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B-Functions and Holonomic Systems

Rationality of Roots of *B*-Functions*

Masaki Kashiwara
284 Harvard St. 64, Cambridge, Mass. 02139, USA

A b-function of an analytic function $f(x)$ is, by definition, a generator of the ideal formed by the polynomials $b(s)$ satisfying

$$P(s, x, D_x) f(x)^{s+1} = b(s) f(x)^s$$

for some differential operator $P(s, x, D_x)$ which is a polynomial on s.

Professor M. Sato introduced the notions of "a-function", "b-function" and "c-function" for relative invariants on prehomogeneous vector spaces, when he studied the fourier transforms and ζ-functions associated with them (see [10, 12]). He defined, in the same time, b-functions for arbitrary holomorphic functions and conjectured their existence and the rationality of their roots.

Professor Bernstein introduced, independently of Prof. Sato, b-functions and proved any polynomial has a non zero b-function [1]. Professor Björk extended this result to an arbitrary analytic functions by the same method [3].

The rationality of roots of b-functions is closely related to the quasi-unipotency of local monodromy. In fact, Professor Malgrange showed that the eigenvalues of local monodromy are $\exp(2\pi\sqrt{-1}\,\alpha)$ for a root α of the b-function when f has an isolated singularity [9].

In this paper, the proof of the existence of b-functions and the rationality of their roots are given. The method employed here is to study the system of differential equations which satisfies $f(x)^s$. First, we will show that $\mathscr{D} f^s$ is a subholonomic system and prove the existence of b-functions as its immediate consequence. Next, we study the rationality of roots of b-functions by using the desingularization theorem due to Hironaka. So, the main result of this paper is the following two theorems.

Theorem. *The characteristic variety of $\mathscr{D} f^s$ is equal to W_f. W_f is, by definition, the closure of* $\{(x, \xi);\ \xi = s \operatorname{grad} \log f(x)$ for some $s \in \mathbb{C}\}$ *in the cotangent vector bundle.*

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Theorem. *Let X', X be two complex manifolds and $F: X' \to X$ be the blowing up with center contained in the zeros of the holomorphic function $f(x)$ on X. Set $f' = f \circ F$. Then $b_f(s)$ is a divisor of $b_{f'}(s) b_{f'}(s+1) \dots b_{f'}(s+N)$ for some integer N. Here, $b_f(s)$ and $b_{f'}(s)$ are the b-functions of $f(x)$ and $f'(x')$ respectively.*

I express my hearty thanks to Professors M. Sato and T. Kawai for many valuable discussions with them.

§ 1. A Statement of the Theorems

Let X be a complex manifold of dimension n and $\mathscr{D}_X$ be a sheaf of differential operators of finite order. We set $\mathscr{D}_X[s] = \mathscr{D}_X \underset{\mathbb{C}}{\otimes} \mathbb{C}[s]$. Here, s is an indeterminate, commuting with all differential operators. $\mathscr{D}_X[s]$ is, therefore, the sheaf of rings whose center is $\mathbb{C}[s]$.

Let $f(x)$ be a non zero holomorphic function on X. Denote by $\mathscr{J}_f$ the Ideal of $\mathscr{D}_X[s]$ consisting of all operators $P(s, x, D)$ in $\mathscr{D}_X[s]$ such that $P(s, x, D) f(x)^s = 0$ holds for a generic x and every s. We set $\mathscr{N}_f = \mathscr{D}_X[s] f^s$. $\mathscr{N}_f$ is therefore isomorphic to $\mathscr{D}_X[s]/\mathscr{J}_f$.

Let t: $\mathscr{N}_f \to \mathscr{N}_f$ be an endomorphism of $\mathscr{N}_f$ defined by

$$P(s) f^s \mapsto P(s+1) f^{s+1} = (P(s+1) f) f^s.$$

t is a $\mathscr{D}_X$-linear homomorphism but not $\mathscr{D}[s]$-linear. We have a commutation relation

$$[t, s] = t \qquad ([t, s] = ts - st).$$

We will denote by $\mathbb{C}[s, t]$ the ring generated by s and t with the fundamental relation $[t, s] = t$. Therefore, we have

(1.1) $\varphi(s) t = t \varphi(s-1)$ in $\mathbb{C}[s, t]$

for any polynomial $\varphi(s)$. Set $\mathscr{D}_X[s, t] = \mathscr{D}_X \underset{\mathbb{C}}{\otimes} \mathbb{C}[s, t]$. $\mathscr{N}_f$ has a structure of $\mathscr{D}_X[s, t]$-Module.

We set $\mathscr{M}_f = \mathscr{N}_f / t \mathscr{N}_f$.

(1.1) **Lemma.** *$\mathscr{M}_f$ is a $\mathscr{D}[s]$-Module.*

In fact, $st\mathscr{N}_f = t(s-1)\mathscr{N}_f \subset t\mathscr{N}_f$.

(1.2) *Definition.* The b-function of $f(x)$ is a generator of the ideal of the polynomials $b(s)$ such that

$$b(s) f^s \in \mathscr{D}[s] f^{s+1} = t \mathscr{N}_f \qquad \text{(or equivalently, } b(s) \mathscr{M}_f = 0).$$

We will denote it by $b_f(s)$. It is clear that $b(s) f^s \in \mathscr{D}[s] f^{s+1}$ implies $b_f(s) | b(s)$. The purpose of this paper is to prove that

(a) *$\mathscr{N}_f$ is a subholonomic system (i.e. a coherent $\mathscr{D}_X$-Module whose characteristic variety has codimension $\geqq n-1$).*

(b) *$b_f(s) \neq 0$ and the roots of it are negative and rational numbers.*

§ 2. Review on the Theory of Systems of Differential Equations

In this paper, the results and the terminologies in S-K-K [11] and [4] are frequently used. Since [4] is written in Japanese and is hardly available, we collect here some of the results in it.

The sheaf of differential operators *of finite order* on a complex manifold X X is denoted by $\mathscr{D}_X$. In this paper, we not use differential operators of infinite order. The sheaf of micro-differential operators of finite order is denoted by $\mathscr{E}_X$.[1] $\mathscr{E}_X$ is a coherent Ring on the cotangent projective bundle P^*X of X. $\mathscr{D}_X^{(m)}$ (resp. $\mathscr{E}_X^{(m)}$) signifies the sheaf of differential (resp. micro-differential) operators of order $\leqq m$. Let $\pi_X: P^*X \to X$ be the canonical projection. Then $\mathscr{E}_X$ contains $\pi_X^{-1}\mathscr{D}_X$ as a sub-Ring and flat over it. Here, π^{-1} means the inverse image in the sheaf theory.

A coherent $\mathscr{D}_X$-Module $\mathscr{M}$ (resp. a coherent $\mathscr{E}_X$-Module $\mathscr{M}$) is called a system of (linear) differential equations (resp. micro-differential equations). The *characteristic variety* of $\mathscr{M}$ is, by definition, the support of $\mathscr{E}_X \underset{\mathscr{D}_X}{\otimes} \mathscr{M}$ and denoted by $SS(\mathscr{M})$.

Let $\gamma = \gamma_X: T^*X - X \to P^*X$ be the canonical projection from the cotangent vector bundle T^*X onto P^*X, defined outside of the zero-section X of T^*X. $\gamma^{-1}(SS(\mathscr{M})) \cup \operatorname{Supp}(\mathscr{M})$ is denoted by $\check{S}S(\mathscr{M})$. Here, the support $\operatorname{Supp}(\mathscr{M})$ of $\mathscr{M}$ is identified with the closed set of the zero section of T^*X. $\check{S}S(\mathscr{M})$ is also called the characteristic variety of $\mathscr{M}$. $\check{S}S(\mathscr{M})$ is an involutory closed analytic set in T^*X, invariant under the action of the multiplicative group $\mathbb{C}^\times$ of non zero complex numbers. Recall that an analytic subset V of T^*X is called involutory if, for any two functions f, g vanishing on V, their poisson bracket $\{f, g\}$ vanishes on V. An involutory analytic subset has always codimension equal or less than $n = \dim X$. It implies, therefore, codim $\check{S}S(\mathscr{M}) \leqq n$.

$\check{S}S(\mathscr{M})$ can be reformulated as follows. Let $\check{\mathscr{E}}_X$ be the Ring of T^*X defined by

$$\check{\mathscr{E}}_X|_{T^*X-X} = \gamma^{-1}\mathscr{E}_X \quad \text{and} \quad \pi_{X*}\check{\mathscr{E}}_X = \mathscr{D}_X,$$

where π_X is the canonical projection from T^*X onto X. By choosing a local coordinate system, for any open set U in T^*X, we have

$$\check{\mathscr{E}}_X(U) = \{(p_j(x, \xi))_{j \in Z}; \quad p_j(x, \xi) \in \mathscr{O}_{T^*X}(U)$$

such that

i) $p_j(x, \xi)$ is homogeneous of degree j with respect to ξ.
ii) $\sup |p_j(x, \xi)| \leqq (-j)!\, R_K^{-j}$ for any $K \subset\subset U$ and $j < 0$.
iii) $p_j(x, \xi) = 0$ for $j \gg 0$.

$\check{\mathscr{E}}_X$ contains $\pi^{-1}\mathscr{D}_X$ and flat over it. $\check{\mathscr{E}}_X|_X$ is isomorphic to $\mathscr{D}_X$.

(2.1) **Lemma.** *$\check{\mathscr{E}}_X$ is a coherent sheaf.*

In fact, $\check{\mathscr{E}}_X$ is coherent on $T^*X - X$, because $\check{\mathscr{E}}_X|_{T^*X-X}$ is isomorphic to $\gamma^{-1}\mathscr{E}_X$ and $\mathscr{E}_X$ is coherent. Let $s_1, \ldots, s_N$ be sections of $\check{\mathscr{E}}_X$ defined in a neighborhood of $(x, \xi) = (x_0, 0)$. Let $\mathscr{N}$ be a kernel of $\check{\mathscr{E}}_X^N \to \check{\mathscr{E}}_X$ defined by s_j. s_j are necessarily dif-

[1] In S-K-K [11] "pseudo-differential" and $\mathscr{P}_X$ are used instead of "micro-differential" and $\mathscr{E}_X$

ferential operators. Let $\mathcal{N}$ be a kernel of $\mathcal{D}_X^N \to \mathcal{D}_X$ defined by s_j. Then, since $\check{\mathcal{E}}_X$ is flat over $\mathcal{D}_X$, $\check{\mathcal{E}}_X \underset{\mathcal{D}_X}{\otimes} \mathcal{N}$ is isomorphic to $\check{\mathcal{N}}$. Therefore, $\mathcal{N}$ is locally of finite type. Q.E.D.

