Quantization of contact manifolds

by

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Abstract

We show the existence of the stack of micro-differential modules on an arbitrary contact manifold, although we cannot expect the global existence of the ring of micro-differential operators.

§0. Introduction

In [SKK], we defined the sheaf of micro-differential operators on the cotangent bundle and we associated a quantized contact transformation with a given contact transformation.

More precisely, for a complex manifold X, let us denote by $\mathcal{E}_X$ the ring of micro-differential operators regarded as a sheaf of rings on the projective cotangent bundle P^*X. Let X and Y be two manifolds with the same dimension. Let U_X and U_Y be open subsets of P^*X and P^*Y respectively, and let $f : U_X \to U_Y$ be a holomorphic map preserving the canonical 1-form. Then for any point $p \in U_X$ there exists an open neighborhood U of p and a $\mathbf{C}$-ring isomorphism $\varphi : f^{-1}\mathcal{E}_Y|_U \to \mathcal{E}_X|_U$. Such a φ is not unique, although with other extra data we can reduce the uniqueness of φ up to the inner automorphism by micro-differential operators with 1 as its principal symbol.

Now let us consider a contact manifold Z with $(2n+1)$ dimension. This means that Z is endowed with an invertible $\mathcal{O}_Z$-module $\mathcal{O}_Z(1)$ and a 1-form $\omega \in \Gamma(Z, \Omega_Z^1 \otimes \mathcal{O}_Z(1))$ such that $\omega \wedge (d\omega)^n$ is a generator of $\Omega_Z^{2n+1} \otimes \mathcal{O}_Z(2n+1)$. Here $\mathcal{O}_Z(k) = \mathcal{O}_Z(1)^{\otimes k}$.

The purpose of this paper is to show that we can naturally define a stack (a sheaf of categories) on Z that is locally isomorphic to the stack of modules over the ring of micro-differential operators.

Let us take an open covering $Z = \bigcup_{i \in I} U_i$ and contact embeddings $f_i : U_i \hookrightarrow P^*X_i$. Set $\mathcal{A}_i = f_i^{-1}((\Omega_{X_i}^n)^{\otimes 1/2} \otimes \mathcal{E}_{X_i} \otimes (\Omega_{X_i}^n)^{\otimes -1/2})$. Then $\mathcal{A}_i$ is a sheaf of $\mathbf{C}$-rings on U_i endowed with an antiautomorphism $*$ such that $*^2 = 1$. The ring $\mathcal{A}_i$ has the order filtration $F(\mathcal{A}_i)$ such that $Gr_k^F \mathcal{A}_i = \mathcal{O}_Z(k)$.

Lemma 1. *Let $\mathcal{G}$ be the sheaf of automorphisms of $\mathcal{A}_i$ commuting with $*$. Then $\{P \in F_0(\mathcal{A}_i); P^*P = 1, \sigma_0(P) = 1\} \to \mathcal{G}$ given by $P \mapsto Ad(P)$ is bijective. Here σ_0 is the symbol map $F_0(\mathcal{A}_i) \to Gr_0^F \mathcal{A}_i = \mathcal{O}_Z$.*

Proof. For $\lambda \in \mathbf{C}$, let $\mathcal{E}(\lambda)$ be the sheaf of micro-differential operators of order $\lambda + \mathbf{Z}_{\leq 0}$. Then any automorphism φ of $\mathcal{E}_X$ is given by $Ad(P)$ for some λ and some invertible element P of $\mathcal{E}(\lambda)$. If φ commutes with $*$, then $Ad(P^*P) = \mathrm{id}$ and hence P^*P must be constant. Hence P is order 0 and we can normalize $P^*P = 1$ and $\sigma_0(P) = 1$ by dividing P by a suitable constant. Q.E.D.

