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On the Holonomic Systems of Linear Differential Equations, II ★

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In this paper we shall study the restriction of holonomic systems of differential equations.

Let X be a complex manifold and Y a submanifold, and let $\mathcal{O}_X$ and $\mathcal{D}_X$ be the sheaf of the holomorphic functions and the sheaf of the differential operators of finite order, respectively. If a function u on X satisfies a system of differential equations, the restriction of u onto Y also satisfies the system of differential equations derived from the system on X. This leads to the following definition. Let $\mathcal{M}$ be a $\mathcal{D}_X$-Module. The restriction of $\mathcal{M}$ onto Y is, by definition, $\mathcal{O}_Y \underset{\mathcal{O}_X}{\otimes} \mathcal{M}$. In [4] it is proved that if $\mathcal{M}$ is a coherent $\mathcal{D}_X$-Module and if Y is non-characteristic to $\mathcal{M}$, then the restriction of $\mathcal{M}$ is also a coherent $\mathcal{D}_Y$-Module. However, if Y is characteristic, the restriction is no longer coherent in general. For examples, if $X = \mathbb{C}^n$ and $Y = \{x = (x_1, \ldots, x_n) \in X;\ x_1 = 0\}$ and $\mathcal{M} = \mathcal{D}_X$, the restriction $\mathcal{M}/x_1\mathcal{M}$ is a free $\mathcal{D}_Y$-Module generated by $D_1^m (m = 0, 1, 2, \ldots)$ and is not coherent.

We shall prove the following theorems in this paper.

Theorem. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-Module on a complex manifold X and f a holomorphic map from Y to X. Then $\mathcal{O}_Y \underset{f^{-1}\mathcal{O}_X}{\otimes} f^{-1}\mathcal{M}$ is a holonomic system on Y.*

This theorem is proved by Bernstein [1] in the polynomial case.

At the same time, we shall prove

Theorem. *If $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-Module, and if $\mathcal{I}$ is a coherent Ideal of $\mathcal{O}_X$, then $\varinjlim_m \mathcal{E}xt^k_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^m; \mathcal{M})$ are also holonomic $\mathcal{D}_X$-Modules.*

Theorem. *If $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-Module defined on X and holonomic outside an analytic subset Y, then $\mathcal{M}/\mathcal{H}^0_Y(\mathcal{M})$ is holonomic on X.*

These theorems imply in particular the following: Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-Module and let ∇ be a meromorphic integrable connection on $\mathcal{F}$ with a pole

★ This is the second of the series of papers which are concerned with holonomic systems. The paper [5] is the first of this series

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on a hypersurface Y. Then, $\mathscr{H}^0_{[X|Y]}(\mathscr{F})$ (i.e., the sheaf of the meromorphic sections of $\mathscr{F}$ with a pole on Y) is a holonomic $\mathscr{D}_X$-Module (in particular, coherent).

Also, we shall prove the following theorem.

Theorem. *For two holonomic $\mathscr{D}_X$-Modules $\mathscr{M}$ and $\mathscr{N}$, $\mathscr{E}xt^j(\mathscr{M};\mathscr{N})$ are constructible (i.e., $\dim_{\mathbb{C}}\mathscr{E}xt^j(\mathscr{M};\mathscr{N})_x<\infty$ for any $x\in X$ and there is a stratification on X on each of whose stratum $\mathscr{E}xt^j_{\mathscr{D}}(\mathscr{M},\mathscr{N})$ is locally constant).*

However, the author does not know how to stratify X so that $\mathscr{E}xt^j_{\mathscr{D}}(\mathscr{M},\mathscr{N})$ is constructible on the strata. This problem is tightly connected with the problem of determining the characteristic variety of $\mathscr{O}_Y \underset{\mathscr{O}_X}{\otimes} \mathscr{M}$.

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§ 1. Algebraic Local Cohomologies

1.1. In this paper we denote by X a complex manifold, by $\mathscr{O}_X$ the sheaf of the holomorphic functions on X and by $\mathscr{D}_X$ the sheaf of the linear differential operators of *finite order*.

1.2. Let $\mathscr{J}$ be a coherent $\mathscr{O}_X$-Ideal and Y the support of $\mathscr{O}_X/\mathscr{J}$. For an $\mathscr{O}_X$-Module $\mathscr{F}$, we define with [2, 3]

(1.2.1) $\Gamma_{[X|Y]}(\mathscr{F})=\varinjlim_m \mathscr{H}om_{\mathscr{O}_X}(\mathscr{J}^m;\mathscr{F})$,

(1.2.2) $\Gamma_{[Y]}(\mathscr{F})=\varinjlim_m \mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{J}^m;\mathscr{F})$.