(2.2) **Lemma.** *Let $\check{\mathcal{M}}$ be a coherent $\check{\mathcal{E}}$-Module defined all over T^*X. Set $\mathcal{M} = \check{\mathcal{M}}|_X$. Then, $\mathcal{M}$ is a coherent $\mathcal{D}_X$-Module and $\check{\mathcal{M}} = \check{\mathcal{E}} \underset{\mathcal{D}}{\otimes} \mathcal{M}$.*

Proof. The first statement is evident. Since $\check{\mathcal{E}}$ is a constant sheaf along the fiber of γ, $\check{\mathcal{M}}$ is also a locally constant sheaf along the fiber of γ. Therefore, $\mathcal{M} = \pi_* \check{\mathcal{M}}$. Thus, we have the canonical homomorphism

$$\check{\mathcal{E}} \underset{\mathcal{D}}{\otimes} \mathcal{M} \to \check{\mathcal{M}},$$

which is isomorphic in a neighborhood of the zero sections. Because they are locally constant along the fiber of γ, it is globally isomorphic. Q.E.D.

By the definition of $\check{\mathcal{E}}_X$, it is evident that, for any coherent $\mathcal{D}_X$-Module $\mathcal{M}$, $\check{SS}(\mathcal{M})$ coincides with the support of $\check{\mathcal{E}}_X \underset{\mathcal{D}_X}{\otimes} \mathcal{M}$.

If $a(x, \xi)$ be a homogeneous functions on T^*X which vanishes on the characteristic variety $\check{SS}(\mathcal{M})$ of $\mathcal{M}$ and u is a section of $\mathcal{M}$, then there exists locally a differential operator $P(x, D)$ such that $P(x, D)u = 0$ and the principal symbol of P is a power of $a(x, \xi)$.

A system is said to be *holonomic* (resp. *subholonomic*) if the codimension of its characteristic variety is $n = \dim X$ (resp. $\geqq n-1$). An n-codimensional involutory subvariety of T^*X (resp. P^*X) is said to be *holonomic*. An analytic subset V in T^*X (resp. P^*X) is called *isotropic*, if the restriction of the fundamental 1-form $\omega = \sum \xi_j dx_j$ onto V vanishes on a non singular locus of V. An isotropic variety has always codimension $\geqq n$. An analytic set is holonomic if and only if it is isotropic and purely of codimension n.

(2.3) **Theorem.** *Let $\mathcal{M}$ be a system of differential (resp. micro-differential) equation. Then, we have*

i) $\mathcal{E}xt^i(\mathcal{M}, \mathcal{D}_X) = 0$ *for* $i < \operatorname{codim} \check{SS}(\mathcal{M})$ (*resp.* $\mathcal{E}xt^i(\mathcal{M}, \mathcal{E}_X) = 0$ *for* $i < \operatorname{codim} \operatorname{Supp} \mathcal{M}$)

ii) $\operatorname{codim} \check{SS}(\mathcal{E}xt^i(\mathcal{M}, \mathcal{D}_X)) \geqq i$ (*resp.* $\operatorname{codim} \operatorname{Supp}(\mathcal{E}xt^i(\mathcal{M}, \mathcal{E}_X)) \geqq i$).

The proof of this theorem can be found in [4] in the case of $\mathcal{D}$-Module. Here, we review the proof in it with a slight modification.

First, we assume $\mathcal{M}$ is a coherent $\mathcal{E}$-Module. Let $\mathcal{M}_0$ be a coherent $\mathcal{E}^{(0)}$-submodule of $\mathcal{M}$ which generates $\mathcal{M}$. By §3, Chapter II if S-K-K [11], we have micro-locally the exact sequence of $\mathcal{E}^{(0)}$-Modules

$$0 \leftarrow \mathcal{M}_0 \leftarrow \mathcal{E}^{(0)r_0} \xleftarrow{P_0} \mathcal{E}^{(0)r_1} \xleftarrow{P_1} \cdots$$

whose symbol sequences

$$0 \leftarrow \overline{\mathcal{M}} = \mathcal{M}_0 / \mathcal{E}^{(-1)} \mathcal{M}_0 \leftarrow \mathcal{O}_{P^*X}^{r_0} \xleftarrow{\sigma(P_0)} \mathcal{O}_{P^*X}^{r_1} \xleftarrow{\sigma(P_1)} \cdots$$

is exact. Therefore we have the free resolution

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{E}^{r_0} \xleftarrow{P_0} \mathcal{E}^{r_1} \xleftarrow{P_1} \cdots$$

of $\mathcal{M}$. $\mathcal{E}xt^i(\overline{\mathcal{M}}, \mathcal{O}_{P^*X})$ is the i-th cohomology of the complex

$$\mathcal{O}^{r_0}_{P^*X} \xrightarrow{\sigma(P_0)} \mathcal{O}^{r_1}_{P^*X} \to \cdots$$

and this is a symbol sequence of

$$\mathcal{E}^{r_0} \xrightarrow{P_0} \mathcal{E}^{r_1} \xrightarrow{P_1} \cdots$$

whose i-th cohomology is $\mathcal{E}xt^i(\mathcal{M}, \mathcal{E})$. Therefore, the vanishing of $\mathcal{E}xt^i(\overline{\mathcal{M}}, \mathcal{O}_{P^*X})$ implies that of $\mathcal{E}xt^i(\mathcal{M}, \mathcal{E})$ by § 3, Chapter II of S-K-K [11]. In other words, we have

$$\operatorname{Supp} \mathcal{E}xt^i(\mathcal{M}, \mathcal{E}) \subset \operatorname{Supp}(\mathcal{E}xt^i(\overline{\mathcal{M}}, \mathcal{O}_{P^*X})).$$

Since $\operatorname{Supp} \overline{\mathcal{M}} = \operatorname{Supp} \mathcal{M}$, a) and b) are the consequence of the well-known theorems of the commutative ring corresponding to them. Now let us prove the case of differential equations as a consequence of micro-differential equations case.

Set $\Omega = \{(t, x; \tau\, dt + \langle \xi, dx \rangle) \in P^*(\mathbb{C} \times X); \tau \neq 0\}$ and h be the projection from Ω onto T^*X defined by $(t, x; \tau\, dt + \langle \xi, dx \rangle) \mapsto (x, \tau + \langle \xi, dx \rangle)$.

Note that $\mathcal{E}_{\mathbb{C}\times X}$ contains $h^{-1}\check{\mathcal{E}}_X$ and faithfully flat over $h^{-1}\check{\mathcal{E}}_X$. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-Module. Set $\mathcal{M}' = \mathcal{E}_{\mathbb{C}\times X} \underset{\mathcal{D}_X}{\otimes} \mathcal{M}$. $\mathcal{M}'$ is a coherent $\mathcal{E}_{\mathbb{C}\times X}$ Module.

(2.4) **Lemma.** $\mathcal{E}xt^i(\mathcal{M}', \mathcal{E}_{\mathbb{C}\times X}) = \mathcal{E}xt^i(\mathcal{M}, \mathcal{D}_X) \underset{\mathcal{D}_X}{\otimes} \mathcal{E}_{\mathbb{C}\times X}$.

This lemma is easily deduced from the fact that $\mathcal{E}_{\mathbb{C}\times X}$ is flat over $\mathcal{D}_X$.

(2.5) **Lemma.** $\operatorname{Supp}(\mathcal{M}') = h^{-1}\check{S}S(\mathcal{M})$.

Because $\mathcal{M}' = \mathcal{E}_{\mathbb{C}\times X} \underset{\check{\mathcal{E}}_X}{\otimes} (\check{\mathcal{E}}_X \underset{\mathcal{D}_X}{\otimes} \mathcal{M})$ and $\mathcal{E}_{\mathbb{C}\times X}$ is faithfully flat over $\check{\mathcal{E}}_X$, this lemma is evident.

Now, Theorem (2.3) for $\mathcal{M}$ is a trivial consequence of that for $\mathcal{M}'$. In fact, we have

$$\operatorname{Supp}(\mathcal{E}xt^i(\mathcal{M}', \mathcal{E}_{\mathbb{C}\times X})) = h^{-1}\check{S}S(\mathcal{E}xt^i(\mathcal{M}, \mathcal{D}_X)).$$

This method is frequently available when we want to get a result on differential equations (resp. hyperfunctions) from the corresponding result on micro-differential equations (resp. microfunctions).

(2.6) **Theorem** [4]. *Let $\mathcal{M}$ be a coherent $\mathcal{D}$-Module (resp. $\mathcal{E}$-Module).*

$$\mathcal{M}_r = \{u \in \mathcal{M}; \operatorname{codim} \check{S}S(\mathcal{D}u) \geqq r\} \qquad (\text{resp. } \{u \in \mathcal{M}; \operatorname{codim} \operatorname{Supp}(\mathcal{E}u) \geqq r)\}$$

is a coherent $\mathcal{D}$-Module (resp. $\mathcal{E}$-Module).

In fact, we can construct $\mathcal{M}_r$ in a cohomological way. After Sato, we will introduce an associated cohomology (see [4]) and express $\mathcal{M}_r$ using this.

For a complex $A^{\bullet}=\{\cdots A^n \xrightarrow{d_A^n} A^{n+1} \to \cdots\}$, we will define $\sigma_{>r}(A^{\bullet})$ and $\sigma_{\leqq r}(A^{\bullet})$ as follows (see [4])

$$\sigma_{>r}(A^{\bullet})\colon \cdots \to 0 \to \operatorname{Im} d_A^r \to A^{r+1} \to A^{r+2} \to \cdots,$$
$$\sigma_{\leqq r}(A^{\bullet})\colon \cdots \to A^{n-2} \to A^{r-1} \to \operatorname{Ker} d_A^r \to 0 \to \cdots.$$

We have

$$(2.1)\qquad \begin{aligned} H^i(\sigma_{>r}(A^{\bullet})) &= \begin{cases} H^i(A^{\bullet}) & \text{for } i>r \\ 0 & \text{for } i\leqq r \end{cases} \\ H^i(\sigma_{\leqq r}(A^{\bullet})) &= \begin{cases} H^i(A^{\bullet}) & \text{for } i\leqq r \\ 0 & \text{for } i>r. \end{cases} \end{aligned}$$

Therefore, $A^{\bullet} \rightsquigarrow \sigma_{>r}(A^{\bullet})$, $\sigma_{\leqq r}(A^{\bullet})$ are functors well defined in a derived category. For a couple of integers (p, q) such that $p\leqq q$ we define

$$_{p<}\sigma_{\leqq q}=\sigma_{>p}\circ\sigma_{\leqq q}=\sigma_{\leqq q}\circ\sigma_{>p}.$$

For a triplet p, q, r such that $p\leqq q\leqq r$, we have a triangle

$$(2.2)\qquad \begin{array}{ccc} & {}_{p<}\sigma_{\leqq q}(A^{\bullet}) & \\ \swarrow & & \nwarrow^{+1} \\ {}_{p<}\sigma_{\leqq r}(A^{\bullet}) & \longrightarrow & {}_{q<}\sigma_{\leqq r}(A^{\bullet}). \end{array}$$

(2.7) *Definition.* For a coherent $\mathscr{D}_X$-Module $\mathscr{M}$, we define

(2.3) $T^i_{pq}(\mathscr{M})=\mathscr{E}xt^i({}_{p<}\sigma_{\leqq q}\,\mathbb{R}\,\mathscr{H}om(\mathscr{M},\mathscr{D}),\mathscr{D})$ for $p\leqq q$.

