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Complex projective manifolds which admit non-isomorphic surjective endomorphisms

By

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Abstract

This article overviews recent progress of the study of endomorphisms of complex projective manifolds from the viewpoint of classification theory of compact complex manifolds, and surveys the papers [19], [21], [46] of the authors.

§ 1. Introduction

A surjective endomorphism of a compact complex manifold X means a surjective morphism (holomorphic map) from X to itself. Typical examples of non-isomorphic surjective endomorphism f and X are as follows:

- X is a projective space $\mathbb{P}^n$ and f is given by
$$(\mathtt{x}_0 : \cdots : \mathtt{x}_n) \mapsto (\mathtt{x}_0^m : \cdots : \mathtt{x}_n^m)$$
for a positive integer $m \geq 2$, where $(\mathtt{x}_0 : \cdots : \mathtt{x}_n)$ is a homogeneous coordinate.
- X is a compact complex torus $\mathbb{C}^n/L$ and f is given by $x \mapsto mx = x + \cdots + x$ for $m \geq 2$, where $L \simeq \mathbb{Z}^{2n}$ is a submodule with $\mathbb{C}^n = L \otimes_{\mathbb{Z}} \mathbb{R}$.

The study of surjective endomorphisms of a given variety X, such as projective spaces $\mathbb{P}^n$, complex tori $\mathbb{C}^n/L$, etc., is a chief concern of complex dynamical systems, which

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is usually done by analytic methods. The complex dynamical systems also treat non-holomorphic cases, e.g., meromorphic endomorphisms $f\colon X \cdots\to X$. For example, some of K3 surfaces admit meromorphic dominant endomorphisms of degree > 1, but any surjective endomorphism of an arbitrary K3 surface is an automorphism.

The concern of our paper is on the other side, i.e., the study of compact complex manifolds X admitting non-isomorphic surjective (holomorphic) endomorphisms. So, we may replace an endomorphism f with another one freely. In particular, f is replaced with a power $f^k = f \circ \cdots f$ for $k > 0$. The existence of non-isomorphic surjective endomorphisms yields strong conditions on the variety X. For example:

- X is not of general type.
- If X has non-negative Kodaira dimension $\kappa(X)$, then the (topological) Euler number $e(X)$ and the Euler–Poincaré characteristic $\chi(X, \mathcal{O}_X)$ are both zero, and the fundamental group $\pi_1(X)$ is infinite.
- If X is a compact smooth surface, then X has at most finitely many irreducible curves whose self-intersection number is negative.
- If X is a smooth projective 3-fold with $\kappa(X) \geq 0$, then X has at most finitely many extremal rays in the sense of Mori [39].

In some cases, we can classify such X only using these conditions. For example, a smooth projective curve admitting a non-isomorphic surjective endomorphism is a rational curve or an elliptic curve (i.e., a one-dimensional compact complex torus).

We note that if $f\colon X \to X$ is a non-isomorphic surjective endomorphism, then so is the product mapping $f \times \mathrm{id}_Y\colon X \times Y \to X \times Y$ for any variety Y. So, in many cases, the classification is done by relating to the direct products of certain varieties of this type. For example: A smooth projective surface X of $\kappa(X) = 1$ admits a non-isomorphic surjective endomorphism if and only if $e(X) = 0$. Moreover this is equivalent to that a finite étale covering of X is isomorphic to $E \times C$ for an elliptic curve E and a curve C of genus ≥ 2 (cf. Proposition 4.1; [19], [20]).

The purpose of this article is to survey the classification of complex projective manifold X admitting non-isomorphic surjective endomorphisms in the case of $\dim X = 2$ and the case of $\dim X = 3$ with $\kappa(X) \geq 0$. The set of irreducible curves with negative self-intersection number plays an important role in the former case, and the theory of extremal rays in the latter case. The classification of such surfaces is given in Theorem 1.1 below, which is proved mainly in [46].

Theorem 1.1. *Let X be a smooth projective surface. Then X admits a non-isomorphic surjective endomorphism if and only if one of the following conditions is satisfied*:

(1) *X is a toric surface* (i.e., there is an equivariant open immersion $(\mathbb{C}^\star)^2 \subset X$ of the two-dimensional algebraic torus $(\mathbb{C}^\star)^2$).

(2) *X is a $\mathbb{P}^1$-bundle over an elliptic curve.*

(3) *X is a $\mathbb{P}^1$-bundle over a curve B of genus ≥ 2 such that $X \times_B B' \simeq \mathbb{P}^1 \times B'$ over B' for a finite étale covering $B' \to B$.*

(4) *X is an abelian surface.*

(5) *X is a hyperelliptic surface.*

(6) *X is a projective surface of $\kappa(X) = 1$ and $e(X) = 0$: This is equivalent to that a finite étale covering of X is isomorphic to $E \times C$ for an elliptic curve E and a curve C of genus ≥ 2.*

We shall explain the idea and the outline of the proof of Theorem 1.1 in Section 4. The authors extended the classification to all the smooth compact complex analytic surfaces in [20]. The classification in the case of smooth projective 3-folds is completed in papers [19] and [21]. The main result of [21] is:

Theorem 1.2. *Let X be a smooth projective* 3*-fold with $\kappa(X) \geq 0$. Then the following two conditions are equivalent to each other*:

(A) *X admits a non-isomorphic surjective endomorphism.*

(B) *There exists a finite étale Galois covering $\tau\colon \widetilde{X} \to X$ and an abelian scheme structure* (i.e., a relative Lie group structure) *$\varphi\colon \widetilde{X} \to T$ over a variety T of dimension ≤ 2 such that the Galois group* $\mathrm{Gal}(\tau)$ *acts on T and φ is* $\mathrm{Gal}(\tau)$*-equivariant.*

The implication (B) $\Rightarrow$ (A) holds in any dimension ([21], Theorem 2.26). The other implication (A) $\Rightarrow$ (B) for 3-folds is proved by considering the minimal models and the structure of Iitaka fibrations, etc. A finer description of the 3-fold X is as follows:

Theorem 1.3. *Let X be a smooth projective* 3*-fold with $\kappa(X) \geq 0$ which admits a non-isomorphic surjective endomorphism. Then there exists a finite étale Galois covering $\widetilde{X} \to X$ satisfying the following conditions*:

(1) *If $\kappa(X) = 0$, then either*

  (*a*) *$\widetilde{X}$ is an abelian* 3*-fold, or*

  (*b*) *$\widetilde{X} \simeq E \times S$ for an elliptic curve E and a surface S birational to a K3 surface or an abelian surface.*

(2) *If* $\kappa(X) = 1$, *then either*

(a) $\widetilde{X}$ *is an abelian scheme over a curve* T *of genus* ≥ 2, *or*

(b) $\widetilde{X} \simeq E \times S$ *for an elliptic curve* E *and a surface* S *with* $\kappa(S) = 1$.

(3) *If* $\kappa(X) = 2$, *then* $\widetilde{X} \simeq E \times S$ *for an elliptic curve* E *and a surface* S *of general type.*

We shall explain the idea and the outline of the proof of Theorems 1.2 and 1.3 in Section 5.

In order to study compact complex manifolds X admitting non-isomorphic surjective endomorphisms, it is important to analyze data of X preserved by the endomorphisms, since they reveal much of the deeper structure of the variety X. The following data are important in this article.

Iitaka fibration: Let $\Phi_X : X \cdots\to W$ be the Iitaka fibration of the variety X. Then for a surjective endomorphism f of X and for a suitable choice of W, there exists an automorphism h of W with $\Phi_X \circ f = h \circ \Phi_X$ (cf. Lemma 3.1 below). Moreover, h is of finite order by [47], Theorem A (cf. Theorem 3.2 below). Hence by replacing f with a suitable power $f^k = f \circ \cdots \circ f$, we may assume that $\Phi_X \circ f = \Phi_X$. Thus, f induces a surjective endomorphism of a fiber of Φ_X.

The set of extremal rays: A surjective endomorphism $f : X \to X$ of X with $\kappa(X) \geq 0$ induces a permutation of the set $\mathrm{ER}(X)$ of extremal rays of X (cf. Lemma 5.1 below; [19], Theorem 4.2). If $\dim(X) = 3$ and $\kappa(X) \geq 0$, then $\mathrm{ER}(X)$ is a finite set; hence, we may assume that $f_* R = R$ for any extremal ray R of X by replacing f with a suitable power f^k.

The set of negative curves: Assume that X is a smooth compact complex analytic surface. An irreducible and reduced curve on X is called *negative* if its self-intersection number is negative. If X admits a non-isomorphic surjective endomorphism f, then the set $\mathcal{S}(X)$ of negative curves is finite and $C \mapsto f(C)$ induces a permutation of $\mathcal{S}(X)$ (cf. Proposition 4.2 below; [46], [20]). The sum $N_X = \sum C$ of all the negative curves plays an important role in determining the structure of X.

Further progress on the study of non-isomorphic surjective endomorphisms f seems to depend on finding such important objects invariant under f_* or f^*.