This definition depends only on Y (not on the choice of $\mathscr{J}$). We have an exact sequence:

(1.2.3) $0\to\Gamma_{[Y]}(\mathscr{F})\to\mathscr{F}\to\Gamma_{[X|Y]}(\mathscr{F})$.

Lemma 1.1. *If $\mathscr{F}$ is a $\mathscr{D}_X$-Module, $\Gamma_{[X|Y]}(\mathscr{F})$ and $\Gamma_{[Y]}(\mathscr{F})$ have a structure of $\mathscr{D}_X$-Modules so that* (1.2.3) *is* $\mathscr{D}_X$-linear.

Proof. We have evidently

$$\Gamma_{[X|Y]}(\mathscr{F})=\varinjlim_m \mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_X\mathscr{J}^m;\mathscr{F})$$

and

$$\Gamma_{[Y]}(\mathscr{F})=\varinjlim_m \mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_X/\mathscr{D}_X\mathscr{J}^m;\mathscr{F})$$

because $\mathscr{D}_X$ is faithfully flat over $\mathscr{O}_X$.

We shall define the multiplication of a differential operator P with $\Gamma_{[X|Y]}(\mathscr{F})$. Suppose that P is of order $\leqq l$. Then we have

$$\mathscr{D}_X\mathscr{J}^m P\subset\mathscr{D}_X\mathscr{J}^{m-l}\quad\text{for } m\geqq l.$$

This gives the $\mathscr{D}_X$-linear homomorphism

$$\mathscr{D}_X \mathscr{I}^m \to \mathscr{D}_X \mathscr{I}^{m-l}$$

by the multiplication of P. Hence, we get the homomorphism

$$\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_X \mathscr{I}^{m-l}; \mathscr{F}) \to \mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_X \mathscr{I}^m; \mathscr{F}).$$

Taking the inductive limit on m, we have the homomorphism $\Gamma_{[X|Y]}(\mathscr{F}) \to \Gamma_{[X|Y]}(\mathscr{F})$, which will be the multiplication by P. It is easy to check that this gives a structure of $\mathscr{D}_X$-Module on $\Gamma_{[X|Y]}(\mathscr{F})$ and that $\mathscr{F} \to \Gamma_{[X|Y]}(\mathscr{F})$ is $\mathscr{D}_X$-linear. Therefore, the kernel $\Gamma_{[Y]}(\mathscr{F})$ of this homomorphism has also a structure of $\mathscr{D}_X$-Module.

We shall denote by $\mathscr{H}^k_{[X|Y]}(\mathscr{F})$ (resp. $\mathscr{H}^k_{[Y]}(\mathscr{F})$) the k-th derived functor of $\Gamma_{[X|Y]}(\mathscr{F})$ (resp. $\Gamma_{[Y]}(\mathscr{F})$).

Since a stalk of an injective $\mathscr{D}_X$-Module is injective over a stalk of $\mathscr{O}_X$, we have

$$\mathscr{H}^k_{[X|Y]}(\mathscr{F}) = \varinjlim_m \mathscr{E}xt^k_{\mathscr{O}_X}(\mathscr{I}^m; \mathscr{F}) \tag{1.2.4}$$

$$\mathscr{H}^k_{[Y]}(\mathscr{F}) = \varinjlim_m \mathscr{E}xt^k_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}^m; \mathscr{F}). \tag{1.2.5}$$

We denote by $\mathbb{R}\Gamma_{[Y]}$, $\mathbb{R}\Gamma_{[X|Y]}$ the right derived functor in the derived category. We have the following triangles:

$$\begin{array}{c} \mathbb{R}\Gamma_{[Y]}\mathscr{F}^\cdot \\ \swarrow \qquad \nwarrow^{+1} \\ \mathscr{F}^\cdot \longrightarrow \mathbb{R}\Gamma_{[X|Y]}(\mathscr{F}^\cdot), \\ \mathbb{R}\Gamma_{[X|Y_1 \cap Y_2]}(\mathscr{F}^\cdot) \\ \swarrow \qquad \nwarrow^{+1} \\ \mathbb{R}\Gamma_{[X|Y_1]}(\mathscr{F}^\cdot) \oplus \mathbb{R}\Gamma_{[X|Y_2]}(\mathscr{F}^\cdot) \to \mathbb{R}\Gamma_{[X|Y_1 \cup Y_2]}(\mathscr{F}^\cdot) \end{array} \tag{1.2.6}$$

