

Let R be a valuation ring over $\mathcal{O}_S$, and K the quotient field of R. In the commutative diagram

$$\begin{array}{ccc}
\mathcal{KS}p(\mathcal{E},\mathcal{F})(\mathrm{Spec}R) & \xrightarrow{(a)} & \mathcal{KGL}(\mathcal{E},\mathcal{F})(\mathrm{Spec}R)\\
\downarrow{\scriptstyle (b)} & & \downarrow{\scriptstyle (d)}\\
\mathcal{KS}p(\mathcal{E},\mathcal{F})(\mathrm{Spec}K) & \xrightarrow{(c)} & \mathcal{KGL}(\mathcal{E},\mathcal{F})(\mathrm{Spec}K),
\end{array}\tag{3.28}$$

(a) and (c) are injective by Proposition 3.15.

If we are given an element Φ of $\mathcal{KSp}(\mathcal{E},\mathcal{F})(\mathrm{Spec}K)$, we know that it extends over $\mathrm{Spec}R$ by choosing a diagonalization as in (2) of Proposition 3.8. Hence (b) is surjective. By [Kausz1], $KGL(\mathcal{E},\mathcal{F})$ is a projective S-scheme, so (d) is bijective by the valuative criterion. Therefore (b) is also bijective. Then $KSp(\mathcal{E},\mathcal{F})$ is a proper S-scheme by the valuative criterion. By Proposition 3.15, the morphism ι is a closed immersion. □

§ 4. Relation with the symplectic Grassmannian

Let $\mathcal{E}$, $\mathcal{F}$ be locally free sheaves of rank $2r$ on a scheme S, and $\pi_{\mathcal{E}} : \mathcal{E}\otimes\mathcal{E}\to\mathcal{P}$, $\pi_{\mathcal{F}} : \mathcal{F}\otimes\mathcal{F}\to\mathcal{P}$ be non-degenerate alternate bilinear forms with values in a line bundle $\mathcal{P}$. We define the non-degenerate alternate bilinear form $\pi_{\mathcal{E}\oplus\mathcal{F}} : (\mathcal{E}\oplus\mathcal{F})\otimes(\mathcal{E}\oplus\mathcal{F})\to\mathcal{P}$ as $\pi_{\mathcal{E}\oplus\mathcal{F}}((e,f)\otimes(e',f')) := \pi_{\mathcal{E}}(e\otimes e') - \pi_{\mathcal{F}}(f\otimes f')$. Let $LGr(\mathcal{E}\oplus\mathcal{F})$ be the symplectic Grassmannian parametrizing rank $2r$ isotropic subbundles of $\mathcal{E}\oplus\mathcal{F}$.

Giving a symplectic isomorphism $\mathcal{E}\xrightarrow{\alpha}\mathcal{F}$ is equivalent to giving a rank $2r$ isotropic subbundle $\mathcal{H}\subset\mathcal{E}\oplus\mathcal{F}$ which projects isomorphically to both $\mathcal{E}$ and $\mathcal{F}$ (Consider the graph of α). Therefore $LGr(\mathcal{E}\oplus\mathcal{F})$ is also a compactification of $Sp(\mathcal{E},\mathcal{F})$.

The relation of the two compactifications $KSp(\mathcal{E},\mathcal{F})$ and $LGr(\mathcal{E}\oplus\mathcal{F})$ is as follows.

Proposition 4.1. *There is a natural morphism $g : KSp(\mathcal{E},\mathcal{F})\to LGr(\mathcal{E}\oplus\mathcal{F})$.*

Proof. Let

$$\begin{aligned}\Phi = &(\mathcal{M}_i, \mu_i, \mathcal{E}_i\to\mathcal{M}_i\otimes\mathcal{E}_{i+1}, \mathcal{E}_i\leftarrow\mathcal{E}_{i+1},\\ &\mathcal{F}_{i+1}\to\mathcal{F}_i, \mathcal{M}_i\otimes\mathcal{F}_{i+1}\leftarrow\mathcal{F}_i \quad (0\le i\le r-1), h:\mathcal{E}_r\xrightarrow{\sim}\mathcal{F}_r)\end{aligned}$$

be the universal generalized symplectic isomorphism from $\mathcal{E}_0=\mathcal{E}_{KSp}$ to $\mathcal{F}_0=\mathcal{F}_{KSp}$. Then by the condition 2 of Definition 3.1, the morphism

$$\beta := (e_0^\sharp\circ\cdots\circ e_{r-1}^\sharp, f_0^\sharp\circ\cdots\circ f_{r-1}^\sharp\circ h) : \mathcal{E}_r\to\mathcal{E}_{KSp}\oplus\mathcal{F}_{KSp}$$

is injective, and its image is a subbundle of $\mathcal{E}_{KSp}\oplus\mathcal{F}_{KSp}$. By the condition 3 of Definition 3.1, this subbundle is isotropic. Hence $\beta(\mathcal{E}_r)\subset\mathcal{E}_{KSp}\oplus\mathcal{F}_{KSp}$ gives us a morphism $KSp(\mathcal{E},\mathcal{F})\to LGr(\mathcal{E}\oplus\mathcal{F})$. □

For later use, we prepare some easy lemmas concerning $LGr(\mathcal{E}\oplus\mathcal{F})$.

Lemma 4.2. *Let $0\to\mathcal{U}\to pr_S^*(\mathcal{E}\oplus\mathcal{F})\to\mathcal{Q}\to 0$ be the universal sequence on $LGr(\mathcal{E}\oplus\mathcal{F})$. Then there is a natural isomorphism*

$$g^*\det\mathcal{Q}\simeq pr_S^*\mathcal{P}^{\otimes r}\otimes\bigotimes_{i=0}^{r-1}\mathcal{M}_i^{\otimes(r-i)}. \tag{4.1}$$

Proof. Let Φ be as in the proof of the above proposition. By the construction of g, we have an isomorphism

$$g^* \det \mathcal{Q} \simeq \det(\mathcal{E} \oplus \mathcal{F})_{KSp} \otimes (\det \mathcal{E}_r)^\vee .$$

By Lemma 2.8 (1), there is a natural isomorphism

$$\det \mathcal{E}_r \simeq \det \mathcal{E}_0 \otimes \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes(i-r)}.$$

Combining these isomorphism together with the isomorphism $\det \mathcal{E} \simeq \det \mathcal{F} \simeq \mathcal{P}^{\otimes r}$, we obtain (4.1). □

Lemma 4.3. *Let V and W be vector spaces of dimension $2r$ over a field K with non-degenerate alternate forms $(-,-)_V$ and $(-,-)_W$. Endow $V \oplus W$ with the non-degenerate alternate form $(-,-)_{V\oplus W}$ given by $((v,w),(v',w'))_{V\oplus W} = (v,v')_V - (w,w')_W$.*

If $U \subset V \oplus W$ is an isotropic subspace of dimension $2r$, then we have $\dim U \cap (V \oplus 0) = \dim U \cap (0 \oplus W)$.

Proof. Easy. □

We denote by $t(U)$ the number $\dim U \cap (V \oplus 0) (= \dim U \cap (0 \oplus W))$, and call it the type of U. We say that U *is of type* $\leq n$ if $t(U) \leq n$.

Notation 4.4. We denote by $LGr(\mathcal{E} \oplus \mathcal{F})_{\leq n}$ the open subscheme of $LGr(\mathcal{E} \oplus \mathcal{F})$ parametrizing rank $2r$ isotropic subbundles of type $\leq n$ of $\mathcal{E} \oplus \mathcal{F}$.

Lemma 4.5. *For $0 \leq n < r$, the codimension of $LGr(\mathcal{E} \oplus \mathcal{F}) \setminus LGr(\mathcal{E} \oplus \mathcal{F})_{\leq n}$ in $LGr(\mathcal{E} \oplus \mathcal{F})$ is greater than or equal to $(n+1)^2$.*

Proof. Easy dimension counting. □

§ 5. Geometry of Strata

If
$$\begin{aligned}\Phi =& (\mathcal{M}_i, \mu_i, \mathcal{E}_i \to \mathcal{M}_i \otimes \mathcal{E}_{i+1}, \mathcal{E}_i \leftarrow \mathcal{E}_{i+1}, \\ & \mathcal{F}_{i+1} \to \mathcal{F}_i, \mathcal{M}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \quad (0 \leq i \leq r-1), h : \mathcal{E}_r \xrightarrow{\sim} \mathcal{F}_r)\end{aligned}$$

with $\mathcal{E}_0 = \mathcal{E}_{KSp(\mathcal{E},\mathcal{F})}$ and $\mathcal{F}_0 = \mathcal{F}_{KSp(\mathcal{E},\mathcal{F})}$ is the universal family on $KSp(\mathcal{E},\mathcal{F})$, then vanishing loci of some μ_i's are closed subschemes of $KSp(\mathcal{E},\mathcal{F})$. In this section we study the closed subschemes just as Kausz did for $KGL(\mathcal{E},\mathcal{F})$ in [Kausz1, §9].

When Kausz studied the strata of $KGL(\mathcal{E},\mathcal{F})$, the scheme $\overline{PGl}$ appeared naturally. The scheme $\overline{PGl}$ also appears in our study of strata of $KSp(\mathcal{E},\mathcal{F})$, but in disguise.

Let S be a scheme, $\mathcal{P}$ a line bundle on S. Let $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{B}$ and $\mathcal{B}'$ be locally free $\mathcal{O}_S$-modules of rank m, and $\pi_{\mathcal{A},\mathcal{B}'}:\mathcal{A}\otimes\mathcal{B}'\to\mathcal{P}$ and $\pi_{\mathcal{B},\mathcal{A}'}:\mathcal{B}\otimes\mathcal{A}'\to\mathcal{P}$ non-degenerate pairings.

The S-groupoid $\mathcal{Q}(\pi_{\mathcal{A},\mathcal{B}'},\pi_{\mathcal{B},\mathcal{A}'})$ is defined as follows. For an S-scheme T, an object of $\mathcal{Q}(\pi_{\mathcal{A},\mathcal{B}'},\pi_{\mathcal{B},\mathcal{A}'})(T)$ is a pair of tuples

$$\begin{aligned}\Phi_{\mathcal{A}} &= \left(\mathcal{M}_i,\mu_i,\mathcal{A}_{i+1}\xrightarrow{a_i^\sharp}\mathcal{A}_i,\mathcal{M}_i\otimes\mathcal{A}_{i+1}\xleftarrow{a_i^\flat}\mathcal{A}_i\quad(0\le i\le m-1)\right)\\ \Phi_{\mathcal{B}} &= \left(\mathcal{M}_i,\mu_i,\mathcal{B}_{i+1}\xrightarrow{b_i^\sharp}\mathcal{B}_i,\mathcal{M}_i\otimes\mathcal{B}_{i+1}\xleftarrow{b_i^\flat}\mathcal{B}_i\quad(0\le i\le m-1)\right)\end{aligned}\tag{5.1}$$

such that $\Phi_{\mathcal{A}}$ and $\Phi_{\mathcal{B}}$ are complete collineations from $(\mathcal{A}')_T=\mathcal{A}_m$ to $(\mathcal{A})_T=\mathcal{A}_0$ and from $(\mathcal{B}')_T=\mathcal{B}_m$ to $(\mathcal{B})_T=\mathcal{B}_0$ respectively, and such that the following diagram is commutative:

$$\begin{array}{ccc} & \left\{\left(\otimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{A}_0\right)\times_{\mathcal{A}_k}\mathcal{A}_m\right\}\otimes\left\{\left(\otimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{B}_0\right)\times_{\mathcal{B}_k}\mathcal{B}_m\right\} & \\ {\scriptstyle q_k^{\mathcal{A}}\otimes p_k^{\mathcal{B}}}\swarrow & & \searrow{\scriptstyle p_k^{\mathcal{A}}\otimes q_k^{\mathcal{B}}} \\ \left(\otimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{A}_0\right)\otimes\mathcal{B}_m & & \mathcal{A}_m\otimes\left(\otimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{B}_0\right) \\ \pi_{\mathcal{A},\mathcal{B}'}\searrow & & \swarrow\pi_{\mathcal{B},\mathcal{A}'} \\ & \left(\otimes_{j=0}^{k-1}\mathcal{M}_j^\vee\right)\otimes(\mathcal{P})_T, & \end{array}\tag{5.2}$$

where $p_k^{\mathcal{A}}$, $q_k^{\mathcal{A}}$, $p_k^{\mathcal{B}}$ and $q_k^{\mathcal{B}}$ are defined by

$$\begin{array}{ccc}\left(\bigotimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{A}_0\right)\times_{\mathcal{A}_k}\mathcal{A}_m & \xrightarrow{p_k^{\mathcal{A}}} & \mathcal{A}_m\\ q_k^{\mathcal{A}}\downarrow & & \downarrow a_k^\sharp\circ\cdots\circ a_{m-1}^\sharp\\ \bigotimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{A}_0 & \xrightarrow{a_{k-1}^\flat\circ\cdots\circ a_0^\flat} & \mathcal{A}_k\end{array}\tag{5.3}$$

and

$$\begin{array}{ccc}\left(\bigotimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{B}_0\right)\times_{\mathcal{B}_k}\mathcal{B}_m & \xrightarrow{p_k^{\mathcal{B}}} & \mathcal{B}_m\\ q_k^{\mathcal{B}}\downarrow & & \downarrow b_k^\sharp\circ\cdots\circ b_{m-1}^\sharp\\ \bigotimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{B}_0 & \xrightarrow{b_{k-1}^\flat\circ\cdots\circ b_0^\flat} & \mathcal{B}_k.\end{array}\tag{5.4}$$

Isomorphisms are defined obviously.

Proposition 5.1. *For any S-scheme T, the functor*

$$Q(\pi_{\mathcal{A},\mathcal{B}'},\pi_{\mathcal{B},\mathcal{A}'})(T)\to\overline{\mathcal{PGL}}(\mathcal{A}',\mathcal{A})(T)\tag{5.5}$$

which associates $\Phi_{\mathcal{A}}$ to an object $(\Phi_{\mathcal{A}}, \Phi_{\mathcal{B}}) \in Q(\pi_{\mathcal{A},\mathcal{B}'}, \pi_{\mathcal{B},\mathcal{A}'})(T)$ is an equivalence. In particular, the functor $Q(\pi_{\mathcal{A},\mathcal{B}'}, \pi_{\mathcal{B},\mathcal{A}'})$ is represented by a scheme which is smooth and projective over S by Theorem 2.7.

Proof. We shall construct the inverse of the functor (5.5).

Given an object

$$\Phi_{\mathcal{A}} = \left(\mathcal{M}_i, \mu_i, \mathcal{A}_{i+1} \xrightarrow{a_i^\sharp} \mathcal{A}_i, \mathcal{M}_i \otimes \mathcal{A}_{i+1} \xleftarrow{a_i^\flat} \mathcal{A}_i \quad (0 \le i \le m-1)\right)$$

of $\overline{\mathcal{PGL}}(\mathcal{A}', \mathcal{A})(T)$, let $\mathcal{B}_k$ be

$$\left\{\left(\otimes_{j=0}^{k-1} \mathcal{M}_j^\vee \otimes \mathcal{A}_0\right) \times_{\mathcal{A}_k} \mathcal{A}_0\right\}^\vee \otimes \otimes_{j=0}^{k-1} \mathcal{M}_j^\vee \otimes (\mathcal{P})_T \quad (0 \le k \le m),$$

and we identify $(\mathcal{B})_T$ and $(\mathcal{B}')_T$ with

$$\{\mathcal{A}_0 \times_{\mathcal{A}_0} \mathcal{A}_m\}^\vee \otimes (\mathcal{P})_T (= \mathcal{A}_m^\vee \otimes (\mathcal{P})_T = (\mathcal{A}'^\vee \otimes \mathcal{P})_T)$$

and

$$\left\{\left(\bigotimes_{j=0}^{m-1} \mathcal{M}_j^\vee \otimes \mathcal{A}_0\right) \times_{\mathcal{A}_m} \mathcal{A}_m\right\}^\vee \otimes \bigotimes_{j=0}^{m-1} \mathcal{M}_j^\vee \otimes (\mathcal{P})_T (= \mathcal{A}_0^\vee \otimes (\mathcal{P})_T = (\mathcal{A}^\vee \otimes \mathcal{P})_T)$$

respectively by $\pi_{\mathcal{B},\mathcal{A}'}$ and $\pi_{\mathcal{A},\mathcal{B}'}$. We have natural morphisms

$$\begin{aligned}\left(\otimes_{j=0}^{k-1} \mathcal{M}_j^\vee \otimes \mathcal{A}_0\right) \times_{\mathcal{A}_k} \mathcal{A}_m &\simeq \left\{\left(\otimes_{j=0}^{k} \mathcal{M}_j^\vee \otimes \mathcal{A}_0\right) \times_{\mathcal{M}_k^\vee \otimes \mathcal{A}_k} (\mathcal{M}_k^\vee \otimes \mathcal{A}_m)\right\} \otimes \mathcal{M}_k \\ &\xrightarrow{(\mathrm{id} \times \mu_k) \otimes \mathrm{id}} \left\{\left(\otimes_{j=0}^{k} \mathcal{M}_j^\vee \otimes \mathcal{A}_0\right) \times_{\mathcal{A}_{k+1}} \mathcal{A}_m\right\} \otimes \mathcal{M}_k\end{aligned}$$

and

$$\left(\otimes_{j=0}^{k} \mathcal{M}_j^\vee \otimes \mathcal{A}_0\right) \times_{\mathcal{A}_{k+1}} \mathcal{A}_m \xrightarrow{\mu_k \times \mathrm{id}} \left(\otimes_{j=0}^{k-1} \mathcal{M}_j^\vee \otimes \mathcal{A}_0\right) \times_{\mathcal{A}_k} \mathcal{A}_m.$$

The duals of these morphisms induce

$$\mathcal{B}_{k+1} \to \mathcal{B}_k \text{ and } \mathcal{B}_k \to \mathcal{M}_k \otimes \mathcal{B}_{k+1}.$$

To complete the proof, we need to verify that

- $\Phi := (\mathcal{M}_i, \mu_i, \mathcal{B}_{i+1} \to \mathcal{B}_i, \mathcal{M}_i \otimes \mathcal{B}_{i+1}\ (0 \le i \le m-1))$ is an object of $\overline{\mathcal{PGL}}(\mathcal{B}', \mathcal{B})(T)$,
- The diagram (5.2) commutes for $(\Phi_{\mathcal{A}}, \Phi_{\mathcal{B}})$,
- This construction gives the inverse of (5.5).