This article is organized as follows: In Section 2, we present briefly the history of the algebraic study of endomorphisms. Basic properties of non-isomorphic surjective endomorphisms related to Kodaira dimension are explained in Section 3. The study on smooth projective surfaces admitting non-isomorphic surjective endomorphisms in papers [46], [20] is surveyed in Section 4 with an outline of the proof of Theorem 1.1. The case of smooth projective 3-folds with $\kappa \geq 0$ is treated in Section 5. After discussion of extremal rays, minimal reductions, and abelian fibrations in Sections 5.1–5.3, we give an outline of the proof of Theorems 1.2 and 1.3. The final Section 6 is devoted to survey the recent result on "building blocks" of étale endomorphisms by Nakayama and Zhang [47].

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§ 2. History of the algebro-geometric study of endomorphisms

We recall briefly the history of the study of endomorphisms of complex projective manifolds by algebro-geometric methods.

§ 2.1. Lazarsfeld conjecture

One of the origins of the study of endomorphisms in algebraic geometry is the following conjecture by Lazarsfeld [35] (1984):

Conjecture 2.1 (Lazarsfeld). Let G be a complex semi-simple algebraic group, $P \subset G$ a maximal parabolic subgroup, and $Y := G/P$. Let $h\colon Y \to X$ be a finite surjective morphism of $\deg(h) > 1$ to a smooth projective variety X. Then $X \simeq \mathbb{P}^n$.

Note that the quotient space $Y = G/P$ is a rational homogeneous Fano manifold with the Picard number one.

In 1989, Paranjape and Srinivas [49] gave partial answers to Conjecture 2.1 and showed that the homogeneous manifold $Y = G/P$ admits a non-isomorphic surjective endomorphism if and only if $Y \simeq \mathbb{P}^n$. This is considered as the first result on endomorphisms of algebraic varieties. Lazarsfeld's conjecture itself was solved affirmatively by Hwang and Mok [23] in 1999. A generalization to the case of compact complex homogeneous manifolds was obtained by Cantat [12] in 2000.

The following related conjecture was proved by [3] and [24] for $n = 3$, but is still open for $n \geq 4$:

Conjecture 2.2. Let X be a Fano manifold of dimension n with Picard number one. If there is a non-isomorphic surjective endomorphism $f\colon X \to X$, then $X \simeq \mathbb{P}^n$.

A stronger conjecture is studied in [3] and [24], where is asked the boundedness of the degrees of surjective morphisms between two given smooth projective manifolds of Picard number one. In [3], it is discovered a formula of Hurwitz type on the top Chern classes for finite morphisms between smooth projective manifolds ([3], Corollary 1.2). Applying the formula, Beauville [5] has succeeded in proving:

Theorem 2.3. *A smooth complex projective hypersurface of dimension* ≥ 2 *and degree* ≥ 3 *does not admit a non-isomorphic surjective endomorphism.*

This result is related also to Conjecture 2.6 below.

§ 2.2. Classification theory

Another aspect of the study of endomorphisms is found in the paper [50] (1998) of Fujimoto and Sato, where they announced some results on the classification of smooth projective varieties of non-negative Kodaira dimension admitting non-isomorphic surjective endomorphisms in dimensions two and three. The results with detailed proofs are given by Fujimoto in [19]. In the paper, Fujimoto almost classified such smooth projective 3-folds by applying the theory of extremal rays developed by Mori [39]. It is proved that there exists a finite étale covering $\widetilde{X} \to X$ such that $\widetilde{X}$ is isomorphic to either an abelian 3-fold or the direct product $E \times S$ of an elliptic curve E and a smooth projective surface S, except for the case where $\kappa(X) = 1$ and the general fiber of the Iitaka fibration of X is an abelian surface.

The exceptional case is treated in a joint paper [21] of the authors, and the classification of the smooth projective 3-folds with $\kappa \geq 0$ admitting non-isomorphic surjective endomorphisms is completed (cf. Theorems 1.2, 1.3, and Section 5).

Before the paper [50] appeared, Sato and his student Segami started to study smooth projective surfaces admitting non-isomorphic surjective endomorphisms. For example, in [51], Segami proved that if such a surface X is irrational and ruled, then X is a $\mathbb{P}^1$-bundle over a curve, and moreover that if the irregularity $q(X) > 1$ in addition, then the $\mathbb{P}^1$-bundle is associated with a semistable vector bundle of rank two. Sato informed the authors the following conjecture in 1998:

Conjecture 2.4 (Sato)**.** A smooth rational surface admitting non-isomorphic surjective endomorphisms is a toric surface.

The converse to Sato's conjecture is true. In fact, every toric variety admits a non-isomorphic surjective endomorphism which is induced from the map

$$(\mathbb{C}^\star)^n \ni (t_1, \dots, t_n) \mapsto (t_1^k, \dots, t_n^k) \in (\mathbb{C}^\star)^n$$

for $k > 1$ for the open torus $(\mathbb{C}^\star)^n$. Conjecture 2.4 was solved affirmatively by Nakayama in [46], where is also given a complete list of smooth projective surfaces of $\kappa = -\infty$ admitting non-isomorphic surjective endomorphisms (cf. Theorem 1.1 and Section 4 below). The key idea is to consider the set of negative curves. Partial classification results for rational surfaces are also given in [5] and [54], where is obtained a list of (weak) del Pezzo surfaces. In 2003, Amerik [1] had another idea in classifying irrational ruled surfaces and generalized it to projective bundles $X \to B$ admitting non-isomorphic surjective endomorphisms *over* B: By the geometric invariant theory, she showed that a $\mathbb{P}^n$-bundle $X \to B$ admits a non-isomorphic surjective endomorphism over B if and only if X is trivialized after a finite base change.

The authors [20] completely classified smooth compact complex analytic surfaces admitting non-isomorphic surjective endomorphisms in 2005. The idea is also to consider the set of negative curves with the help of classification theory of elliptic surfaces and VII_0-surfaces.

§ 2.3. Polarized endomorphisms

Endomorphisms, especially *polarized* endomorphisms, are studied by many researchers of arithmetic geometry (cf. [9], [14], [27], [56]), where the *canonical height functions* play an important role. A surjective endomorphism $f\colon X \to X$ of a projective variety X is called *polarized* if there is an ample divisor H such that $f^*H \sim qH$ for some $q > 0$. The following are some of geometric results in [14] and [56], which are not related to the canonical height functions:

Theorem 2.5.

(1) *If $f\colon X \to X$ is a polarized endomorphism, then there exist a closed immersion $i\colon X \subset \mathbb{P}^N$ and a surjective endomorphism $g\colon \mathbb{P}^N \to \mathbb{P}^N$ such that $g \circ i = i \circ f$* ([14], Corollary 2.2).

(2) *Let X be a smooth projective variety admitting a non-isomorphic polarized endomorphism. If $\kappa(X) \geq 0$, then a finite étale covering of X is an abelian variety* ([14], Theorem 4.2). *If $\kappa(X) < 0$, then X is uniruled* ([56], Proposition 2.2.1).

In particular, the study of polarized endomorphisms is reduced to that of surjective endomorphisms g of $\mathbb{P}^N \supset X$ such that $g(X) = X$.

Endomorphisms of projective spaces $\mathbb{P}^N$ are studied by many researches of complex analysis as a subject of complex analytic dynamical systems. For the invariant subvarieties, the following conjecture is studied (cf. [15], §4):

Conjecture 2.6. Let $g\colon \mathbb{P}^N \to \mathbb{P}^N$ be a non-isomorphic surjective endomorphism, and $V \subset \mathbb{P}^N$ a subvariety with $g^{-1}(V) = V$. Then V is linear.

This is solved affirmatively in case $N = 2$ with $\deg(V) \geq 3$ ([15], §4), and in case V is a smooth hypersurface with $(N, \deg(V)) \neq (2, 2)$ ([13]). Theorem 2.3 gives another proof in the case of smooth hypersurfaces V of degree > 2. Note that the arguments on Conjecture 2.6 in [15] and [13] are algebraic. In 2004, the paper [8] announced a proof of Conjecture 2.6 in any case, but unfortunately, the proof seems to have a gap, so the conjecture is still open.

§ 2.4. Building blocks

Inspired by dynamical study of automorphisms of projective varieties (cf. [55]), D.-Q. Zhang started to consider "building blocks" of surjective endomorphisms of projective algebraic varieties in 2006. The building blocks are obtained through the Iitaka fibration, the Albanese map, and the maximal rationally connected fibration (cf. [10], [11], [34], [22]). Nakayama and Zhang [47] gave a weak answer to the question what are the building blocks for étale endomorphisms, assuming good minimal model conjectures, etc. They asserted that the study of étale endomorphisms is reduced in some sense to that of étale endomorphisms of abelian varieties or that of nearly étale rational endomorphisms of weak Calabi–Yau varieties (cf. Section 6). However, the result only gives a perspective of classification; for example, even if we know each of the building blocks of an étale endomorphism f very well, it is rather difficult to recover the structure of f as in [19], [21]. We overview some of the results of [47] in Section 6.

§ 3. Basic properties related to Kodaira dimension

We discuss elementary properties related to non-isomorphic surjective endomorphisms and the Kodaira dimension of compact complex manifolds. All the results in this section are well-known except for Theorem 3.2.