Now, shrinking U_i if necessary, we may assume that there exists a $\mathbf{C}$-ring isomorphism $f_{ij} : \mathcal{A}_j|_{U_{ij}} \xrightarrow{\sim} \mathcal{A}_i|_{U_{ij}}$ which commutes with $*$. Here we employed the notation

$$U_{i_0 i_1 \cdots i_p} = U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_p}.$$

Then $f_{ij} \circ f_{jk} : \mathcal{A}_k|_{U_{ijk}} \to \mathcal{A}_i|_{U_{ijk}}$ is not equal to $f_{ik}|_{U_{ik}}$ in general. Hence we cannot patch $\mathcal{A}_i$ together to get a ring globally defined on Z.

By Lemma 1, there exists $P_{ijk} \in \Gamma(U_{ijk}; F_0(\mathcal{A}_i))$ such that

$$f_{ij} \circ f_{jk} = Ad(P_{ijk}) \circ f_{ik} \quad \text{and} \quad P_{ijk}^* P_{ijk} = 1,\ \sigma_0(P_{ijk}) = 1. \tag{0.1}$$

For $i, j, k, l \in I$, we have

$$(f_{ij} \circ f_{jk}) \circ f_{kl} = Ad(P_{ijk}) \circ f_{ik} \circ f_{kl} = Ad(P_{ijk} P_{ikl}) \circ f_{il}$$

and

$$f_{ij} \circ (f_{jk} \circ f_{kl}) = f_{ij} \circ Ad(P_{jkl}) \circ f_{jl} = Ad(f_{ij}(P_{jkl})) \circ f_{ij} \circ f_{jl} = Ad(f_{ij}(P_{jkl}) P_{ijl}) \circ f_{il}.$$

Hence by Lemma 1, we obtain

$$P_{ijk}P_{ikl} = f_{ij}(P_{jkl})P_{ijl}. \tag{0.2}$$

This cocycle relation permits us to patch the categories of $\mathcal{A}_i$-modules to get a stack globally defined over Z.

§1. Stack

Let us recall the definition of a stack on a topological space X. A prestack $\mathcal{C}$ on X consists of following data:

(1.1) a category $\mathcal{C}(U)$ for any open subset U of X,

(1.2) A functor $r_{VU} : \mathcal{C}(U) \to \mathcal{C}(V)$ for open subsets V and U with $V \subset U$.

(1.3) An isomorphism of functors $\theta_{WVU} : r_{WV} \circ r_{VU} \to r_{WU}$ for open subsets $W \subset V \subset U$.

They are assumed to satisfy the following axioms.

$(PS1)$ $r_{UU} = id.$

$(PS2)$ $\theta_{UUU} = id.$

$(PS3)$ For open subsets $U_1 \subset U_2 \subset U_3 \subset U_4$, the diagram

$$\begin{tikzcd}
r_{U_1U_2} \circ r_{U_2U_3} \circ r_{U_3U_4} \arrow[r, "\theta_{U_2U_3U_4}"] \arrow[d, "\theta_{U_1U_2U_3}"] & r_{U_1U_2} \circ r_{U_2U_4} \arrow[d, "\theta_{U_1U_2U_4}"] \\
r_{U_1U_3} \circ r_{U_3U_4} \arrow[r, "\theta_{U_1U_3U_4}"] & r_{U_1U_4}
\end{tikzcd}$$

commutes.

A prestack $\mathcal{C}$ is called a stack if it satisfies furthermore the following axioms.

$(S1)$ For any open subset U and $A, B \in Ob(\mathcal{C}(U))$, the presheaf on U

$\mathcal{H}om(A, B) : U \supset V \mapsto \mathrm{Hom}_{\mathcal{C}(V)}(r_{VU}(A), r_{VU}(B))$

is a sheaf.