and we have also the relations

$$\begin{aligned} \mathbb{R}\Gamma_{[Y_1 \cap Y_2]}(\mathscr{F}^\cdot) &= \mathbb{R}\Gamma_{[Y_1]}\mathbb{R}\Gamma_{[Y_2]}(\mathscr{F}^\cdot), \\ \mathbb{R}\Gamma_{[X|Y_1]}\mathbb{R}\Gamma_{[Y_2]}(\mathscr{F}^\cdot) &= \mathbb{R}\Gamma_{[Y_2]}\mathbb{R}\Gamma_{[X|Y_1]}(\mathscr{F}^\cdot), \\ \mathbb{R}\Gamma_{[X|Y]}\mathbb{R}\Gamma_{[Y]}(\mathscr{F}^\cdot) &= \mathbb{R}\Gamma_{[Y]}\mathbb{R}\Gamma_{[X|Y]}(\mathscr{F}^\cdot) = 0. \\ \mathbb{R}\Gamma_{[X|Y_1]}\mathbb{R}\Gamma_{[X|Y_2]}(\mathscr{F}^\cdot) &= \mathbb{R}\Gamma_{[X|Y_1 \cup Y_2]}(\mathscr{F}^\cdot). \end{aligned} \tag{1.2.7}$$

1.3. Suppose Y is a hypersurface defined by $f=0$ with a holomorphic function f. For an $\mathscr{O}_X$-Module $\mathscr{F}$, we shall denote by $\mathscr{F}_f$ the $\mathscr{O}_X$-Module associated with the presheaf $U \mapsto \Gamma(U; \mathscr{F})_f$; here $\Gamma(U; \mathscr{F})_f$ is a localization by f. Then it is easy to see that

$$\mathbb{R}\Gamma_{[X|Y]}(\mathscr{F}) = \mathscr{F}_f = \mathscr{O}_{X_f} \underset{\mathscr{O}_X}{\otimes} \mathscr{F}. \tag{1.3.1}$$

$\mathscr{D}_{X,f}$ is nothing but the Ring of differential operators with pole on Y. Although $\mathscr{D}_X$ has two structures of $\mathcal{O}_X$-Modules (by the left and the right multiplications), we obtain the same $\Gamma_{[X|Y]}(\mathscr{D}_X)$.

1.4. We shall investigate the meaning of $\Gamma_{[X|Y]}$ and $\Gamma_{[Y]}$ from the viewpoint of systems of differential equations.

Theorem 1.2. *Let $\mathscr{F}^\cdot$ be a complex of right $\mathscr{D}_X$-Modules and $\mathscr{G}^\cdot$ a complex of left $\mathscr{D}_X$-Modules. Then, for any analytic subset Y, we have*

$$(1.4.1)\quad \mathbb{R}\Gamma_{[X|Y]}(\mathscr{F}^\cdot)\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathscr{G}^\cdot \xrightarrow{\sim} \mathbb{R}\Gamma_{[X|Y]}(\mathscr{F}^\cdot)\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathbb{R}\Gamma_{[X|Y]}(\mathscr{G}^\cdot) \xleftarrow{\sim} \mathscr{F}^\cdot\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathbb{R}\Gamma_{[X|Y]}(\mathscr{G}^\cdot)$$

$$(1.4.2)\quad \mathbb{R}\Gamma_{[Y]}(\mathscr{F}^\cdot)\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathscr{G}^\cdot \xleftarrow{\sim} \mathbb{R}\Gamma_{[Y]}(\mathscr{F}^\cdot)\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathbb{R}\Gamma_{[Y]}(\mathscr{G}^\cdot) \xrightarrow{\sim} \mathscr{F}^\cdot\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathbb{R}\Gamma_{[Y]}(\mathscr{G}^\cdot).$$

Here $\overset{L}{\otimes}$ is the left derived functor of $\otimes$ in the derived category.

Proof. First we shall observe that (1.4.1) and (1.4.2) are equivalent. In fact, if (1.4.1) holds, then

$$\mathbb{R}\Gamma_{[Y]}(\mathscr{F}^\cdot)\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathbb{R}\Gamma_{[X|Y]}(\mathscr{G}^\cdot)=\mathbb{R}\Gamma_{[X|Y]}\mathbb{R}\Gamma_{[Y]}(\mathscr{F}^\cdot)\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathscr{G}^\cdot=0.$$

This implies $\mathbb{R}\Gamma_{[Y]}(\mathscr{F}^\cdot)\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathscr{G}^\cdot \xleftarrow{\sim} \mathbb{R}\Gamma_{[Y]}(\mathscr{F}^\cdot)\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathbb{R}\Gamma_{[Y]}(\mathscr{G}^\cdot)$. Thus, we obtain (1.4.2.). Conversely, if (1.4.2) holds, then

$$\mathbb{R}\Gamma_{[X|Y]}(\mathscr{F}^\cdot)\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathbb{R}\Gamma_{[Y]}(\mathscr{G}^\cdot)=\mathbb{R}\Gamma_{[Y]}\mathbb{R}\Gamma_{[X|Y]}(\mathscr{F}^\cdot)\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathscr{G}^\cdot=0,$$

which implies (1.4.1).