The Kodaira dimension $\kappa(X)$ is one of the most important bimeromorphic invariants of compact complex manifolds X. For a positive integer m, the m-th pluricanonical linear system $|mK_X|$ is defined to be the set of divisors $\operatorname{div}(\eta)$ associated to non-zero m-ple holomorphic n-forms η, where $n = \dim X$. Thus, $|mK_X|$ is identified with the projective space

$$\mathbb{P}(\mathrm{H}^0(X, \omega_X^{\otimes m})^\vee) = \left(\mathrm{H}^0(X, \omega_X^{\otimes m}) \setminus \{0\}\right) / \mathbb{C}^\star,$$

where $\omega_X = \Omega^n_X$ is the sheaf of germs of holomorphic n-forms. The canonical divisor K_X is a divisor with $\mathcal{O}_X(K_X) \simeq \omega_X$. Even if ω_X has no non-zero meromorphic sections, we use K_X symbolically and call it the canonical divisor. Then, $\omega_X^{\otimes m} \simeq \mathcal{O}_X(mK_X)$.

Suppose that $|mK_X| \neq \emptyset$. Then the base locus $\operatorname{Bs}|mK_X| = \bigcap_{D \in |mK_X|} \operatorname{Supp} D$ is a proper subset of X. For $x \in X \setminus \operatorname{Bs}|mK_X|$, the subset

$$H_x = \{D \in |mK_X|\,;\, x \in \operatorname{Supp} D\}$$

is a hyperplane of $|mK_X|$. By $x \mapsto [H_x]$, we have a meromorphic map

$$\Phi_m = \Phi_{|mK_X|} \colon X \cdots\!\to |mK_X|^\vee = \mathbb{P}(\mathrm{H}^0(X, \mathcal{O}_X(mK_X)))$$

to the dual projective space $|mK_X|^\vee$, which is holomorphic on $X \setminus \operatorname{Bs}|mK_X|$. The map Φ_m is called the m-th pluricanonical map. The Kodaira dimension $\kappa(X)$ is defined by:

$$\kappa(X) := \begin{cases} -\infty, & \text{if } |mK_X| = \emptyset \text{ for any } m > 0; \\ \max\{\dim \Phi_m(X)\,;\, |mK_X| \neq \emptyset, m \in \mathbb{N}\}, & \text{otherwise.} \end{cases}$$

This is a bimeromorphic invariant of compact complex manifolds. For a singular compact complex variety, its Kodaira dimension is defined as that of a compact complex manifold bimeromorphic to it. Iitaka [25] (cf. [26]) proved that, in case $\kappa(X) > 0$, if m is sufficiently large and divisible, then $\Phi_X = \Phi_m \colon X \cdots\!\to \Phi_m(X)$ is uniquely determined up to bimeromorphic equivalence and a very general fiber F of Φ_X is connected with $\kappa(F) = 0$. The map Φ_X is called the Iitaka fibration of X. Iitaka also showed the following asymptotic behavior of $P_m(X) = \dim \mathrm{H}^0(X, \mathcal{O}_X(mK_X))$: There exist an integer m_0, positive numbers $\alpha < \beta$ such that, for any $m \gg 0$,

$$\alpha m^{\kappa(X)} \leq P_{mm_0}(X) \leq \beta m^{\kappa(X)}.$$

Let $g \colon X \to Y$ be a generically finite surjective morphism of compact complex manifolds. Then we have a natural injection $g^*\Omega^1_Y \to \Omega^1_X$ for the sheaf Ω^1 of germs of holomorphic one-forms. Taking the wedge product, we have also an injection $g^* \colon g^*\omega_Y \to \omega_X$. Since ω_X and ω_Y are invertible, the cokernel of g^* is expressed as $\omega_X \otimes \mathcal{O}_R$ for an effective divisor $R = R_g$. Note that, locally on X, g^* is expressed as the determinant of Jacobian matrix of g, and R is defined as the zero locus of the function g^*. Therefore, $R = 0$ if and only if g is étale. As a divisor, we can write $K_X = g^*K_Y + R$, which is called the ramification formula. Here, we use $=$ rather than the linear equivalence $\sim$, since the formula means that $\operatorname{div}(g^*\eta) = g^* \operatorname{div}(\eta) + R$ for a meromorphic n-form η on Y. For every integer $m > 0$, we also have an injection $g^*\omega_Y^{\otimes m} \to \omega_X^{\otimes m}$. Taking global sections, we have an inequality: $\kappa(X) \geq \kappa(Y)$.

Lemma 3.1. *Let $\Phi_X \colon X \dashrightarrow W$ be the Iitaka fibration of a compact complex manifold X with $\kappa(X) > 0$. Suppose that X admits a surjective endomorphism $f \colon X \to X$. Then, for a suitable choice of W, there exists a biregular automorphism h of W such that $\Phi_X \circ f = h \circ \Phi_X$.*

Proof. The injective homomorphism

$$f^* \colon \mathrm{H}^0(X, \mathcal{O}_X(mK_X)) \to \mathrm{H}^0(X, \mathcal{O}_X(mK_X))$$

is an isomorphism, since $\mathrm{H}^0(X, \mathcal{O}_X(mK_X))$ is finite-dimensional. For the Iitaka fibration $\Phi_m \colon X \dashrightarrow |mK_X|^\vee$ and for the pull-back homomorphism f^* above, we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle \Phi_m}\big\downarrow & & {\scriptstyle \Phi_m}\big\downarrow \\ |mK_X|^\vee & \xrightarrow{(f^*)^\vee} & |mK_X|^\vee. \end{array}$$

Therefore, the assertion holds for $W = \Phi_m(X)$. □

The following was conjectured for several years, which has recently been proved in [47], Theorem A:

Theorem 3.2. *The automorphism h of W in* Lemma 3.1 *is of finite order, if X is a compact Kähler manifold.*

Remark 3.3.

(1) In [47], Theorem A, the result holds also for dominant meromorphic endomorphisms $X \dashrightarrow X$ of compact Kähler manifolds. The proof involves an argument on variation of Hodge structures and on the automorphism group of W.

(2) If Φ_X is holomorphic, then, by Theorem 3.2, a suitable power f^k is an endomorphism of X over W, i.e., $\Phi_X \circ f^k = \Phi_X$.

Lemma 3.4. *Let $f \colon X \to X$ be a surjective endomorphism of a compact complex manifold X.*

(1) *If X is Kähler, then f is a finite morphism.*

(2) *If $\kappa(X) \geq 0$ or if X is a non-uniruled projective variety, then f is finite and étale* (cf. [26], Theorem 11.7)*.*

(3) *If $\kappa(X) = \dim(X)$* (i.e., X is of general type)*, then f is an automorphism* (cf. [26], Proposition 10.10)*.*

Proof. (1): The pull-back homomorphism $f^*\colon \mathrm{H}^2(X,\mathbb{R}) \to \mathrm{H}^2(X,\mathbb{R})$ is an automorphism preserving the subspace $\mathrm{H}^{1,1}(X,\mathbb{R}) := \mathrm{H}^2(X,\mathbb{R}) \cap \mathrm{H}^{1,1}(X)$. Thus, any Kähler form of X is cohomologous to the pullback by f^* of another Kähler form of X. Hence, no fiber of f has positive dimension, i.e., f is a finite morphism.

(2): By the ramification formula $K_X = f^*K_X + R$, we have the ramification formula

$$(*) \qquad K_X = (f^k)^*K_X + (f^{k-1})^*R + \cdots + f^*R + R$$

for the k-th power $f^k = f \circ \cdots \circ f$ for any k. Suppose that $\kappa(X) \geq 0$, i.e., $|mK_X| \neq \emptyset$ for some $m > 0$. Then any member $D \in |mK_X|$ is written as

$$D = (f^k)^*D_k + (f^{k-1})^*(mR) + \cdots + f^*(mR) + mR$$

for another member $D_k \in |mK_X|$. Thus, $R = 0$, since D has finitely many irreducible components. If X is projective and not uniruled, then $K_X A^{n-1} \geq 0$ for any ample divisor A by [38], where $n = \dim X$. Hence, (*) implies $R = 0$; otherwise

$$K_X A^{n-1} \geq \left((f^{k-1})^*R + \cdots + f^*R + R\right) A^{n-1} \geq k \longrightarrow \infty.$$

Since f is proper, $R = 0$ means that f is finite and étale.

(3): By assumption, the Iitaka fibration $\Phi_X\colon X \cdots\to W$ is a birational map. Thus the assertion is derived by Lemma 3.1 and (2). □

Remark 3.5. If $\dim X = 2$, then f is finite by [20] even if X is not Kähler. For $\dim X \geq 3$, the finiteness of f is unknown.