Here we shall just check that if a pair of tuples

$$(5.6) \quad \begin{aligned}\Phi_{\mathcal{A}} &= \left(\mathcal{M}_i, \mu_i, \mathcal{A}_{i+1} \xrightarrow{a_i^\sharp} \mathcal{A}_i, \mathcal{M}_i \otimes \mathcal{A}_{i+1} \xleftarrow{a_i^\flat} \mathcal{A}_i \quad (0 \le i \le m-1)\right) \\ \Phi_{\mathcal{B}} &= \left(\mathcal{M}_i, \mu_i, \mathcal{B}_{i+1} \xrightarrow{b_i^\sharp} \mathcal{B}_i, \mathcal{M}_i \otimes \mathcal{B}_{i+1} \xleftarrow{b_i^\flat} \mathcal{B}_i \quad (0 \le i \le m-1)\right)\end{aligned}$$

is an object of $\mathcal{Q}(\pi_{\mathcal{A},\mathcal{B}'},\pi_{\mathcal{B},\mathcal{A}'})(T)$, then there is an isomorphism

$$\mathcal{B}_k \simeq \left\{\left(\otimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{A}_0\right)\times_{\mathcal{A}_k}\mathcal{A}_m\right\}^\vee\otimes\otimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes(\mathcal{P})_T, \tag{5.7}$$

leaving other verification to the reader.

Let $\beta:\left(\bigotimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{B}_0\right)\times\mathcal{B}_m\to\mathcal{B}_k$ be the morphism which sends $(y_0,y_m)\in\left(\bigotimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{B}_0\right)\times\mathcal{B}_m$ to $(b_{k-1}^\flat\circ\cdots\circ b_0^\flat)(y_0)+(b_k^\sharp\circ\cdots\circ b_{m-1}^\sharp)(y_m)\in\mathcal{B}_k$. By the definition of collineation, β is surjective. We define a bilinear form

$$\left\{\left(\otimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{A}_0\right)\times_{\mathcal{A}_k}\mathcal{A}_m\right\}\otimes\mathcal{B}_k\to\otimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes(\mathcal{P})_T \tag{5.8}$$

by $(x_0,x_m)\otimes\beta(y_0,y_m)\mapsto\pi_{\mathcal{A},\mathcal{B}'}(x_0,y_m)+\pi_{\mathcal{B},\mathcal{A}'}(y_0,x_m)$. Note that if $\beta(y_0,y_m)=0$, then $(y_0,-y_m)\in\left(\bigotimes_{j=0}^{k-1}\mathcal{M}_j^\vee\otimes\mathcal{B}_0\right)\times_{\mathcal{B}_k}\mathcal{B}_m$ so we have $\pi_{\mathcal{A},\mathcal{B}'}(x_0,y_m)=-\pi_{\mathcal{B},\mathcal{A}'}(y_0,x_m)$ by the commutativity of (5.2). Therefore (5.8) is well-defined. Since $\pi_{\mathcal{A},\mathcal{B}'}$ and $\pi_{\mathcal{B}\mathcal{A}'}$ are non-degenerate, (5.8) is also non-degenerate. Hence we have the isomorphism (5.7). □

Definition 5.2. Let

$$\begin{aligned}\Phi=&(\mathcal{M}_i,\mu_i,\mathcal{E}_i\to\mathcal{M}_i\otimes\mathcal{E}_{i+1},\mathcal{E}_i\leftarrow\mathcal{E}_{i+1},\\ &\mathcal{F}_{i+1}\to\mathcal{F}_i,\mathcal{M}_i\otimes\mathcal{F}_{i+1}\leftarrow\mathcal{F}_i\quad(0\le i\le r-1),h:\mathcal{E}_r\xrightarrow{\sim}\mathcal{F}_r),\end{aligned} \tag{5.9}$$

be the universal generalized symplectic isomorphism from $\mathcal{E}_0=(\mathcal{E})_{KSp(\mathcal{E},\mathcal{F})}$ to $\mathcal{F}_0=(\mathcal{F})_{KSp(\mathcal{E},\mathcal{F})}$. For a subset $I\subset\{0,\dots,r-1\}$, we denote by X_I the subscheme $\bigcap_{i\in I}\{\mu_i=0\}\subset KSp(\mathcal{E},\mathcal{F})$.

Definition 5.3. For a subset $I=\{i_1<\cdots<i_l\}\subset\{0,\dots,r-1\}$, let $\mathcal{S}p\mathcal{F}l_I(\mathcal{E})$ be the functor from the category of S-schemes to the category of sets that associates to an S-scheme T the set of filtrations

$$0\subset\mathbb{F}_l(\mathcal{E}_T)\subset\mathbb{F}_{l-1}(\mathcal{E}_T)\subset\cdots\subset\mathbb{F}_1(\mathcal{E}_T)\subset\mathcal{E}_T$$

of isotropic subbundles with rank $\mathbb{F}_j(\mathcal{E}_T)=r-i_j$. We understand that $\mathbb{F}_{l+1}(\mathcal{E}_T)=0$.

We denote by $\mathrm{SpFl}_I(\mathcal{E})$ the S-scheme that represents $\mathcal{S}p\mathcal{F}l(\mathcal{E})$.

Put $\mathbf{SpFl}_I:=\mathrm{SpFl}_I(\mathcal{E})\times_S\mathrm{SpFl}_I(\mathcal{F})$, $\widetilde{\mathcal{E}}:=(\mathcal{E})_{\mathrm{SpFl}_I}$, $\widetilde{\mathcal{F}}:=(\mathcal{F})_{\mathrm{SpFl}_I}$ and $\widetilde{\mathcal{P}}:=(\mathcal{P})_{\mathrm{SpFl}_I}$. Let

$$\begin{aligned}&0\subset\mathbb{F}_l(\widetilde{\mathcal{E}})\subset\cdots\subset\mathbb{F}_1(\widetilde{\mathcal{E}})\subset\widetilde{\mathcal{E}},\\ &0\subset\mathbb{F}_l(\widetilde{\mathcal{F}})\subset\cdots\subset\mathbb{F}_1(\widetilde{\mathcal{F}})\subset\widetilde{\mathcal{F}}\end{aligned} \tag{5.10}$$

be the pull-backs to $\mathbf{SpFl}_I$ of the universal filtrations of $\mathcal{E}$ and $\mathcal{F}$ on $\mathrm{SpFl}_I(\mathcal{E})$ and $\mathrm{SpFl}_I(\mathcal{F})$ respectively. The non-degenerate alternate bilinear forms $\pi_{\mathcal{E}}:\mathcal{E}\otimes\mathcal{E}\to\mathcal{P}$ and $\pi_{\mathcal{F}}:\mathcal{F}\otimes\mathcal{F}\to\mathcal{P}$ induce nondegenerate alternate bilinear forms

$$\begin{aligned}&\widetilde{\pi}_{\mathcal{E}}:\mathbb{F}_1(\widetilde{\mathcal{E}})^\perp/\mathbb{F}_1(\widetilde{\mathcal{E}})\otimes\mathbb{F}_1(\widetilde{\mathcal{E}})^\perp/\mathbb{F}_1(\widetilde{\mathcal{E}})\to\widetilde{\mathcal{P}},\\ &\widetilde{\pi}_{\mathcal{F}}:\mathbb{F}_1(\widetilde{\mathcal{F}})^\perp/\mathbb{F}_1(\widetilde{\mathcal{F}})\otimes\mathbb{F}_1(\widetilde{\mathcal{F}})^\perp/\mathbb{F}_1(\widetilde{\mathcal{F}})\to\widetilde{\mathcal{P}}\end{aligned}$$

and non-degenerate bilinear forms

$$\widetilde{\pi}_{\mathcal{E},i} : \mathbb{F}_{i+1}(\widetilde{\mathcal{E}})^{\perp}/\mathbb{F}_i(\widetilde{\mathcal{E}})^{\perp} \otimes \mathbb{F}_i(\widetilde{\mathcal{E}})/\mathbb{F}_{i+1}(\widetilde{\mathcal{E}}) \to \widetilde{\mathcal{P}},$$
$$\widetilde{\pi}_{\mathcal{F},i} : \mathbb{F}_{i+1}(\widetilde{\mathcal{F}})^{\perp}/\mathbb{F}_i(\widetilde{\mathcal{F}})^{\perp} \otimes \mathbb{F}_i(\widetilde{\mathcal{F}})/\mathbb{F}_{i+1}(\widetilde{\mathcal{F}}) \to \widetilde{\mathcal{P}} \quad (1 \le i \le l).$$

Then the stratum X_I is described as follows. This is a symplectic analogue of [Kausz1, Themorem 9.3]:

Proposition 5.4. *There is an isomorphism*

$$X_I \to \mathrm{KSp}(\mathbb{F}_1(\widetilde{\mathcal{E}})^{\perp}/\mathbb{F}_1(\widetilde{\mathcal{E}}), \mathbb{F}_1(\widetilde{\mathcal{F}})^{\perp}/\mathbb{F}_1(\widetilde{\mathcal{F}})) \times_{\mathbf{SpFl}_I} \mathcal{Q} \tag{5.11}$$

of S-schemes, where $\mathcal{Q} = \mathcal{Q}(\widetilde{\pi}_{\mathcal{E},1}, \widetilde{\pi}_{\mathcal{F},1}) \times_{\mathbf{SpFl}_I} \cdots \times_{\mathbf{SpFl}_I} \mathcal{Q}(\widetilde{\pi}_{\mathcal{E},l}, \widetilde{\pi}_{\mathcal{F},l})$.

Proof. For an S-scheme T, we shall give a bijective correspondence between the sets of T-valued points of both sides of (5.11). For simplicity of notation we assume that $T{=}S$.

An S-valued point of X_I is a generalized symplectic isomorphism $\mathcal{E}$ to $\mathcal{F}$

$$\begin{aligned}\Phi =(&\mathcal{M}_i, \mu_i, \mathcal{E}_i \xrightarrow{e_i^{\flat}} \mathcal{M}_i \otimes \mathcal{E}_{i+1}, \mathcal{E}_i \xleftarrow{e_i^{\sharp}} \mathcal{E}_{i+1},\\ &\mathcal{F}_{i+1} \xrightarrow{f_i^{\sharp}} \mathcal{F}_i, \mathcal{M}_i \otimes \mathcal{F}_{i+1} \xleftarrow{f_i^{\flat}} \mathcal{F}_i \quad (0 \le i \le r-1), h : \mathcal{E}_r \xrightarrow{\sim} \mathcal{F}_r),\end{aligned} \tag{5.12}$$

such that $\mu_i = 0$ for $i \in I$. For $i < j$, we put

$$\mathcal{E}_i^{[j]} := \mathrm{Ker}(\mathcal{E}_i \xrightarrow{e_{j-1}^{\flat} \circ \cdots \circ e_i^{\flat}} \bigotimes_{k=i}^{j-1} \mathcal{M}_k \otimes \mathcal{E}_j),$$
$$\mathcal{F}_i^{[j]} := \mathrm{Ker}(\mathcal{F}_i \xrightarrow{f_{j-1}^{\flat} \circ \cdots \circ f_i^{\flat}} \bigotimes_{k=i}^{j-1} \mathcal{M}_k \otimes \mathcal{F}_j),$$
$$\mathcal{E}_j^{[i]} := \mathrm{Ker}(\mathcal{E}_j \xrightarrow{e_i^{\sharp} \circ \cdots \circ e_{j-1}^{\sharp}} \mathcal{E}_i), \quad \mathcal{F}_j^{[i]} := \mathrm{Ker}(\mathcal{F}_j \xrightarrow{f_i^{\sharp} \circ \cdots \circ f_{j-1}^{\sharp}} \mathcal{F}_i).$$

For $i < k < j$, we put

$$\mathcal{E}_k^{[i][j]} := \mathcal{E}_k^{[i]} \cap \mathcal{E}_k^{[j]} \text{ and } \mathcal{F}_k^{[i][j]} := \mathcal{F}_k^{[i]} \cap \mathcal{F}_k^{[j]}. \tag{5.13}$$

Claim 5.4.1. $\mathcal{E}_r \supset \mathcal{E}_r^{[i_k]}$ and $\mathcal{F}_r \supset \mathcal{F}_r^{[i_k]}$ are subbundles of rank $r{-}i_k$ $(1 \le k \le l)$.

Proof of Claim 5.4.1. By Lemma 2.8 (2), $\mathrm{Im}(\mathcal{E}_{i_k+1} \xrightarrow{e_{i_k}^{\sharp}} \mathcal{E}_{i_k})$ is a rank $r + i_k$ subbundle of $\mathcal{E}_{i_k}$. By the condition 1 of Definition 3.1,

$$\mathcal{E}_r \xrightarrow{e_{i_k}^{\sharp} \circ \cdots \circ e_{r-1}^{\sharp}} \mathrm{Im}(\mathcal{E}_{i_k+1} \xrightarrow{e_{i_k}^{\sharp}} \mathcal{E}_{i_k})$$

is surjective. Hence $\mathcal{E}_r^{[i_k]}$ is a subbundle of rank $r - i_k$ of $\mathcal{E}_r$. $\square$

Put $\mathcal{E}_k \supset (\mathcal{F}_r^{[j]})_{<k>} := (e_k^\sharp \circ \cdots \circ e_{r-1}^\sharp \circ h^{-1})(\mathcal{F}_r^{[j]})$ and $\mathcal{F}_k \supset (\mathcal{E}_r^{[j]})_{<k>} := (f_k^\sharp \circ \cdots \circ f_{r-1}^\sharp \circ h)(\mathcal{E}_r^{[j]})$. By the condition 2 of Definition 3.1, $(\mathcal{F}_r^{[i_k]})_{<0>}$ and $(\mathcal{E}_r^{[i_k]})_{<0>}$ are subbundles of rank $r - i_k$ of $\mathcal{E}$ and $\mathcal{F}$ respectively. By the same reasoning in the proof of Claim 5.4.1, $\mathcal{E}_0^{[i_k+1]}$ and $\mathcal{F}_0^{[i_k+1]}$ are subbundles of rank $r+i_k$ of $\mathcal{E}$ and $\mathcal{F}$ respectively. So we obtained filtrations

$$\mathcal{E} \supset \mathcal{E}_0^{[i_l+1]} \supset \cdots \supset \mathcal{E}_0^{[i_1+1]} \supset (\mathcal{F}_r^{[i_1]})_{<0>} \supset \cdots \supset (\mathcal{F}_r^{[i_l]})_{<0>} \supset 0,$$
$$\mathcal{F} \supset \mathcal{F}_0^{[i_l+1]} \supset \cdots \supset \mathcal{F}_0^{[i_1+1]} \supset (\mathcal{E}_r^{[i_1]})_{<0>} \supset \cdots \supset (\mathcal{E}_r^{[i_l]})_{<0>} \supset 0.$$

Claim 5.4.2. $(\mathcal{F}_r^{[i_k]})_{<0>}^\perp = \mathcal{E}_0^{[i_k+1]}$ and $(\mathcal{E}_r^{[i_k]})_{<0>}^\perp = \mathcal{F}_0^{[i_k+1]}$ $(1 \le k \le l)$.

Proof of Claim 5.4.2. We shall check that the morphism

$$\left(\otimes_{j=1}^{i_k-1}\mathcal{M}_j^\vee \otimes \mathcal{E}_0^{[i_k+1]}\right) \otimes (\mathcal{F}_r^{[i_k]})_{<0>} \to \otimes_{j=1}^{i_k-1}\mathcal{M}_j^\vee$$

induced by $\pi_\mathcal{E}$ is zero. Take sections $x \in \bigotimes_{j=1}^{i_k-1}\mathcal{M}_j^\vee \otimes \mathcal{E}_0^{[i_k+1]}$ and $(e_0^\sharp \circ \cdots \circ e_{r-1}^\sharp \circ h^{-1})(y') \in (\mathcal{F}_r^{[i_k]})_{<0>}$ with $y' \in \mathcal{F}_r^{[i_k]}$. Since $\mathcal{E}_r \xrightarrow{e_{i_k}^\sharp \circ \cdots \circ e_{r-1}^\sharp} \mathcal{E}_{i_k}^{[i_k+1]}$ is surjective, we can find $x' \in \mathcal{E}_r$ such that $(x, x') \in \left(\bigotimes_{j=1}^{i_k-1}\mathcal{M}_j^\vee \otimes \mathcal{E}_0\right) \times_{\mathcal{E}_{i_k}} \mathcal{E}_r$. Since $y' \in \mathcal{F}_r^{[i_k]}$, we have $(0, y') \in \left(\bigotimes_{j=1}^{i_k-1}\mathcal{M}_j^\vee \otimes \mathcal{F}_0\right) \times_{\mathcal{F}_{i_k}} \mathcal{F}_r$. By the commutativity of the diagram 3.2, we have $\pi_\mathcal{E}(x \otimes (e_0^\sharp \circ \cdots \circ e_{r-1}^\sharp \circ h^{-1}(y')) = \pi_\mathcal{F}(x' \otimes 0) = 0$. Therefore $(\mathcal{F}_r^{[i_k]})_{<0>}^\perp \supset \mathcal{E}_0^{[i_k+1]}$. Both $(\mathcal{F}_r^{[i_k]})_{<0>}^\perp$ and $\mathcal{E}_0^{[i_k+1]}$ are subbundles of rank $r + i_k$, hence $(\mathcal{F}_r^{[i_k]})_{<0>}^\perp = \mathcal{E}_0^{[i_k+1]}$. $\square$