Now, we shall prove this theorem. The question being local, we may assume that Y is a finite intersection of hypersurfaces $Y_1,\ldots,Y_l$. We shall prove it by induction on l.

a) When $l=1$ (i.e., Y is a hypersurface), suppose that Y is defined by $f=0$. We may assume that any stalk $\mathscr{F}^j_x$ and $\mathscr{G}^j_x$ are free $\mathscr{D}_{X,x}$-modules. Thus, it is enough to show (1.4.1) when $\mathscr{F}=\mathscr{D}_X$ and $\mathscr{G}=\mathscr{D}_X$. Then we have $\mathbb{R}\Gamma_{[X|Y]}(\mathscr{F})=\mathscr{D}_{X,f}$ and $\mathbb{R}\Gamma_{[X|Y]}(\mathscr{G})=\mathscr{D}_{X,f}$. We have also $\mathscr{D}_{X,f}\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathscr{D}_{X,f}=\mathscr{D}_{X,f}$. This shows (1.4.1).

b) When $l\geqq 2$. Set $Y'=Y_2\cap\ldots\cap Y_l$. By the hypothesis of the induction, the theorem is true for Y'. Therefore, we have

$$\begin{aligned}&\mathbb{R}\Gamma_{[Y]}(\mathscr{F}^\cdot)\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathscr{G}^\cdot\\&\quad=\mathbb{R}\Gamma_{[Y_1]}\mathbb{R}_{[Y']}(\mathscr{F}^\cdot)\underset{\mathscr{D}_X}{\otimes}\mathscr{G}^\cdot=\mathbb{R}\Gamma_{[Y']}(\mathscr{F}^\cdot)\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathbb{R}\Gamma_{[Y_1]}(\mathscr{G}^\cdot)\\&\quad=\mathscr{F}^\cdot\overset{L}{\otimes}\mathbb{R}\Gamma_{[Y']}\mathbb{R}\Gamma_{[Y_1]}(\mathscr{G}^\cdot)=\mathscr{F}^\cdot\overset{L}{\otimes}\mathbb{R}\Gamma_{[Y]}(\mathscr{G}^\cdot).\end{aligned}$$

This shows (1.4.2). Q.E.D.

We shall prove the following two theorems in this paper.

Theorem 1.3. *Let Y be an analytic subset of a complex manifold X, and $\mathscr{M}$ a coherent $\mathscr{D}_X$-Module which is holonomic on $X-Y$. Then $\mathscr{H}^k_{[X|Y]}(\mathscr{M})$ are holonomic $\mathscr{D}_X$-Modules.*

Theorem 1.4. *Under the same assumption as above, if $\mathscr{M}$ is holonomic on X, then $\mathscr{H}^k_{[Y]}(\mathscr{M})$ are holonomic $\mathscr{D}_X$-Modules.*

Together with Theorem 1.2, we have the following theorem.

Theorem 1.5. *Let Y be an analytic subset of a complex manifold X, $\mathscr{M}$ a coherent $\mathscr{D}_X$-Module and $\mathscr{N}$ a $\mathscr{D}_X$-Module.*

a) *If $\mathscr{M}$ is holonomic on $X-Y$, then*

$$\begin{aligned}&\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathbb{R}\Gamma_{[X|Y]}\mathbb{R}\mathscr{H}om(\mathscr{M};\mathscr{D}_X);\mathscr{D}_X);\mathscr{N})\\&\quad=\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M};\mathbb{R}\Gamma_{[X|Y]}(\mathscr{N})).\end{aligned}$$

b) *If $\mathscr{M}$ is holonomic on X, then*

$$\begin{aligned}&\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathbb{R}\Gamma_{[Y]}\mathbb{R}\mathscr{H}om(\mathscr{M};\mathscr{D}_X);\mathscr{D}_X);\mathscr{N})\\&\quad=\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M};\mathbb{R}\Gamma_{[Y]}(\mathscr{N})).\end{aligned}$$

Proof. Let us prove a). We have

$$\begin{aligned}&\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathbb{R}\Gamma_{[X|Y]}\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M};\mathscr{D}_X);\mathscr{D}_X);\mathscr{N})\\&\quad=\mathbb{R}\Gamma_{[X|Y]}\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M};\mathscr{D}_X)\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathscr{N}\\&\quad=\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M};\mathscr{D}_X)\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathbb{R}\Gamma_{[X|Y]}(\mathscr{N})\\&\quad=\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M};\mathbb{R}\Gamma_{[X|Y]}(\mathscr{N})).\end{aligned}$$

b) is obtained in the same way. Q.E.D.