Corollary 3.6. *Let X be a compact complex manifold admitting non-isomorphic surjective endomorphisms. If $\kappa(X) \geq 0$ or if X is non-uniruled projective, then the Euler number $e(X) = \sum_{i \geq 0}(-1)^i b_i(X)$ and the Euler-Poincaré characteristic $\chi(X, \mathcal{O}_X) = \sum_{i \geq 0}(-1)^i \dim \mathrm{H}^i(X, \mathcal{O}_X)$ are both zero; moreover, the fundamental group $\pi_1(X)$ is infinite.*

Remark 3.7. If X is a non-uniruled smooth projective 3-fold, then $\kappa(X) \geq 0$ by Mori [40] and by Miyaoka [36]. However, if we assume that $\chi(X, \mathcal{O}_X) = 0$ for the 3-fold X, in addition, then $\kappa(X) \geq 0$ is easily derived as follows: If the irregularity $q(X)$ is zero, then $p_g(X) = 1 + \dim \mathrm{H}^2(X, \mathcal{O}_X) \geq 1$; thus $\kappa(X) \geq 0$. If $q(X) > 0$, then, for the Stein factorization $X \to V$ of the Albanese map $X \to \mathrm{Alb}(X)$ and for a general fiber F of $X \to V$, we have $\kappa(X) \geq \kappa(F) + \kappa(V)$ as Iitaka's addition formula of κ valid for 3-folds (cf. [28]). Here, $\kappa(F) \geq 0$ since X is not uniruled. Further $\kappa(V) \geq 0$ by [52]. Thus, $\kappa(X) \geq 0$. Therefore, by Corollary 3.6, we have a simple proof of the assertion that for a smooth projective 3-folds X with non-isomorphic surjective endomorphism, $\kappa(X) \geq 0$ if and only if X is not uniruled.

§ 4. Smooth projective surfaces

In this section, we consider a smooth projective surface X admitting a non-isomorphic surjective endomorphism $f\colon X \to X$. As is shown in Section 3, X is not of general type; If $\kappa(X) \geq 0$, then $e(X) = \chi(X, \mathcal{O}_X) = 0$ and f is étale. Moreover, we see that if $\kappa(X) \geq 0$, then X is minimal. In fact, if X has a (-1)-curve (= the exceptional curve of the first kind), then $f^{-1}(C)$ is a disjoint union of (-1)-curves which are copies of C, thus X has infinitely many (-1)-curves, a contradiction. The classification in the case of $\kappa(X) \geq 0$ is done by:

Proposition 4.1 ([19], Proposition 3.3; [20], Appendix to Section 4)**.** *Let X be a smooth projective surface with $0 \leq \kappa(X) \leq 1$. Then the following conditions are mutually equivalent*:

(1) *X admits a non-isomorphic surjective endomorphism.*

(2) $e(X) = 0$.

(3) $\chi(X, \mathcal{O}_X) = 0$.

(4) *If $\kappa(X) = 0$, then X is an abelian surface or a hyperelliptic surface. If $\kappa(X) = 1$, then a finite étale Galois covering of X is isomorphic to the product $E \times C$ of an elliptic curve E and a curve C of genus ≥ 2.*

§ 4.1. Negative curves

To proceed the classification in the case of ruled surfaces, the set $\mathcal{S}(X)$ of negative curves plays an important role. Here, an irreducible and reduced curve is called negative if its self-intersection number is negative.

Proposition 4.2 ([46], Proposition 11; [20], Proposition 3.5)**.** *$\mathcal{S}(X)$ is a finite set and the map $C \mapsto f(C)$ induces a permutation of $\mathcal{S}(X)$.*

Proof. Let C be a negative curve. Suppose that $f(C) = f(C')$ for an irreducible curve C'. Then $f_*C = \alpha f_*(C')$ for some $\alpha > 0$. For the Néron–Severi group $\mathrm{NS}(X)$, the push-forward map $f_*\colon \mathrm{NS}(X) \otimes \mathbb{Q} \to \mathrm{NS}(X) \otimes \mathbb{Q}$ is bijective. Hence, $C = \alpha C'$ in $\mathrm{NS}(X) \otimes \mathbb{Q}$. Then $C = C'$ by $CC' < 0$. This observation implies that $C \mapsto f(C)$ gives a bijection $\mathcal{S}(X) \to \mathcal{S}(X)$. Let $\mathcal{S}(X)_0$ be the subset of negative curves which are irreducible components of the ramification divisor R_f. If $C \in \mathcal{S}(X) \setminus \mathcal{S}(X)_0$, then $|C^2| > |f(C)^2|$ by $C = f^*(f(C))$. Thus, for any $C \in \mathcal{S}(X)$, there exists an integer $k > 0$ such that $f^k(C) \in \mathcal{S}(X)_0$. Then Lemma 4.3 below on the set theory completes the proof. $\square$

Lemma 4.3 ([20], Lemma 3.4; [46], Proposition 11). *Let $\mathcal{S}$ be a set, $\mathcal{S}_0$ a finite subset, and $h\colon \mathcal{S} \to \mathcal{S}$ an injection such that*

$$\mathcal{S} = \bigcup\nolimits_{m=1}^{\infty} (h^m)^{-1}(\mathcal{S}_0).$$

Then $\mathcal{S}$ is finite.

Therefore, by Proposition 4.2, we may assume from the beginning that $f^{-1}(C) = C$ for any negative curve C by replacing f with a power f^k. Then there is a positive integer a such that $\deg f = a^2$, $f^*C = aC$, and $f_*(C) = aC$ for any $C \in \mathcal{S}(X)$. We define

$$N_X := \sum\nolimits_{C \in \mathcal{S}(X)} C.$$

Then the ramification formula of f can be replaced with

$$K_X + N_X = f^*(K_X + N_X) + \Delta$$

for an effective divisor Δ whose irreducible components are not negative. Investigating the ramification of $f|_C\colon C \to C$, we have:

Lemma 4.4 ([20], Lemma 3.7; [46], Lemma 13). *A connected component of N_X is one of the following*:

(1) *An elliptic curve.*

(2) *A* straight chain of $\mathbb{P}^1$, *i.e., a reduced divisor $\sum_{i=1}^{l} C_i$ with irreducible components $C_i \simeq \mathbb{P}^1$ satisfying*

$$C_iC_j = \begin{cases} 1, & \text{if } |i-j| = 1; \\ 0, & \text{if } |i-j| > 1. \end{cases}$$

(3) *A* cycle of rational curves, *i.e., either a rational curve with exactly one node, or a connected reduced normal crossing divisor $\sum_{i=1}^{l} C_i$ $(l \geq 2)$ with irreducible components $C_i \simeq \mathbb{P}^1$ satisfying*

$$C_i \Big(\sum\nolimits_{j \neq i} C_j \Big) = 2.$$

§ 4.2. Ruled surfaces

The classification of smooth rational surfaces X is reduced to proving Conjecture 2.4. This is proved in [46] by using the properties of the set $\mathcal{S}(X)$ of negative curves shown in Section 4.1 and the following:

Proposition 4.5 ([46], Theorem 17). *Let X be a smooth rational surface with finitely many negative curves. For the sum N_X of all the negative curves, assume that any connected component of N_X is either a straight chain of $\mathbb{P}^1$ or a cycle of rational curves. Then X is a toric surface.*

Next, we consider the case where X is irrational and ruled. Let $\pi\colon X \to T$ be the ruling to a smooth projective curve T of genus $g(T) = q(X) \geq 1$. Then $\pi \circ f = h \circ \pi$ for a surjective étale endomorphism $h\colon T \to T$. An irreducible component C of a reducible fiber of $X \to T$ is a negative curve, and hence $f^*C = aC$ by Section 4.1. Therefore, $h^*(\pi(C)) = a\pi(C)$, which contradicts that h is étale. Hence, π is smooth, i.e., a $\mathbb{P}^1$-bundle. This argument was essentially used in [51]. The classification in the case $g(T) = q(X) = 1$ is done by:

Proposition 4.6 ([46], Propositions 5 and 14). *An elliptic ruled surface, i.e., a smooth projective ruled surface with the irregularity one, admits a non-isomorphic surjective endomorphism if and only if it is a $\mathbb{P}^1$-bundle over an elliptic curve.*

In case $g(T) \geq 2$, there is no negative curve of X dominating T by Lemma 4.4. Thus, π is associated with a semi-stable vector bundle of rank two on T. With more arguments, we have:

Theorem 4.7 ([46], Theorems 8 and 15). *Let $X \to T$ be a $\mathbb{P}^1$-bundle over a smooth projective curve T of genus ≥ 2. Then, the following three conditions are mutually equivalent*:

(1) *X admits a non-isomorphic surjective endomorphism.*

(2) *$-K_{X/T}$ is semi-ample.*

(3) *$X \times_T T' \simeq \mathbb{P}^1 \times T'$ for a finite étale covering $T' \to T$.*

Here, for the proof of (3) $\Rightarrow$ (1), we apply the following:

Lemma 4.8 ([46], Lemma 6). *For a finite subgroup $G \subset \mathrm{Aut}(\mathbb{P}^1)$, there exists a G-equivariant non-isomorphic surjective endomorphism $f\colon \mathbb{P}^1 \to \mathbb{P}^1$, i.e.,*

$$f(\sigma \cdot x) = \sigma \cdot f(x)$$

for $\sigma \in \mathrm{Aut}(\mathbb{P}^1)$ and $x \in \mathbb{P}^1$.

Remark 4.9. A generalization of Theorem 4.7 to the case of $\mathbb{P}^n$-bundles over a higher dimensional base is obtained by Amerik in [1], Theorem 1. In the proof, Lemma 4.8 is also generalized to the case of $\mathbb{P}^n$, and the geometric invariant theory is used instead of properties of semistable vector bundles.

Remark 4.10. A smooth projective surface X admits a non-isomorphic étale endomorphism if and only if X is the $\mathbb{P}^1$-bundle over an elliptic curve associated with a semi-stable vector bundle of rank two. In fact, the "if" part is shown in [46], Propositions 5, and the "only if" part is derived essentially from the absence of negative curves, i.e., $N_X = 0$ (cf. Section 4.1).