(*S*2) Let $\{U_i\}$ be an open covering of an open set U, $A_i \in Ob(\mathcal{C}(U_i))$ and let $\varphi_{ij} : r_{U_{ij}U_j}(A_j) \to r_{U_{ij}U_i}(A_i)$ be an isomorphism. Assume the commutativity of the following diagram for any i, j, k:

$$\begin{tikzcd}
r_{U_{ijk}U_{jk}} r_{U_{jk}U_k} A_k \arrow[r, "\varphi_{jk}"] \arrow[d, "\theta_{U_{ijk}U_{jk}U_k}"] & r_{U_{ijk}U_{jk}} r_{U_{jk}U_j} A_j \arrow[r, "\theta_{U_{ijk}U_{jk}U_j}"] & r_{U_{ijk}U_j} A_j \\
r_{U_{ijk}U_k} A_k & & r_{U_{ijk}U_{ij}} r_{U_{ij}U_j} A_j \arrow[u, "\theta_{U_{ijk}U_{ij}U_j}"', "\wr"] \arrow[d, "\varphi_{ij}"] \\
r_{U_{ijk}U_{ik}} r_{U_{ik}U_k} A_k \arrow[u, "\theta_{U_{ijk}U_{ik}U_k}", "\wr"'] \arrow[d, "\varphi_{ik}"] & & r_{U_{ijk}U_{ij}} r_{U_{ij}U_i} A_i \arrow[d, "\theta_{U_{ijk}U_{ij}U_i}"] \\
r_{U_{ijk}U_{ik}} r_{U_{ik}U_i} A_i \arrow[rr, "\theta_{U_{ijk}U_{ik}U_i}"] & & r_{U_{ijk}U_i} A_i \,.
\end{tikzcd}$$

Then there exist an object A of $\mathcal{C}(U)$ and a family of isomorphisms $\psi_i : r_{U_iU}(A) \xrightarrow{\sim} A_i$ such that

$$\begin{tikzcd}
r_{U_{ij}U_j} r_{U_jU} A \arrow[r, "\theta_{U_{ij}U_jU}"] \arrow[d, "\psi_j"] & r_{U_{ij}U_j} A & r_{U_{ij}U_i} r_{U_iU} A \arrow[l, "\theta_{U_{ij}U_iU}"', "\sim"] \arrow[d, "\psi_i"] \\
r_{U_{ij}U_j} A_j \arrow[rr, "\varphi_{ij}"] & & r_{U_{ij}U_i} A_i
\end{tikzcd}$$

commutes.

For an open subset U of X, we can define the restriction $\mathcal{C}|_U$ to U, which is a stack on U.

For two stacks $\mathcal{C}_1, \mathcal{C}_2$ on X, we can define the notion of functors from $\mathcal{C}_1$ to $\mathcal{C}_2$ and for two functors f, g from $\mathcal{C}_1$ to $\mathcal{C}_2$, we can define the notion of morphisms from f to g. We call a functor $f : \mathcal{C}_1 \to \mathcal{C}_2$ an equivalence if there exists a functor $g : \mathcal{C}_2 \to \mathcal{C}_1$ such that $f \circ g$ and $g \circ f$ are isomorphic to the identity respectively.

§2. Patching of stacks

Let $\{U_i\}$ be an open covering of X and $\mathcal{C}_i$ a stack on U_i. Let $\varphi_{ij} : \mathcal{C}_j|_{U_{ij}} \to \mathcal{C}_i|_{U_{ij}}$ be an equivalence of stacks. Let $\psi_{ijk} : \varphi_{ij} \circ \varphi_{jk} \xrightarrow{\sim} \varphi_{ik}$ be an isomorphism of functors from $\mathcal{C}_k|_{U_{ijk}}$ to $\mathcal{C}_i|_{U_{ijk}}$. Assume that

(PC) For any i, j, k, l the diagram

$$\begin{array}{ccc} \varphi_{ij} \circ \varphi_{jk} \circ \varphi_{kl} & \xrightarrow{\psi_{jkl}} & \varphi_{ij} \circ \varphi_{jl} \\ \downarrow{\scriptstyle \psi_{ijk}} & & \downarrow{\scriptstyle \psi_{ijl}} \\ \varphi_{ik} \circ \varphi_{kl} & \xrightarrow{\psi_{ikl}} & \varphi_{il} \end{array}$$

commutes.