Remark. In [7] we will see that if $\mathscr{M}$ has regular singularity, then $\mathscr{D}^\infty\underset{\mathscr{D}}{\otimes}\mathbb{R}\Gamma_{[Y]}(\mathscr{M})=\mathbb{R}\Gamma_Y(\mathscr{D}^\infty\underset{\mathscr{D}}{\otimes}\mathscr{M})$, where $\mathscr{D}^\infty$ is the sheaf of the differential operators of infinite order. However, this relation does not hold when $\mathscr{M}$ has irregular singularity.

1.5. Let Θ be the sheaf of the vector fields. Then $\mathscr{D}_X$ is an $\mathscr{O}_X$-Algebra generated by Θ. Therefore, it is easy to see the following lemma.

Lemma 1.6. *Let $\mathscr{F}$ be an $\mathcal{O}_X$-Module. Suppose that a sheaf homomorphism $\psi\colon \Theta \underset{\mathbb{C}}{\otimes} \mathscr{F} \to \mathscr{F}$ satisfies the following conditions:*

(i) $\psi(av\otimes s)=a\psi(v\otimes s)$ *(resp.* $\psi(av\otimes s)=\psi(v\otimes as)$ *for* $a\in\mathcal{O}_X$, $v\in\Theta_X$ *and* $s\in\mathscr{F}$.

(ii) $\psi(v\otimes as)=a\psi(v\otimes a)+v(a)\psi(v\otimes s)$ *(resp.* $\psi(av\otimes s)=a\psi(v\otimes s)$ $-v(a)\psi(v\otimes s))$ *for* $a\in\mathcal{O}_X$, $v\in\Theta_X$ *and* $s\in\mathscr{F}$.

(iii) $\psi([v_1 v_2]\otimes s)=\psi(v_1\otimes\psi(v_2\otimes s))-\psi(v_2\otimes\psi(v_1\otimes s))$ *(resp.* $\psi([v_1,v_2]\otimes s)=\psi(v_2\otimes\psi(v_1\otimes s))-\psi(v_1\otimes\psi(v_2\otimes s))$ *for* $v_1,v_2\in\Theta_X$ *and* $s\in\mathscr{F}$.

Then there is a unique structure of the left (resp. right) $\mathscr{D}_X$-Module on $\mathscr{F}$ such that $\psi(v\otimes s)=vs$ (resp. $\psi(v\otimes s)=sv$) and that the induced structure of the $\mathcal{O}_X$-Module coincides with the original one of $\mathscr{F}$.

1.6. Let $\mathscr{M}$ and $\mathscr{N}$ be two left $\mathscr{D}_X$-Modules. Then $\mathscr{M}\underset{\mathcal{O}_X}{\otimes}\mathscr{N}$ has the structure of a left $\mathscr{D}_X$-Module by $v(s\otimes t)=vs\otimes t+s\otimes vt$ for $v\in\Theta_X$, $s\in\mathscr{M}$, $t\in\mathscr{N}$. If $\mathscr{M}$ is a right $\mathscr{D}_X$-Module and $\mathscr{N}$ is a left $\mathscr{D}_X$-Module, $\mathscr{M}\underset{\mathcal{O}_X}{\otimes}\mathscr{N}$ has the structure of a right $\mathscr{D}_X$-Module by $(s\otimes t)v=sv\otimes t-s\otimes vt$. If $\mathscr{M}$ and $\mathscr{N}$ are right $\mathscr{D}_X$-Modules, then $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{M};\mathscr{N})$ has the structure of a left $\mathscr{D}_X$-Module by $(vf)(s)$ $=f(sv)-f(s)v$ for $f\in\mathscr{H}om_{\mathcal{O}_X}(\mathscr{M};\mathscr{N})$, $v\in\Theta_X$ and $s\in\mathscr{M}$. If $\mathscr{M}$ is a left $\mathscr{D}_X$-Module and $\mathscr{N}$ is a right $\mathscr{D}_X$-Module, then $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{M};\mathscr{N})$ has the structure of a right $\mathscr{D}_X$-Module by $(fv)(s)=f(vs)+f(s)v$ for $f\in\mathscr{H}om_{\mathcal{O}_X}(\mathscr{M};\mathscr{N})$, $v\in\Theta_X$ and $s\in\mathscr{M}$.