These results complete the classification of smooth projective surfaces admitting non-isomorphic surjective endomorphisms.

§ 5. Projective threefolds of non-negative Kodaira dimension

In this section, we explain the classification in [19], [21] of smooth projective 3-folds admitting non-isomorphic surjective endomorphisms. Applying the theory of extremal rays by Mori [39], we can reduce the classification problem to the case of smooth minimal models. The theory of elliptic fibrations and abelian fibrations enable us to describe the structure of the minimal models in detail.

§ 5.1. Extremal contraction

We study the set of extremal rays and the associated contraction morphisms for smooth projective 3-folds admitting non-isomorphic surjective endomorphisms.

First, we recall some basics on the theory of extremal rays by Mori [39]. For a smooth projective n-fold X, the Picard number $\rho(X)$ is defined as the rank of the Néron-Severi group $\mathrm{NS}(X)$. We set

$$\mathrm{N}^1(X) := \mathrm{NS}(X) \otimes \mathbb{R} \quad \text{and} \quad \mathrm{N}_1(X) := \mathrm{Hom}(\mathrm{NS}(X), \mathbb{R}).$$

A divisor D is numerically trivial if the class $\mathrm{cl}(D) \in \mathrm{N}^1(X)$ is zero. We can regard $\mathrm{N}^1(X)$ as a vector subspace of $\mathrm{H}^2(X, \mathbb{R})$. There is a natural perfect pairing $\mathrm{N}^1(X) \times \mathrm{N}_1(X) \to \mathbb{R}$ induced from the intersection theory. Hence, for an algebraic 1-cycle $Z = \sum n_i Z_i$, the numerical equivalence class $\mathrm{cl}(Z) \in \mathrm{N}_1(X)$ corresponds to the map $D \mapsto DZ = \sum n_i DZ_i$ for divisors D.

Let $\mathrm{NE}(X) \subset \mathrm{N}_1(X)$ be the cone generated by $\mathrm{cl}(Z)$ for all the irreducible curves Z, and let $\overline{\mathrm{NE}}(X)$ be the closure of $\mathrm{NE}(X)$ in $\mathrm{N}_1(X)$ with respect to the metric topology. The Kleiman criterion of ampleness is that a divisor D is ample if and only if the functional D on $\mathrm{N}_1(X)$ is positive on $\overline{\mathrm{NE}}(X) \setminus \{0\}$. A divisor D of X is *nef* if and only if D is non-negative on $\overline{\mathrm{NE}}(X)$. The $\overline{\mathrm{NE}}(X)$ is called the Kleiman–Mori cone. An *extremal ray* (more precisely, a K_X-negative extremal ray) is a 1-dimensional face R of $\overline{\mathrm{NE}}(X)$ with $K_X R < 0$. An extremal ray R defines a proper surjective morphism $\mathrm{Cont}_R \colon X \to Y$ onto a normal projective variety Y with connected fibers such that for an irreducible curve $C \subset X$, $\mathrm{Cont}_R(C)$ is a point if and only if $\mathrm{cl}(C) \in R$. In higher dimensional case, this is a consequence of the cone theorem and the base point free theorem (cf. [31], [33], etc.). The morphism Cont_R is characterized by the property and is called the *contraction morphism* of R. We denote by $\mathrm{ER}(X)$ the set of extremal rays of X.

Second, we study $\overline{\mathrm{NE}}(X)$ and $\mathrm{ER}(X)$ for X admitting non-isomorphic surjective endomorphisms. The following result is proved in [19], Propositions 4.2 and 4.12.

Lemma 5.1. *Let $f\colon Y \to X$ be a finite surjective morphism between smooth projective n-folds with $\rho(X) = \rho(Y)$.*

(1) *The push-forward map $f_*\colon \mathrm{N}_1(Y) \to \mathrm{N}_1(X)$ is an isomorphism and $f_* \overline{\mathrm{NE}}(Y) = \overline{\mathrm{NE}}(X)$.*

(2) *Let $f_*\colon \mathrm{N}^1(Y) \to \mathrm{N}^1(X)$ be the map induced from the push-forward map $D \mapsto f_*D$ of divisors D. Then the dual $f^*\colon \mathrm{N}_1(X) \to \mathrm{N}_1(Y)$ is an isomorphism and $f^* \overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(Y)$.*

(3) *If f is étale and the canonical divisor K_X is not nef, then f^* and f_* above give a one-to-one correspondence between* $\mathrm{ER}(X)$ *and* $\mathrm{ER}(Y)$.

(4) *Under the same assumption as in* (3)*, for an extremal ray $R \in \mathrm{ER}(X)$ and for the contraction morphisms $\mathrm{Cont}_R\colon X \to X'$, $\mathrm{Cont}_{f^*R}\colon Y \to Y'$, there exists a finite surjective morphism $f'\colon Y' \to X'$ such that $\mathrm{Cont}_R \circ f = f' \circ \mathrm{Cont}_{f^*R}$.*

Lemma 5.1 is applied to the following fundamental result on $\mathrm{ER}(X)$ for smooth projective 3-folds X, which is proved in [19], Proposition 4.6 and Theorem 4.8.

Theorem 5.2. *Let $f\colon X \to X$ be a non-isomorphic surjective endomorphism of a smooth projective* 3*-fold X with $\kappa(X) \geq 0$. If K_X is not nef, then the following assertions hold*:

(1) $\mathrm{ER}(X)$ *is a finite set and f_* induces a permutation of* $\mathrm{ER}(X)$. *In particular, there is a positive integer k such that the power $f^k = f \circ \cdots \circ f$ satisfies $f^k_* R = R$ for any $R \in \mathrm{ER}(X)$.*

(2) *The contraction morphism $\mathrm{Cont}_R\colon X \to X'$ associated to any extremal ray $R \in \mathrm{ER}(X)$ is a divisorial contraction, and is* (the inverse of) *the blowing up of X' along an* elliptic *curve C.*

(3) *In the situation of* (2) *above, assume that $f_*R = R$. Let $f'\colon X' \to X'$ be the endomorphism induced from f such that $\mathrm{Cont}_R \circ f = f' \circ \mathrm{Cont}_R$ as in* Lemma 5.1, (4)*. Then $f'^{-1}(C) = C$.*

Proof. (1): The contraction morphism Cont_R for $R \in \mathrm{ER}(X)$ is birational since $\kappa(X) \geq 0$. Furthermore, Cont_R contracts a prime divisor E to a point or a curve, by [39]. Hence, E is contained in the fixed part of any non-empty linear system $|mK_X|$. Therefore, $\mathrm{ER}(X)$ is finite. The f_* gives a permutation by Lemma 5.1.

(2) and (3): Replacing f with a power f^k, by (1), we may assume that $f_*R = R$. Let $\varphi := \mathrm{Cont}_R\colon X \to X'$ be the contraction morphism. Then the exceptional divisor E satisfies $f^{-1}(E) = E$ set-theoretically, by Lemma 5.1.

Assume that $\varphi(E)$ is a point. Then, E is isomorphic to $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or a singular quadric surface, by [39]. In particular, E is simply connected. Since $f^{-1}(E) = E$ and f is étale, we have $\deg(f|_E) = \deg(f) \geq 2$. Hence $E = f^{-1}(E)$ is not connected; this is a contradiction.

Therefore, $C = \varphi(E)$ is a curve. Furthermore, by [39], X' and C are smooth, and φ is (the inverse of) the blowing up of X' along C. For the endomorphism $f' \colon X' \to X'$, the scheme-theoretic inverse image $f'^{-1}(C)$ is just C, since f' is étale. In particular, $f'|_C \colon C \to C$ is étale and non-isomorphic. Thus, C is an elliptic curve. □

§ 5.2. Minimal reduction of an endomorphism

Theorem 5.2 enables us to apply the minimal model program to a smooth projective 3-fold X with $\kappa(X) \geq 0$ and a non-isomorphic surjective endomorphism f of X. Thus, we can reduce the study of $f \colon X \to X$ to an endomorphism $f_{\min} \colon X_{\min} \to X_{\min}$ of a minimal model $X_{\min}$ of X. We shall explain the reduction, which is called the minimal reduction.

Recall that a normal projective variety Y is called a minimal model if Y has only terminal singularities and the canonical divisor K_Y is nef. If Z is a projective variety birational to the minimal model Y, then Y is called a minimal model of Z; however Y is not necessarily uniquely determined up to isomorphism in case $\dim Z \geq 3$.

Let $f \colon X \to X$ be a non-isomorphic surjective endomorphism of a smooth projective 3-fold X with $\kappa(X) \geq 0$. If K_X is nef, then X is minimal, so we do not need to consider the reduction. Assume that K_X is not nef. Then $\mathrm{ER}(X) \neq \emptyset$. We may assume that $f_* R = R$ for any extremal ray $R \in \mathrm{ER}(X)$ as before. Let us choose an extremal ray $R \in \mathrm{ER}(X)$ and consider the contraction morphism $\mu_0 := \mathrm{Cont}_R \colon X_0 = X \to X_1$. Then, by Theorem 5.2, μ_0 is the blowing up of a smooth projective 3-fold X_1 along an elliptic curve $C_1 \subset X_1$. Moreover, an étale endomorphism $f_1 \colon X_1 \to X_1$ is induced which satisfies $\mu_0 \circ f = f_1 \circ \mu_0$ and $f_1^{-1}(C_1) = C_1$.