In particular $\mathcal{E} \supset (\mathcal{F}_r^{[i_k]})_{<0>}$ and $\mathcal{F} \supset (\mathcal{E}_r^{[i_k]})_{<0>}$ are isotropic subbundles, therefore the filtrations

$$\begin{aligned} &\mathcal{E} \supset (\mathcal{F}_r^{[i_1]})_{<0>} \supset \cdots \supset (\mathcal{F}_r^{[i_l]})_{<0>} \supset 0, \\ &\mathcal{F} \supset (\mathcal{E}_r^{[i_1]})_{<0>} \supset \cdots \supset (\mathcal{E}_r^{[i_l]})_{<0>} \supset 0 \end{aligned} \tag{5.14}$$

determine an S-valued point of $\mathbf{SpFl}_I$, and induce non-degenerate pairings

$$\begin{aligned} &\pi_{\mathcal{E},k} : \mathcal{E}_0^{[i_{k+1}+1]}/\mathcal{E}_0^{[i_k+1]} \otimes (\mathcal{F}_r^{[i_k]})_{<0>}/(\mathcal{F}_r^{[i_{k+1}]})_{<0>} \to \mathcal{P}, \\ &\pi_{\mathcal{F},k} : \mathcal{F}_0^{[i_{k+1}+1]}/\mathcal{F}_0^{[i_k+1]} \otimes (\mathcal{E}_r^{[i_k]})_{<0>}/(\mathcal{E}_r^{[i_{k+1}]})_{<0>} \to \mathcal{P}. \end{aligned} \tag{5.15}$$

The bf-morphisms of rank $j+i_1$

$$\Big(\mathcal{M}_j,\mu_j,\mathcal{E}_{j+1}^{[i_1+1]}/(\mathcal{F}_r^{[i_1]})_{<j+1>}\to\mathcal{E}_j^{[i_1+1]}/(\mathcal{F}_r^{[i_1]})_{<j>},$$
$$\mathcal{M}_j\otimes\mathcal{E}_{j+1}^{[i_1+1]}/(\mathcal{F}_r^{[i_1]})_{<j+1>}\leftarrow\mathcal{E}_j^{[i_1+1]}/(\mathcal{F}_r^{[i_1]})_{<j>}\Big),$$
$$\Big(\mathcal{M}_j,\mu_j,\mathcal{F}_{j+1}^{[i_1+1]}/(\mathcal{E}_r^{[i_1]})_{<j+1>}\to\mathcal{F}_j^{[i_1+1]}/(\mathcal{E}_r^{[i_1]})_{<j>},$$
$$\mathcal{M}_j\otimes\mathcal{F}_{j+1}^{[i_1+1]}/(\mathcal{E}_r^{[i_1]})_{<j+1>}\leftarrow\mathcal{F}_j^{[i_1+1]}/(\mathcal{E}_r^{[i_1]})_{<j>}\Big)$$

($0\le j\le i_1-1$) together with the isomorphism

$$\mathcal{E}_{i_1}^{[i_1+1]}/(\mathcal{F}_r^{[i_1]})_{<i_1>}\simeq\mathcal{E}_r/\Big(\mathcal{E}_r^{[i_1]}+h^{-1}(\mathcal{F}_r^{[i_1]})\Big)\simeq\mathcal{F}_r/\Big(h(\mathcal{E}_r^{[i_1]})+\mathcal{F}_r^{[i_1]}\Big)$$
$$\simeq\mathcal{F}_{i_1}^{[i_1+1]}/(\mathcal{E}_r^{[i_1]})_{<i_1>}$$

determine an S-valued point of $\mathrm{KSp}(\mathcal{E}_0^{[i_1+1]}/(\mathcal{F}_r^{[i_1]})_{<0>},\mathcal{F}_0^{[i_1+1]}/(\mathcal{E}_r^{[i_1]})_{<0>})$. For $i_k<j<i_{k+1}$ ($1\le k\le l$ and $i_{l+1}=r$ by convention), we can see that the induced tuples

$$\Big(\mathcal{M}_j,\mu_j,\mathcal{E}_{j+1}^{[i_k][i_{k+1}+1]}\to\mathcal{E}_j^{[i_k][i_{k+1}+1]},\mathcal{M}_j\otimes\mathcal{E}_{j+1}^{[i_k][i_{k+1}+1]}\leftarrow\mathcal{E}_j^{[i_k][i_{k+1}+1]}\Big),$$
$$\Big(\mathcal{M}_j,\mu_j,\mathcal{F}_{j+1}^{[i_k][i_{k+1}+1]}\to\mathcal{F}_j^{[i_k][i_{k+1}+1]},\mathcal{M}_j\otimes\mathcal{F}_{j+1}^{[i_k][i_{k+1}+1]}\leftarrow\mathcal{F}_j^{[i_k][i_{k+1}+1]}\Big) \tag{5.16}$$

are bf-morphisms of rank $j-i_k$. The isomorphisms

$$(\mathcal{E}_0^{[i_{k+1}+1]}/\mathcal{E}_0^{[i_k+1]})\otimes\mathcal{M}_{i_k}^\vee\otimes\cdots\otimes\mathcal{M}_1^\vee\otimes\mathcal{M}_0^\vee\simeq\mathcal{E}_{i_k+1}^{[i_k][i_{k+1}+1]},$$
$$(\mathcal{F}_0^{[i_{k+1}+1]}/\mathcal{F}_0^{[i_k+1]})\otimes\mathcal{M}_{i_k}^\vee\otimes\cdots\otimes\mathcal{M}_1^\vee\otimes\mathcal{M}_0^\vee\simeq\mathcal{F}_{i_k+1}^{[i_k][i_{k+1}+1]}$$

induce bf-morphisms of rank 0

$$\Big(\otimes_{a=0}^{i_k}\mathcal{M}_a,0,\mathcal{E}_{i_k+1}^{[i_k][i_{k+1}+1]}\to\mathcal{E}_0^{[i_{k+1}+1]}/\mathcal{E}_0^{[i_k+1]},$$
$$\otimes_{a=0}^{i_k}\mathcal{M}_a\otimes\mathcal{E}_{i_k+1}^{[i_k][i_{k+1}+1]}\leftarrow\mathcal{E}_0^{[i_{k+1}+1]}/\mathcal{E}_0^{[i_k+1]}\Big),$$
$$\Big(\otimes_{a=0}^{i_k}\mathcal{M}_a,0,\mathcal{F}_{i_k+1}^{[i_k][i_{k+1}+1]}\to\mathcal{F}_0^{[i_{k+1}+1]}/\mathcal{F}_0^{[i_k+1]}, \tag{5.17}$$
$$\otimes_{a=0}^{i_k}\mathcal{M}_a\otimes\mathcal{F}_{i_k+1}^{[i_k][i_{k+1}+1]}\leftarrow\mathcal{F}_0^{[i_{k+1}+1]}/\mathcal{F}_0^{[i_k+1]}\Big).$$

We also have isomorphisms

$$\mathcal{E}_{i_{k+1}}^{[i_k][i_{k+1}+1]}\simeq\mathcal{E}_r^{[i_k]}/\mathcal{E}_r^{[i_{k+1}]}\simeq(\mathcal{E}_r^{[i_k]})_{<0>}/(\mathcal{E}_r^{[i_{k+1}]})_{<0>},$$
$$\mathcal{F}_{i_{k+1}}^{[i_k][i_{k+1}+1]}\simeq\mathcal{F}_r^{[i_k]}/\mathcal{F}_r^{[i_{k+1}]}\simeq(\mathcal{F}_r^{[i_k]})_{<0>}/(\mathcal{F}_r^{[i_{k+1}]})_{<0>}. \tag{5.18}$$

The data (5.16), (5.17) and (5.18) determine an S-valued point of $\mathcal{Q}(\pi_{\mathcal{E},1},\pi_{\mathcal{F},1})\times_{\mathbf{SpFl}_I}\cdots\times_{\mathbf{SpFl}_I}\mathcal{Q}(\pi_{\mathcal{E},l},\pi_{\mathcal{F},l})$. This defines the morphism (5.11).

Now we shall construct the inverse of (5.11). An S-valued point of

$$\mathrm{KSp}(\mathbb{F}_1(\widetilde{\mathcal{E}})^{\perp}/\mathbb{F}_1(\widetilde{\mathcal{E}}),\mathbb{F}_1(\widetilde{\mathcal{F}})^{\perp}/\mathbb{F}_1(\widetilde{\mathcal{F}}))\times_{\mathbf{SpFl}_I}\mathcal{Q}$$

is data:

- $\mathcal{E}\supset\mathbb{F}_1(\mathcal{E})\supset\cdots\supset\mathbb{F}_l(\mathcal{E})\supset 0$, $\mathcal{F}\supset\mathbb{F}_1(\mathcal{F})\supset\cdots\supset\mathbb{F}_l(\mathcal{F})\supset 0$, where $\mathbb{F}_j(\mathcal{E})$ and $\mathbb{F}_j(\mathcal{F})$ are isotropic subbundles of rank $r-i_j$ of $\mathcal{E}$ and $\mathcal{F}$ respectively,
- a generalized symplectic isomorphism from $\mathbb{F}_1(\mathcal{E})^{\perp}/\mathbb{F}_1(\mathcal{E})$ to $\mathbb{F}_1(\mathcal{F})^{\perp}/\mathbb{F}_1(\mathcal{F})$

$$\Big(\mathcal{M}'_j,\mu'_j,\mathcal{G}_{j+1}\xrightarrow{g_j^{\sharp}}\mathcal{G}_j,\mathcal{M}'_j\otimes\mathcal{G}_{j+1}\xleftarrow{g_j^{\flat}}\mathcal{G}_j,$$
$$\mathcal{H}_{j+1}\xrightarrow{h_j^{\sharp}}\mathcal{H}_j,\mathcal{M}'_j\otimes\mathcal{H}_{j+1}\xleftarrow{h_j^{\flat}}\mathcal{H}_j,\overline{h}:\mathcal{G}_{i_1}\to\mathcal{H}_{i_1}\ (0\le j\le i_1-1)\Big),$$

- an object of $\mathcal{Q}(\pi_{\mathcal{E},k},\pi_{\mathcal{F},k})$ $(1\le k\le l)$

$$\Big(\mathcal{M}'_j,\mu'_j,\mathcal{G}^{(k)}_{j+1}\to\mathcal{G}^{(k)}_j,\mathcal{M}'_j\otimes\mathcal{G}^{(k)}_{j+1}\leftarrow\mathcal{G}^{(k)}_j,$$
$$\mathcal{H}^{(k)}_{j+1}\to\mathcal{H}^{(k)}_j,\mathcal{M}'_j\otimes\mathcal{H}^{(k)}_{j+1}\leftarrow\mathcal{H}^{(k)}_j\ (i_k\le j\le i_{k+1}-1)\Big)$$

with $\mathcal{G}^{(k)}_{i_k}=\mathbb{F}_{k+1}(\mathcal{E})^{\perp}/\mathbb{F}_k(\mathcal{E})^{\perp}$, $\mathcal{H}^{(k)}_{i_k}=\mathbb{F}_{k+1}(\mathcal{F})^{\perp}/\mathbb{F}_k(\mathcal{F})^{\perp}$, $\mathcal{G}^{(k)}_{i_{k+1}}=\mathbb{F}_k(\mathcal{F})/\mathbb{F}_{k+1}(\mathcal{F})$ and $\mathcal{H}^{(k)}_{i_{k+1}}=\mathbb{F}_k(\mathcal{E})/\mathbb{F}_{k+1}(\mathcal{E})$, where

$$\pi_{\mathcal{E},k}:\mathbb{F}_{k+1}(\mathcal{E})^{\perp}/\mathbb{F}_k(\mathcal{E})^{\perp}\otimes\mathbb{F}_{k+1}(\mathcal{E})/\mathbb{F}_k(\mathcal{E})\to\mathcal{P},$$
$$\pi_{\mathcal{F},k}:\mathbb{F}_{k+1}(\mathcal{F})^{\perp}/\mathbb{F}_k(\mathcal{F})^{\perp}\otimes\mathbb{F}_{k+1}(\mathcal{F})/\mathbb{F}_k(\mathcal{F})\to\mathcal{P}.$$

Then we put $\mathcal{M}_i:=\mathcal{M}'_i$, $\mu_i:=\mu'_i$ for $i\notin I$. For $i=i_k$, $\mathcal{M}_{i_k}:=\mathcal{M}'_{i_k}\otimes\bigotimes_{j=0}^{i_k-1}\mathcal{M}_j^{\vee}$ and $\mu_{i_k}=0$. For $0\le j\le i_1$, put $\widetilde{\mathcal{G}}_j:=\mathbb{F}_1(\mathcal{E})^{\perp}\times_{\mathcal{G}_0}\mathcal{G}_j$ and $\widetilde{\mathcal{H}}_j:=\mathbb{F}_1(\mathcal{F})^{\perp}\times_{\mathcal{H}_0}\mathcal{H}_j$. Then for $0\le j\le i_1-1$, we have bf-morphisms of rank $r+j$

$$\Big(\mathcal{M}_j,\mu_j,\widetilde{\mathcal{G}}_{j+1}\to\widetilde{\mathcal{G}}_j,\mathcal{M}_j\otimes\widetilde{\mathcal{G}}_{j+1}\leftarrow\widetilde{\mathcal{G}}_j\Big),$$
$$\Big(\mathcal{M}_j,\mu_j,\widetilde{\mathcal{H}}_{j+1}\to\widetilde{\mathcal{H}}_j,\mathcal{M}_j\otimes\widetilde{\mathcal{H}}_{j+1}\leftarrow\widetilde{\mathcal{H}}_j\Big).$$

For $0\le j\le i_1$, we define $\mathcal{E}_j$ and $\mathcal{F}_j$ so that the diagrams

$$\begin{array}{ccc} \widetilde{\mathcal{G}}_0 & \to & \widetilde{\mathcal{G}}_j\otimes\bigotimes_{a=0}^{j-1}\mathcal{M}_a \\ \downarrow & & \downarrow \\ \mathcal{E} & \to & \bigotimes_{a=0}^{j-1}\mathcal{M}_a\otimes\mathcal{E}_j, \end{array}\qquad \begin{array}{ccc} \widetilde{\mathcal{H}}_0 & \to & \widetilde{\mathcal{H}}_j\otimes\bigotimes_{a=0}^{j-1}\mathcal{M}_a \\ \downarrow & & \downarrow \\ \mathcal{F} & \to & \bigotimes_{a=0}^{j-1}\mathcal{M}_a\otimes\mathcal{F}_j \end{array}$$

are cocartesian.

Then for $0 \le j \le i_1 - 1$, we have bf-morphisms of rank $r + j$

$$(\mathcal{M}_j, \mu_j, \mathcal{E}_{j+1} \to \mathcal{E}_j, \mathcal{M}_j \otimes \mathcal{E}_{j+1} \leftarrow \mathcal{E}_j),$$
$$(\mathcal{M}_j, \mu_j, \mathcal{F}_{j+1} \to \mathcal{F}_j, \mathcal{M}_j \otimes \mathcal{F}_{j+1} \leftarrow \mathcal{F}_j).$$

We define $\mathcal{E}_r = \mathcal{F}_r$ by the cartesian diagram:

$$\begin{array}{ccc} \mathcal{E}_r = \mathcal{F}_r & \to & \widetilde{\mathcal{G}}_{i_1} \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{H}}_{i_1} & \to & \mathcal{G}_{i_1} \simeq \mathcal{H}_{i_1}. \end{array}$$

Then we have

$$\mathrm{Ker}(\mathcal{E}_r \to \widetilde{\mathcal{G}}_{i_1}) \simeq \mathbb{F}_1(\mathcal{F}) \text{ and } \mathrm{Ker}(\mathcal{F}_r \to \widetilde{\mathcal{H}}_{i_1}) \simeq \mathbb{F}_1(\mathcal{E}).$$

By this we can consider $\mathbb{F}_1(\mathcal{F}) \supset \cdots \supset \mathbb{F}_l(\mathcal{F}) \supset 0$ and $\mathbb{F}_1(\mathcal{E}) \supset \cdots \supset \mathbb{F}_l(\mathcal{E}) \supset 0$ as filtrations of $\mathcal{E}_r$ and $\mathcal{F}_r$ respectively.