$T^i_{pq}(\mathscr{M})$ possesses the following properties

(2.8) **Proposition.** *$T^i_{pq}(\mathscr{M})$ is a covariant functor on $\mathscr{M}$ and*

(0) *$T^i_{pq}(\mathscr{M})$ is a coherent $\mathscr{D}$-Module.*

(i) *For a triplet $p\leqq q\leqq r$, we have the long exact sequence*

$$\cdots \to T^i_{qr}(\mathscr{M}) \to T^i_{pr}(\mathscr{M}) \to T^i_{pq}(\mathscr{M}) \to T^{i+1}_{qr}(\mathscr{M}) \to \cdots$$

(ii) *$T^i_{pq}(\mathscr{M})=0$ for $p=q$, $T^i_{pq}(\mathscr{M})=T^i_{-1q}(\mathscr{M})$ for $p<0$ and $T^i_{pq}(\mathscr{M})=0$ for $q<0$,*

(iii) *$T^i_{pq}(\mathscr{M})=0$ for $i+q<0$,*

(iv) *$T^i_{q-1,q}(\mathscr{M})=\mathscr{E}xt^{i+q}_{\mathscr{D}}(\mathscr{E}xt^q_{\mathscr{D}}(\mathscr{M},\mathscr{D}),\mathscr{D}))$,*

(v) *$T^i_{pq}(\mathscr{M})=0$ for $p\geqq n$, and $T^i_{pq}(\mathscr{M})=T^i_{pn}(\mathscr{M})$ for $q\geqq n$,*

(vi) *$T^i_{pq}(\mathscr{M})=0$ for $i+p\geqq n$,*

(vii) *$T^i_{pq}(\mathscr{M})=0$ for $i<0$,*

(viii) *$T^i_{pq}(\mathscr{M})=0$ for $i\neq 0$, $p<0$, $q\geqq n$,*

(ix) *$T^i_{pq}(\mathscr{M})=\mathscr{M}$ for $i=0$, $p<0$, $q\geqq n$,*

(x) *$T^n_{p0}(\mathscr{M})=0$ for $p<0$, $n\geqq 1$,*

(xi) *$T^{n-1}_{p0}(\mathscr{M})=0$ for $p<0$, $n\geqq 2$,*

(xii) *$\check{S}S(T^i_{pq}(\mathscr{M}))\subset \check{S}S(\mathscr{M})$.*

Proof. The property i) is evident by a triple (2.2). iv) is a consequence of

$${}_{q-1<}\sigma_{\leqq q}\,\mathbb{R}\,\mathscr{H}om(\mathscr{M},\mathscr{D})=\mathscr{E}xt^q(\mathscr{M},\mathscr{D})[-q].$$

By the induction on $q-p$, 0) and iii) can be reduced to the case in which $p=q-1$. In this case, they are trivial.

(v) is a trivial consequence of the fact

$$\mathscr{E}xt^i(\mathscr{M},\mathscr{D})=0 \quad \text{for } i>n.$$

Let us prove vi) and vii). By the induction on $q-p$, by using i), we may assume $p=q-1$. In this case

$$T^i_{pq}(\mathscr{M})=\mathscr{E}xt^{i+q}(\mathscr{E}xt^q(\mathscr{M},\mathscr{D}),\mathscr{D})=0 \quad \text{for } i+q=i+p+1>n.$$

Since codim $\check{S}S(\mathscr{E}xt^q(\mathscr{M},\mathscr{D}))\geqq q$, $T^i_{pq}(\mathscr{M})=0$ for $i<0$ and $p=q-1$. It implies (vii).

$$\mathbb{R}\,\mathscr{H}om(\mathbb{R}\,\mathscr{H}om(\mathscr{M},\mathscr{D}),\mathscr{D})=\mathscr{M} \quad \text{implies viii) and ix).}$$

For $p<0$, $T^i_{p0}(\mathscr{M})=\mathscr{E}xt^i_{\mathscr{D}}(\mathscr{H}om(\mathscr{M},\mathscr{D}),\mathscr{D})$. Let $0\leftarrow\mathscr{M}\leftarrow\mathscr{L}_0\leftarrow\mathscr{L}_1$ be a free resolution of $\mathscr{M}$. Then

$$0\to\mathscr{H}om(\mathscr{M},\mathscr{D})\to\mathscr{H}om(\mathscr{L}_0,\mathscr{D})\to\mathscr{H}om(\mathscr{L}_1,\mathscr{D})$$

is exact. Let $\mathscr{N}$ be a cokernel of $\mathscr{H}om(\mathscr{L}_0,\mathscr{D})\to\mathscr{H}om(\mathscr{L}_1,\mathscr{D})$. Since

$$\text{proj dim}\,\mathscr{H}om(\mathscr{M},\mathscr{D})\leqq\max(\text{proj dim}\,\mathscr{N}-2,0)=\max(n-2,0),$$

we have (x) and (xi). (xii) is a trivial consequence of

$$\check{\mathscr{E}}\underset{\mathscr{D}}{\otimes}T^i_{pq}(\mathscr{M})=\mathscr{E}xt^i_{\check{\mathscr{E}}}({}_{p<}\sigma_{\leqq q}\,\mathbb{R}\,\mathscr{H}om_{\check{\mathscr{E}}}(\check{\mathscr{E}}\underset{\mathscr{D}}{\otimes}\mathscr{M},\check{\mathscr{E}}),\check{\mathscr{E}}). \quad \text{Q.E.D.}$$

(2.9) **Proposition.** (a) *For a coherent $\mathscr{D}$-Module $\mathscr{M}$, we have*

$$\text{codim}\,\check{S}S(T^i_{pq}(\mathscr{M}))>i+p$$

for any i and p.

(b) *If* codim $\check{S}S(\mathscr{M})>q$, *then* $T^i_{pq}(\mathscr{M})=0$.

Proof. By the induction on $q-p$, we may suppose that $q=p+1$. In this case,

$$T^i_{pq}(\mathscr{M})=\mathscr{E}xt^{i+p+1}(\mathscr{E}xt^{p+1}(\mathscr{M},\mathscr{D}),\mathscr{D}).$$

Theorem (2.3) immediately implies the proposition.

(2.10) **Theorem.** *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-Module. Then*

$$0=T^0_{nn}(\mathscr{M})\subset T^0_{n-1,n}(\mathscr{M})\subset\cdots\subset T^0_{0n}(\mathscr{M})\subset T^0_{-1,n}(\mathscr{M})=\mathscr{M}$$

and

$$T^0_{q,n}(\mathscr{M})=\{s\in\mathscr{M};\ \text{codim}\,\check{S}S(\mathscr{D}s)>q\}.$$

Proof. $T^0_{-1,n}(\mathscr{M})=\mathscr{M}$ and $T_{nn}{}^0(\mathscr{M})=0$ are already shown. In the exact sequence

$$T^{-1}_{q-1,q}(\mathscr{M})\to T^0_{q,n}(\mathscr{M})\to T^0_{q-1,n}(\mathscr{M})$$

the first term $T^{-1}_{q-1,q}(\mathscr{M})$ vanishes by vii) in Proposition (2.8). Therefore $T^0_{q,n}(\mathscr{M})\to T^0_{q-1,n}(\mathscr{M})$ is injective. Thus we have the filtration

$$0=T^0_{n,n}(\mathscr{M})\subset T^0_{n-1,n}(\mathscr{M})\subset\cdots\subset T^0_{-1,n}(\mathscr{M})=\mathscr{M}.$$

By Proposition (2.9), we have

$$\operatorname{codim} \check{S}S(T^0_{q,n}(\mathscr{M})) > q.$$

Therefore, it suffices to show that any coherent sub $\mathscr{D}$-Module $\mathscr{M}'$ of $\mathscr{M}$ such that codim $\check{S}S(\mathscr{M}') > q$ is contained in $T^0_{q,n}(\mathscr{M})$. By the exact sequence

$$T^{-1}_{-1,q}(\mathscr{M}') \to T^0_{q,n}(\mathscr{M}') \to T^0_{-1,n}(\mathscr{M}') \to T^0_{-1,q}(\mathscr{M}').$$

By Proposition (2.8), $T^{-1}_{-1,q}(\mathscr{M}')=0$. Proposition (2.9) implies that $T^0_{-1,q}(\mathscr{M}')=0$. Thus we have $T^0_{q,n}(\mathscr{M}')=\mathscr{M}'$. Therefore, we have a diagram

$$\mathscr{M}' = T^0_{q,n}(\mathscr{M}') \to T^0_{q,n}(\mathscr{M}) \subset \mathscr{M}, \quad \text{and} \quad \mathscr{M}' \subset T^0_{q,n}(\mathscr{M}). \quad \text{Q.E.D.}$$

The above propositions and theorems are also valid for $\mathscr{E}$-Modules or $\check{\mathscr{E}}$-Modules by suitable modifications.

(2.11) **Proposition.** *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-Module, and V be an irreducible component of the characteristic variety of $\mathscr{M}$ of codimension r. Then, the characteristic variety of $\mathscr{E}xt^r(\mathscr{M},\mathscr{D})$ contains V.*

Proof. If $\mathscr{E}xt^r(\check{\mathscr{E}} \underset{\mathscr{D}}{\otimes} \mathscr{M}, \check{\mathscr{E}})=0$ at a generic point of V, then $\mathscr{E}xt^j(\mathscr{M},\check{\mathscr{E}})$ vanishes there for all j. It implies that $\mathscr{M} = \mathbb{R}\mathscr{H}om(\mathbb{R}\mathscr{H}om(\mathscr{M},\check{\mathscr{E}}),\check{\mathscr{E}})$ vanishes, which is a contradiction. Q.E.D.

(2.12) **Theorem.** *Let $\mathscr{M}$ be a coherent $\mathscr{D}$-Module, r be an integer. Suppose that*

$$\mathscr{E}xt^j(\mathscr{M},\mathscr{D})=0 \qquad \text{for } j \neq r.$$

Then, for any non zero sub-Module $\mathscr{M}'$ of $\mathscr{M}$, $\check{S}S(\mathscr{M}')$ is r codimensional at any point in it.

Proof. Since ${}_{r<}\sigma_{\leqq n}\mathbb{R}\mathscr{H}om(\mathscr{M},\mathscr{D})=0$ we have $T^0_{r,n}(\mathscr{M})=0$. This implies Theorem (2.12), together with Proposition (2.11). Q.E.D.

Let $\mathscr{M}$ be a coherent $\mathscr{E}$-Module whose support is contained in an analytic set V, and V_0 be an irreducible component of V. Let us choose a coherent $\mathscr{E}^{(0)}$ sub-Module $\mathscr{M}_0$ of $\mathscr{M}$ which generate $\mathscr{M}$ at $\mathscr{E}$-Module. The multiplicity of the coherent $\mathscr{O}_{P^*X}$-Module $\mathscr{M}_0/\mathscr{E}^{(-1)}\mathscr{M}_0$ at the generic point of V_0 is called *the multiplicity of $\mathscr{M}$ along V_0*.