If K_{X_1} is nef, then we stop here. Otherwise, we consider the same thing to (X_1, f_1) as (X, f). Namely, we first replace f_1 with a power f_1^k so that f_{1*} acts trivially on $\mathrm{ER}(X_1)$, second, choose an extremal ray $R_1 \in \mathrm{ER}(X_1)$, and consider the contraction morphism $\mu_1 := \mathrm{Cont}_{R_1} \colon X_1 \to X_2$.

In this way, we have successive contractions of extremal rays $X = X_0 \to X_1 \to X_2 \to \cdots$ with a strictly decreasing sequence $\rho(X) > \rho(X_1) > \cdots$ of Picard numbers. Note that no flipping contractions can occur in our minimal model program. Thus, X_k is a smooth minimal model for some k. Here, we have a non-isomorphic surjective endomorphism $f_k \colon X_k \to X_k$ which commutes with a power of f. To sum up, after replacing f by a suitable power f^l, we have a sequence of extremal contractions

$$X = X_0 \xrightarrow{\mu_0} X_1 \xrightarrow{\mu_1} \cdots \xrightarrow{\mu_{k-1}} X_k$$

and non-isomorphic surjective endomorphisms $f_i\colon X_i \to X_i$ for $0 \le i \le k$ such that

(1) $\mu_0 = \mu, f_0 = f, \mu_i \circ f_i = f_{i+1} \circ \mu_i$ for $0 \le i < k$,

(2) $\mu_{i-1}\colon X_{i-1} \to X_i$ is (the inverse of) the blowing up along an elliptic curve C_i on X_i with $f_i^{-1}(C_i) = C_i$ for all $1 \le i \le k$,

(3) X_k is a smooth minimal model of X.

We set $X_{\min} := X_k$ and $f_{\min} := f_k$. The endomorphism $f_{\min}\colon X_{\min} \to X_{\min}$ is called the minimal reduction of $f\colon X \to X$. We know that $K_{X_{\min}}$ is semi-ample, i.e., $\operatorname{Bs}|mK_{X_{\min}}| = \emptyset$ for some $m > 0$ by the abundance theorem for projective 3-folds (cf. [36], [37], [30]). In particular, the Iitaka fibration $\Phi_X\colon X \to W$ is holomorphic for the canonical model

$$W = \operatorname{Proj} \oplus_{m\ge 0} \mathrm{H}^0(X, \mathcal{O}_X(mK_X)),$$

where $\Phi_X = \Phi_{X_k} \circ \mu_{k-1} \circ \cdots \mu_0$ for the Iitaka fibration $\Phi_{X_k}\colon X_k \to W$. By Lemma 3.1 and Theorem 3.2, there is an automorphism h of W of finite order such that $\Phi_X \circ f = h \circ \Phi_X$. So, we may assume $\Phi_X \circ f = \Phi_X$ by replacing f with a suitable power f^l.

The smooth minimal model $X_{\min}$ is a unique minimal model of X and has a strong property as in Lemma 5.3 below, which was proved implicitly in papers [19], [21]; In fact, this is derived from the classification results of the minimal model X_k. In particular, the birational morphism $X \to X_k$ is unique up to isomorphism.

Lemma 5.3. *If X is a 3-dimensional smooth projective minimal model, then there is no non-isomorphic birational morphism $\psi\colon X \to V$ onto a normal projective variety V such that $K_X = \psi^*K_V$ and ψ is an isomorphism in codimension one. In particular, X is a unique minimal model in the birational equivalence class, and the birational automorphism group* $\operatorname{Bir}(X)$ *coincides with the automorphism group* $\operatorname{Aut}(X)$.

Proof. Let H be the pullback of an ample $\mathbb{Q}$-divisor of V by ψ. Then $H \sim_{\mathbb{Q}} f^*H_1$ for another semi-ample $\mathbb{Q}$-divisor H_1 since $f^*\colon \mathrm{N}^1(X) \to \mathrm{N}^1(X)$ is isomorphic. Then we have a birational morphism $\psi_1\colon X \to V_1$ and a finite morphism $g_0\colon V \to V_1$ such that $\psi_1 \circ f = g_0 \circ \psi$. In particular, ψ_1 is also an isomorphism in codimension one. Considering the same thing to ψ_1 and repeating, we have infinitely many birational morphisms $\psi_i\colon X \to V_i$ and finite morphisms $g_i\colon V_i \to V_{i+1}$ with $\psi_{i+1} \circ f = g_i \circ \psi_i$ for $i \ge 0$, where $V_0 = V$ and $\psi_0 = \psi$.

Let C be an irreducible curve contained in a fiber of ψ. Then $C \simeq \mathbb{P}^1$ and $K_XC = 0$. Furthermore, for any $i > 0$, $C_i := f^i(C)$ is also a smooth rational curve contained in a fiber of ψ_i with $K_XC_i = 0$. Here, $(f^i)^{-1}(C_i)$ is contracted to points by ψ and is a disjoint copies of C since f is étale. Hence, ψ contracts infinitely many $\mathbb{P}^1$ to points. This is a contradiction.

If $\nu\colon X \cdots\to X'$ is a non-isomorphic birational map to another minimal model X', then ν is isomorphic in codimension one, and there is a birational morphism $\psi\colon X \to V$ as above (cf. [29]). Thus, X is a unique minimal model and $\mathrm{Bir}(X) = \mathrm{Aut}(X)$. □

In Section 5.4, we explain the structure of the minimal model $X_{\min}$ in detail. From an non-isomorphic surjective endomorphism $f_{\min}\colon X_{\min} \to X_{\min}$, an endomorphism of original X is recovered by the following:

Lemma 5.4. *Let $\sigma\colon \widehat{X} \to X$ be the blowing up along an elliptic curve $C \subset X$ with $f^{-1}(C) = C$. Then there is an endomorphism $\hat{f}$ of $\widehat{X}$ with $\sigma \circ \hat{f} = f \circ \sigma$. Moreover, $K_X C = 0$ and C is contained in a fiber of the Iitaka fibration Φ_X.*

Proof. For the defining ideal $\mathcal{I}_C$, we have $f^*\mathcal{I}_C^j = \mathcal{I}_C^j$ for any j, since f is étale. Thus, there is an endomorphism $\hat{f}$ of $\widehat{X}$ compatible with f. We also have an isomorphism $f_C^*(N_{C/X}) \simeq N_{C/X}$ for the normal bundle $N_{C/X}$ and for the induced morphism $f_C = f|_C\colon C \to C$. Thus, $\deg N_{C/X} = -K_X C = 0$. If $\Phi_X(C)$ is not a point, then the finite morphism $\phi = \Phi|_C\colon C \to \Phi_X(C)$ satisfies $\phi \circ f_C = \phi$ contradicting $\deg(\phi \circ f_C) = (\deg \phi)(\deg f) > \deg(\phi)$. Thus, $\Phi_X(C)$ is a point. □

§ 5.3. Seifert abelian fibrations and simple abelian fibrations

In many cases, the minimal model $X_{\min}$ has a structure of an abelian fibration $X_{\min} \to T$ over a lower dimensional variety T. Here, we explain some basic facts on elliptic fibrations, abelian fibrations, especially Seifert abelian fibrations and simple abelian fibrations that are used in the sequel (cf. [21], Section 2.1).

A projective surjective morphism $\pi\colon V \to S$ of normal algebraic varieties is called an *abelian fibration* (or *abelian fiber space*) if a general fiber of π is an abelian variety. An abelian fibration of relative dimension one is called an *elliptic fibration* (or an *elliptic fiber space*). If π is smooth and has a structure of S-group scheme, then it is called an *abelian* scheme.

Definition 5.5 (cf. [21], Definition 2.3 and Lemma 2.4)**.** Let $V \to S$ be an abelian fibration from a smooth variety V to a normal variety S. It is called a *Seifert* abelian fiber space if there exist finite Galois surjective morphisms $W \to V$ and $T \to S$ satisfying the following conditions:

(1) W and T are smooth varieties.

(2) W is isomorphic to the normalization of $V \times_S T$ over T.

(3) $W \to V$ is étale.

(4) $W \to T$ is an abelian scheme.

If $V \to S$ is a Seifert abelian fiber space, then V is a unique relative minimal model over S, since K_V is relatively numerically trivial and there are no rational curves contained in fibers. If S is projective and $\dim(V) = \dim(S) + 1$, then we may replace the condition (4) with

(4′) $W \simeq E \times T$ over T for an elliptic curve E.