For $i_k < p \le i_{k+1}$ $(1 \le k \le l)$, we define $\mathcal{G}_p^\star$, $\mathcal{H}_p^\star$, $\mathcal{G}_p^\circ$ and $\mathcal{H}_p^\circ$ by the cocartesian diagrams:

$$\begin{array}{ccc} \mathcal{G}_{i_{k+1}}^{(k)} = \mathbb{F}_k(\mathcal{F})/\mathbb{F}_{k+1}(\mathcal{F}) & \hookrightarrow & \mathcal{E}_r/\mathbb{F}_{k+1}(\mathcal{F}) \\ \downarrow & & \downarrow \\ \mathcal{G}_p^{(k)} & \to & \mathcal{G}_p^\star, \end{array}$$

$$\begin{array}{ccc} \mathcal{H}_{i_{k+1}}^{(k)} = \mathbb{F}_k(\mathcal{E})/\mathbb{F}_{k+1}(\mathcal{E}) & \hookrightarrow & \mathcal{F}_r/\mathbb{F}_{k+1}(\mathcal{E}) \\ \downarrow & & \downarrow \\ \mathcal{H}_p^{(k)} & \to & \mathcal{H}_p^\star, \end{array}$$

$$\begin{array}{ccc} \mathcal{G}_{i_k+1}^{(k)} \otimes \bigotimes_{a=i_k+1}^{p-1} \mathcal{M}_a^\vee & \hookrightarrow & \left(\mathcal{E}/\mathbb{F}_k(\mathcal{E})^\perp \otimes \bigotimes_{a=0}^{p-1} \mathcal{M}_a^\vee\right) \\ \downarrow & & \downarrow \\ \mathcal{G}_p^{(k)} & \to & \mathcal{G}_p^\circ, \end{array}$$

$$\begin{array}{ccc} \mathcal{H}_{i_k+1}^{(k)} \otimes \bigotimes_{a=i_k+1}^{p-1} \mathcal{M}_a^\vee & \hookrightarrow & \left(\mathcal{F}/\mathbb{F}_k(\mathcal{F})^\perp \otimes \bigotimes_{a=0}^{p-1} \mathcal{M}_a^\vee\right) \\ \downarrow & & \downarrow \\ \mathcal{H}_p^{(k)} & \to & \mathcal{H}_p^\circ, \end{array}$$

and $\mathcal{E}_p$ and $\mathcal{F}_p$ by the cocartesian diagrams:

$$\begin{array}{ccc} \mathcal{G}_p^{(k)} & \to & \mathcal{G}_p^\circ \\ \downarrow & & \downarrow \\ \mathcal{G}_p^\star & \to & \mathcal{E}_p, \end{array} \qquad \begin{array}{ccc} \mathcal{H}_p^{(k)} & \to & \mathcal{H}_p^\circ \\ \downarrow & & \downarrow \\ \mathcal{H}_p^\star & \to & \mathcal{F}_p. \end{array}$$

Then for $i_k < p \le i_{k+1} - 1$, we have bf-morphisms of rank $r + p$

$$(\mathcal{M}_p, \mu_p, \mathcal{E}_{p+1} \to \mathcal{E}_p, \mathcal{M}_p \otimes \mathcal{E}_{p+1} \leftarrow \mathcal{E}_p),$$
$$(\mathcal{M}_p, \mu_p, \mathcal{F}_{p+1} \to \mathcal{F}_p, \mathcal{M}_p \otimes \mathcal{F}_{p+1} \leftarrow \mathcal{F}_p). \tag{5.19}$$

Moreover for $1 \le k < l$ we have morphisms

$$\begin{aligned}\mathcal{E}_{i_{k+1}} \to \mathcal{E}_{i_{k+1}}/\mathcal{G}^{\star}_{i_{k+1}} &\simeq \mathcal{G}^{\circ}_{i_{k+1}}/\mathcal{G}^{(k)}_{i_{k+1}} \\ &\simeq (\mathcal{E}/\mathbb{F}_{k+1}(\mathcal{E})^{\perp}) \otimes \otimes_{a=0}^{i_{k+1}-1}\mathcal{M}_a^{\vee} \\ &\simeq \{(\mathcal{E}/\mathbb{F}_{k+1}(\mathcal{E})^{\perp}) \otimes \otimes_{a=0}^{i_{k+1}}\mathcal{M}_a^{\vee}\} \otimes \mathcal{M}_{i_{k+1}} \\ &\simeq \mathcal{G}^{\circ}_{i_{k+1}+1} \otimes \mathcal{M}_{i_{k+1}} \hookrightarrow \mathcal{E}_{i_{k+1}+1} \otimes \mathcal{M}_{i_{k+1}}\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}_{i_{k+1}+1} \to \mathcal{E}_{i_{k+1}+1}/\mathcal{G}^{\circ}_{i_{k+1}+1} &\simeq \mathcal{G}^{\star}_{i_{k+1}+1}/\mathcal{G}^{(k+1)}_{i_{k+1}+1} \\ &\simeq \mathcal{E}_r/\mathbb{F}_{k+1}(\mathcal{F}) = \mathcal{G}^{\star}_{i_{k+1}} \hookrightarrow \mathcal{E}_{i_{k+1}}.\end{aligned}$$

So we have bf-morphism of $r + i_{k+1}$

$$\begin{aligned}&\big(\mathcal{M}_{i_{k+1}}, \mu_{i_{k+1}} = 0, \mathcal{E}_{i_{k+1}+1} \to \mathcal{E}_{i_{k+1}}, \mathcal{M}_{i_{k+1}} \otimes \mathcal{E}_{i_{k+1}+1} \leftarrow \mathcal{E}_{i_{k+1}}\big) \\ &\big(\mathcal{M}_{i_{k+1}}, \mu_{i_{k+1}} = 0, \mathcal{F}_{i_{k+1}+1} \to \mathcal{F}_{i_{k+1}}, \mathcal{M}_{i_{k+1}} \otimes \mathcal{F}_{i_{k+1}+1} \leftarrow \mathcal{F}_{i_{k+1}}\big) \quad (1 \le k < l).\end{aligned} \tag{5.20}$$

We also have morphisms

$$\mathcal{E}_{i_1} \to \mathcal{E}_{i_1}/\widetilde{\mathcal{G}}_{i_1} \simeq \mathcal{E}/\mathbb{F}_1(\mathcal{E})^{\perp} \otimes \otimes_{a=0}^{i_1-1}\mathcal{M}_a^{\vee} \simeq \mathcal{E}_{i_1+1} \otimes \mathcal{M}_{i_1}$$
$$\mathcal{E}_{i_1+1} \to \mathcal{E}_{i_1+1}/\mathcal{G}^{\circ}_{i_1+1} \simeq \mathcal{G}^{\star}_{i_1+1}/\mathcal{G}^{(1)}_{i_1+1} \simeq \mathcal{E}_r/\mathbb{F}_1(\mathcal{F}) \simeq \widetilde{\mathcal{G}}_{i_1} \hookrightarrow \mathcal{E}_{i_1}.$$

Hence we have bf-morphisms of rank $r + i_1$

$$\begin{aligned}&(\mathcal{M}_{i_1}, \mu_{i_1} = 0, \mathcal{E}_{i_1+1} \to \mathcal{E}_{i_1}, \mathcal{M}_{i_1} \otimes \mathcal{E}_{i_1+1} \leftarrow \mathcal{E}_{i_1}), \\ &(\mathcal{M}_{i_1}, \mu_{i_1} = 0, \mathcal{F}_{i_1+1} \to \mathcal{F}_{i_1}, \mathcal{M}_{i_1} \otimes \mathcal{F}_{i_1+1} \leftarrow \mathcal{F}_{i_1}).\end{aligned} \tag{5.21}$$

Then the data (5.19), (5.20), (5.21) and $\mathcal{E}_r = \mathcal{F}_r$ determine an S-valued point of X_I. $\square$

We denote by ι_I the inclusion $X_I \hookrightarrow \mathrm{KSp}(\mathcal{E}, \mathcal{F})$. We denote the set$\{0, 1, \ldots, r-1\}$ by $[0, r-1]$. When $I = [0, r-1]$, the isomorphism (5.11) is

$$X_{[0,r-1]} \simeq \mathbf{SpFl}_{[0,r-1]}, \tag{5.22}$$

and for the universal filtrations (5.10) on $\mathbf{SpFl}_{[0,r-1]}$, we have $l = r$ and $\mathrm{rank}\mathbb{F}_j(\widetilde{\mathcal{E}}) = \mathrm{rank}\mathbb{F}_j(\widetilde{\mathcal{F}}) = r + 1 - j$.

Notation 5.5. For tuples $(a_1, \ldots, a_r)$ and $(b_1, \ldots, b_r)$ of integers, we denote by $\mathcal{O}(a_1, \ldots, a_r; b_1, \ldots, b_r)$ the line bundle

$$\bigotimes_{j=1}^{r}\Big(\mathbb{F}_{r+2-j}(\widetilde{\mathcal{E}})^{\perp}/\mathbb{F}_{r+1-j}(\widetilde{\mathcal{E}})^{\perp}\Big)^{\otimes a_j} \otimes \bigotimes_{j=1}^{r}\Big(\mathbb{F}_{r+2-j}(\widetilde{\mathcal{F}})^{\perp}/\mathbb{F}_{r+1-j}(\widetilde{\mathcal{F}})^{\perp}\Big)^{\otimes b_j}$$

on $\mathbf{SpFl}_{[0,r-1]}(= \mathrm{SpFl}_{[0,r-1]}(\mathcal{E}) \times_S \mathrm{SpFl}_{[0,r-1]}(\mathcal{F}))$.

We often identify $X_{[0,r-1]}$ with $\mathbf{SpFl}_{[0,r-1]}$ by the isomorphism (5.22).

Lemma 5.6. *Let*

$$\begin{aligned}\Phi =&(\mathcal{M}_i, \mu_i, \mathcal{E}_i \to \mathcal{M}_i \otimes \mathcal{E}_{i+1}, \mathcal{E}_i \leftarrow \mathcal{E}_{i+1}, \\ &\mathcal{F}_{i+1} \to \mathcal{F}_i, \mathcal{M}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \quad (0 \leq i \leq r-1), h : \mathcal{E}_r \to \mathcal{F}_r),\end{aligned} \tag{5.23}$$

be the universal generalized symplectic isomorphism from $\mathcal{E}_0 = (\mathcal{E})_{KSp(\mathcal{E},\mathcal{F})}$ *to* $\mathcal{F}_0 = (\mathcal{F})_{KSp(\mathcal{E},\mathcal{F})}$.

There are natural isomorphisms

$$\iota^*_{[0,r-1]}\mathcal{M}_0 \simeq \mathcal{O}(\mathbf{e}_r; \mathbf{e}_r) \otimes pr_S^*\mathcal{P}^\vee,$$

and for $1 \leq j \leq r-1$

$$\iota^*_{[0,r-1]}\mathcal{M}_j \simeq \mathcal{O}(\mathbf{e}_{r-j} - \mathbf{e}_{r-j+1}; \mathbf{e}_{r-j} - \mathbf{e}_{r-j+1})$$

of line bundles on $X_{[0,r-1]} \simeq \mathbf{SpFl}_{[0,r-1]}$, *where*

$$\mathbf{e}_i := (0, \ldots, 0, \overset{i\text{-}th}{1}, 0, \ldots, 0).$$

Proof. This lemma follows from the correspondence of scheme-valued points of $X_{[0,r-1]}$ and $\mathbf{SpFl}_{[0,r-1]}$ given in Proposition 5.4:

Using the notation of the proof of Proposition 5.4, we have

$$\bigotimes_{a=0}^{j} \mathcal{M}_a^\vee \otimes \left(\frac{\mathbb{F}_{j+2}(\mathcal{E})^\perp}{\mathbb{F}_{j+1}(\mathcal{E})^\perp}\right) \simeq \mathcal{E}_{j+1}^{[j][j+2]} \simeq \frac{\mathbb{F}_{j+1}(\mathcal{F})}{\mathbb{F}_{j+2}(\mathcal{F})}. \tag{5.24}$$

□

§ 6. Global sections

Let S be a scheme over $\operatorname{Spec} k$ with k an algebraically closed field of characteristic zero. Let $\mathcal{P}$ be a line bundle on S, and $\mathcal{E}$, $\mathcal{F}$ locally free $\mathcal{O}_S$-modules of rank $2r$ with non-degenerate alternate bilinear forms $\pi_\mathcal{E} : \mathcal{E} \otimes \mathcal{E} \to \mathcal{P}$ and $\pi_\mathcal{F} : \mathcal{F} \otimes \mathcal{F} \to \mathcal{P}$.

If $g : \mathcal{E} \to \mathcal{F}$ is a symplectic isomorphism, then composing it with symplectic isomorphisms $\gamma : \mathcal{E} \to \mathcal{E}$ and $\delta : \mathcal{F} \to \mathcal{F}$, we obtain a symplectic isomorphism $\delta \circ g \circ \gamma^{-1} : \mathcal{E} \to \mathcal{F}$. This induces a left action on $Sp(\mathcal{E}, \mathcal{F})$ of the group S-scheme $Sp(\mathcal{E}) \times_S Sp(\mathcal{F})$.

For a generalized symplectic isomorphism Φ from $\mathcal{E}$ to $\mathcal{F}$, we can also consider the composition $\delta \circ \Phi \circ \gamma^{-1}$ (See Paragraph 3.2). So the action of $Sp(\mathcal{E}) \times_S Sp(\mathcal{F})$ extends to $KSp(\mathcal{E}, \mathcal{F})$. Moreover the action naturally lifts to the line bundles $\bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes c_i}$ ($c_i \in \mathbb{Z}$).

The subschemes $X_I \subset KSp(\mathcal{E},\mathcal{F})$ $(I \subset [0, r-1])$ are stable under the action. Thus vector bundles $pr_{S*}\iota_I^* \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes c_i}$ $(c_i \in \mathbb{Z})$ on S have an action of $Sp(\mathcal{E}) \times_S Sp(\mathcal{F})$ (Here we consider left action). The goal of this section is to describe this action.

The arguments in this section are straightforward translation of the corresponding arguments in [Kausz2] to the symplectic case.

We shall use the following well-known theorem in the sequel.

Theorem 6.1. *If $S = \mathrm{Spec}K$ with K a field of characteristic zero, then for tuples of integers $\overrightarrow{a} = (a_1, \dots, a_r)$ and $\overrightarrow{b} = (b_1, \dots, b_r)$,*

$$\mathrm{H}^0\left(\mathbf{SpFl}_{[0,r-1]}, \mathcal{O}(\overrightarrow{a}; \overrightarrow{b})\right) \neq 0$$

if and only if $a_1 \geq \cdots \geq a_r \geq 0$ and $b_1 \geq \cdots \geq b_r \geq 0$. When it is nonzero, it is an irreducible $Sp(\mathcal{E}) \times_S Sp(\mathcal{F})$ -module.

Definition 6.2. For a tuple of integers $(c_0, \dots, c_{r-1}) \in \mathbb{Z}^{\oplus r}$ and a subset $I \subset [0, r-1]$, the set $\mathbb{A}(c_0, \dots, c_{r-1})_I$ is defined to consist of tuples of integers $\overrightarrow{q} = (q_1, \dots, q_r)$ such that

(i) $q_1 \geq \cdots \geq q_r \geq 0$,
(ii) $\sum_{i=1}^{l} q_i \leq c_{r-l}$ if $r - l \notin I$ and $\sum_{i=1}^{l} q_i = c_{r-l}$ if $r - l \in I$.

For $\overrightarrow{q} = (q_1, \dots, q_r)$, we denote by $|\overrightarrow{q}|$ the sum $\sum_{i=1}^{r} q_i$.

Theorem 6.3. (1) *Let $(c_0, \dots, c_{r-1})$ be a tuple of integers. There is a unique direct sum decomposition of the vector bundle $pr_{S*}\iota_I^* \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes c_i}$ indexed by $\mathbb{A}(c_0, \dots, c_{r-1})_I$*

$$pr_{S*}\iota_I^* \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes c_i} = \bigoplus_{\overrightarrow{q} \in \mathbb{A}(c_0, \dots, c_{r-1})_I} \mathcal{V}_{\overrightarrow{q}}^{(c_0, \dots, c_{r-1})}$$

such that

(*a*) *$\mathcal{V}_{\overrightarrow{q}}^{(c_0, \dots, c_{r-1})}$ is a $Sp(\mathcal{E}) \times_S Sp(\mathcal{F})$-stable vector subbundle of $pr_{S*}\iota_I^* \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes c_i}$,*

(*b*) *For every $\overrightarrow{q} \in \mathbb{A}(c_0, \dots, c_{r-1})_I$, the direct summand $\mathcal{V}_{\overrightarrow{q}}^{(c_0, \dots, c_{r-1})}$ is included in the subbundle $pr_{S*}\iota_I^* \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes \sum_{j=1}^{r-i} q_j} \subset pr_{S*}\iota_I^* \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes c_i}$,*

(*c*) *The composite of $Sp(\mathcal{E}) \times_S Sp(\mathcal{F})$-equivariant morphisms*

$$\mathcal{V}_{\overrightarrow{q}}^{(c_0, \dots, c_{r-1})} \hookrightarrow pr_{S*}\iota_I^* \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes \sum_{j=1}^{r-i} q_j} \to pr_{S*}\iota_{[0,r-1]}^* \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes \sum_{j=1}^{r-i} q_j}$$

is an isomorphism.

(2) *For two tuples $(c_0, \dots, c_{r-1})$ and $(c'_0, \dots, c'_{r-1})$ with $c_j \geq c'_j$ for $0 \leq j \leq r-1$, the subbundle*

$$\bigoplus_{\overrightarrow{q} \in \mathbb{A}(c'_0, \dots, c'_{r-1})_I} \mathcal{V}_{\overrightarrow{q}}^{(c_0, \dots, c_{r-1})} \subset \bigoplus_{\overrightarrow{q} \in \mathbb{A}(c_0, \dots, c_{r-1})_I} \mathcal{V}_{\overrightarrow{q}}^{(c_0, \dots, c_{r-1})} = pr_{S*}\iota_I^* \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes c_i}$$

is equal to the subbundle $pr_{S*}\iota_I^* \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes c_i'} \subset pr_{S*}\iota_I^* \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes c_i}$.

The direct sum decomposition $\bigoplus_{\overrightarrow{q} \in \mathbb{A}(c_0',\dots,c_{r-1}')_I} \mathcal{V}_{\overrightarrow{q}}^{(c_0,\dots,c_{r-1})}$ *gives the direct sum decomposition of* $pr_{S*}\iota_I^* \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes c_i'}$ *satisfying* (*a*), (*b*), (*c*) *in* (1), *that is,* $\mathcal{V}_{\overrightarrow{q}}^{(c_0,\dots,c_{r-1})} = \mathcal{V}_{\overrightarrow{q}}^{(c_0',\dots,c_{r-1}')}$ *for* $\overrightarrow{q} \in \mathbb{A}(c_0',\dots,c_{r-1}')_I$.

Before starting the proof of the theorem, we present two corollaries.

Corollary 6.4. *There is a natural isomorphism*

$$pr_{S*} \otimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes n(r-i)} \simeq \bigoplus_{\overrightarrow{q}} pr_{S*}\mathcal{O}(\overrightarrow{q};\overrightarrow{q}) \otimes \mathcal{P}^{-|\overrightarrow{q}|}$$

of $Sp(\mathcal{E}) \times_S Sp(\mathcal{F})$*-equivariant vector bundles on* S*, where* $\overrightarrow{q} = (q_1,\dots,q_r)$ *varies through all tuples of integers with* $n \geq q_1 \geq \cdots \geq q_r \geq 0$*, and* pr_S *on the left is the projection of* $KSp(\mathcal{E},\mathcal{F})$ *to* S*, and* pr_S *on the right is the projection of* $\mathrm{SpFl}_{[0,r-1]}(\mathcal{E}) \times_S \mathrm{SpFl}_{[0,r-1]}(\mathcal{F})$ *to* S.

Proof. Take $I = \emptyset$ in the above theorem, and use Lemma 5.6. □

Corollary 6.5. *Let* $0 \to \mathcal{U} \to pr_S^*(\mathcal{E} \oplus \mathcal{F}) \to \mathcal{Q} \to 0$ *be the universal sequence on* $LGr(\mathcal{E} \oplus \mathcal{F})$*. Then there is a natural isomorphism*

$$pr_{S*}(\det \mathcal{Q})^{\otimes n} \simeq \bigoplus_{\overrightarrow{q}} pr_{S*}\mathcal{O}(\overrightarrow{q};\overrightarrow{q}) \otimes \mathcal{P}^{\otimes(nr-|\overrightarrow{q}|)}$$

of $Sp(\mathcal{E}) \times_S Sp(\mathcal{F})$*-equivariant vector bundles on* S*, where* $\overrightarrow{q} = (q_1,\dots,q_r)$ *varies through all tuples of integers with* $n \geq q_1 \geq \cdots \geq q_r \geq 0$*, and* pr_S *on the left is the projection of* $LGr(\mathcal{E} \oplus \mathcal{F})$ *to* S*, and* pr_S *on the right is the projection of* $\mathrm{SpFl}_{[0,r-1]}(\mathcal{E}) \times_S \mathrm{SpFl}_{[0,r-1]}(\mathcal{F})$ *to* S.