(2.13) **Proposition.** *The definition of the multiplicity does not depend on the choice of $\mathscr{M}_0$.*

Proof. Take another $\mathscr{M}_0'$. Then, we may assume $\mathscr{M}_0 \supset \mathscr{M}_0'$ replacing $\mathscr{M}_0$ with $\mathscr{E}^{(m)}\mathscr{M}_0$ for $m \gg 0$. In the same way, we may assume

$$\mathscr{M}_0 \subset \mathscr{E}^{(m)}\mathscr{M}_0' \qquad \text{for } m \gg 0.$$

Now, we will prove by the induction on m. If $m=1$, we have the exact sequences

$$0 \to \mathscr{M}_0'/\mathscr{E}^{(-1)}\mathscr{M}_0 \to \mathscr{M}_0/\mathscr{E}^{(-1)}\mathscr{M}_0 \to \mathscr{M}_0/\mathscr{M}_0' \to 0$$

and

$$0 \to \mathscr{E}^{(-1)}\mathscr{M}_0/\mathscr{E}^{(-1)}\mathscr{M}_0' \to \mathscr{M}_0'/\mathscr{E}^{(-1)}\mathscr{M}_0' \to \mathscr{M}_0'/\mathscr{E}^{(-1)}\mathscr{M}_0 \to 0.$$

Thus, we have

$$\begin{aligned}
&\operatorname{mult}(\mathcal{M}_0/\mathcal{E}^{(-1)}\mathcal{M}_0)\\
&\quad=\operatorname{mult}(\mathcal{M}_0/\mathcal{M}_0')+\operatorname{mult}(\mathcal{M}_0'/\mathcal{E}^{(-1)}\mathcal{M}_0)\\
&\quad=\operatorname{mult}(\mathcal{E}^{(-1)}\mathcal{M}_0/\mathcal{E}^{(-1)}\mathcal{M}_0')+\operatorname{mult}(\mathcal{M}_0'/\mathcal{E}^{(-1)}\mathcal{M}_0)\\
&\quad=\operatorname{mult}(\mathcal{M}_0'/\mathcal{E}^{(-1)}\mathcal{M}_0'),
\end{aligned}$$

and the proposition is proved. Suppose $m>1$. Set $\mathcal{M}_0''=\mathcal{M}_0+\mathcal{E}^{(1)}\mathcal{M}_0'$. Since $\mathcal{E}^{(1)}\mathcal{M}_0'\subset\mathcal{M}_0''\subset\mathcal{E}^{(m-1)}(\mathcal{E}^{(1)}\mathcal{M}_0)$ and $\mathcal{M}_0\subset\mathcal{M}_0''\subset\mathcal{E}^{(1)}\mathcal{M}_0$, $\operatorname{mult}(\mathcal{M}_0/\mathcal{E}^{(-1)}\mathcal{M}_0)=\operatorname{mult}(\mathcal{M}_0''/\mathcal{E}^{(-1)}\mathcal{M}_0'')=\operatorname{mult}(\mathcal{E}^{(1)}\mathcal{M}_0/\mathcal{M}_0)=\operatorname{mult}(\mathcal{M}_0/\mathcal{E}^{(-1)}\mathcal{M}_0)$, by the hypothesis of the induction. Q.E.D.

The multiplicity is an additive quantity. That is, we have the following

(2.14) **Proposition.** *Let $0\to\mathcal{M}'\to\mathcal{M}\to\mathcal{M}''\to 0$ be an exact sequence. Then the multiplicity of $\mathcal{M}$ is a sum of those of $\mathcal{M}'$ and $\mathcal{M}''$.*

Proof. Let $\mathcal{M}_0$ be a coherent $\mathcal{E}^{(0)}$ sub-Module of $\mathcal{M}$ which generates $\mathcal{M}$. Let $\mathcal{M}_0'=\mathcal{M}_0\cap\mathcal{M}'$, $\mathcal{M}_0''$ be the image of $\mathcal{M}$ in $\mathcal{M}''$. Then, we have the exact sequence

$$0\to\mathcal{M}_0'/\mathcal{E}^{(-1)}\mathcal{M}_0'\to\mathcal{M}_0/\mathcal{E}^{(-1)})\mathcal{M}_0\to\mathcal{M}_0''/\mathcal{E}^{(-1)}\mathcal{M}_0''\to 0.$$

Therefore, the multiplicity of $\mathcal{M}_0/\mathcal{E}^{(-1)}\mathcal{M}_0$ is a sum of those of $\mathcal{M}_0'/\mathcal{E}^{(-1)})\mathcal{M}_0'$ and $\mathcal{M}_0''/\mathcal{E}^{(-1)}\mathcal{M}_0''$. It implies Proposition (2.14). Q.E.D.

§ 3. The Existence of *b*-Functions

In this section, we will show that there exists locally a non zero polynomial $b(s)$ such that

$$b(s)f^s\in\mathcal{D}[s]f^{s+1},$$

for any holomorphic function $f(x)$. The question being only in the neighborhood of zeros of f, we will assume

(3.1) $\{x;\partial f/\partial x_1=\cdots=\partial f/\partial x_n=0\}$ is contained in $f^{-1}(0)$.

Moreover, we assume, in the first step, that

(3.2) $f(x)$ is quasi-homogeneous, that is, there is a vector field X_0 such that $X_0 f=f$.

Under this condition, $\mathcal{N}_f$ is generated by f^s because $s^m f^s=X_0^m f^s$. Therefore, $\mathcal{N}_f$ is a coherent $\mathcal{D}_X$-Module. Later we will prove that $\mathcal{N}_f$ is always coherent $\mathcal{D}$-Module without the assumption (3.2).

(3.1) **Proposition.** *$\mathcal{N}_f$ is subholonomic (i.e. the codimension of the characteristic variety is $n-1$).*

Proof. Let $\mathcal{N}'=\{u\in\mathcal{N}_f;\ \operatorname{codim}\check{S}S(\mathcal{D}u)\geqq n-1\}$, $\mathcal{N}'$ is a coherent sub-Module of $\mathcal{N}_f$. If a derivative of $f(x)$ does not vanish, $\mathcal{N}_f$ is evidently subholonomic. Therefore, $\mathcal{N}_f$ and $\mathcal{N}'$ coincides outside of zero of f. Let u_0 be the section of $\mathcal{N}_f/\mathcal{N}'$ corresponding to $f^s \in \mathcal{N}_f$. Since the support of $\mathcal{O}_X u_0$ is contained in $f^{-1}(0)$, there is an integer m such that $f^m u_0 = 0$ by the Nullstellensatz of Hilbert. It means that $\mathcal{D} f^m \cdot f^s$ is a subholonomic system. Since $\mathcal{D} f^m \cdot f^s$ is isomorphic to $\mathcal{N}_f$ by the homomorphism t^m, $\mathcal{N}_f$ is a subholonomic system. Q.E.D.

(3.2) **Corollary.** *$\mathcal{M}_f$ is holonomic.*

Proof. We have the exact sequence

$$0 \to \mathcal{N}_f \xrightarrow{t} \mathcal{N}_f \to \mathcal{M}_f \to 0.$$

At each irreducible component of the characteristic variety of $\mathcal{N}_f$, the multiplicity of $\mathcal{M}_f$ is the difference of that of $\mathcal{N}_f$ and the same one, that is, zero. Therefore the characteristic variety of $\mathcal{M}_f$ does not contain any irreducible component of that of $\mathcal{N}_f$. Hence, the codimension of the characteristic variety is strictly greater than that of $\mathcal{N}_f$. Q.E.D.

(3.3) **Theorem.** *For any holomorphic function $f(x)$, there exists locally a non zero polynomial $b(s)$ such that*

$$b(s) f^s \in \mathcal{D}[s] f^{s+1}.$$

Proof. First, we will assume Condition (3.2). In this case, $\mathcal{M}_f$ is a holonomic system. For any point x, $\mathcal{H}om_{\mathcal{D}}(\mathcal{M}_f, \mathcal{M}_f)_x$ has finite dimension over $\mathbb{C}$ by [5]. Therefore, there is a non zero polynomial $b(s)$ which is zero in $\mathcal{E}nd_{\mathcal{D}}(\mathcal{M}_f)_x$. It is equivalent to say that $b(s) f^s$ belongs to $\mathcal{D}[s] f^{s+1}$.

Next, we will prove the general case. Let $f'(t, x)$ be a holomorphic function on $\mathbb{C} \times X$ defined by $tf(x)$. It is evidently quasi-homogeneous. By the preceding result, there is a non zero polynomial $b(s)$ and a differential operator $P(t, x, D_t, D_x)$ such that

$$(3.3) \quad P(t, x, D_t, D_x) f'(t, x)^{s+1} = b(s) f'(t, x)^s.$$

Let $Q(t, x, D_t, D_x)$ be the homogeneous part of degree (-1) with respect to t. Note that t is of degree 1 and D_t is of degree -1. Then, comparing the homogeneity of the both sides of (3.3), we have $Q(t, x, D_t, D_x)\, t^{s+1} f(x)^{s+1} = b(s)\, t^s f^s_{(x)}$. $Q(t, x, D_t, D_x)$ has the form

$$Q(t, x, D_t, D_x) = \sum Q_j(x, D_x)(tD_t)^j D_t.$$

Hence, we obtain

$$(s+1) \sum s^j Q_j(x, D_x) f(x)^{s+1} = b(s) f(x)^s. \quad \text{Q.E.D.}$$

§ 4. Integration of Systems of Differential Equations

In order to obtain more precise informations of $b(s)$ and $\mathcal{N}_f$, we employ Hironaka's desingularization theorem. Therefore, we must study the relationship between $\mathcal{N}_f$ and $\mathcal{N}_{f'}$ where $f' = f \circ F$ and $F\colon X' \to X$ is a monoidal transform.

Remember that Bernstein and Gelfand [2] considered the integral relation

$$(4.1)\quad f^s(x)=\int f'(x')^s\,\delta(x-F(x'))\frac{\partial x}{\partial x'}\,dx'$$

when they proved that $f(x)^s$ is a meromorphic function of s. Following their line, we understand $\mathcal{N}_f$ as "an integration" of $\mathcal{N}_{f'}$, along fibers of F. In this section, as its preparation, we will study the integration of systems of differential equations.