These facts are easily checked by using Lemma 1.6. Since the sheaf Ω_X^n of the n-forms ($n=\dim X$) is a right $\mathscr{D}_X$-Module, $\mathscr{M}\mapsto\Omega_X^n\otimes\mathscr{M}$ and $\mathscr{N}\mapsto\mathscr{H}om_{\mathcal{O}_X}(\Omega_X^n;\mathscr{N})$ give the equivalence of the category of left $\mathscr{D}_X$-Modules and the category of right $\mathscr{D}_X$-Modules.

The following lemma being easily checked, we leave the proof to the reader.

Lemma 1.7. (i) *Let $\mathscr{M}$ be a right (resp. left) $\mathscr{D}_X$-Module, $\mathscr{N}$ a left (resp. right) $\mathscr{D}_X$-Module and $\mathscr{L}$ a right $\mathscr{D}_X$-Module. Then*

$$\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}\underset{\mathcal{O}_X}{\otimes}\mathscr{N};\mathscr{L})\cong\mathscr{H}om_{\mathscr{D}_X}(\mathscr{N};\mathscr{H}om_{\mathcal{O}_X}(\mathscr{M},\mathscr{L})).$$

(ii) *If $\mathscr{M}$ is a right $\mathscr{D}_X$-Module and if $\mathscr{N}$ and $\mathscr{L}$ are left $\mathscr{D}_X$-Modules, then*

$$(\mathscr{M}\underset{\mathcal{O}_X}{\otimes}\mathscr{N})\underset{\mathscr{D}_X}{\otimes}\mathscr{L}\cong\mathscr{M}\underset{\mathscr{D}_X}{\otimes}(\mathscr{N}\underset{\mathcal{O}_X}{\otimes}\mathscr{L}).$$

Lemma 1.8. *Let* $\mathscr{M}^\cdot$ (resp. $\mathscr{N}^\cdot$) *be a complex of right (resp. left) $\mathscr{D}_X$-Modules. Then*

$$\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\Omega_X^n;\mathscr{M}^\cdot\overset{L}{\underset{\mathcal{O}_X}{\otimes}}\mathscr{N}^\cdot)=\mathscr{M}^\cdot\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathscr{N}^\cdot[-n]$$

where $n=\dim X$.

Proof. We have

$$\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}\left(\Omega_X^n; \mathscr{M}^{\boldsymbol{\cdot}} \overset{L}{\underset{\mathscr{O}_X}{\otimes}} \mathscr{N}^{\boldsymbol{\cdot}}\right)$$
$$= \left(\mathscr{M}^{\boldsymbol{\cdot}} \overset{L}{\underset{\mathscr{O}_X}{\otimes}} \mathscr{N}^{\boldsymbol{\cdot}}\right) \overset{L}{\underset{\mathscr{D}_X}{\otimes}} \mathbb{R}\mathscr{H}om(\Omega_X^n; \mathscr{D}_X)$$
$$= \left(\mathscr{M}^{\boldsymbol{\cdot}} \overset{L}{\underset{\mathscr{O}_X}{\otimes}} \mathscr{N}^{\boldsymbol{\cdot}}\right) \overset{L}{\underset{\mathscr{D}_X}{\otimes}} \mathscr{O}_X[-n]$$
$$= \mathscr{M}^{\boldsymbol{\cdot}} \overset{L}{\underset{\mathscr{D}_X}{\otimes}} \left(\mathscr{N}^{\boldsymbol{\cdot}} \underset{\mathscr{O}_X}{\otimes} \mathscr{O}_X\right)[-n]$$
$$= \mathscr{M}^{\boldsymbol{\cdot}} \overset{L}{\underset{\mathscr{D}_X}{\otimes}} \mathscr{N}^{\boldsymbol{\cdot}}[-n]. \quad \text{Q.E.D.}$$

Lemma 1.9. *For a coherent left $\mathscr{D}_X$-Module $\mathscr{M}$ and a $\mathscr{D}_X$-Module $\mathscr{N}$,*

$$\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}; \mathscr{N}) = \mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\Omega_X^n; \mathbb{R}\mathscr{H}om(\mathscr{M}; \mathscr{D}_X) \overset{L}{\underset{\mathscr{O}_X}{\otimes}} \mathscr{N})[n].$$

where $n = \dim X$, and Ω_X^n is the sheaf of n-forms on X.