Then there exists a stack $\mathcal{C}$ and an equivalence $\mathcal{C}|_{U_i} \to \mathcal{C}_i$ satisfying the plausible compatibility conditions. Such a $\mathcal{C}$ is unique up to equivalence.

§3. Patching of stacks of modules

In this paper, a ring means a (not necessarily commutative) ring with 1. Let $\{U_i\}$ be an open covering of X and let $\mathcal{A}_i$ be a sheaf of rings on U_i. Assume that there is given a ring isomorphism $f_{ij} : \mathcal{A}_j|_{U_{ij}} \to \mathcal{A}_i|_{U_{ij}}$ and $a_{ijk} \in \Gamma(U_{ijk}; \mathcal{A}_i^\times)$ such that

$$(C1) \qquad f_{ij} \circ f_{jk} = Ad(a_{ijk}) f_{ik} \quad \text{in} \quad \mathrm{Hom}(\mathcal{A}_k|_{U_{ijk}}, \mathcal{A}_i|_{U_{ijk}})$$

and

$$(C2). \qquad a_{ijk} a_{ikl} = f_{ij}(a_{jkl}) a_{ijl} \quad \text{in} \quad \Gamma(U_{ijk}; \mathcal{A}_i^\times)$$

Here $\mathcal{A}_i^\times$ denotes the sheaf of invertible sections of $\mathcal{A}_i$.
Note that if the $\mathcal{A}_i$ are commutative, then $\{f_{ij}\}$ satisfies the chain conditions and hence we can define the globally defined ring $\mathcal{A}$ such that $\mathcal{A}|_{U_i} \simeq \mathcal{A}_i$. In a non-commutative case, we cannot construct such an $\mathcal{A}$ in general, but we can construct a stack locally isomorphic to the stack of $\mathcal{A}_i$-modules.

Let $\mathrm{Mod}(\mathcal{A}_i)$ be the stack of left $\mathcal{A}_i$-modules on U_i. In order to patch $\mathrm{Mod}(\mathcal{A}_i)$ together, let us apply the result in §2.

For $M \in \mathrm{Mod}(\mathcal{A}_j)$, let $\varphi_{ij}(M)$ be the $\mathcal{A}_i$-module with a sheaf isomorphism $\alpha_{ij} : M \to \varphi_{ij}(M)$ such that

$$a\alpha_{ij}(u) = \alpha_{ij}(f_{ji}(a)u) \quad \text{for} \quad a \in \mathcal{A}_i \quad \text{and} \quad u \in M.$$

This defines the functor $\varphi_{ij} : \mathrm{Mod}(\mathcal{A}_j)|_{U_{ij}} \to \mathrm{Mod}(\mathcal{A}_i)|_{U_{ij}}$.
Let us define an isomorphism of functors

$$\psi_{ijk} : \varphi_{ij} \circ \varphi_{jk} \to \varphi_{ik}$$

as follows. For $M \in \mathrm{Mod}(\mathcal{A}_k)|_{U_{ijk}}$, we define

$$\psi_{ijk}(M) : \varphi_{ij} \circ \varphi_{jk}(M) \to \varphi_{ik}(M)$$

by $\alpha_{ij}\alpha_{jk}(u) \mapsto \alpha_{ik}(a_{kji}^{-1}u)$ for $u \in M$. Let us check that $\psi_{ijk}(M)$ is $\mathcal{A}_i$- linear. For $a \in \mathcal{A}_i$ and $u \in M$, we have

$$a\alpha_{ij}\alpha_{jk}(u) = \alpha_{ij}(f_{ji}(a)\alpha_{jk}(u)) = \alpha_{ij}\alpha_{jk}(f_{kj}f_{ji}(a)u) = \alpha_{ij}\alpha_{jk}(a_{kji}f_{ki}(a)a_{kji}^{-1}u).$$