Now, let X and Y be two complex manifold and $F: X \to Y$ be a holomorphic map. $\mathcal{D}_{Y\leftarrow X}$ is, by definition,

$$F^{-1}(\mathcal{D}_Y \underset{\mathcal{O}_Y}{\otimes} (\Omega_Y)^{\otimes -1}) \underset{F^{-1}\mathcal{O}_Y}{\otimes} \Omega_X.$$

Here, Ω_X (resp. Ω_Y) is the sheaf of holomorphic forms of degree dim X (resp. dim Y) on X (resp. Y). $\mathcal{D}_{Y\leftarrow X}$ has a structure of $(F^{-1}\mathcal{D}_Y, \mathcal{D}_X)$-bi-Module. That is, $F^{-1}\mathcal{D}_Y$ operates $\mathcal{D}_{Y\leftarrow X}$ from the left and $\mathcal{D}_X$ from the right.

(4.1) *Definition.* For a system $\mathcal{M}$ on X, we define the integration $\int\mathcal{M}$ by

$$(4.2)\quad \int\mathcal{M}=R^0F_*\left(\mathcal{D}_{Y\leftarrow X}\overset{L}{\underset{\mathcal{D}_X}{\otimes}}\mathcal{M}\right).$$

and $\int^i\mathcal{M}$ by

$$(4.3)\quad \int^i\mathcal{M}=R^iF_*\left(\mathcal{D}_{Y\leftarrow X}\overset{L}{\underset{\mathcal{D}_X}{\otimes}}\mathcal{M}\right).$$

Here, $\overset{L}{\otimes}$ is a left derived functor in the derived category.

The purpose of this section is to prove the following finiteness theorem.

(4.2) **Theorem.** *Let $F: X \to Y$ be a holomorphic map and $\mathcal{M}$ be a coherent $\mathcal{D}_X$-Module satisfying the following conditions*

(i) *F is a projective morphism (i.e. F can be imbedded into $Y \times \mathbb{P}^N \to Y$).*

(ii) *There exists a coherent sub $\mathcal{O}_X$-Module $\mathcal{M}_0$ of $\mathcal{M}$ which generates $\mathcal{M}$ as a $\mathcal{D}_X$-Module.*

Then we have

(a) *$\int^i\mathcal{M}$ is a coherent $\mathcal{D}_Y$-Module for each i, and*

(b) $\check{S}S(\int^i\mathcal{M})\subset\tilde{\omega}\rho^{-1}\check{S}S(\mathcal{M})$

where $\tilde{\omega}$ and ρ are the canonical morphisms $X \underset{Y}{\times} T^*Y \to T^*Y$ and $X \underset{Y}{\times} T^*Y \to T^*X$, respectively.

This section is spent to the proof of this theorem. We will prove this theorem in two steps. First, we prove it in the case where $X = Y \times \mathbb{P}^N$ and nextly in the general case.

For a submanifold Z of Y, the definition of a coherent $\mathcal{E}_Y$-Module $\mathcal{C}_{Z|Y}$ is given in S-K-K [11] (there, it is denoted by $\mathcal{C}^f_{Z|Y}$). Choosing a local coordinate system $(y_1, \ldots, y_n)$ of Y such that $Z=\{y_1=\ldots=y_l=0\}$,

$$\mathcal{C}_{Z|Y}=\mathcal{E}_y/\mathcal{E}_Y y_1+\cdots+\mathcal{E}_Y y_l+\mathcal{E}_Y D_{y_{l+1}}+\cdots+\mathcal{E}_Y D_{y_n}.$$

We define $\check{\mathscr{C}}_{Z|Y}$ by

$$\check{\mathscr{C}}_{Z|Y}=\check{\mathscr{E}}_Y/\check{\mathscr{E}}_Y y_1+\cdots+\check{\mathscr{E}}_Y y_l+\check{\mathscr{E}}_Y D_{y_{l+1}}+\cdots+\check{\mathscr{E}}_Y D_{y_n}.$$

$\check{\mathscr{C}}_{Z|Y}$ is equal to $\check{\mathscr{E}}_Y \underset{\mathscr{D}_Y}{\otimes} \mathscr{B}_{Z|Y}$. $\mathscr{B}_{Z|Y}$ is a $\mathscr{D}_Y$-Module defined intrinsicly by

$$\varinjlim_{\nu} \mathscr{E}xt^l_{\mathscr{O}_Y}(\mathscr{O}_Z/\mathscr{J}^\nu, \mathscr{O}_Y),$$

where $\mathscr{J}$ is the defining ideal of Z. We define

$$\check{\mathscr{E}}_{Y\leftarrow X}=\check{\mathscr{C}}_{X|X\times Y}\underset{\mathscr{O}_X}{\otimes}\Omega_X.$$

Here, X is identified to the graph of F. By the canonical embedding

$$X\underset{Y}{\times} T^*Y \xrightarrow{\sim} T^*_X(X\times Y)\to T^*(X\times Y),$$

we regard $\check{\mathscr{E}}_{Y\leftarrow X}$ as the sheaf on $X\underset{Y}{\times} T^*Y$. Let S be a compact complex manifold.

(4.3) **Proposition.** *For any coherent $\mathscr{O}_S$-Module $\mathscr{F}$, we have*

$$R^i\tilde{\omega}_*(\check{\mathscr{E}}_{Y\leftarrow Y\times S}\underset{\mathscr{O}_S}{\otimes}\mathscr{F})=\check{\mathscr{E}}_Y\underset{\mathbb{C}}{\otimes}H^i(S;\,\Omega_S\otimes\mathscr{F}).$$

Here, $\tilde{\omega}$ is the canonical projection from $(Y\times S)\underset{Y}{\times} T^*Y\to T^*Y$.

Proof. We have

$$\begin{aligned}\check{\mathscr{E}}_{Y\leftarrow Y\times S}\underset{\mathscr{O}_S}{\otimes}\mathscr{F}&=\check{\mathscr{C}}_{Y\times S|Y\times Y\times S}\otimes\Omega_{Y\times S}\underset{\mathscr{O}_S}{\otimes}\mathscr{F}\\&=\check{\mathscr{C}}_{Y\times S|Y\times Y\times S}\otimes\Omega_Y\otimes(\Omega_S\underset{\mathscr{O}_S}{\otimes}\mathscr{F})\end{aligned}$$

and

$$\check{\mathscr{E}}_Y=\check{\mathscr{C}}_{Y|Y\times Y}\otimes\Omega_Y.$$

Hence, this proposition is equivalent to

$$R^i\tilde{\omega}_*(\check{\mathscr{C}}_{Y\times S|Y\times YS}\otimes(\Omega_S\underset{\mathscr{O}_S}{\otimes}\mathscr{F}))=\check{\mathscr{C}}_{Y|Y\times Y}\otimes H^i(S{:}\ \Omega_S\underset{\mathscr{O}_S}{\otimes}\mathscr{F}).$$

Thus, the proposition is the consequence of the following

(4.4) **Proposition.** *Let Z be a submanifold of Y, then*

$$R^i\tilde{\omega}_*(\check{\mathscr{C}}_{Z\times S|Y\times S}\underset{\mathscr{O}_S}{\otimes}\mathscr{F})=\check{\mathscr{C}}_{Z|Y}\underset{\mathbb{C}}{\otimes}H^i(S{:}\mathscr{F}).$$

Proof. We have the exact sequence

$$0\to\mathscr{C}_{\{0\}\times Z|\mathbb{C}\times Y}\xrightarrow{t}\mathscr{C}_{\{0\}\times Z|\mathbb{C}\times Y}\to\check{\mathscr{C}}_{Z|Y}\to 0$$

and

$$0\to\mathscr{C}_{\{0\}\times Z\times S|\mathbb{C}\times Y\times S}\xrightarrow{t}\mathscr{C}_{\{0\}\times Z\times S|\mathbb{C}\times Y\times S}\to\check{\mathscr{C}}_{Z\times S|Y\times S}\to 0.$$

Here, T^*Y is considered as a subset of $T^*(\mathbb{C}\times Y)$ and t is a coordinate function on $\mathbb{C}$, etc. Hence, this proposition follows from the following

(4.5) **Proposition.**

$$R^i\tilde{\omega}_*(\mathscr{C}_{Z\times S|Y\times S}\underset{\mathcal{O}_S}{\otimes}\mathscr{F})=\mathscr{C}_{Z|Y}\otimes H^i(S:\mathscr{F}).$$

Proof. By the quantized contact transformation of P^*Y, there is a hypersurface Z' of Y such that $\mathscr{C}_{Z|Y}$ is transformed to $\mathscr{C}_{Z'|Y}$, and $\mathscr{C}_{Z\times S|Y\times S}$ to $\mathscr{C}_{Z'\times S|Y\times S}$. Therefore, we may assume, from the first time, that Z is a hypersurface. $\mathscr{C}_{Z|Y}$ is a union of $\mathscr{C}^{(m)}_{Z|Y}$. Here, $\mathscr{C}^{(m)}_{Z|Y}=\mathscr{E}^{(m)}_Y\delta(\varphi)$. ($\delta(\varphi)$ is a delta function with support on Z). Thus it suffices to show that

$$(4.4)\quad R^i\tilde{\omega}_*(\mathscr{C}^{(m)}_{Z\times S|Y\times S}\otimes\mathscr{F})=\mathscr{C}^{(m)}_{Z|S}\otimes H^i(S:\mathscr{F}).$$

$\mathscr{C}^{(m)}_{Z|S}$ is isomorphic to $\mathscr{C}^{(-1)}_{Z|S}$ by the elliptic operator of order $m+1$. Therefore, it suffices to show (4.4) when $m=-1$. $\mathscr{C}^{(-1)}_{Z|Y}$ and $\mathscr{C}^{(-1)}_{Z\times S|Y\times S}$ is isomorphic to $\mathcal{O}_Y|_Z$ and $\mathcal{O}_{Z\times S}|_{Y\times S}$. Therefore, (4.4) is a consequence of

$$R^iF_*(\mathcal{O}_{Y\times S}\underset{\mathcal{O}_S}{\otimes}\mathscr{F})=\mathcal{O}_Y\underset{\mathbb{C}}{\otimes}H^i(S:\mathscr{F})$$

where F is the projection $Y\times S\to Y$. Q.E.D.