In fact, we have

$$\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}; \mathscr{N}) = \mathbb{R}\mathscr{H}om(\mathscr{M}; \mathscr{D}_X) \overset{L}{\underset{\mathscr{D}_X}{\otimes}} \mathscr{N}.$$

§ 2. *b*-Functions

2.1. Let f be a holomorphic function on X and Y the zeros of f. As we mentioned, $\mathscr{M}_f$ is not necessarily a coherent $\mathscr{D}_X$-Module even if $\mathscr{M}$ is a coherent $\mathscr{D}_X$-Module. We shall show that $\mathscr{M}_f$ is holonomic if $\mathscr{M}$ is holonomic outside $f^{-1}(0)$. Also, we shall show the existence of b-functions, i.e., for a section u of $\mathscr{M}$, there is a nonzero polynomial $b(s)$ and a differential operator $P(s)$ which is a polynomial on s satisfying $P(s)f^{s+1}u = b(s)f^s u$.

We use the same technique as in [6].

2.2. Let s be an indeterminate. The sheaf $\mathscr{D}_X[s]$ is, by definition, the sheaf of rings $\mathscr{D}_X \underset{\mathbb{C}}{\otimes} \mathbb{C}[s]$, where s commutes with the sections of $\mathscr{D}_X$. Let $\mathbb{C}[s,t]$ be the ring generated by s and t with the fundamental commutation relation

$$[t, s] = t.$$

We denote by $\mathscr{D}_X[s,t]$ the ring $\mathscr{D}_X \underset{\mathbb{C}}{\otimes} \mathbb{C}[s,t]$, in which s and t commute with the sections of $\mathscr{D}_X$.

Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-Module holonomic outside $f^{-1}(0)$ and u a section of $\mathscr{M}$. Let $\mathscr{J}$ be the Ideal of $\mathscr{D}[s]$ consisting of the $P(s)$ in $\mathscr{D}[s]$ such that

(2.2.1) $\quad f^{mn-s}P(s)f^s u = 0$

for a sufficiently large m.

Note that $f^{m-s}P(s)f^s$ belongs to $\mathscr{D}[s]$ for a sufficiently large m, and the identity (2.2.1) should be understood to hold in $\mathbb{C}[s] \underset{\mathbb{C}}{\otimes} \mathscr{M}$. We will denote by

$\mathcal{N}$ the $\mathcal{D}[s]$-Module $\mathcal{D}[s]/\mathcal{J}$ and the modulo class [1] is denoted symbolically by $f^s u$. Therefore, $\mathcal{N}$ is generated by $f^s u$ as a $\mathcal{D}[s]$-Module.

The following lemma is evident.

Lemma 2.1. *The system $\mathcal{N}$ has a structure of a $\mathcal{D}[s,t]$-Module by*

$$t\colon P(s)f^s u \mapsto P(s+1)f^{s+1}u.$$

For any complex number λ, $\mathcal{N}/(s-\lambda)\mathcal{N}$ is denoted by $\mathcal{N}_\lambda$, and $f^s u$ modulo $(s-\lambda)\mathcal{N}$ is denoted by $f^\lambda u$. $\mathcal{N}_\lambda$ is a $\mathcal{D}_X$-Module generated by $f^\lambda u$.

Lemma 2.2. *$\mathcal{D}f^s u$ and $\mathcal{N}_\lambda$ are coherent $\mathcal{D}_X$-Modules.*

This lemma is an immediate consequence of the following proposition proved in [4]. (See also [8].)

Proposition 2.3 ([4]). *Let $\mathcal{D}_m$ be the sheaf of differential operators of order $\leqq m$. An Ideal $\mathcal{J}$ of $\mathcal{D}_X$ is coherent if $\mathcal{J}\cap\mathcal{D}_m$ is a coherent $\mathcal{O}_X$-Module for any m.*

2.3. We will take a stratification $\{X_\alpha\}_{\alpha\in A}$ of X such that

(2.3.1) $SS(\mathcal{M}) \subset \bigsqcup_{\alpha\in A} T^*_{X_\alpha}X \cup \pi^{-1}(f^{-1}(0))$

Here, $T^*_{X_\alpha}X$ signifies the conormal bundle of X_α.

(2.3.2) Any X_α is either disjoint from $f^{-1}(0)$ or contained in $f^{-1}(0)$.

It is clear that there exists such a stratification.

Lemma 2.4. *There exists a neighborhood Ω of $f^{-1}(0)$ such that, for any X_α disjoint from $f^{-1}(0)$, $d(f|X_\alpha)$ does not vanish at any point in $\Omega\cap X_\alpha$.*

Proof. If it fails, there exists an analytic path $x(t)$ such that $x(0)\in f^{-1}(0)$, $x(t)\in X_\alpha$ for $0<|t|\ll 1$ and that $d(f|X_\alpha)$ vanishes at $x(t)$ for $0<|t|\ll 1$. Therefore, $f(x(t))$ is a constant function of t, which implies that $f(x(t))=0$. This leads to contradiction. Q.E.D.