Proof. Let $g : KSp(\mathcal{E},\mathcal{F}) \to LGr(\mathcal{E} \oplus \mathcal{F})$ be the morphism in Proposition 4.1. We have the pull-back morphism

$$g^* : pr_{S*}(\det \mathcal{Q})^{\otimes n} \to pr_{S*}g^*(\det \mathcal{Q})^{\otimes n},$$

where pr_S on the right-hand side is the projection of $KSp(\mathcal{E},\mathcal{F})$. To show that g^* is an isomorphism, we may assume that $S = \operatorname{Spec} k$ because locally on S, the bundles $\mathcal{E}$, $\mathcal{F}$, $\mathcal{P}$ and the bilinear forms are pull-backs of those on Spec$k$. Since g is proper birational and $LGr(\mathcal{E} \oplus \mathcal{F})$ is smooth (hence normal), we have $g_*\mathcal{O}_{KSp(\mathcal{E},\mathcal{F})} \simeq \mathcal{O}_{LGr(\mathcal{E}\oplus\mathcal{F})}$ by [EGAIII, Corollaire 4.3.12]. From this and the projection formula, it follows that g^* is an isomorphism.

By Lemma 4.2, we have a natural isomorphism

$$g^*(\det \mathcal{Q})^{\otimes n} \simeq pr_S^* \mathcal{P}^{\otimes nr} \otimes \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes n(r-i)}.$$

Now the corollary follows from Corollary 6.4. □

Now we move on to the proof of Theorem 6.3. Since locally on S, the bundles $\mathcal{E}$, $\mathcal{F}$, $\mathcal{P}$ and the bilinear forms are pull-backs of those on $\operatorname{Spec} k$, we have only to prove the theorem for $S = \operatorname{Spec} k$. We may assume that $\mathcal{E} = \mathcal{F} = k^{\oplus 2r}$ and the nondegenerate bilinear forms of $\mathcal{E}$ and $\mathcal{F}$ are given by the matrix J_{2r}. In the rest of this section, we write E and F instead of $\mathcal{E}$ and $\mathcal{F}$.

Let $\mathbb{T}_{\mathrm{Sp}_{2r}} \subset \mathrm{Sp}_{2r}(k)$ be the subgroup of consisting of diagonal matrices in $\mathrm{Sp}_{2r}(k)$. Put $\mathbb{B}_E := \mathrm{U}_{2r}^+ \mathbb{T}_{\mathrm{Sp}_{2r}} \subset \mathrm{Sp}(E) = \mathrm{Sp}_{2r}(k)$ and $\mathbb{B}_F := \mathrm{U}_{2r}^- \mathbb{T}_{\mathrm{Sp}_{2r}} \subset \mathrm{Sp}(F) = \mathrm{Sp}_{2r}(k)$. Let

$$\mathrm{U}_{2r}^+ \times \mathbb{A}^r \times \mathrm{U}_{2r}^- \simeq KSp(E,F)^{(\mathrm{id},\mathrm{id})} \tag{6.1}$$

be the isomorphism (3.23). The restriction of (6.1) to the open subscheme $Sp(E,F)^{(\mathrm{id},\mathrm{id})} := KSp(E,F)^{(\mathrm{id},\mathrm{id})} \cap Sp(E,F)$ gives an isomorphism

$$\mathrm{U}_{2r}^+ \times (\mathbb{A} \setminus \{0\})^r \times \mathrm{U}_{2r}^- \simeq Sp(E,F)^{(\mathrm{id},\mathrm{id})},$$

which is given by $\mathrm{U}_{2r}^+ \times (\mathbb{A}\setminus\{0\})^r \times \mathrm{U}_{2r}^- \ni (z, \mathbf{y}, x) \mapsto x \circ D_{\mathbf{y}} \circ z^{-1}$, where $\mathbf{y} = (y_0, \ldots, y_{r-1})$ and

$$D_{\mathbf{y}} = \operatorname{diag}\left((\prod_{i=0}^{r-1} y_i)^{-1}, \prod_{i=0}^{r-1} y_i, (\prod_{i=0}^{r-2} y_i)^{-1}, \prod_{i=0}^{r-2} y_i, \ldots, y_0^{-1}, y_0 \right).$$

For $\rho = \operatorname{diag}(\rho_1, \rho_1^{-1}, \ldots, \rho_r, \rho_r^{-1})$, $\tau = \operatorname{diag}(\tau_1^{-1}, \tau_1, \ldots, \tau_r^{-1}, \tau_r) \in \mathbb{T}_{\mathrm{Sp}_{2r}}$, and $u_E \in \mathrm{U}_{2r}^+$ and $u_F \in \mathrm{U}_{2r}^-$, we have

$$(u_F \tau) \circ x \circ D_{\mathbf{y}} \circ z^{-1} \circ (u_E \rho)^{-1} = (u_F \circ \tau \circ x \circ \tau^{-1}) \circ (\tau \circ D_{\mathbf{y}} \circ \rho^{-1}) \circ (\rho \circ z^{-1} \circ \rho^{-1} \circ u_E^{-1})$$

with $u_F \circ \tau \circ x \circ \tau^{-1} \in \mathrm{U}_{2r}^-$ and $\rho \circ z^{-1} \circ \rho^{-1} \circ u_E^{-1} \in \mathrm{U}_{2r}^+$. We have $\tau \circ D_{\mathbf{y}} \circ \rho^{-1} = D_{\mathbf{y}'}$ with

$$\mathbf{y}' = \operatorname{diag}(\tau_r y_0 \rho_r, \ldots, \tau_{r-j} \tau_{r-j+1}^{-1} y_j \rho_{r-j} \rho_{r-j+1}^{-1}, \ldots). \tag{6.2}$$

By this we know that $KSp(E,F)^{(\mathrm{id},\mathrm{id})} \subset KSp(E,F)$ is a $\mathbb{B}_E \times \mathbb{B}_F$-stable open subscheme such that under the isomorphism (6.1), the action of $(u_E \rho, u_F \tau)$ on $KSp(E,F)^{(\mathrm{id},\mathrm{id})}$ is expressed by

$$(z, \mathbf{y}, x) \mapsto (u_E \rho z \rho^{-1}, \mathbf{y}', u_F \tau x \tau^{-1}) \tag{6.3}$$

with $\mathbf{y}'$ as in (6.2).

Corollary 6.6. *For $I \subset [0, r-1]$, the scheme $X_I \cap KSp(E,F)^{(\mathrm{id},\mathrm{id})}$ has an open dense $\mathbb{B}_E \times \mathbb{B}_F$-orbit.*

Proof. Under the isomorphism (6.1), a point $(z, \mathbf{y}, x) \in \mathrm{U}_{2r}^{+} \times \mathbb{A}^r \times \mathrm{U}_{2r}^{-}$ lies in $X_I \cap KSp(E,F)^{(\mathrm{id},\mathrm{id})}$ if and only if $y_i = 0$ for $i \in I$, where $\mathbf{y} = (y_0, \ldots, y_{r-1})$. By the description (6.3) of $\mathbb{B}_E \times \mathbb{B}_F$-action, the open dense subset

$$X_I \cap \mathrm{Sp}(E,F)^{(\mathrm{id},\mathrm{id})} \subset X_I \cap KSp(E,F)$$

is a $\mathbb{B}_E \times \mathbb{B}_F$-orbit. □

Proposition 6.7. *If W is a finite dimensional irreducible $\mathrm{Sp}(E)\times\mathrm{Sp}(F)$-module, then* $\dim \mathrm{Hom}(W, \mathrm{H}^0(X_I, \iota_I^* \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes c_i})) \leq 1$.

Proof. If $\mathbb{B}_E \times \mathbb{B}_F$ acts on nonzero sections $s_1, s_2 \in \mathrm{H}^0(X_I, \iota_I^* \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes c_i})$ by the same character, then s_1/s_2 is a $\mathbb{B}_E \times \mathbb{B}_F$-invariant meromorphic function of X_I. Since X_I has an open dense $\mathbb{B}_E \times \mathbb{B}_F$-orbit, s_1/s_2 is a constant. □

Proposition 6.8. *If $W \subset \mathrm{H}^0(X_I, \iota_I^* \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes c_i})$ is an irreducible $\mathrm{Sp}(E) \times \mathrm{Sp}(F)$-submodule, then for some $\overrightarrow{q} \in \mathbb{A}(c_0, \ldots, c_{r-1})_I$, we have the inclusion $W \subset \mathrm{H}^0(X_I, \iota_i^* \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes \sum_{j=1}^{r-i} q_j})$ and the composite of morphisms*

$$W \hookrightarrow \mathrm{H}^0\left(X_I, \iota_I^* \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes \sum_{j=1}^{r-i} q_j}\right) \to \mathrm{H}^0\left(X_{[0,r-1]}, \iota_{[0,r-1]}^* \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes \sum_{j=1}^{r-i} q_j}\right)$$

is an isomorphism.

Proof. The restriction of the isomorphism (6.1) induces an isomorphism

$$\mathrm{U}_{2r}^{+} \times \mathbb{A}^{r-|I|} \times \mathrm{U}_{2r}^{-} \simeq X_I \cap KSp(E,F)^{(\mathrm{id},\mathrm{id})} =: X_I^{(\mathrm{id},\mathrm{id})},$$

where $\mathbb{A}^r \supset \mathbb{A}^{r-|I|} = \{y_i = 0; i \in I\}$.

Since a line bundle on $\mathbb{A}^N$ is trivial, we can find a nowhere vanishing section $s_0 \in \iota_I^* \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes c_i}|_{X_I^{(\mathrm{id},\mathrm{id})}}$. The section s_0 is unique up to scalar, so $\mathbb{B}_E \times \mathbb{B}_F$ acts on s_0 as a character. Since $\mathbb{B}_E \times \mathbb{B}_F$ acts on a highest weight vector $s \in W$ as a character, it acts on the algebraic function $(s|_{X_I^{(\mathrm{id},\mathrm{id})}})/s_0$ on $X_I^{(\mathrm{id},\mathrm{id})}$ as a character. Hence we find that $(s|_{X_I^{(\mathrm{id},\mathrm{id})}})/s_0 = \prod_{i \in [0,r-1] \setminus I} y_i^{\alpha_i}$ with $\alpha_i \geq 0$. For $i \in I$ we put $\alpha_i = 0$. Then s is a global section of $\iota_I^* \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes c_i - \alpha_i}$ which is nowhere vanishing on $X_I^{(\mathrm{id},\mathrm{id})}$. Thus the composite of morphisms

$$W \to \mathrm{H}^0\left(X_I, \iota_I^* \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes c_i - \alpha_i}\right) \to \mathrm{H}^0\left(X_{[0,r-1]}, \iota_{[0,r-1]}^* \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes c_i - \alpha_i}\right)$$

is nonzero, hence an isomorphism because both W and $\mathrm{H}^0\left(X_{[0,r-1]}, \iota^*_{[0,r-1]} \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes c_i-\alpha_i}\right)$ are irreducible $\mathrm{Sp}(E)\times\mathrm{Sp}(F)$-modules.

It remains to show that if we define $\overrightarrow{q}=(q_1,\dots,q_r)$ by the equation $c_i-\alpha_i=\sum_{j=1}^{r-i} q_j$, then $\overrightarrow{q}\in\mathbb{A}(c_0,\dots,c_{r-1})_I$. Since

$$\mathrm{H}^0\left(X_{[0,r-1]}, \iota^*_{[0,r-1]} \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{c_i-\alpha_i}\right) \simeq \mathrm{H}^0\left(\mathbf{SpFl}_{[0,r-1]}, \mathcal{O}(\overrightarrow{q};\overrightarrow{q})\right)$$

is nonzero, (i) of Definition 6.2 is satisfied. Since $\alpha_i\geq 0$ and $\alpha_i=0$ for $i\in I$, (ii) of Definition 6.2 is satisfied. □

Proposition 6.9. *For integers $q_1\geq\cdots\geq q_r\geq 0$, the morphism*

$$\mathrm{H}^0\left(KSp(E,F), \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes\sum_{j=1}^{r-i} q_j}\right) \to \mathrm{H}^0\left(X_{[0,r-1]}, \iota^*_{[0,r-1]} \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes\sum_{j=1}^{r-i} q_j}\right) \tag{6.4}$$

is surjective.

Proof. Since $\mathrm{H}^0\left(X_{[0,r-1]}, \iota^*_{[0,r-1]} \bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes\sum_{j=1}^{r-i} q_j}\right)$ is an irreducible $\mathrm{Sp}(E)\times\mathrm{Sp}(F)$-representation, it suffices to prove that (6.4) is nonzero. It suffices to prove that for

$$(q_1,\dots,q_r)=(\overbrace{1,\dots,1}^{l\text{ times}},0,\dots,0) \qquad (1\leq l\leq r),$$

(6.4) is nonzero.

Let $\gamma_l:\mathcal{O}_{KSp(E,F)}^{\oplus l}\hookrightarrow\mathcal{O}_{KSp(E,F)}^{\oplus 2r}=E_{KSp(E,F)}$ be the inclusion of direct sum of $(2i-1)$-th component for $1\leq i\leq l$, and $\delta_l: F_{KSp(E,F)}=\mathcal{O}_{KSp(E,F)}^{\oplus 2r}\twoheadrightarrow\mathcal{O}_{KSp(E,F)}^{\oplus l}$ the projection to the direct sum of $(2i-1)$-th component for $1\leq i\leq l$. The determinant of the morphism of rank l vector bundles

$$\delta_l\circ f_0^\sharp\circ\cdots\circ f_{r-1}^\sharp\circ h\circ e_{r-1}^\flat\circ\cdots\circ e_0^\flat\circ\gamma_l:\mathcal{O}_{KSp(E,F)}^{\oplus l}\to\left(\otimes_{i=0}^{r-1}\mathcal{M}_i\right)^{\oplus l}$$

defines a section $\sigma_l\in\bigotimes_{i=0}^{r-1}\mathcal{M}_i^{\otimes l}$. By using diagonalization, you can see that $\sigma_l|_{KSp(E,F)^{(\mathrm{id},\mathrm{id})}}$ vanishes along the divisor $\sum_{j=r-l+1}^{r-1} X_{\{j\}}$, and that σ_l gives a section of $\bigotimes_{i=0}^{r-l}\mathcal{M}_i^{\otimes l}\otimes\bigotimes_{i=r-l+1}^{r-1}\mathcal{M}_i^{\otimes r-i}$ which is nowhere vanishing on $KSp(E,F)^{(\mathrm{id},\mathrm{id})}$. So $\sigma_l\in\bigotimes_{i=0}^{r-l}\mathcal{M}_i^{\otimes l}\otimes\bigotimes_{i=r-l+1}^{r-1}\mathcal{M}_i^{\otimes r-i}$ induces a nonzero section of $\iota^*_{[0,r-1]}\left(\bigotimes_{i=0}^{r-l}\mathcal{M}_i^{\otimes l}\otimes\bigotimes_{i=r-l+1}^{r-1}\mathcal{M}_i^{\otimes r-i}\right)$. This completes the proof of the proposition. □

§ 7. Factorization of generalized theta functions

In this section we shall apply the results about the compactification KSp obtained in the previous sections to the study of the generalized theta functions on the moduli of (parabolic) symplectic bundles on an algebraic curve. More precisely, we shall prove the so-called factorization theorem of generalized theta functions on the moduli of stack of symplectic bundles. For ordinary vector bundles, the factorization theorem has been proved by Narasimhan-Ramadas, Sun and Kausz ([N-Rd], [S1], [S2], [Kausz3]).

Let us start with the definition of the moduli stack of (parabolic) symplectic bundles.

Let C be a connected projective nodal curve over an algebraically closed field k, $P^{(1)}, \dots, P^{(m)}$ distinct smooth points of C, and L a line bundle on C. Put $\overrightarrow{P} = (P^{(1)}, \dots, P^{(m)})$.

Definition 7.1. (1) We define the moduli stack $\overline{M}(C, \overrightarrow{P}; L)$ as follows. For an affine k-scheme T, an object of the groupoid $\overline{M}(C, \overrightarrow{P}; L)(T)$ is the following data:

• a T-flat coherent $\mathcal{O}_{C\times T}$-module $\mathcal{G}$ whose restriction to every geometric fiber $C \times \operatorname{Spec} \overline{k(t)}$ $(t \in T)$ is a rank $2r$ torsion-free sheaf,

• a non-degenerate bilinear alternate form $\mathcal{G} \otimes \mathcal{G} \to pr_C^* L$, (Here "non-degenerate" means that the induced morphism $\mathcal{G} \to \mathcal{H}om(\mathcal{G}, pr_C^* L)$ is an isomorphism.)

• for every point $P^{(j)}$ $(1 \le j \le m)$, a filtration

$$\mathcal{G}|_{P^{(j)}\times T} \supset \mathbb{F}_1(\mathcal{G}|_{P^{(j)}\times T}) \supset \cdots \supset \mathbb{F}_r(\mathcal{G}|_{P^{(j)}\times T}) \supset 0$$

of isotropic vector subbundles with $\operatorname{rank}\mathbb{F}_i(\mathcal{G}|_{P^{(j)}\times T}) = r + 1 - i$.

Isomorphisms of the groupoid $\overline{M}(C, \overrightarrow{P}; L)(T)$ are defined obviously.

(2) The substack $M(C, \overrightarrow{P}; L)$ of $\overline{M}(C, \overrightarrow{P}; L)$ is defined such that an object of $\overline{M}(C, \overrightarrow{P}; L)(T)$ described above is in $M(C, \overrightarrow{P}; L)(T)$ if and only if $\mathcal{G}$ is locally free. Clearly if C is smooth, then $M(C, \overrightarrow{P}; L) = \overline{M}(C, \overrightarrow{P}; L)$.