Now, let us prove the theorem when $X=Y\times\mathbb{P}^N$. Set $S=\mathbb{P}^N$. By Condition (ii), there is a resolution

$$0\leftarrow\mathscr{M}\leftarrow\mathscr{D}_X\underset{\mathcal{O}_S}{\otimes}\mathscr{F}_0\leftarrow\mathscr{D}_X\underset{\mathcal{O}_S}{\otimes}\mathscr{F}_1\leftarrow\cdots$$

of $\mathscr{M}$ for coherent $\mathcal{O}_S$-Modules $\mathscr{F}_j$, locally on Y. Because there is a coherent $\mathcal{O}_S$-Module $\mathscr{F}_0$ and a surjective homomorphism $\mathscr{M}_0\leftarrow\mathcal{O}_X\underset{\mathcal{O}_S}{\otimes}\mathscr{F}_0$. Therefore, we have $0\leftarrow\mathscr{M}\leftarrow\mathscr{D}_X\underset{\mathcal{O}_S}{\otimes}\mathscr{F}_0$. Let $\mathscr{L}$ be the kernel of this homomorphism, and $\mathscr{D}_k$ be the sheaf of differential operators of order $\leqq k$. Since a coherent $\mathcal{O}_X$-Module, $\mathscr{L}\cap\mathscr{D}_k\otimes\mathscr{F}_0$ generates $\mathscr{L}$ as $\mathscr{D}_X$-Module for $k\gg 0$, we have a surjective homomorphism $\mathscr{L}\leftarrow\mathscr{D}_X\underset{\mathcal{O}_S}{\otimes}\mathscr{F}_1$. Continuing this process, we get a resolution of $\mathscr{M}$. Thus $\check{\mathscr{E}}_{Y\leftarrow X}\overset{L}{\underset{\mathscr{D}X}{\otimes}}\mathscr{M}$ is quasi-isomorphic to

$$\check{\mathscr{E}}_{Y\leftarrow X}\underset{\mathcal{O}_S}{\otimes}\mathscr{F}_0\leftarrow\check{\mathscr{E}}_{Y\leftarrow X}\otimes\mathscr{F}_1\leftarrow\cdots.$$

(4.6) **Lemma.** $R^i\tilde{\omega}_*(\check{\mathscr{E}}_{Y\leftarrow X}\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathscr{M})$ *are coherent* $\check{\mathscr{E}}_Y$*-Modules.*

In fact, we have the spectral sequence

$$\mathscr{E}_2^{pq}=\mathscr{H}_{-p}(R^q\tilde{\omega}_*(\check{\mathscr{E}}_{Y\leftarrow X}\otimes\mathscr{F}))\underset{p}{\Rightarrow}R^{p+q}\tilde{\omega}_*\left(\check{\mathscr{E}}_{Y\leftarrow X}\overset{L}{\underset{\mathscr{D}X}{\otimes}}\mathscr{M}\right).$$

By Proposition (4.3), all $\mathscr{E}_2^{pq}$ are coherent $\check{\mathscr{E}}_Y$-Modules, which implies Lemma (4.6).

$R^i\tilde{\omega}_*\left(\check{\mathscr{E}}_{Y\leftarrow X}\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathscr{M}\right)_Y$ is evidently isomorphic to $R^iF_*\left(\mathscr{D}_{Y\leftarrow X}\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathscr{M}\right)$. Therefore, this is a coherent $\mathscr{D}_P$-Module. Moreover, we have

$$R^i\tilde{\omega}_*\left(\check{\mathscr{E}}_{Y\leftarrow X}\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathscr{M}\right)=\check{\mathscr{E}}_Y\underset{\mathscr{D}_Y}{\otimes}R^iF_*\left(\mathscr{D}_{Y\leftarrow X}\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathscr{M}\right)$$

by Lemma (2.2). Therefore,

(4.5) $\check{S}S(\int^{i}\mathcal{M})=\operatorname{Supp}(R^i\tilde{\omega}_*(\check{\mathscr{E}}_{Y\leftarrow X}\underset{\mathscr{D}_X}{\otimes}\mathcal{M}))$.

Since $R^i\tilde{\omega}_*$ is a functor with a local nature, the right hand side of (4.5) is contained in

$$\tilde{\omega}(\operatorname{Supp}(\check{\mathscr{E}}_{Y\leftarrow X}\underset{\mathscr{D}_X}{\otimes}\mathcal{M}))=\tilde{\omega}(\operatorname{Supp}(\check{\mathscr{E}}_{Y\leftarrow X}\underset{\mathscr{E}_X}{\otimes}(\check{\mathscr{E}}_X\underset{\mathscr{D}_X}{\otimes}\mathcal{M})))\subset\tilde{\omega}\rho^{-1}(\operatorname{Supp}(\check{\mathscr{E}}_X\underset{\mathscr{D}_X}{\otimes}\mathcal{M}))$$
$$=\tilde{\omega}\rho^{-1}\check{S}S(\mathcal{M}).$$

Thus, we proved Theorem (4.2) when $X=Y\times\mathbb{P}^N$.

Now, let us prove Theorem (4.2) in a general case.

By Condition (ii), F is decomposed into

$$X\hookrightarrow X'=Y\times\mathbb{P}^N\to Y.$$

(4.7) **Lemma.** *Let $X\to Y$ be an imbedding and $Y\to Z$ be a smooth morphism, then*

$$\mathscr{D}_{Z\leftarrow Y}\underset{\mathscr{D}_Y}{\overset{L}{\otimes}}\mathscr{D}_{Y\leftarrow X}=\mathscr{D}_{Z\leftarrow Y}.$$

Proof. Since

$$\mathscr{D}_{Z\leftarrow Y}\underset{\mathscr{D}_Y}{\overset{L}{\otimes}}\mathscr{D}_{Y\leftarrow X}=\mathscr{D}_{Z\leftarrow Y}\underset{\mathscr{D}_Y}{\overset{L}{\otimes}}\mathscr{B}_{X|Y\times X}\underset{\mathscr{O}_X}{\otimes}\Omega_X=\mathscr{D}_{Z\leftarrow Y}\underset{\mathscr{D}_Y}{\overset{L}{\otimes}}\mathscr{D}_{Y\times X}\underset{\mathscr{D}_{Y\times X}}{\overset{L}{\otimes}}\mathscr{B}_{X|Y\times X}\underset{\mathscr{O}_X}{\otimes}\Omega_X$$

and

$$\mathscr{D}_{Z\leftarrow Y}\underset{\mathscr{D}_Y}{\overset{L}{\otimes}}\mathscr{D}_{Y\times X}=\mathscr{D}_{Z\times X\leftarrow Y\times X},$$

it suffices to show that

$$\mathscr{D}_{Z\times X\leftarrow Y\times X}\underset{\mathscr{D}_{Y\times X}}{\overset{L}{\otimes}}\mathscr{B}_{X|Y\times X}=\mathscr{B}_{X|Z\times X}.$$

Thus, the lemma is a corollary of the following

(4.8) **Lemma.** *Let $X\to Y$ be a smooth morphism and Z be a submanifold of X such that $Z\to Y$ is an embedding. Then*

$$\mathscr{D}_{Y\leftarrow X}\overset{L}{\otimes}\mathscr{B}_{Z|X}=\mathscr{B}_{Z|Y}.$$

Proof. Locally, $Z\subset X\to Y$ has the form $X=Y\times S$, $Z\subset Y$ and $Z\times\{0\}\subset Y\times S$ for $0\in S$. Thus, it suffces to show that

$$\mathscr{D}_{Y\leftarrow Y\times S}\underset{\mathscr{D}_{Y\times S}}{\overset{L}{\otimes}}\mathscr{B}_{Z\times\{0\}|Y\times S}=\mathscr{B}_{Z|Y}.$$

Since we have

$$\mathscr{D}_{Y\leftarrow Y\times S}\underset{\mathscr{D}_{Y\times S}}{\overset{L}{\otimes}}\mathscr{B}_{Z\times\{0\}|Y\times S}=\mathscr{D}_{pt\leftarrow S}\underset{\mathscr{D}_S}{\overset{L}{\otimes}}\mathscr{B}_{Z\times\{0\}|Y\times S}=\Omega_S\underset{\mathscr{D}_S}{\overset{L}{\otimes}}\mathscr{B}_{Z\times\{0\}|Y\times S}$$

and since it is isomorphic to $\mathscr{B}_{Z|Y}$, we obtain the desired result.

Now, we can prove Theorem (3.2) in a general case.

Remember that F is decomposed into $X \hookrightarrow X' = Y \times \mathbb{P}^N \xrightarrow{F'} Y$. We have

$$\mathscr{D}_{Y\leftarrow X}\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathscr{M}=\mathscr{D}_{Y\leftarrow X'}\overset{L}{\underset{\mathscr{D}_{X'}}{\otimes}}\left(\mathscr{D}_{X'\leftarrow Y}\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathscr{M}\right).$$

Because $\mathscr{D}_{X'\leftarrow X}$ is flat over $\mathscr{D}_X$ and

$$\mathscr{D}_{X'\leftarrow X}=\mathscr{D}_{X'}\otimes(\Omega_X\otimes\Omega_{X'}^{-1}),$$

$\Omega_X\otimes\Omega_{X'}^{-1}\otimes\mathscr{M}_0$ generates $\mathscr{D}_{X'\leftarrow X}\underset{\mathscr{D}_X}{\otimes}\mathscr{M}$. Therefore $\mathscr{M}'=\mathscr{D}_{X'\leftarrow X}\underset{\mathscr{D}_X}{\otimes}\mathscr{M}$ satisfies Condition (ii). Since $R^iF_*\left(\mathscr{D}_{Y\leftarrow X}\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathscr{M}\right)=R^iF'_*\left(\mathscr{D}_{Y\leftarrow X'}\overset{L}{\underset{\mathscr{D}_{X'}}{\otimes}}\mathscr{M}'\right)$, they are coherent $\mathscr{D}_Y$-Modules. The characteristic variety of $\mathscr{M}'$ is equal to $\tilde{\omega}'\rho'^{-1}\check{S}S(\mathscr{M})$. Here ρ' and $\tilde{\omega}'$ are the canonical maps $X\underset{X'}{\times}T^*X'\to T^*X$ and $X\underset{X'}{\times}T^*X'\to T^*X'$. By the following diagram, Statement (b) on $\mathscr{M}$ is a consequence of that for $\mathscr{M}'$.

$$\begin{array}{ccccc}
X\underset{Y}{\times}T^*Y & \longrightarrow & X\underset{X'}{\times}T^*X' & \longrightarrow & T^*X\\
\downarrow & & \downarrow & & \\
X'\underset{Y}{\times}T^*Y & \longrightarrow & T^*X' & & \\
\downarrow & & & & \\
T^*Y. & & & &
\end{array}$$

At the end of this section, we will give the following proposition used later.

(4.9) **Proposition.** *Let $X\to Y$ be a proper morphism and V be an isotropic subvariety of T^*X, then so is $\tilde{\omega}\rho^{-1}V$.*

Proof. Let ω_X(resp. ω_Y) be the fundamental 1-form on T^*X (resp. T^*Y). An isotropic subvariety V is, by definition, a subvariety on which ω_X vanishes at its non singular locus.(For the brevity, in this case, we will say that the restriction of ω_X to V vanishes.) Since $\rho^*\omega_X=\tilde{\omega}^*\omega_Y$ and $\rho^*\omega_X|_{\rho^{-1}V}=0$, we have $\varpi^*\omega_Y|_{\rho^{-1}V}=0$. It implies $\omega_Y|_{\tilde{\omega}\rho^{-1}V}=0$. Q.E.D.

This proposition implies immediately the following corollary.

(4.10) **Corollary.** *Under the condition of Theorem* (4.2), *if $\mathscr{M}$ is a holonomic system on X, $\int^i\mathscr{M}$ are holonomic systems on Y.*

§ 5. Rationality of Roots of *b*-Functions

Let $f(x)$ be a holomorphic function on a complex manifold X of dimension n. There is an interest only with a neighborhood of the zeros of f when we consider b-function. Therefore, we assume hereafter that.