The following gives a sufficient condition for elliptic fibrations to be Seifert (cf. [41]; [42], Theorems 1.2 and 4.2):

Proposition 5.6. *Let $\pi\colon V \to S$ be an elliptic fibration from a smooth projective n-fold V onto a normal projective variety S. If the following conditions are satisfied, then π is a Seifert elliptic fibration*:

(1) *π is equi-dimensional.*

(2) *K_V is π-numerically trivial.*

(3) *For any prime divisor $\Gamma \subset S$, the support of a general fiber of $\pi^{-1}(\Gamma) \to \Gamma$ is an elliptic curve* (*In other words, the singular fiber type of π along Γ is of ${}_m\mathrm{I}_0$ for some $m \geq 1$*)*.*

The following gives a sufficient condition for abelian fibrations over curves to be Seifert, which is derived from arguments in [32], §6, and in [43], §7:

Proposition 5.7. *Let $\pi\colon X \to C$ be an abelian fiber space from a smooth projective variety X onto a smooth projective curve C such that K_X is π-nef, i.e., $K_X\gamma \geq 0$ for any curve $\gamma \subset X$ contracted to a point by π. If there is a point $t \in C$ such that, for the fiber $X_t = \pi^{-1}(t)$, the kernel of the natural homomorphism $\pi_1(X_t) \to \pi_1(X)$ of fundamental groups contains no nonzero proper Hodge substructure of $\mathrm{H}_1(X_t, \mathbb{Z}) \simeq \pi_1(X_t)$, then π is a Seifert abelian fibration.*

By some arguments of Ueno [53] on Hilbert schemes, we have the following characterization of abelian fibrations whose very general fiber is a simple abelian variety (cf. [21], §2.3).

Proposition 5.8. *Let $\varphi\colon M \to T$ be an abelian fibration between smooth quasi-projective varieties. Then the following conditions are equivalent to each other*:

(1) *One smooth fiber of φ is a simple abelian variety.*

(2) *A very general fiber of φ is a simple abelian variety.*

(3) *If* $A_t \subset M_t$ *is a positive-dimensional proper abelian subvariety of a smooth fiber* $M_t = \varphi^{-1}(t)$*, then an irreducible component* S *of* $\mathrm{Hilb}(M/T)$ *containing the point* $[A_t]$ *does not dominate* T*.*

Combining the argument on Hilbert schemes with a discussion on variation of Hodge structures, we have:

Theorem 5.9 ([21], Theorem 2.23)**.** *Let* $\varphi\colon M \to T$ *be a smooth abelian fibration over a quasi-projective variety* T *and* $f\colon M \to M$ *be a non-isomorphic surjective endomorphism over* T*. Suppose that there is a simple abelian subvariety* A *of codimension one in a fiber* $M_o = \varphi^{-1}(o)$ *satisfying* $f^{-1}(A) = A$*. Then* φ *has a factorization* $M \xrightarrow{\alpha} S \xrightarrow{\beta} T$ *such that*

(1) $\alpha\colon M \to S$ *is a smooth abelian fibration and* A *is a fiber of* α*,*

(2) $\beta\colon S \to T$ *is a smooth elliptic fibration,*

(3) $\alpha \circ f = v \circ \alpha$ *for an automorphism* v *of* S *over* T*.*

In particular, φ *is a non-simple abelian fibration.*

§ 5.4. Proof of Theorems 1.2 and 1.3

We shall explain an outline of the proof of Theorems 1.2 and 1.3.

The implication (B) $\Rightarrow$ (A) in Theorem 1.2 holds in any dimension by [21], Theorem 2.26, where a Galois cohomology group is considered. Hence, it is enough to show (A) $\Rightarrow$ (B); roughly speaking, for a given smooth projective 3-fold X with a non-isomorphic surjective endomorphism f, we will construct a finite étale Galois covering $\widetilde{X}$ over X which has a structure of an abelian scheme over a lower dimensional variety. The proof is divided into the following cases:

(i) $\kappa(X) = 0$.

(ii) $\kappa(X) = 1$ and a general fiber of the Iitaka fibration Φ_X is an abelian surface.

(iii) $\kappa(X) = 1$ and a general fiber of the Iitaka fibration Φ_X is not an abelian surface.

(iv) $\kappa(X) = 2$.

In papers [19], [21], the proof treats first the minimal model $X_{\min}$, and later the original X by applying results similar to Lemma 5.4. Here, an expected étale covering $\widetilde{X} \to X$ is obtained as the pullback of a similarly expected étale covering $\widetilde{X}_{\min} \to X_{\min}$ by the birational morphism $X \to X_{\min}$. We need to check $\widetilde{X}$ to have the same property as $\widetilde{X}_{\min}$. However, for the sake of simplicity, in this article, we explain only the case of minimal models. The readers interested in recovering from $X_{\min}$ to X refer to [19], Section 6, and [21], Sections 4 and 5.

Case (i): Then $K_X \sim_{\mathbb{Q}} 0$. Moreover, the fundamental group $\pi_1(X)$ is infinite by Lemma 3.4. By Bogomolov's decomposition theorem (cf. [7], [4]), there exists a finite étale Galois covering $\widetilde{X} \to X$ such that $\widetilde{X}$ is isomorphic to an abelian 3-fold or the direct product $E \times S$ of an elliptic curve E and a K3 surface S. Thus, we are done.

Case (iv): In this case, the Iitaka fibration $\Phi_X \colon X \to W$ is a minimal elliptic fibration over a normal surface W. Here, it is known that W has only quotient singularities. It suffices to show Proposition 5.10 below, which is originally proved in [19], Theorem 5.1; we shall give a simple proof using Lemma 5.3.

Proposition 5.10. *There exists a finite étale covering $\widetilde{X} \to X$ such that $\widetilde{X}$ is isomorphic to the direct product $E \times W'$ of an elliptic curve E and a surface W' of general type.*

Proof. Note that Φ_X is not necessarily equi-dimensional. By Lemma 5.3 and by applying results in [44], Appendix A, we infer that there is an equi-dimensional elliptic fibration $\pi \colon X \to T$ onto a normal projective surface T such that Φ_X is the composition of π and a birational morphism $T \to W$. Since T is uniquely determined, we have $\pi \circ f = \pi$. It is enough to prove that $\pi \colon X \to T$ is a Seifert elliptic fibration. Assume the contrary. Then, by Proposition 5.6, there exists a prime divisor $B \subset X$ such that any irreducible component of general fibers of $B \to \pi(B)$ is a rational curve. Since B is an irreducible component of $\pi^{-1}(\pi(B))$, $(f^k)^{-1}(B) = B$ for some $k > 0$. Thus, f^k induces a non-isomorphic étale endomorphism $B \to B$ over $\pi(B)$. Here, $f^{-1}(\gamma)$ of a rational curve γ in a fiber of $B \to \pi(B)$ is a union of rational curves, and the number of irreducible components of $f^{-1}(\gamma)$ is $\deg f > 1$. Hence, a general fiber of $B \to \pi(B)$ contains infinitely many rational curves; this is a contradiction. Thus, we are done. $\square$

Case (iii): A general fiber of the Iitaka fibration Φ_X is a hyperelliptic surface, since f induces a non-isomorphic surjective endomorphism of the fiber. For a hyperelliptic surface F, the quotient map $F \to F/\operatorname{Aut}^0(F) \simeq \mathbb{P}^1$ is an elliptic fibration with only multiple singular fibers. We can consider a relative version of the quotient map by the theory of "relative generic quotients" by Fujiki [17], [18]. Thus, Φ_X is the composition of a rational map $X \cdots\to S$ to a normal projective surface S and a fibration $S \to C$ such that a general fiber of S/C is $\mathbb{P}^1$ and a general fiber of X/S is an elliptic curve. By Lemma 5.3 and by applying results in [44], Appendix A, we may replace S to satisfy that $X \to S$ is a holomorphic equi-dimensional elliptic fibration (cf. Proof of [19], Theorem 5.10). By the existence of f, it is shown that the singular fiber type of X/S along the discriminant locus is ${}_m\mathrm{I}_0$ for some m. Thus, X/S is Seifert by Proposition 5.6, and we have an expected étale covering $\widetilde{X}$.

Case (ii): The remaining case is treated in [21]. We write the Iitaka fibration as $\Phi_X : X \to C$ instead of using W. Since $\deg(f) \geq 2$, we have:

Lemma 5.11 ([21], Lemma 3.8)**.** *The natural homomorphism $\pi_1(X_t) \to \pi_1(X)$ of fundamental groups has infinite image for a general fiber $X_t = \Phi_X^{-1}(t)$.*

Definition 5.12. Let $\varphi \colon M \to S$ be an abelian fiber space between smooth varieties. If $\pi(M_s) \to \pi_1(M)$ is injective for a general fiber $M_s = \varphi^{-1}(s)$, then φ is called a *primitive* abelian fiber space. If φ is not injective, then it is called *imprimitive.*

Suppose that Φ_X is a primitive abelian fibration. Then, Φ_X is a Seifert abelian fibration by Proposition 5.7. Thus, we have an expected finite étale Galois covering $\widetilde{X}$. If Φ_X is simple, then there is no elliptic curve $E \subset X$ with $f^{-1}(E) = E$ by Theorem 5.9; in particular, even if we consider X to be not necessarily minimal, X is shown to be minimal (cf. Theorem 5.2). If Φ_X is not simple and if X is not necessarily minimal, then we need some more arguments in showing $\widetilde{X}$ to have an expected property (cf. [21], Section 4.2).