Let

$$\Bigg(\mathcal{G}^{univ}, \mathcal{G}^{univ} \otimes \mathcal{G}^{univ} \to pr_C^* L,$$
$$\mathcal{G}^{univ}|_{P^{(j)}\times \overline{M}(C,\overrightarrow{P};L)} \supset \mathbb{F}_1(\mathcal{G}^{univ}|_{P^{(j)}\times \overline{M}(C,\overrightarrow{P};L)}) \supset \dots \quad (1 \le j \le m)\Bigg)$$

be the universal object of the moduli stack $\overline{M}(C, \overrightarrow{P}; L)$.

Definition 7.2. Let n be an integer. If each point $P^{(j)}$ $(1 \le j \le m)$ is given a tuple of integers $\overrightarrow{\lambda}^{(j)} = (\lambda_1^{(j)}, \dots, \lambda_r^{(j)})$, we denote by $\Xi^{(n;\overrightarrow{\lambda}^{(1)},\dots,\overrightarrow{\lambda}^{(m)})}_{\overline{M}(C,\overrightarrow{P},L)}$, or simply $\Xi^{(n;\overrightarrow{\lambda}^{(1)},\dots,\overrightarrow{\lambda}^{(m)})}$, the line bundle

$$\left(\det \mathbb{R}pr_* \mathcal{G}^{univ}\right)^{\otimes(-n)} \otimes \bigotimes_{j=1}^{m} \bigotimes_{i=1}^{r} \left(\frac{\mathbb{F}_{r+2-i}\left(\mathcal{G}^{univ}|_{P^{(j)}\times \overline{M}(C,\overrightarrow{P},L)}\right)^{\perp}}{\mathbb{F}_{r+1-i}\left(\mathcal{G}^{univ}|_{P^{(j)}\times \overline{M}(C,\overrightarrow{P},L)}\right)^{\perp}} \right)^{\otimes \lambda_i^{(j)}}$$

on $\overline{M}(C,\overrightarrow{P};L)$, where pr is the projection $C\times\overline{M}(C,\overrightarrow{P};L)\to\overline{M}(C,\overrightarrow{P};L)$.

In this paper, for simplicity of notation, we restrict ourselves to the case of a nodal curve with only one singular point. In this case, C is either irreducible or having two irreducible components. We first state and prove the factorization theorem for the irreducible case, and later we shall comment on how to modify the argument for the reducible case.

Let C be an irreducible projective nodal curve with only one singular point P, and $\mathfrak{n}:\widetilde{C}\to C$ the normalization. Put $\{P_1,P_2\}:=\mathfrak{n}^{-1}(P)$. Let $P_3,\dots,P_m\in C\setminus\{P\}$ be distinct points. We denote by the same letters $P_3,\dots,P_m$ the corresponding points of $\widetilde{C}$. Put $\widetilde{L}:=\mathfrak{n}^*L$ and $\overrightarrow{P}=(P_3,\dots,P_m)$.

Theorem 7.3. *Let $\overrightarrow{\lambda}^{(j)}=(\lambda_1^{(j)},\dots,\lambda_r^{(j)})$ $(3\le j\le m)$ be tuples of integers. Then we have a canonical isomorphism*

$$\mathrm{H}^0\left(M(C,\overrightarrow{P};L),\Xi^{(n;\overrightarrow{\lambda}^{(3)},\dots,\overrightarrow{\lambda}^{(m)})}_{M(C,\overrightarrow{P};L)}\right)$$
$$\simeq\bigoplus_{\overrightarrow{q}=(q_1,\dots,q_r)}\mathrm{H}^0\left(M(\widetilde{C},\{P_1,P_2\}\cup\overrightarrow{P};\widetilde{L}),\Xi^{(n;\overrightarrow{q},\overrightarrow{q},\overrightarrow{\lambda}^{(3)},\dots,\overrightarrow{\lambda}^{(m)})}_{M(\widetilde{C},\{P_1,P_2\}\cup\overrightarrow{P};\widetilde{L})}\right)\otimes_k(L|_P)^{\otimes(rn-|\overrightarrow{q}|)},$$

where $\overrightarrow{q}=(q_1,\dots,q_r)$ varies through all tuples of integers with $n\ge q_1\ge\cdots\ge q_r\ge 0$.

For simplicity of notation, we shall deal with the case $\overrightarrow{P}=\emptyset$. Let

$$(\widetilde{\mathcal{G}}^{univ},\widetilde{\pi}^{univ}:\widetilde{\mathcal{G}}^{univ}\otimes\widetilde{\mathcal{G}}^{univ}\to pr_{\widetilde{C}}^*\widetilde{L})$$

be the universal object of the stack $M(\widetilde{C};\widetilde{L})$. Let σ_i be the section $(P_i,\mathrm{id}):M(\widetilde{C};\widetilde{L})\to\widetilde{C}\times M(\widetilde{C};\widetilde{L})$ $(i=1,2)$. There is a non-degenerate alternate bilinear form $\sigma_i^*\widetilde{\mathcal{G}}^{univ}\otimes\sigma_i^*\widetilde{\mathcal{G}}^{univ}\to\sigma_i^*pr_{\widetilde{C}}^*\widetilde{L}$ induced by $\widetilde{\pi}^{univ}$. Since there are natural isomorphisms

$$\sigma_1^*pr_{\widetilde{C}}^*\widetilde{L}\simeq\widetilde{L}|_{P_1}\otimes\mathcal{O}_{M(\widetilde{C};\widetilde{L})}\simeq L|_P\otimes\mathcal{O}_{M(\widetilde{C};\widetilde{L})}\simeq\widetilde{L}|_{P_2}\otimes\mathcal{O}_{M(\widetilde{C};\widetilde{L})}\simeq\sigma_2^*pr_{\widetilde{C}}^*\widetilde{L},$$

we can consider the stacks

$$Sp:=Sp(\sigma_1^*\widetilde{\mathcal{G}}^{univ},\sigma_2^*\widetilde{\mathcal{G}}^{univ})\quad\text{and}\quad LGr:=LGr(\sigma_1^*\widetilde{\mathcal{G}}^{univ}\oplus\sigma_2^*\widetilde{\mathcal{G}}^{univ}).$$

Let $g':LGr\to M(\widetilde{C},\widetilde{L})$ be the projection. Let

$$\mathcal{U}\subset g'^*\left(\sigma_1^*\widetilde{\mathcal{G}}^{univ}\oplus\sigma_2^*\widetilde{\mathcal{G}}^{univ}\right)$$

be the universal isotropic rank $2r$ subbundle. We denote by $\widetilde{\mathcal{G}'}^{univ}$ the vector bundles $(\mathrm{id}_{\widetilde{C}}\times g')^*\widetilde{\mathcal{G}}^{univ}$ on $\widetilde{C}\times LGr$. Put $\sigma_i':=(P_i,\mathrm{id}):LGr\to\widetilde{C}\times LGr$.

We define a sheaf $\mathcal{H}$ on $C \times LGr$ to be the kernel of the composite of morphisms

$$(\mathfrak{n}')_* \widetilde{\mathcal{G}'}^{univ} \to \eta_*(\sigma_1'^* \widetilde{\mathcal{G}'}^{univ} \oplus \sigma_2'^* \widetilde{\mathcal{G}'}^{univ}) \to \eta_* \left(\frac{\sigma_1'^* \widetilde{\mathcal{G}'}^{univ} \oplus \sigma_2'^* \widetilde{\mathcal{G}'}^{univ}}{\mathcal{U}} \right),$$

where $\eta = (P, \mathrm{id}_{LGr}) : LGr \to C \times LGr$ and $\mathfrak{n}' := \mathfrak{n} \times \mathrm{id}_{LGr} : \widetilde{C} \times LGr \to C \times LGr$. Then $\mathcal{H}$ is flat over LGr and the restriction to every geometric fiber is torsion-free of rank $2r$. You can easily see that there is a unique non-degenerate alternate bilinear form $\mathcal{H} \otimes \mathcal{H} \to pr_C^* L$ such that the diagram

$$\begin{array}{ccc} \mathfrak{n}'_*(\widetilde{\mathcal{G}'}^{univ}) \otimes \mathfrak{n}'_*(\widetilde{\mathcal{G}'}^{univ}) & \xrightarrow{\mathfrak{n}'_*((\mathrm{id}_{\widetilde{C}} \times g')^* \widetilde{\pi})} & \mathfrak{n}'_*(pr_{\widetilde{C}}^* \widetilde{L}) \\ \uparrow & & \uparrow \\ \mathcal{H} \otimes \mathcal{H} & \longrightarrow & pr_C^* L \end{array}$$

commutes. Then $(\mathcal{H}, \mathcal{H} \otimes \mathcal{H} \to pr_C^* L)$ is an object of $\overline{M}(C; L)$. This gives rise to a morphism $\overline{f} : LGr \to \overline{M}(C; L)$. We have a commutative diagram:

$$\begin{array}{ccc} M(\widetilde{C}; \widetilde{L}) & & \\ g' \uparrow & & \\ LGr & \xrightarrow{\overline{f}} & \overline{M}(C; L) \\ \cup & & \cup \\ Sp & \xrightarrow{f} & M(C; L), \end{array}$$

where f, the restriction of $\overline{f}$, is an isomorphism of stacks.

Lemma 7.4. *If $\mathcal{A}$ is a line bundle on $\overline{M}(C; L)$, then we have isomorphisms*

$$\mathrm{H}^0\left(LGr, \overline{f}^* \mathcal{A}\right) \xrightarrow{\sim} \mathrm{H}^0\left(Sp, \overline{f}^* \mathcal{A}\right) \xleftarrow{f^*} \mathrm{H}^0\left(M(C; L), \mathcal{A}\right)$$

of vector spaces.

Proof. Since f is an isomorphism, f^* is clearly bijective. Let us prove that the restriction map

$$\mathrm{H}^0\left(LGr, \overline{f}^* \mathcal{A}\right) \to \mathrm{H}^0\left(Sp, \overline{f}^* \mathcal{A}\right) \tag{7.1}$$

is an isomorphism.

If H is a rank $2r$ torsion-free sheaf on C, then it is known that H_P, the stalk of H at P, is isomorphic to $\mathfrak{m}^a \oplus \mathcal{O}_{C,P}^{2r-a}$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{C,P}$. If H has a non-degenerate alternate bilinear form $H \otimes H \to L$, then the restriction of the bilinear form to the free summand of H_P is non-degenerate. Hence H_P is isomorphic to $\mathfrak{m}^{2i(H)} \oplus \mathcal{O}_{C,P}^{2(r-i(H))}$ for some $0 \leq i(H) \leq r$. We denote by $\overline{M}(C;L)_{\leq n}$ the open substack of $\overline{M}(C;L)$ parametrizing H's with $i(H) \leq n$. We put $LGr_{\leq n} := LGr(\sigma_1^* \widetilde{\mathcal{G}}^{univ} \oplus \sigma_2^* \widetilde{\mathcal{G}}^{univ})_{\leq n}$ (See Notation 4.4). We have a commutative diagram:

$$\begin{array}{ccc} LGr_{\leq 1} & \xrightarrow{\overline{f}} & \overline{M}(C;L)_{\leq 1} \\ \cup & & \cup \\ Sp & \xrightarrow{f} & M(C;L), \end{array}$$

here by abuse of notation, the restriction of $\overline{f}$ to $LGr_{\leq 1}$ is also denoted by $\overline{f}$. Correspondingly we have a commutative diagram of vector spaces

$$\begin{array}{ccc} \mathrm{H}^0(LGr_{\leq 1}, \overline{f}^* \mathcal{A}) & \xleftarrow{f''^*} & \mathrm{H}^0(\overline{M}(C;L)_{\leq 1}, \mathcal{A}) \\ {\scriptstyle (a)}\downarrow & & \downarrow{\scriptstyle (b)} \\ \mathrm{H}^0(Sp, f^* \mathcal{A}) & \xleftarrow{f^*} & \mathrm{H}^0(M(C;L), \mathcal{A}). \end{array} \tag{7.2}$$

By Lemma 4.5, the restriction map $\mathrm{H}^0(LGr, \overline{f}^* \mathcal{A}) \to \mathrm{H}^0(LGr_{\leq 1}, \overline{f}^* \mathcal{A})$ is an isomorphism. Therefore in order to prove the bijectivity of (7.1), it suffices to prove that the morphism (a) in the diagram (7.2) is an isomorphism.

By the claim below, $\overline{M}(C;L)_{\leq 1}$ is normal. Using [EGAIV, Théorème 5.10.5] on the atlas on the moduli stack $\overline{M}(C;L)_{\leq 1}$, we know that the map (b) in the diagram (7.2) is an isomorphism. This and the bijectivity of f^* imply that the map (a) is surjective. Since (a) is clearly injective, it is bijective.

Claim 7.4.1. $\overline{M}(C;L)_{\leq 1}$ is normal.

Proof of Claim. The claim is a consequence of Faltings's description of the versal deformation of symplectic sheaves on a nodal point as follows.

Fix an isomorphism $\widehat{\mathcal{O}_{C,P}} \simeq k[[x,y]]/(xy)$ of k-algebras, and an isomorphism $L_P \simeq \mathcal{O}_{C,P}$ of $\mathcal{O}_{C,P}$-modules. Fix a rank 2r torsion-free sheaf H on C and a non-degenerate alternate bilinear form $H \otimes H \to L$.

We define the two functors D and D_{loc} from the category of local artinian k-algebras to the category of sets as follows. D is the deformation functor of the symplectic sheaf $(H, H \otimes H \to L)$, i.e., for an artinian local R, $D(R)$ is the set of equivalece classes of the data $(\widetilde{H}, \widetilde{H} \otimes \widetilde{H} \to L_R, \widetilde{H} \otimes_R k \simeq H)$, where $(\widetilde{H}, \widetilde{H} \otimes \widetilde{H} \to L_R)$ is a family of symplectic sheaves on C over $\operatorname{Spec} R$ such that the restriction of the bilinear form $\widetilde{H} \otimes \widetilde{H} \to L_R$ over $\operatorname{Spec} k$ is, by the isomorphism $\widetilde{H} \otimes_R k \simeq H$, equal to $H \otimes H \to L$. D_{loc} is the deformation functor of the symplectic $k[[x,y]]/(xy)$-module $\widehat{H_P} (:= H_P \otimes_{\mathcal{O}_{C,P}} \widehat{\mathcal{O}_{C,P}})$, i.e., for an artinian local k-algebra R, $D_{loc}(R)$ is the set of equivalence classes of the data

$(\widetilde{E}, \widetilde{E} \otimes_{R[[x,y]]/(xy)} \widetilde{E} \to R[[x,y]]/(xy), \widetilde{E} \otimes_R k \simeq \widehat{H_P})$, where $\widetilde{E}$ is an R-flat $R[[x,y]]/(xy)$-module, and $\widetilde{E} \otimes_{R[[x,y]]/(xy)} \widetilde{E} \to R[[x,y]]/(xy)$ is a non-degenerate alternate bilinear form whose restriction over $\operatorname{Spec} k$ is, by the isomorphism $\widetilde{E} \otimes_R k \simeq \widehat{H_P}$, equal to $\widehat{H_P} \otimes \widehat{H_P} \to \widehat{L_P} \simeq k[[x,y]]/(xy)$.

Given an element of $D(R)$, its completion at P is an element of D_{loc}. So we have a natural transformation $D \to D_{loc}$, which is smooth (cf. [BL]).

As noted at the beginning of the proof of the lemma, we have an isomorphism $\widehat{H_P} \simeq (k[[x,y]]/(xy))^{\oplus 2(r-i)} \oplus ((x,y)/(xy))^{\oplus 2i}$ of $k[[x,y]]/(xy)$-modules. If $E \subset \widehat{H_P}$ is the orthogonal complement of the direct summand $(k[[x,y]]/(xy))^{\oplus 2(r-i)}$, the restriction to E of the bilinear form $\widehat{H_P} \otimes \widehat{H_P} \to \simeq k[[x,y]]/(xy)$ is non-degenerate. Let D_E be the deformation functor of the symplectic $k[[x,y]]/(xy)$-module $(E, E \otimes_{k[[x,y]]/(xy)} E \to k[[x,y]]/(xy))$. Given a deformation of $(E, E \otimes_{k[[x,y]]/(xy)} E \to k[[x,y]]/(xy))$ over an artinian local k-algebra R, by taking the direct sum with $R[[x,y]]/(xy)^{\oplus 2(r-i)}$ we obtain an element of $D_{loc}(R)$. So we have a natural transformation $D_E \to D_{loc}$, which is an isomorphism (cf. the argument in the last paragraph of [Fal, page 492]).

The hull of the deformation functor D_E has been calculated by Faltings. Let $Y = (y_{\alpha\beta})$ be a $2i \times 2i$ matrix, where $y_{\alpha\beta}$ is an indeterminate. Put $Z := J_{2i}^{-1}\,{}^tPJ_{2i}$. ($Z$ is the adjoint of Y with respect to the alternate form determined by J_{2i}.) Let I be the ideal of $k[[y_{\alpha\beta}\,|1 \le \alpha, \beta \le 2i]]$ generated by all the entries of the matrix YZ. Faltings constructed a deformation of the symplectic $k[[x,y]]/(xy)$-module E over $\operatorname{Spf} k[[y_{\alpha\beta}]]/I$ and proved its versality (cf. [Fal, Theorem 3.7 and Remark 3.8]). In particular, when $i = 1$, the ideal I is generated by $y_{11}y_{22} - y_{12}y_{21}$. So the singularity of $\overline{M}(C;L)_{\le 1}$ is of the form $uv - zw = 0$ for a local coordinate $(u, v, z, w, \dots)$. Hence $\overline{M}(C;L)_{\le 1}$ is normal. This completes the proof of the claim. $\square$

This is the end of proof of the lemma. $\square$

Lemma 7.5. *We have a natural isomorphism*

$$\overline{f}^* \Xi^{(n)}_{\overline{M}(C,L)} \simeq g'^* \Xi^{(n)}_{M(\widetilde{C},\widetilde{L})} \otimes (\det \mathcal{Q})^{\otimes n} \tag{7.3}$$

of line bundles on LGr, where $\mathcal{Q} := (\sigma_1'^* \widetilde{\mathcal{G}'}^{univ} \oplus \sigma_2'^* \widetilde{\mathcal{G}'}^{univ})/\mathcal{U}$.