Hence we obtain

$$\psi_{ijk}(M)(a\alpha_{ij}\alpha_{jk}(u)) = \alpha_{ik}(f_{ki}(a)a_{kji}^{-1}u) = a\alpha_{ik}(a_{kji}^{-1}u) = a\psi_{ijk}(M)\big(\alpha_{ij}\alpha_{jk}(u)\big).$$

Thus, $\psi_{ijk}(M)$ is $\mathcal{A}_i$-linear and hence ψ_{ijk} is a well-defined morphism of functors.

Next, we shall check the chain condition (PC). The composition $\psi_{ikl} \circ \psi_{ijk}$ is calculated as follows:

$$\psi_{ikl}\psi_{ijk}(\alpha_{ij}\alpha_{jk}\alpha_{kl}(u)) = \psi_{ikl}\alpha_{ik}(a_{kji}^{-1}\alpha_{kl}(u)) = \psi_{ikl}\alpha_{ik}\alpha_{kl}(f_{lk}(a_{kji}^{-1})u) = \alpha_{il}(a_{lki}^{-1}f_{lk}(a_{kji}^{-1})u).$$

Similarly, we have

$$\psi_{ijl}\psi_{jkl}(\alpha_{ij}\alpha_{jk}\alpha_{kl}(u)) = \psi_{ijl}\alpha_{ij}\big(\psi_{jkl}(\alpha_{jk}\alpha_{kl}(u))\big) = \psi_{ijl}\alpha_{ij}\alpha_{jl}(a_{lkj}^{-1}u) = \alpha_{il}(a_{lji}^{-1}a_{lkj}^{-1}u).$$

Then $\psi_{ikl} \circ \psi_{ijk} = \psi_{ijl} \circ \psi_{jkl}$ follows from (C2).

By the arguments in §2, we can patch $\mathrm{Mod}(\mathcal{A}_i)$ together and we obtain a globally defined stack.

Now, we can apply this result to the situation in §1, and we obtain

Theorem 2. *For any contact manifold Z, we can define canonically a stack* $\mathrm{Mod}(Z)$ *on Z, which is locally equivalent to the stack of $\mathcal{E}_X$-modules.*

We call an object L of $\mathrm{Mod}(Z)$ invertible if it is locally isomorphic to $\mathcal{A}_i$. If there is an invertible object L, then $\mathcal{A} = \mathrm{End}(L)$ is a sheaf of rings locally isomorphic to the sheaf of micro-differential operators and $\mathrm{Mod}(Z)$ is equivalent to $\mathrm{Mod}(\mathcal{A})$. Hence the existence of a globally defined ring of micro-differential operators is equivalent to the existence of an invertible object.

§4. Sheaf of Microfunctions

Let Z be a contact manifold and let $Z_{\mathbf{R}}$ be a real analytic submanifold such that Z is a complexification of $Z_{\mathbf{R}}$. Let $\bar{Z}$ be the complex conjugate of Z. By shrinking Z if necessary, we may assume that there is an isomorphism of complex manifolds $\bar{Z} \to Z$ that is set-theoretically the identity on $Z_{\mathbf{R}}$. Assume that and $\mathcal{O}_Z(1)$ has a complex conjugation and $\sqrt{-1}\omega$ is invariant by the complex conjugation. Let Λ_x be the set of oriented Lagrangian vector subspaces in $T_x(Z_{\mathbf{R}})$. Then $\Lambda = \cup \Lambda_x$ is a fiber bundle over $Z_{\mathbf{R}}$. Let $\pi : \Lambda \to Z_{\mathbf{R}}$ be the projection.

Since $\pi_1(\Lambda_x) \cong \mathbf{Z}$, there is a canonical double covering $p : \tilde{\Lambda} \to \Lambda \times_{Z_{\mathbf{R}}} \Lambda$ over $\Lambda \times_{Z_{\mathbf{R}}} \Lambda$ with a canonical map $i : \Lambda \to \tilde{\Lambda}$ such that $p \circ i$ is the diagonal embedding.