The zero of $df(x)$ is contained in the zero of $f(x)$. Let Y be a closed analytic set of X which is contained in $f^{-1}(0)$, and $F: X' \to X$ be a projective holomorphic map such that $X' - Y' \xrightarrow{\sim} X - Y$, where $Y' = F^{-1}(Y)$. Set $f' = f \circ F$. Let $b_f(s)$ and $b_{f'}(s)$ be a b-functions for f and f', respectively. Then we have the following theorem

(5.1) **Theorem.** *$b_f(s)$ is a divisor of $b_{f'}(s) b_{f'}(s+1) \dots b_{f'}(s+N)$ for a sufficiently large N.*

This theorem implies immediately the following

(5.2) **Corollary.** *The roots of $b_f(s)$ are strictly negative rational numbers.*

In fact, by Hironaka's desingularization theorem, there exists F such that f' is of the form

(5.1) $f' = t_1^{r_1} t_2^{r_2} \dots t_l^{r_l}$

for a suitable local coordinate system $(t_1, \dots, t_n)$. The b-function of this function is easily calculated to be

$$\prod_{\nu=1}^{l} [(r_\nu s + 1)(r_\nu s + 2) \dots (r_\nu s + r_\nu)].$$

In particular, the roots of $b_{f'}(s)$ are negative rational numbers, which implies so are the roots of $b_f(s)$ by Theorem (5.1). In this section, we also prove the following theorem.

(5.3) **Theorem.** *Let W_f be the closure of $\{(x, s\, d \log f(x)) \in T^*X; f(x) \neq 0, s \in \mathbb{C}\}$ in T^*X. Then, $\mathcal{N}_f$ is a coherent $\mathcal{D}_X$-Module and its characteristic variety is W_f.*

Before starting the proof of Theorems, we show several geometric properties of W_f. Let $\tilde{W}$ be the closure of

$$\{(s, x, s\, d \log f(x)) \in \mathbb{C} \times T^*X;\ f(x) \neq 0\}$$

in $\mathbb{C} \times T^*X$.

(5.4) **Lemma.** *The canonical projection $\tilde{W} \to T^*X$ is a finite map and its image is W.*

In fact, it follows immediately from the fact that $f(x)$ is integral over the ideal generated by the derivatives of $f(x)$.

(5.5) **Lemma.** *Set $\tilde{W}_0 = \tilde{W} \cap s^{-1}(0)$. Then $\tilde{W}_0$ is the inverse image of $W_0 = \{(x, \xi) \in W; f(x)\xi = 0\}$ by the map $\tilde{W} \to W$.*

Proof. Both coincide evidently where $f(x)$ is not zero. Suppose $f(x_0) = 0$, and $(s_0, x_0, \xi_0) \in \tilde{W}$. Then there is a path $(s(t), x(t), \xi(t))$ such that $s(0) = s_0$, $x(0) = x_0$, $\xi(0) = \xi_0$ and $s(t) d \log f(x(t)) = \xi(t)$. Thus, $s(t) df(x(t)) = f(x(t)) \xi(t)$ and

$$|s(t)|\,|df(x(t))| = |\xi(t)|\,|f(x(t))| \leqq C |\xi(t)|\,|df(x(t))|\,|x(t) - x_0|.$$

Thus, we have $|s(t)| \leqq |\xi(t)|\,|x(t) - x_0|$, and therefore, $s_0 = s(0) = 0$. Q.E.D.

(5.6) **Proposition.** *W_0 is holonomic.*

Proof. It suffices to show that W_0 is isotropic, that is, the fundamental form ω vanishes at the generic point of W_0. Let W' be a normalization of $\tilde{W}$, W_0' be an inverse image of W_0 by the projection $W' \to W$. At a generic point of W_0', $W_0' \to W_0$ is a local isomorphism. On W', $\omega = s d \log f(x)$ and s and $f(x)$ are functions on W'. Note that at a generic point of W_0', W' is non singular. Proposition being clear outside of the zero of f, it is sufficient to consider only the component of W_0' where f vanishes. Let ψ be a defining function of W_0' at its generic point. Then, s and f can be written by

$$s = g\psi^m, \qquad f = h\psi^l$$

for some functions g, h which do not vanish at generic point of W_0'. Then

$$\omega = sd \log f = sd \log h + l g \psi^{m-1} d\psi.$$

On W_0', $s = \psi = 0$, we have $\omega|_{W_0'} = 0$. Hence $\omega|_{W_0} = 0$. Q.E.D.

Now let us prove Theorem (5.1) and Theorem (5.3) at the same time. Return to the situation given in the beginning of this section. Assume that Theorem (5.3) is valid for (X', f'). This is easily verified if f' has normally crossing, that is, of the form (5.1).

Set $\mathcal{N}' = \int \mathcal{N}_{f'}$. By Theorem (4.2), $\mathcal{N}'$ is a coherent $\mathcal{D}_X$-Module with the structure of $\mathcal{D}_X[s, t]$-Module. Note that $\mathcal{N}'$ is isomorphic to $\mathcal{N}$ outside Y.

(5.7) **Lemma.** *$\mathcal{N}'$ is a subholonomic system. More precisely, we have $\check{SS}(\mathcal{N}') = W_f \cup \Lambda$ for some holonomic variety Λ.*

Proof. It is evident that $\check{SS}(\mathcal{N}')$ contains W_f. By Theorem (4.2), $\check{SS}(\mathcal{N}')$ is contained in $\tilde{\omega}\rho^{-1}(\check{SS}(\mathcal{N}_{f'})) = \tilde{\omega}\rho^{-1}(W_{f'})$. But, $\tilde{\omega}\rho^{-1}(W_{f'}) = \tilde{\omega}\rho^{-1}(W_{f'} \underset{X}{\times} (X - Y)) \cup \tilde{\omega}\rho^{-1}(W_{f'} \underset{X}{\times} Y) \subset W_f \cup \tilde{\omega}\rho^{-1}(W_{f'} \underset{X}{\times} Y)$. Since $W_{f'} \underset{X}{\times} Y$ is isotropic by Proposition (5.6), so is $\tilde{\omega}\rho^{-1}(W_{f'} \underset{X}{\times} Y)$ by Proposition (4.9). Thus we have the desired result.

Q.E.D.

$\mathcal{D}_{X \leftarrow X'}$ has the canonical section $1_{X \leftarrow X'}$ corresponding to $\delta(x - F(x')) \dfrac{\partial x}{\partial x'} dx'$ where $\partial x / \partial x'$ is a jacobian. If $\mathcal{D}_{X \leftarrow X'}$ is understood as a subsheaf of $\mathrm{Hom}(F^{-1}\Omega_X, \Omega_{X'})$, $1_{X \leftarrow X'}$ is nothing but the inverse image F^*. Therefore, this $1_{X \leftarrow X'}$ is defined canonically because $\dim X = \dim X'$. We define $\mathbb{C}_X \to R^0 f_*(\mathbb{C}_{X'}) \to R^0 F_* \left(\mathcal{D}_{X \leftarrow X'} \overset{L}{\underset{\mathcal{D}_{X'}}{\otimes}} \mathcal{N}_f\right)$ by $\mathbb{C}_{X'}\ 1 \mapsto 1_{X \leftarrow X'} \otimes f'^s$. We denote by u the image of $1 \in \mathbb{C}_X$ by this homomorphism. Set $\mathcal{N}'' = \mathcal{D}_X[s] u \subset \mathcal{N}'$.

(5.8) **Lemma.** *$\mathcal{N}''$ is a coherent $\mathcal{D}_X$-Module with the structure of $\mathcal{D}_X[s, t]$-Module.*

Proof. Since $t(1_{X \leftarrow X'} \otimes f'^s) = 1_{X \leftarrow X'} \otimes f'^{s+1} = f(1_{X \leftarrow X'} \otimes f's)$, we have $tu = fu$. Therefore, $t\mathcal{N}'' \subset \mathcal{N}''$, from which $\mathcal{N}''$ has a structure of $\mathcal{D}_X[s, t]$-Module. Since $\mathcal{N}''$ is a union of increasing coherent sub-Modules in $\mathcal{N}'$, it is coherent.

Now consider the $\mathcal{D}[s]$-linear homomorphism

$$\mathcal{N}'' \ni P(s) u \mapsto P(s) f^s \in \mathcal{N}_f.$$

(5.9) **Lemma.** *The above homomorphism is well defined.*

In fact suppose that $P(s)u=0$. Then, at a generic point u being nothing but f^s, $P(s)f^s=0$ at a generic point. It follows $P(s)f^s=0$ in $\mathcal{N}_f$.

Thus, we have the diagram of $\mathcal{D}_X[s,t]$-Modules: $\mathcal{N}_f \twoheadleftarrow \mathcal{N}'' \subset \mathcal{N}'$. Here, $\twoheadrightarrow$ means a surjective homomorphism. Therefore, $\mathcal{N}_f$ is finitely generated and hence coherent. Thus, we obtain, as an intermediate result, the following

(5.10) **Proposition.** *$\mathcal{N}_f$ is a coherent subholonomic $\mathcal{D}_X$-Module whose characteristic variety is a union of W_f and a holonomic variety.*

In order to complete Theorem (5.3) and prove Theorem (5.1), we give several properties of $\mathcal{D}[s,t]$-Modules.

Let $\mathcal{L}$ be a $\mathcal{D}_X[s,t]$-Module, which is coherent over $\mathcal{D}_X$ such that $\mathcal{L}/t\mathcal{L}$ is holonomic. By [5], $\mathcal{E}nd_{\mathcal{D}_X}(\mathcal{L}/t\mathcal{L})$ has a finite dimensional stalk at any point. Therefore, there is a non zero polynomial $b(s)$ such that $b(s)(\mathcal{L}/t\mathcal{L})=0$. We denote by $b(s,\mathcal{L})$ the largest common divisor of them. By the definition, the b-function $b_f(s)$ of $f(x)$ is nothing but $b(s,\mathcal{N}_f)$.