Proof. We have isomorphisms

(7.4)
$$\begin{aligned}\overline{f}^* \Xi^{(n)}_{\overline{M}(C,L)} &\simeq (\det \mathbb{R}pr_{LGr*}\mathcal{H})^{\otimes(-n)} \\ &\simeq g'^* \left(\det \mathbb{R}pr_{M(\widetilde{C},\widetilde{L})*} \widetilde{\mathcal{G}}^{univ}\right)^{\otimes(-n)} \otimes \det \left(\frac{\sigma_1'^* \widetilde{\mathcal{G}'}^{univ} \oplus \sigma_2'^* \widetilde{\mathcal{G}'}^{univ}}{\mathcal{U}}\right)^{\otimes n},\end{aligned}$$

and we have $\left(\det \mathbb{R}pr_{M(\widetilde{C};\widetilde{L})*}\widetilde{\mathcal{G}}^{univ}\right)^{\otimes(-n)} \simeq \Xi^{(n)}_{M(\widetilde{C},\widetilde{L})}$ by definition. $\square$

Proof of Theorem 7.3. We put

$$\mathbf{SpFl} := \mathrm{SpFl}_{[0,r-1]}(\sigma_1^*\widetilde{\mathcal{G}}^{univ}) \times_{M(\widetilde{C};\widetilde{L})} \mathrm{SpFl}_{[0,r-1]}(\sigma_2^*\widetilde{\mathcal{G}}^{univ}),$$

and let $g'' : \mathbf{SpFl} \to M(\widetilde{C};\widetilde{L})$ be the projection.

Applying Corollary 6.5, we have a canonical isomorphism

$$g'_*(\det \mathcal{Q})^{\otimes n} \simeq \bigoplus_{\overrightarrow{q}=(q_1,\dots,q_r)} g''_*\mathcal{O}(\overrightarrow{q};\overrightarrow{q}) \otimes_k (L|_P)^{\otimes(nr-|\overrightarrow{q}|)}, \tag{7.5}$$

where $n \geq q_1 \geq \cdots \geq q_r \geq 0$.

We have isomorphisms

$$\begin{aligned}
&\mathrm{H}^0\left(M(C;L), \Xi^{(n)}_{M(C;L)}\right) \simeq \mathrm{H}^0\left(LGr, \overline{f}^*\Xi^{(n)}_{\overline{M}(C;L)}\right) \quad \text{by Lemma 7.4}\\
&\simeq \mathrm{H}^0\left(LGr, g'^*\Xi^{(n)}_{M(\widetilde{C};\widetilde{L})} \otimes (\det \mathcal{Q})^{\otimes n}\right) \quad \text{by (7.3)}\\
&\simeq \mathrm{H}^0\left(M(\widetilde{C};\widetilde{L}), \Xi^{(n)}_{M(\widetilde{C},\widetilde{L})} \otimes g'_*(\det \mathcal{Q})^{\otimes n}\right) \quad \text{by projection formula}\\
&\simeq \bigoplus_{\overrightarrow{q}} \mathrm{H}^0\left(M(\widetilde{C};\widetilde{L}), \Xi^{(n)}_{M(\widetilde{C},\widetilde{L})} \otimes g''_*\mathcal{O}(\overrightarrow{q};\overrightarrow{q})\right) \otimes_k (L|_P)^{\otimes nr-|\overrightarrow{q}|} \quad \text{by (7.5)}\\
&\simeq \bigoplus_{\overrightarrow{q}} \mathrm{H}^0\left(M(\widetilde{C},\{P_1,P_2\};\widetilde{L}), \Xi^{(n;\overrightarrow{q},\overrightarrow{q})}_{M(\widetilde{C},\{P_1,P_2\};\widetilde{L})}\right) \otimes_k (L|_P)^{\otimes nr-|\overrightarrow{q}|},
\end{aligned}$$

where $\overrightarrow{q} = (q_1,\dots,q_r)$ varies through all tuples of integers with $n \geq q_1 \geq \cdots \geq q_r \geq 0$. $\square$

Reducible case. Let C be a connected reducible nodal curve with only one singular point P. Then C is a union of smooth curves C_1 and C_2 intersecting at P. Let $\mathfrak{n} : \widetilde{C} = C_1 \sqcup C_2 \to C$ be the normalization. Put $L_i := L|_{C_i}$ for $i = 1,2$. Put $\{Q_1, R_1\} := \mathfrak{n}^{-1}(P)$ such that $Q_1 \in C_1$ and $R_1 \in C_2$. Let $Q_2,\dots,Q_m \in C_1 \setminus \{Q_1\}$ and $R_2,\dots,R_l \in C_2 \setminus \{R_1\}$ be distinct points. Put $\overrightarrow{Q} := (Q_2,\dots,Q_m)$ and $\overrightarrow{R} := (R_2,\dots,R_l)$. Let $\overrightarrow{\lambda}^{(j)} = (\lambda_1^{(j)},\dots,\lambda_r^{(j)})$ $(2 \leq j \leq m)$ and $\overrightarrow{\mu}^{(j)} := (\mu_1^{(j)},\dots,\mu_r^{(j)})$ $(2 \leq j \leq l)$ be tuples of integers.

With these notations prepared, in the reducible case, the counterpart of Theorem 7.3 is the following:

Theorem 7.6. *There is a canonical isomorphism*

$$\begin{aligned}
&\mathrm{H}^0\left(M(C,\overrightarrow{Q}\cup\overrightarrow{R};L),\Xi^{(n;\overrightarrow{\lambda}^{(2)},\dots,\overrightarrow{\lambda}^{(m)},\overrightarrow{\mu}^{(2)},\dots,\overrightarrow{\mu}^{(l)})}_{M(C,\overrightarrow{Q}\cup\overrightarrow{R};L)}\right)\\
&\simeq\bigoplus_{\overrightarrow{q}=(q_1,\dots,q_r)}\mathrm{H}^0\left(M(C_1,\{Q_1\}\cup\overrightarrow{Q};L_1),\Xi^{(n;\overrightarrow{q},\overrightarrow{\lambda}^{(2)},\dots,\overrightarrow{\lambda}^{(m)})}_{M(C_1,\{Q_1\}\cup\overrightarrow{Q};L_1)}\right)\\
&\qquad\otimes\mathrm{H}^0\left(M(C_2,\{R_1\}\cup\overrightarrow{R};L_2),\Xi^{(n;\overrightarrow{q},\overrightarrow{\mu}^{(2)},\dots,\overrightarrow{\mu}^{(l)})}_{M(C_2,\{R_1\}\cup\overrightarrow{R};L_2)}\right)\otimes(L|_P)^{(\otimes rn-|\overrightarrow{q}|)},
\end{aligned}$$

where $\overrightarrow{q}=(q_1,\dots,q_r)$ varies through all the tuples of integers with $n\geq q_1\geq\cdots\geq q_r\geq 0$.

You can prove the above theorem by similar argument as in the proof of Theorem 7.3. Let us mention how to modify the argument. For simplicity, we assume that $\overrightarrow{Q}=\emptyset$ and $\overrightarrow{R}=\emptyset$. Let

$$\left(\widetilde{\mathcal{G}}_i^{univ},\widetilde{\pi}_i^{univ}:\widetilde{\mathcal{G}}_i^{univ}\otimes\widetilde{\mathcal{G}}_i^{univ}\to pr_{C_i}^*L_i\right)$$

be the universal object of the stack $M(C_i;L_i)$. Let σ_1 and σ_2 be the morphisms

$$\begin{aligned}
(Q_1,\mathrm{id})&:M(C_1;L_1)\to C_1\times M(C_1;L_1),\\
(R_1,\mathrm{id})&:M(C_2;L_2)\to C_2\times M(C_2;L_2)
\end{aligned}$$

respectively. Let $\phi_i:M(C_1;L_1)\times M(C_2;L_2)\to M(C_i;L_i)$ be the projection.

Then we can consider the stacks

$$\begin{aligned}
Sp&:=Sp\left(\phi_1^*\sigma_1^*\widetilde{\mathcal{G}}_1^{univ},\phi_2^*\sigma_2^*\widetilde{\mathcal{G}}_2^{univ}\right),\\
LGr&:=LGr\left(\phi_1^*\sigma_1^*\widetilde{\mathcal{G}}_1^{univ}\oplus\phi_2^*\sigma_2^*\widetilde{\mathcal{G}}_2^{univ}\right),
\end{aligned}\tag{7.6}$$

which are stacks over $M(C_1;L_1)\times M(C_2;L_2)$.

If in the proof of Theorem 7.3, you substitute $M(C_1;L_1)\times M(C_2;L_2)$ for $M(\widetilde{C};\widetilde{L})$ and understand that Sp and LGr are given by (7.6), then you will obtain a proof of Theorem 7.6.

§ 8. A result on the multiplication pull-back

The purpose of this section is to prove Proposition 8.1. Its importance might not be clear at the moment. But it will be used in [A] at a crucial point.

Orthogonal Grassmannian. Let $(V,(-,-)_V)$ be a $2n$-dimensional k-vector space with a non-degenerate symmetric bilinear form. Assume that n is even. Let $\mathbf{OGr}_n(V)$ be the orthogonal Grassmannian parametrizing isotropic subspaces of V of dimension

n. Then $\mathbf{OGr}_n(V)$ has two connected components $\mathbf{OGr}_n^+(V)$ and $\mathbf{OGr}_n^-(V)$; U and $U' \in \mathbf{OGr}_n(V)$ lie in the same connected component if and only if $\dim U \cap U'$ is even.

On $\mathbf{OGr}_n(V)$, there is a short exact sequence

$$0 \to \mathcal{U} \to V \otimes \mathcal{O}_{\mathbf{OGr}_n(V)} \xrightarrow{\pi} \mathcal{Q} \to 0$$

given by the universal subbundle $\mathcal{U}$ and the universal quotient bundle $\mathcal{Q}$.

Let $\bullet \in \{+, -\}$, and take a point $[U \subset V] \in \mathbf{OGr}_n^\bullet(V)$. Taking the determinat of the composed morphism

$$U \otimes \mathcal{O}_{\mathbf{OGr}_n^\bullet(V)} \hookrightarrow V \otimes \mathcal{O}_{\mathbf{OGr}_n^\bullet(V)} \xrightarrow{\pi} \mathcal{Q},$$

we have the morphism

$$\delta : \wedge^n U \otimes \mathcal{O}_{\mathbf{OGr}_n^\bullet(V)} \to \det \mathcal{Q}.$$

It is well-known that the zero divisor $\mathrm{div}(\delta)$ is divisible by two, i.e., $\mathrm{div}(\delta) = 2 \cdot (\frac{1}{2}\mathrm{div}(\delta))$ with $\frac{1}{2}\mathrm{div}(\delta)$ a Cartier divisor. (In fact, take a point $p = [W \subset V] \in \mathbf{OGr}_n^\bullet(V)$. We can find a form-preserving isomorphism

$$\alpha : \mathbb{C}^n \oplus (\mathbb{C}^n)^\vee \to V$$

such that $\alpha(\mathbb{C}^n) = U$ and $\alpha((\mathbb{C}^n)^\vee) \cap W = 0$, where $\mathbb{C}^n \oplus (\mathbb{C}^n)^\vee$ is endowed with the standard symmetric form. In a neighborhood of p, the composed morphism

$$\tau := \pi \circ ((\alpha|_{(\mathbb{C}^n)^\vee}) \otimes \mathrm{id}_{\mathcal{O}}) : (\mathbb{C}^n)^\vee \otimes \mathcal{O} \to \mathcal{Q}$$

is an isomorphism. Then the composed morphism

$$\tau^{-1} \circ \pi \circ ((\alpha|_{\mathbb{C}^n}) \otimes \mathrm{id}_{\mathcal{O}}) : \mathbb{C}^n \otimes \mathcal{O} \to (\mathbb{C}^n)^\vee \otimes \mathcal{O}$$

is represented by a skew-symmetric matrix, whose determinant is a square of the phaffian.)

Note that $\mathcal{Q} \simeq \mathcal{O}(\mathrm{div}(\delta))$. Thus we can take a square root of the line bundle $\det \mathcal{Q}$, i.e, we define $(\det \mathcal{Q})^{\otimes \frac{1}{2}}$ to be $\mathcal{O}(\frac{1}{2}\mathrm{div}(\delta))$. Then $(\det \mathcal{Q})^{\otimes \frac{1}{2}}$ does not depend on the choice of the point $[U \subset V] \in \mathbf{OGr}_n^\bullet(V)$, and it is well-define.

Note that by definition, $(\det \mathcal{Q})^{\otimes \frac{1}{2}}$ has a section whose zero divisor is $\frac{1}{2}\mathrm{div}(\delta)$, which is set-theoretically the locus of $[W \subset V] \in \mathbf{OGr}_n^\bullet(V)$ with $W \cap U \neq 0$.

Multiplication pull-back. Let $E^{(1)} = E^{(2)} = \oplus_{i=1}^{2r} k\mathbf{e}_i$ and $G^{(1)} = G^{(2)} = \oplus_{j=1}^{2s} k\mathbf{g}_j$ be k-vector spaces endowed with the symplectic forms $\langle -, - \rangle_{E^{(i)}}$ and $\langle -, - \rangle_{G^{(i)}}$ given by the matrices J_{2r} and J_{2s}. We give the tensor product $E^{(i)} \otimes G^{(i)}$ the symmetric bilinear form $\langle -, - \rangle_{E^{(i)} \otimes G^{(i)}}$ determined by $\langle e \otimes g, e' \otimes g' \rangle_{E^{(i)} \otimes G^{(i)}} := \langle e, e' \rangle_{E^{(i)}} \langle g, g' \rangle_{G^{(i)}}$.

We give the vector space $(E^{(1)} \otimes G^{(1)}) \oplus (E^{(2)} \otimes G^{(2)})$ the symmetric bilinear form $\langle -, - \rangle_{(E^{(1)} \otimes G^{(1)}) \oplus (E^{(2)} \otimes G^{(2)})}$ determined by

$$\langle (a_1, a_2), (a'_1, a'_2) \rangle_{(E^{(1)} \otimes G^{(1)}) \oplus (E^{(2)} \otimes G^{(2)})} = \langle a_1, a'_1 \rangle_{E^{(1)} \otimes G^{(1)}} - \langle a_2, a'_2 \rangle_{E^{(2)} \otimes G^{(2)}}$$

for $a_i, a'_i \in E^{(i)} \otimes G^{(i)}$.

Let $\mathbf{OGr}_{4rs} := \mathbf{OGr}_{4rs}\left((E^{(1)} \otimes G^{(1)}) \oplus (E^{(2)} \otimes G^{(2)})\right)$ be the orthogonal Grassmannian parametrizing $4rs$-dimensional isotropic subspaces of $(E^{(1)} \otimes G^{(1)}) \oplus (E^{(2)} \otimes G^{(2)})$.

For symplectic isomorphisms $\alpha : E^{(1)} \to E^{(2)}$ and $\beta : G^{(1)} \to G^{(2)}$, the tensor product $\alpha \otimes \beta : E^{(1)} \otimes G^{(1)} \to E^{(2)} \otimes G^{(2)}$ is an isomorphism preserving the symmetric bilinear forms of $E^{(1)} \otimes G^{(1)}$ and $E^{(2)} \otimes G^{(2)}$. The graph $\Gamma_{\alpha \otimes \beta}$ of $\alpha \otimes \beta$ determines a point of $\mathbf{OGr}_{4rs}$. We denote by m the morphism

$$\mathrm{Sp}(E^{(1)}, E^{(2)}) \times \mathrm{Sp}(G^{(1)}, G^{(2)}) \to \mathbf{OGr}^{+}_{4rs} \subset \mathbf{OGr}_{4rs}$$

given by $(\alpha, \beta) \mapsto \Gamma_{\alpha,\beta}$, where $\mathbf{OGr}^{+}_{4rs}$ is the one of the two components of $\mathbf{OGr}_{4rs}$ that contains the image of $\mathrm{Sp}(E^{(1)}, E^{(2)}) \times \mathrm{Sp}(G^{(1)}, G^{(2)})$.

We denote by $\mathbf{LGr}^{\circ}$ the open subset

$$(LGr(E^{(1)} \oplus E^{(2)}) \times \mathrm{Sp}(G^{(1)}, G^{(2)})) \cup (\mathrm{Sp}(E^{(1)}, E^{(2)}) \times LGr(G^{(1)} \oplus G^{(2)}))$$

of $\mathbf{LGr} := LGr(E^{(1)} \oplus E^{(2)}) \times LGr(G^{(1)} \oplus G^{(2)})$. The morphism m extends to a morphism

$$\widetilde{m} : \mathbf{LGr}^{\circ} \to \mathbf{OGr}^{+}_{4rs}.$$

In fact, for maximal isotropic subspaces $U \subset E^{(1)} \oplus E^{(2)}$ and $V \subset G^{(1)} \oplus G^{(2)}$, the map

$$U \otimes V \to (E^{(1)} \otimes G^{(1)}) \oplus (E^{(2)} \otimes G^{(2)})$$

is injective if either $[U \subset E^{(1)} \oplus E^{(2)}] \in \mathrm{Sp}(E^{(1)}, E^{(2)})$ or $[V \subset G^{(1)} \oplus G^{(2)}] \in \mathrm{Sp}(G^{(1)}, G^{(2)})$.