Then, there remains the case where Φ_X is imprimitive. Let $C^\star \subset C$ be the maximum open subset over which Φ_X is smooth. Then, for any $t \in C^\star$ and for the fiber $X_t = \Phi_X^{-1}(t)$, the kernel of $\pi_1(X_t) \to \pi_1(X)$ contains a Hodge structure H_t of rank two by Lemma 5.11 and Proposition 5.7. By gathering $\{H_t\}$, we have a variation of Hodge substructure H of $\mathrm{R}_1\,\Phi_{X*}\mathbb{Z}_X|_{C^\star}$ of rank two (cf. [21], Corollary 3.9). Then we have a factorization $X \dashrightarrow Z \to C$ of Φ_X such that $q \colon Z \to C$ and $\pi \colon X \dashrightarrow Z$ are smooth elliptic fibrations over $C^\star$ and over $q^{-1}(C^\star)$, respectively, and that $\mathrm{H}_1(X_z, \mathbb{Z}) = H_{q(z)}$ for $z \in q^{-1}(C^\star)$. The factorization is called an H-factorization and its existence is proved in [21], Proposition 2.20, by an argument similar to the proof of Proposition 5.8 and Theorem 5.9.

By Lemma 5.3 and by applying results in [44], Appendix A, we may replace Z to be a normal projective surface satisfying the condition that $\pi \colon X \to Z$ is a holomorphic equi-dimensional elliptic fibration (cf. [21], Theorem 5.6). Here, note that π is not Seifert since Φ_X is imprimitive. So, we need more arguments than the previous cases. The following is proved in [21], Proposition 5.7:

Proposition 5.13. *There is a non-isomorphic surjective endomorphism β of Z with $\beta \circ \pi = \pi \circ f$. Moreover, there is a finite Galois covering $\widetilde{C} \to C$ such that:*

(1) *$\widetilde{Z} \simeq E \times \widetilde{C}$ over $\widetilde{C}$ for the normalization $\widetilde{Z}$ of $Z \times_C \widetilde{C}$ and for an elliptic curve E.*

(2) *β lifts to an endomorphism of $E \times \widetilde{C}$ of the form $\phi \times \mathrm{id}_{\widetilde{C}}$ for an endomorphism ϕ of E.*

(3) *The normalization $\widetilde{X}$ of $X \times_C \widetilde{C}$ is étale over X, and $\widetilde{X} \to \widetilde{Z}$ is a non-Seifert minimal elliptic fibration.*

Let us consider the composite $\widetilde{X} \to \widetilde{Z} \to E$. Then it is shown to be a holomorphic fiber bundle. Moreover, the following stronger result is obtained by applying the ∂-étale cohomological description of global structures of elliptic fibrations developed in [45]:

Proposition 5.14 ([21], Theorem 5.10). *There exist a non-Seifert minimal elliptic surface $S \to \widetilde{C}$ and a finite étale covering $E' \to E$ such that:*

(1) $\widetilde{X}' := E' \times_E \widetilde{X}$ *is isomorphic to* $E' \times S$ *over* E'.

(2) ϕ *lifts to an endomorphism* ϕ' *of* E'.

(3) $\widetilde{X}' \to \widetilde{X} \to X$ *is a finite étale Galois covering.*

(4) f *lifts to an endomorphism of* $\widetilde{X}' \simeq E' \times S$ *of the form* $\phi' \times v$ *for an automorphism* v *of* S.

(5) *The Galois group* $\mathrm{Gal}(\widetilde{X}'/X)$ *acts on* S *and the projection* $\widetilde{X'} \to S$ *is equivariant.*

Therefore, the étale covering $\widetilde{X}' \to X$ satisfies the required conditions. By additional arguments recovering from objects on $X_{\min}$ to that on X, we complete the proof of Theorems 1.2 and 1.3.

§ 6. Building blocks of étale endomorphisms

We shall overview the paper [47] on building blocks of étale endomorphisms. In the birational classification theory of projective varieties, we study a variety by analyzing the Iitaka fibration, the Albanese map, and the maximal rationally connected fibration. Then the classification is reduced in some sense to the following varieties assuming the good minimal model conjecture, etc.

(i) A rationally connected variety (cf. [10], [11], [34]).

(ii) An abelian variety.

(iii) A weak Calabi–Yau variety in the sense of [47], i.e., a minimal projective variety F with only terminal (or canonical) singularities, $K_F \sim_{\mathbb{Q}} 0$, and

$$q^{\max}(F) := \max\{q(F') \mid F' \to F \text{ is a finite étale covering}\} = 0.$$

(iv) A minimal variety of general type.

The reduction is given as follows:

Step 1. $(\kappa > 0) \Rightarrow (\kappa = 0) \cup \text{(iv)}$: For a variety X with $0 < \kappa(X) < \dim X$, we have the Iitaka fibration $\Phi_X : X \dashrightarrow W$. Then a very general fiber F has $\kappa(F) = 0$.

Step 2. $(\kappa = 0) \Rightarrow$ (ii), (iii): Assume that X has a good minimal model $X_{\min}$, i.e., $K_{X_{\min}} \sim_{\mathbb{Q}} 0$. Then, by [28], Corollary 8.4, there is a finite étale covering $F \times A \to X_{\min}$ for an abelian variety A and a weak Calabi–Yau variety F.

Step 3. $(\kappa = -\infty) \Rightarrow (\kappa \geq 0) \cup$ (i): For a variety X, a weak version of abundance conjecture says that $\kappa(X) = -\infty$ if and only if X is uniruled. If X is uniruled, then there exists uniquely up to birational equivalence a *maximal rationally connected fibration* $\pi\colon X \dashrightarrow S$, which satisfies the following conditions (cf. [10], [11], [34], [22]):

- $\pi^{-1}(U) \to U$ is holomorphic for an open dense subset $U \subset S$.
- A general fiber of π is rationally connected.
- S is not uniruled.

Varieties in the four classes (i)–(iv) are considered to be the building blocks of projective varieties.

The authors of [47] want to have the similar building blocks for varieties with surjective endomorphisms. But they restricted to the case of étale endomorphisms. One reason is that for an étale endomorphism $g\colon V \to V$ of a singular variety V, there is an equivariant resolution of singularities $\mu\colon X \to V$, i.e., $\mu^{-1} \circ g \circ \mu\colon X \to V \to V \dashrightarrow X$ is étale. Another reason is that a finite étale morphism corresponds to a finite index subgroup of the fundamental group of the base space. We explain the reduction in [47] along the same steps. Let $f\colon X \to X$ be an étale endomorphism of a smooth projective variety X.

Step 1: Assume that $\kappa(X) > 0$. By the equivariant resolution, we may assume that the Iitaka fibration $\Phi_X\colon X \to W$ is holomorphic. Then $\Phi_X \circ f^k = \Phi_X$ for a suitable power f^k by [47], Theorem A. Thus, f^k induces an étale endomorphism of a very general fiber F of Φ_X, where $\kappa(F) = 0$.

Step 2: Let X be a smooth projective variety of $\kappa(X) = 0$ with a good minimal model $X_{\min}$. Then, the étale endomorphism f descends to a nearly étale (cf. [47], §3) rational endomorphism $f_{\min}\colon X_{\min} \dashrightarrow X_{\min}$ of a minimal model $X_{\min}$. There is a finite étale covering $F \times A \to X_{\min}$ for a weak Calabi–Yau variety F and an abelian variety A as above. Then, for a suitable choice of A, $f_{\min}$ induces a rational endomorphism of $F \times A$ of the form $f_F \times f_A$, where $f_A\colon A \to A$ is étale and $f_F\colon F \dashrightarrow F$ is nearly étale (cf. [47], §4). Thus, we have the following commutative diagram of rational maps:

$$\begin{tikzcd} X \arrow[r, "\text{birational}"] \arrow[d, "f"'] & X_{\min} \arrow[d, "f_{\min}"'] & F \times A \arrow[l, "\text{étale}"'] \arrow[d, "f_F \times f_A"] \\ X \arrow[r] & X_{\min} & F \times A. \arrow[l] \end{tikzcd}$$

On the other hand, there is a conjecture that the fundamental group $\pi_1(F)$ is finite for a weak Calabi–Yau variety F. If the conjecture is true (which is confirmed if $\dim F \leq 3$ by [48]), then f_F is birational and hence, the study of f is reduced to that of f_A.

Step 3. For a uniruled X, the étale endomorphism f descends to an étale endomorphism h of the base space S of the maximal rationally connected fibration $\pi \colon X \cdots\to S$, for a suitable choice of S (cf. [47], §5), where the proof needs the existence of relative minimal models for resolutions of singularities proved in [6] in order to show the étaleness of h. The commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ S & \xrightarrow[h]{} & S \end{array}$$

of rational maps is birationally Cartesian, since a rationally connected manifold is simply connected.

Therefore, the conclusion of [47] is that if we admit many conjectures and if we consider modulo birational equivalence, then the building blocks of étale endomorphisms are the endomorphisms of abelian varieties (and the nearly étale rational endomorphisms of weak Calabi–Yau varieties).

Remark 6.1.

(1) The story of [47] gives only a perspective of classification of projective varieties admitting non-isomorphic surjective endomorphisms. For example, in *Step* 1, even if we know very well the structure of $f|_F \colon F \to F$, it is usually very difficult to recover the original $f \colon X \to X$ as we have seen in Section 5.

(2) If $\dim X = 3$, then the argument in *Step* 2 covers almost all the results in Section 5 concerning the case of $\kappa(X) = 0$. In *Step* 2, Kawamata's result in [28] is used instead of Bogomolov's decomposition theorem.

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