(5.11) **Proposition.** *Let $\mathcal{L}$ be a coherent holonomic $\mathcal{D}_X$-Module wih a structure of $\mathcal{D}_X[s,t]$-Module. Then, $t^N\mathcal{L}=0$ for a sufficiently large N.*

Proof. First note that a decreasing sequence $\mathcal{L}_j$ of a holonomic system $\mathcal{L}$ is stationary. In fact, the kernel of the surjective homomorphisms $\mathcal{E}xt^n(\mathcal{L},\mathcal{D}_X)\to \mathcal{E}xt^n(\mathcal{L}_j,\mathcal{D}_X)$ forms an increasing sequence of a coherent $\mathcal{D}_X$-Module $\mathcal{E}xt^n(\mathcal{L},\mathcal{D}_X)$. Therefore, it is stationary which implies that $\mathcal{E}xt^n(\mathcal{L}_j,\mathcal{D}_X)\to\mathcal{E}xt^n(\mathcal{L}_{j+1},\mathcal{D}_X)$ is an isomorphism for $j\gg 0$. Since $\mathcal{L}_j=\mathcal{E}xt^n(\mathcal{E}xt^n(\mathcal{L}_j,\mathcal{D}_X)\mathcal{D}_X)$, $\mathcal{L}_{j+1}\to\mathcal{L}_j$ is an isomorphism. Let $\mathcal{L}'$ be the intersection of all $t^N\mathcal{L}$. Since $t^N\mathcal{L}$ is a decreasing sequence, it is stationary and $\mathcal{L}'=t^N\mathcal{L}$ for some N. Thus $t\mathcal{L}'=\mathcal{L}'$. $\mathcal{L}'$ is also a $\mathcal{D}_X[s,t]$-Module such that $t\colon \mathcal{L}'\to\mathcal{L}'$ is an isomorphism.

$$b(s,\mathcal{L}')t=t\,b(s-1,\mathcal{L}') \quad \text{in } \mathbb{C}[s,t].$$

It follows that $b(s-1,\mathcal{L}')$ and $b(s,\mathcal{L}')$ must coincide up to constant multiplication. Therefore, $b(s,\mathcal{L}')$ is a constant function and consequently $\mathcal{L}'=0$. Q.E.D.

(5.12) **Corollary.** *$\mathcal{E}xt^j(\mathcal{N}_f,\mathcal{D})=0$ for $j\neq n-1$.*

Proof. By Theorem (2.3), $\mathcal{E}xt^j(\mathcal{N}_f,\mathcal{D})=0$ for $j<n-1$. Therefore, it is sufficient to show that $\mathcal{L}=\mathcal{E}xt^n(\mathcal{N}_f,\mathcal{D})$ is zero. $\mathcal{L}$ is a holonomic system with a structure of $\mathcal{D}[s,t]$-Module. $t^N\mathcal{L}=0$ for some N by the preceding proposition. The exact sequence

$$0\to\mathcal{N}_f\xrightarrow{t}\mathcal{N}_f\to\mathcal{M}_f\to 0$$

brings the exact sequence

$$\mathcal{L}\xrightarrow{t}\mathcal{L}\to\mathcal{E}xt^{n+1}(\mathcal{M}_f,\mathcal{D})=0.$$

It follows that $\mathcal{L}=t\mathcal{L}$. Therefore, $\mathcal{L}=t^N\mathcal{L}=0$. Q.E.D.

As its corollary, we can prove Theorem (5.3).

(5.13) **Corollary.** $\check{S}S(\mathcal{N}_f) = W_f$.

Proof. By Proposition (5.10), $W_f \subset \check{S}S(\mathcal{N}_f) \subset W_f \cup \Lambda$. Theorem (2.12) implies $\check{S}S(\mathcal{N}_f) = W_f$, because it is purely $(n-1)$-codimensional. Q.E.D.

Using Hironaka's desingularization theorem, there exists some F such that f' is of the form (5.1). And in this case, Theorem (5.3) is easily verified for f'. Therefore, Theorem (5.3) is proved for any f.

(5.14) **Corollary.** *Let $\mathcal{L}$ be a $\mathcal{D}[s,t]$-Module and $\mathcal{L}'$ be a $\mathcal{D}[s,t]$-sub-Modules of $\mathcal{L}$. Assume that $\mathcal{L}$, $\mathcal{L}'$ are coherent over $\mathcal{D}_X$ and $\mathcal{L}/t\mathcal{L}'$ is a holonomic system. Then $b(s,\mathcal{L}')$ is a divisor of $b(s,\mathcal{L})b(s+1,\mathcal{L})\ldots b(s+N,\mathcal{L})$ for $N \gg 0$.*

Proof. Set $\mathcal{L}'' = \mathcal{L}/\mathcal{L}'$. $\mathcal{L}''$ is holonomic. By Proposition (5.11), $t^N\mathcal{L}''=0$, or equivalently, $\mathcal{L}' \supset t^N\mathcal{L}$. Using the relation,

$$b(s+j,\mathcal{L})t^j\mathcal{L} = t^j b(s,\mathcal{L})\mathcal{L} \subset t^{j+1}\mathcal{L},$$

we obtain

$$b(s+N,\mathcal{L})\ldots b(s,\mathcal{L})\mathcal{L} \subset t^{N+1}\mathcal{L} \subset t\mathcal{L}'$$

Hence, $b(s,\mathcal{L})\ldots b(s+N,\mathcal{L})\mathcal{L}' \subset t\mathcal{L}'$ which implies the desired result. Q.E.D.

Now, we can prove Theorem (5.1). It is evident that $b_f(s) = b(s,\mathcal{N}_f)$ is a divisor of $b(s,\mathcal{N}'')$. By the preceding corollary, we have

$$b_f(s) | b(s,\mathcal{N}')b(s+1,\mathcal{N}')\ldots b(s+N,\mathcal{N}')$$

for a sufficiently large N. Thus, it suffices to show

$$b(s,\mathcal{N}') | b_{f'}(s).$$

Since $b_{f'}(s)\mathcal{N}_{f'} \subset t\mathcal{N}_{f'}$, there is a homomorphism $g\colon \mathcal{N}_{f'} \to \mathcal{N}_{f'}$ such that $b_{f'}(s) = t\circ g$, $b_{f'}(s) = t\circ \int g$ in $\mathcal{N}'$. It follows that $b_{f'}(s)$ is a multiple of $b(s,\mathcal{N}')$. It completes the proof of Theorem (5.1).

§ 6. Miscellaneous Results

Let α be a complex number. Set

$$\mathcal{N}_\alpha = \mathcal{N}_f/(s-\alpha)\mathcal{N}_f.$$

By Theorem (5.3), $\check{S}S(\mathcal{N}_\alpha)$ is contained in W_f. Since $\check{S}S(\mathcal{N}_\alpha)$ is the zero section outside the zero of $f(x)$, $\check{S}S(\mathcal{N}_\alpha)$ is contained in W_0. In particular, $\mathcal{N}_\alpha$ is a holonomic system. Now, consider $\mathcal{D}f^\alpha$. $\mathcal{D}f^\alpha$ is, by definition, $\mathcal{D}/\{P\in\mathcal{D};\ Pf^\alpha=0$ at a generic point$\}$. Then we have the surjective map $\mathcal{N}_\alpha \to \mathcal{D}f^\alpha$. It implies

(6.1) **Proposition.** *$\mathcal{D}f^\alpha$ is also a holonomic system whose characteristic variety is contained in W_0.*

We obtain further the following result.

(6.2) **Proposition.** *If $b_f(\alpha-j)\neq 0$ for $j=1,2,\ldots$, then $\mathcal{N}_\alpha \xrightarrow{\sim} \mathcal{D}f^\alpha$.*

Proof. It suffices to show that, if $Pf^\alpha=0$, then $P\in\mathcal{J}_f+(s-\alpha)\mathcal{D}[s]$. We will prove it by the induction of the order m of P. Pf^s can be written in $a(s,x)f^{s-m}$. Here $a(s,x)$ is a holomorphic function and polynomial on s. $Pf^\alpha=0$ implies $a(\alpha,x)=0$. Therefore, Pf^{s+m} is contained in $(s-\alpha+m)\mathcal{N}_f$. In other words, Pf^{s+m} is contained in $(s-\alpha+m)\mathcal{N}_f\cap t^m\mathcal{N}_f$. First we will show that it is contained in $(s-\alpha+m)t^m\mathcal{N}_f$. Since $b(\alpha-j)\neq 0$ for $j=1,\ldots,$

$$s-\alpha+j:\mathcal{M}_f\to\mathcal{M}_f$$

is an isomorphism, in particular, injective. Therefore, we have

$$(s-\alpha+j)\mathcal{N}_f\cap t\mathcal{N}_f\subset(s-\alpha+j)t\mathcal{N}_f.$$

Since

$$(s-\alpha+m)\mathcal{N}_f\cap t^m\mathcal{N}_f\subset(s-\alpha+m)t\mathcal{N}_f\cap t^m\mathcal{N}_f=t((s-\alpha+m-1)\mathcal{N}_f\cap t^{m-1}\mathcal{N}_f),$$

and since the induction on m says that

$$(s-\alpha+m-1)\mathcal{N}_f\cap t^{m-1}\mathcal{N}_f\subset(s-\alpha+m-1)t^{m-1}\mathcal{N}_f$$

we have $(s-\alpha+m)\mathcal{N}_f\cap t^m\mathcal{N}_f\subset t(s-\alpha+m-1)t^{m-1}\mathcal{N}_f=(s-\alpha+m)t^m\mathcal{N}_f$. It follows that Pf^{s+m} can be written as $Pf^{s+m}=(s-\alpha+m)\,Q(s)\,f^{s+m}$, or equivalently $Pf^s=(s-\alpha)\,Q(s-m)\,f^s$. Q.E.D.

(6.3) **Theorem.** *There is $P(s)$ in $\mathcal{D}[s]$ such that*

(1) *it can be written in the form $P(s)=s^m+A_1(x,D)s^{m-1}+\cdots+A_m(x,D)$ where $A_j(x,D)$ is a differential operator of order at most j and*

(2) $P(s)f^s=0$.

Proof. Consider a function $f'(t,x)=tf(x)$ on $X'=\mathbb{C}\times X$. $\mathcal{N}_{f'}$ is a coherent Module and $\mathcal{N}_{f'}=\mathcal{D}_{X'}f'^s$. In fact, $sf'^s=(tD_t)f'^s$. Since $f(x)$ is integral on $(\partial f/\partial x_1,\ldots,\partial f/\partial x_n)$, there exists a function $a(s,x,\xi)$ which is homogeneous on $(s,\xi)=(s,\xi_1,\ldots,\xi_n)$ of degree m such that

$$a(f(x),x,\partial f/\partial x_1,\ldots,\partial f/\partial x_n)=0$$

and

$$a(s,x,0)=s^m.$$

Then

$$a(t\tau,x,\xi)=0 \quad \text{for } (t,x;\tau dt+\langle\xi,dx\rangle) \text{ in } W_{f'}.$$

Since the characteristic variety of $\mathcal{N}_{f'}$ is $W_{f'}$, there is a differential operator $P(t,x,D_t,D_x)$ defined in a neighborhood of $t=0$ such that

$$P(t,x,D_t,D_x)f'^s=0$$

and

$$\sigma(P)=a(t\tau,x,\xi)^N$$

for a sufficiently large N. P can be written in the form

$$P=\tilde{P}(tD_t, x, D_x)+\sum_{j=1}^{\infty} t^j P_j(tD_t, x, D_x)+\sum_{j=1}^{\infty} D_t^j Q_j(tD_t, x, D_x).$$

By considering a homogeneity of Pf'^s with respect to t, we have $\tilde{P}f'^s=0$, that is $\tilde{P}(s, x, D_x)f^s=0$. $\tilde{P}(s, x, D_x)$ has evidently the desired properties. Q.E.D.

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