Let p_1 and p_2 be the first and the second projection from $\Lambda \times_{Z_{\mathbf{R}}} \Lambda$ onto Λ. Let σ be the covering automorphism of $p : \tilde{\Lambda} \to \Lambda \times_{Z_{\mathbf{R}}} \Lambda$ and let L be the subsheaf of $p_* \mathbf{C}_{\tilde{\Lambda}}$ consisting of sections s such that $\sigma^* s = -s$. Then L is locally isomorphic to $\mathbf{C}_{\Lambda \times_{Z_{\mathbf{R}}} \Lambda}$ and $i^{-1}L$ is canonically isomorphic to $\mathbf{C}_\Lambda$. Let $\mathcal{C}$ be the stack on $Z_{\mathbf{R}}$ defined by : for any open subset U of $Z_{\mathbf{R}}, \mathcal{C}(U) = \{(F, \varphi); F$ is a sheaf on $\pi^{-1}(U)$ and φ is an automorphism $p_2^{-1} F \otimes L \simeq p_1^{-1} F$ such that $i^{-1}\varphi : F \to F$ is equal to the identity $\}$.

Then $\mathcal{C}$ is a stack locally equivalent to the stack of sheaves on $Z_{\mathbf{R}}$.

We can define the stack $\mathcal{C} \otimes \mathrm{Mod}(Z)$ over $Z_{\mathbf{R}}$ in an obvious way. Then for $M \in \mathrm{Mod}(Z)$ and $F \in \mathcal{C} \otimes \mathrm{Mod}(Z)$, $\mathcal{H}om(M, F)$ belongs to $\mathcal{C}$.

Now, we have

Proposition 3. *We can define canonically an object* $\mathcal{C}_{Z_{\mathbf{R}}}$ *of* $\mathcal{C} \otimes \mathrm{Mod}(Z)$, *which is locally isomorphic to the sheaf of microfunctions.*

§5. Regular holonomic systems

Since the notion of regular holonomic $\mathcal{E}$-modules is invariant by the quantized contact transformations, we can define the notion of regular holonomic systems for objects in $\mathrm{Mod}(Z)$. The subcategory $\mathrm{Reg}(Z)$ of regular holonomic systems in $\mathrm{Mod}(Z)$ forms a full abelian subcategory of $\mathrm{Mod}(Z)$.

Let Λ be a Lagrangian submanifold of Z. Then $(\Omega_\Lambda^{\dim \Lambda})^{\otimes 1/2}$ defines the stack $\mathcal{C}_\Lambda$ of twisted sheaves (cf. e.g. [K1]). The stack $\mathcal{C}_\Lambda$ is locally isomorphic to the stack of sheaves on Λ and it contains $(\Omega_\Lambda^{\dim \Lambda})^{\otimes 1/2}$ as an object. Then we have the following proposition, which is a translation of Theorem (10.3) [K2]

Proposition 4. *The category of regular holonomic systems with support in* Λ *is equivalent to the category of locally constant objects in* $\mathcal{C}_\Lambda$.

Here a locally constant object L in $\mathcal{C}_\Lambda$ is an object in $\mathcal{C}_\Lambda$ locally isomorphic to a constant sheaf of finite rank.

§6. Discussion

We know by the Riemann-Hilbert correspondence, the category of perverse sheaves is equivalent to the category of regular holonomic D_X-modules. We can ask what is the stack of “perverse sheaves on Z”, which is equivalent to the stack $\mathrm{Reg}(Z)$ of regular holonomic systems on Z.

Another question is : we defined $\mathrm{Mod}(Z)$ for a contact manifold Z. Is there an analogue of $\mathrm{Mod}(Z)$ on any Poisson manifold Z ?

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