Let

$$0 \to \mathcal{U} \to \left\{ \left(E^{(1)} \otimes G^{(1)}\right) \oplus \left(E^{(2)} \otimes G^{(2)}\right) \right\} \otimes \mathcal{O}_{\mathbf{OGr}^{+}_{4rs}} \to \mathcal{Q} \to 0$$

be the universal sequence on $\mathbf{OGr}^{+}_{4rs}$. Let $\mathcal{Q}_E$ and $\mathcal{Q}_G$ be the universal quotient bundles on $LGr(E^{(1)} \oplus E^{(2)})$ and $LGr(G^{(1)} \oplus G^{(2)})$ respectively. You can easily obtain an isomorphism

$$\widetilde{m}^*(\det \mathcal{Q})^{\otimes \frac{1}{2}} \simeq (\det \mathcal{Q}_E)^{\otimes s} \boxtimes (\det \mathcal{Q}_G)^{\otimes r}\big|_{\mathbf{LGr}^{\circ}}.$$

So we have morphisms of vector spaces

$$\begin{aligned} &\mathrm{H}^0(\mathbf{OGr}^{+}_{4rs}, (\det \mathcal{Q})^{\otimes \frac{1}{2}}) \xrightarrow{\widetilde{m}^*} \mathrm{H}^0(\mathbf{LGr}^{\circ}, (\det \mathcal{Q}_E)^{\otimes s} \boxtimes (\det \mathcal{Q}_G)^{\otimes r}) \\ &\simeq \mathrm{H}^0(LGr(E^{(1)} \oplus E^{(2)}), (\det \mathcal{Q}_E)^{\otimes s}) \otimes \mathrm{H}^0(LGr(G^{(1)} \oplus G^{(2)}), (\det \mathcal{Q}_G)^{\otimes r}). \end{aligned} \tag{8.1}$$

By Corollary 6.5, there are natural isomorphisms

$$\begin{aligned}(8.2)\qquad &\mathrm{H}^0(LGr(E^{(1)}\oplus E^{(2)}),(\det\mathcal{Q}_E)^{\otimes s})\\ &\simeq\bigoplus_{\overrightarrow{\lambda}}\mathrm{H}^0\left(\mathrm{SpFl}_{[0,r-1]}(E^{(1)})\times\mathrm{SpFl}_{[0,r-1]}(E^{(2)}),\mathcal{O}(\overrightarrow{\lambda},\overrightarrow{\lambda})\right)\end{aligned}$$

and

$$\begin{aligned}(8.3)\qquad &\mathrm{H}^0(LGr(G^{(1)}\oplus G^{(2)}),(\det\mathcal{Q}_G)^{\otimes r})\\ &\simeq\bigoplus_{\overrightarrow{\mu}}\mathrm{H}^0\left(\mathrm{SpFl}_{[0,s-1]}(G^{(1)})\times\mathrm{SpFl}_{[0,s-1]}(G^{(2)}),\mathcal{O}(\overrightarrow{\mu},\overrightarrow{\mu})\right),\end{aligned}$$

where $\overrightarrow{\lambda}=(\lambda_1,\dots,\lambda_r)$ runs through all tuples of integers with $s\geq\lambda_1\geq\cdots\geq\lambda_r\geq 0$, and $\overrightarrow{\mu}=(\mu_1,\dots,\mu_s)$ with $r\geq\mu_1\geq\cdots\geq\mu_s\geq 0$. Composing (8.1) with the tensor product of (8.2) and (8.3), we have a morphism

$$\begin{aligned}\varphi:&\mathrm{H}^0\left(\mathbf{OGr}^+_{4rs},(\det\mathcal{Q})^{\otimes\frac{1}{2}}\right)\\ &\to\bigoplus_{\overrightarrow{\lambda},\overrightarrow{\mu}}\Big\{\mathrm{H}^0\left(\mathrm{SpFl}_{[0,r-1]}(E^{(1)})\times\mathrm{SpFl}_{[0,r-1]}(E^{(2)}),\mathcal{O}(\overrightarrow{\lambda};\overrightarrow{\lambda})\right)\\ &\qquad\otimes\mathrm{H}^0\left(\mathrm{SpFl}_{[0,s-1]}(G^{(1)})\times\mathrm{SpFl}_{[0,s-1]}(G^{(2)})\mathcal{O}(\overrightarrow{\mu};\overrightarrow{\mu})\right)\Big\}.\end{aligned}$$

We denote by $\pi_{\overrightarrow{\lambda},\overrightarrow{\mu}}$ the projection of the target of φ to the $(\overrightarrow{\lambda},\overrightarrow{\mu})$-component.

For $\overrightarrow{\lambda}=(\lambda_1,\dots,\lambda_r)$ with $s\geq\lambda_1\geq\cdots\geq\lambda_r\geq 0$, we denote by $\overrightarrow{\lambda^*}$ the tuple $(\lambda_1^*,\dots,\lambda_s^*)$ of integers such that $r\geq\lambda_1^*\geq\cdots\geq\lambda_s^*\geq 0$ and

$$\{\lambda_1^*+s,\lambda_2^*+s-1,\dots,\lambda_s^*+1\}\cup\{\lambda_1+r,\lambda_2+r-1,\lambda_r+1\}=[1,r+s].$$

Proposition 8.1. *For $\overrightarrow{\lambda}=(\lambda_1,\dots,\lambda_r)$ with $s\geq\lambda_1\geq\cdots\geq\lambda_r\geq 0$, the composed morphism $\pi_{\overrightarrow{\lambda},\overrightarrow{\lambda^*}}\circ\varphi$ is non-zero.*

Proof. We shall find explicitly an element of $\mathrm{H}^0(\mathbf{OGr}^+_{4rs},(\det\mathcal{Q})^{\otimes\frac{1}{2}})$ the image of which by $\pi_{\overrightarrow{\lambda},\overrightarrow{\lambda^*}}\circ\varphi$ is non-zero.

Let L be the subset of $[1,2r]\times[1,2s]$ consisting of all pairs (a,b) satisfying one of the following conditions.

- Both a and b are odd.
- a is odd, and b is even, and $s+1-(b/2)\leq\lambda_{(a+1)/2}$.
- b is odd, and a is even, and $r+1-(a/2)\leq\lambda^*_{(b+1)/2}$.

Let $V_1 \subset E^{(1)} \otimes G^{(1)}$ be the $2rs$-dimensional subspace spanned by $\mathbf{e}_a \otimes \mathbf{g}_b$ with $(a,b) \in L$. Let $V_2 \subset E^{(2)} \otimes G^{(2)}$ be the $2rs$-dimensional subspace spanned by $\mathbf{e}_a \otimes \mathbf{g}_b$ with $(a,b) \in [1,2r] \times [1,2s] \setminus L$. You can check easily that V_1 and V_2 are isotropic.

The subset of $\mathbf{OGr}^{+}_{4rs}$

$$\left\{ W \subset (E^{(1)} \otimes G^{(1)}) \oplus (E^{(2)} \otimes G^{(2)}) \middle| W \cap (V_1 \oplus V_2) \neq 0 \right\}$$

is a support of a zero-divisor of some section of $(\det \mathcal{Q})^{\otimes \frac{1}{2}}$, which we denote by $\sigma_{V_1 \oplus V_2}$. We shall check that $(\pi_{\overrightarrow{\lambda}, \overrightarrow{\lambda^*}} \circ \varphi)(\sigma_{V_1 \oplus V_2}) \neq 0$. For this, we need to recall the construction of the isomorphisms (8.2) and (8.3).

We denote by $\mathbf{KSp}^{\circ}$ the open subset

$$(KSp(E^{(1)}, E^{(2)}) \times Sp(G^{(1)}, G^{(2)})) \cup (Sp(E^{(1)}, E^{(2)}) \times KSp(G^{(1)}, G^{(2)}))$$

of $\mathbf{KSp} := KSp(E^{(1)}, E^{(2)}) \times KSp(G^{(1)}, G^{(2)})$.

By Proposition 4.1, there is a morphism $\mathbf{KSp}^{\circ} \to \mathbf{LGr}^{\circ}$, which we denote by $\widetilde{g}$.

Let

$$\begin{aligned}
&(\mathcal{M}_i, \mu_i, \mathcal{E}^{(1)}_i \to \mathcal{M}_i \otimes \mathcal{E}^{(1)}_{i+1}, \mathcal{E}^{(1)}_i \leftarrow \mathcal{E}^{(1)}_{i+1},\\
&\mathcal{E}^{(2)}_{i+1} \to \mathcal{E}^{(2)}_i, \mathcal{M}_i \otimes \mathcal{E}^{(2)}_{i+1} \leftarrow \mathcal{E}^{(2)}_i \quad (0 \leq i \leq r-1), \mathcal{E}^{(1)}_r \xrightarrow{\sim} \mathcal{E}^{(2)}_r)
\end{aligned}$$

and

$$\begin{aligned}
&(\mathcal{N}_i, \nu_i, \mathcal{G}^{(1)}_i \to \mathcal{N}_i \otimes \mathcal{G}^{(1)}_{i+1}, \mathcal{G}^{(1)}_i \leftarrow \mathcal{G}^{(1)}_{i+1},\\
&\mathcal{G}^{(2)}_{i+1} \to \mathcal{G}^{(2)}_i, \mathcal{N}_i \otimes \mathcal{G}^{(2)}_{i+1} \leftarrow \mathcal{G}^{(2)}_i \quad (0 \leq i \leq s-1), \mathcal{G}^{(1)}_r \xrightarrow{\sim} \mathcal{G}^{(2)}_r)
\end{aligned}$$

be the universal generalized symplectic isomorphism on $KSp(E^{(1)}, E^{(2)})$ and $KSp(G^{(1)}, G^{(2)})$ respectively. By Lemma 4.2, we have

$$\widetilde{g}^* \left((\det \mathcal{Q}_E)^{\otimes s} \boxtimes (\det \mathcal{Q}_G)^{\otimes r} \right) \simeq \left(\bigotimes_{i=0}^{r-1} \mathcal{M}_i^{\otimes s(r-i)} \right) \boxtimes \left(\bigotimes_{j=0}^{s-1} \mathcal{N}_j^{\otimes r(s-j)} \right).$$

Put $\widetilde{m}' := \widetilde{m} \circ \widetilde{g}$. In order to compute $(\pi_{\overrightarrow{\lambda}, \overrightarrow{\lambda^*}} \circ \varphi)(\sigma_{V_1 \oplus V_2})$, we first determine with how many orders the section $\widetilde{m}'^*(\sigma_{V_1 \oplus V_2})$ vanishes along divisors $\{\mu_i = 0\}$ and $\{\nu_j = 0\}$.

Let us see how the morphism $\widetilde{m}'$ is expressed in the open subset $KSp(E^{(1)}, E^{(2)})^{(\mathrm{id},\mathrm{id})} \times KSp(G^{(1)}, G^{(2)})^{(\mathrm{id},\mathrm{id})} \cap \mathbf{KSp}^{\circ}$.

Let

$$\mathrm{U}^{+}_{2r} \times \mathbb{A}^r \times \mathrm{U}^{-}_{2r} \simeq KSp(E^{(1)}, E^{(2)})^{(\mathrm{id},\mathrm{id})}$$

and

$$\mathrm{U}^{+}_{2s} \times \mathbb{A}^s \times \mathrm{U}^{-}_{2s} \simeq KSp(G^{(1)}, G^{(2)})^{(\mathrm{id},\mathrm{id})}$$

be the chart given in (3.23). For $\mathbf{y}_E = (y_{E,1}, \dots, y_{E,r}) \in \mathbb{A}^r$, we define the $2r \times 2r$ matrices $(\mathbf{y}_E)'$ and $(\mathbf{y}_E)''$ as follows:

$$(\mathbf{y}_E)' := \operatorname{diag}\left(\prod_{i=0}^{r-1} y_{E,i}, 1, \prod_{i=0}^{r-2} y_{E,i}, 1, \dots, y_{E,0}, 1\right)$$
$$(\mathbf{y}_E)'' := \operatorname{diag}\left(1, \prod_{i=0}^{r-1} y_{E,i}, 1, \prod_{i=0}^{r-2} y_{E,i}, \dots, 1, y_{E,0}\right).$$

We define $(\mathbf{y}_G)'$ and $(\mathbf{y}_G)''$ for $\mathbf{y}_G = (y_{G,1}, \dots, y_{G,s}) \in \mathbb{A}^s$ similarly.

Recall that the point $(\mathbf{z}_E, \mathbf{y}_E, \mathbf{x}_E) \in \mathrm{U}_{2r}^+ \times (\mathbb{A} \setminus \{0\})^r \times \mathrm{U}_{2r}^- = Sp(E^{(1)}, E^{(2)})$ corresponds to the symplectic isomorphism $\mathbf{x}_E \circ (\mathbf{y}_E)'' \circ (\mathbf{y}_E)'^{-1} \circ \mathbf{z}_E^{-1}$.

For $((\mathbf{z}_E, \mathbf{y}_E, \mathbf{x}_E), (\mathbf{z}_G, \mathbf{y}_G, \mathbf{x}_G)) \in Sp(E^{(1)}, E^{(2)}) \times Sp(G^{(1)}, G^{(2)})$, consider the tensored morphism

$$(\mathbf{x}_E \otimes \mathbf{x}_G) \circ ((\mathbf{y}_E)'' \otimes (\mathbf{y}_G)'') \circ ((\mathbf{y}_E)' \otimes (\mathbf{y}_G)')^{-1} \circ (\mathbf{z}_E \otimes \mathbf{z}_G)^{-1} : E^{(1)} \otimes G^{(1)} \to E^{(2)} \otimes G^{(2)}.$$

Its graph is equal to the image of

$$\zeta := ((\mathbf{z}_E \otimes \mathbf{z}_G) \circ ((\mathbf{y}_E)' \otimes (\mathbf{y}_G)'), (\mathbf{x}_E \otimes \mathbf{x}_G) \circ ((\mathbf{y}_E)'' \otimes (\mathbf{y}_G)''))$$
$$: E^{(1)} \otimes G^{(1)} \to \left(E^{(1)} \otimes G^{(1)}\right) \oplus \left(E^{(2)} \otimes G^{(2)}\right).$$

Now let us calculate the zero-divisor of the section $\widetilde{m}'^*(\sigma_{V_1 \oplus V_2})$.

Let $V_1^c \subset E^{(1)} \otimes G^{(1)}$ be the $2rs$-dimensional subspace spanned by $\mathbf{e}_a \otimes \mathbf{g}_b$ with $(a,b) \in [1,2r] \times [1,2s] \setminus L$. Let $V_2^c \subset E^{(2)} \otimes G^{(2)}$ be the $2rs$-dimensional subspace spanned by $\mathbf{e}_a \otimes \mathbf{g}_b$ with $(a,b) \in L$ ("c" means the complement). Let τ be the projection

$$\left(E^{(1)} \otimes G^{(1)}\right) \oplus \left(E^{(2)} \otimes G^{(2)}\right) \to V_1^c \oplus V_2^c.$$

The intersection of the image of ζ and $V_1 \oplus V_2$ is non-zero if and only if the composed morphism $\tau \circ \zeta$ is not an isomorphism. The zero-divisor defined by the determinant of $\tau \circ \zeta$ is the twice of the zero-divisor defined by $\widetilde{m}'^*(\sigma_{V_1 \oplus V_2})$.

Since $V_1 \oplus V_2$ is $\mathrm{U}_{2r}^+ \times \mathrm{U}_{2r}^- \times \mathrm{U}_{2s}^+ \times \mathrm{U}_{2s}^-$-invariant (easily checked), the zero-divisor of $\widetilde{m}'^*(\sigma_{V_1 \oplus V_2})$ is a pull-back of some divisor on $\mathbb{A}^r \times \mathbb{A}^s$ by the projection $(\mathrm{U}_{2r}^+ \times \mathbb{A}^r \times \mathrm{U}_{2r}^-) \times (\mathrm{U}_{2s}^+ \times \mathbb{A}^s \times \mathrm{U}_{2s}^-) \to \mathbb{A}^r \times \mathbb{A}^s$. When $\mathbf{z}_E = \mathbf{x}_E = \mathrm{id}_{2r}$ and $\mathbf{z}_G = \mathbf{x}_G = \mathrm{id}_{2s}$, the morphism $\tau \circ \zeta$ is expressed by a diagonal matrix with respect to the basis $\{\mathbf{e}_a \otimes \mathbf{g}_b\}$, and its determinant is easily computed to be

$$\left(\prod_{i=0}^{r-1} y_{E,i}^{(r-i)s - \sum_{l=1}^{r-i} \lambda_l} \times \prod_{j=0}^{s-1} y_{G,j}^{(s-j)r - \sum_{m=1}^{s-j} \lambda_m^*}\right)^2.$$

Therefore on $KSp(E^{(1)},E^{(2)})^{(\mathrm{id},\mathrm{id})}\times KSp(G^{(1)},G^{(2)})^{(\mathrm{id},\mathrm{id})}$, the zero-divisor of $\widetilde{m}'^*(\sigma_{V_1\oplus V_2})$ is defined by

$$\prod_{i=0}^{r-1} y_{E,i}^{(r-i)s-\sum_{l=1}^{r-i}\lambda_l} \times \prod_{j=0}^{s-1} y_{G,j}^{(s-j)r-\sum_{m=1}^{s-j}\lambda^*_m} = 0.$$

This implies that on $KSp(E^{(1)},E^{(2)})^{(\mathrm{id},\mathrm{id})}\times KSp(G^{(1)},G^{(2)})^{(\mathrm{id},\mathrm{id})}$, $\widetilde{m}'^*(\sigma_{V_1\oplus V_2})$ becomes a nowhere vanishing section of the line bundle $\bigotimes_{i=1}^{r-1}\mathcal{M}_i^{\otimes\sum_{l=1}^{r-i}\lambda_l}\boxtimes\bigotimes_{j=1}^{s-1}\mathcal{N}_j^{\otimes\sum_{m=1}^{s-j}\lambda^*_m}$. Since $(\pi_{\vec{\lambda},\vec{\lambda}^*}\circ\varphi)(\sigma_{V_1\oplus V_2})$ is nothing but the restriction of

$$\widetilde{m}'^*(\sigma_{V_1\oplus V_2}) \in \bigotimes_{i=1}^{r-1}\mathcal{M}_i^{\otimes\sum_{l=1}^{r-i}\lambda_l}\boxtimes\bigotimes_{j=1}^{s-1}\mathcal{N}_j^{\otimes\sum_{m=1}^{s-j}\lambda^*_m}$$

to the closed subscheme

$$\mathrm{SpFl}_{[0,r-1]}(E^{(1)})\times\mathrm{SpFl}_{[0,r-1]}(E^{(2)})\times\mathrm{SpFl}_{[0,s-1]}(G^{(1)})\times\mathrm{SpFl}_{[0,s-1]}(G^{(2)}),$$

it is non-zero. (Note that the intersection of the closed subscheme and the open subset $KSp(E^{(1)},E^{(2)})^{(\mathrm{id},\mathrm{id})}\times KSp(G^{(1)},G^{(2)})^{(\mathrm{id},\mathrm{id})}$ is not empty. In fact, the closed subset is defined by $y_{E,i}=y_{G,j}=0$ for all i,j on the open subset.) □

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