Theorem 2.5. *On some neighborhood Ω of $f^{-1}(0)$, $\mathcal{D}(f^s u)$ (resp. $\mathcal{N}_\lambda$) is a subholonomic (resp. holonomic) $\mathcal{D}_X$-Module. (A coherent $\mathcal{D}_X$-Module is called holonomic (resp. subholonomic) if the codimension of the characteristic variety is at least* $\dim X$ *(resp.* $\dim X-1$*).)*

In order to prove this theorem, we note the following proposition.

Proposition 2.6. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two coherent $\mathcal{D}_X$-Modules. Suppose that $SS(\mathcal{L}_1)\cap SS(\mathcal{L}_2)$ is contained in the zero section of the cotangent bundle T^*X. Then $\mathcal{L}_1 \underset{\mathcal{O}_X}{\otimes} \mathcal{L}_2$ is also a coherent $\mathcal{D}_X$-Module and its characteristic variety is contained in*

$$\{(x,\xi_1+\xi_2)\in T^*X;\ (x,\xi_1)\in SS(\mathcal{L}_1) \textit{ and } (x,\xi_2)\in SS(\mathcal{L}_2)\}.$$

Especially, if $\mathcal{L}_1$ is holonomic (resp. subholonomic) and $\mathcal{L}_2$ is holonomic, then $\mathcal{L}_1 \underset{\mathcal{O}_X}{\otimes} \mathcal{L}_2$ is holonomic (resp. subholonomic).

Since $\mathscr{L}_1 \underset{\mathcal{O}_X}{\otimes} \mathscr{L}_2$ is obtained as the restriction of the system $\mathscr{L}_1 \hat{\otimes} \mathscr{L}_2$ on $X \times X$ onto the diagonal set. (See Proposition 4.7.) This proposition is a consequence of Chapter II, Theorem 3.5.3 and Theorem 3.5.9 of [9].

Now, let us prove Theorem 2.5. We take Ω as in Lemma 2.4. Since $SS(\mathscr{D}u) \cap SS(\mathscr{D}f^s)$ (resp. $SS(\mathscr{D}u) \cap SS(\mathscr{D}f^\lambda)$) is contained in the zero section of T^*X on $\Omega - f^{-1}(0)$, $\mathscr{D}f^s \otimes \mathscr{D}u$ (resp. $\mathscr{D}f^\lambda \otimes \mathscr{D}u$) is subholonomic (resp. holonomic) on $\Omega - f^{-1}(0)$. Since there are surjective homomorphisms $\mathscr{D}f^s \otimes \mathscr{D}u \supset \mathscr{D}(f^s \otimes u) \to \mathscr{D}(f^s u)$ (resp. $\mathscr{D}f^\lambda \otimes \mathscr{D}u \supset \mathscr{D}(f^\lambda \otimes u) \to \mathscr{N}_\lambda$), we can conclude that $\mathscr{D}(f^s u)$ (resp. $\mathscr{N}_\lambda$) is subholonomic (resp. holonomic) on $\Omega - f^{-1}(0)$.

Let $\mathscr{L}$ (resp. $\mathscr{L}'$) be the sub-Module of $\mathscr{D}(f^s u)$ (resp. $\mathscr{N}_\lambda$) consisting of all w such that $\mathscr{D}w$ is subholonomic (resp. holonomic). By [4] (cf. [6]), $\mathscr{L}$ (resp. $\mathscr{L}'$) is subholonomic (resp. holonomic) on Ω. Therefore, $\mathscr{D}(f^s u)/\mathscr{L}$ and $\mathscr{N}_\lambda/\mathscr{L}'$ are coherent $\mathscr{D}_X$-Modules supported in $f^{-1}(0)$. Therefore, by Hilbert's Nullstelensatz, there exists an integer m such that $f^m \cdot f^s u \in \mathscr{L}$ (resp. $f^m \cdot f^\lambda u \in \mathscr{L}'$). Therefore, $\mathscr{D}(f^m \cdot f^s u)$ (resp. $\mathscr{D}(f^m \cdot f^\lambda u)$) is a subholonomic (resp. holonomic) system on Ω. However, $\mathscr{D}(f^m \cdot f^s u)$ is isomorphic to $\mathscr{D}(f^s u)$ by the homomorphism t^m. Hence, it follows that $\mathscr{D}(f^s u)$ is subholonomic.

$\mathscr{D}(f^m \cdot f^s u)$ and $\mathscr{D}(f^s u)$ have the same multiplicity at the irreducible components of the characteristic variety of $\mathscr{D}(f^s u)$. Since the multiplicity is an additive quantity, the characteristic variety of $\mathscr{D}f^s u/\mathscr{D}f^m \cdot f^s u$ does not contain any irreducible component of that of $\mathscr{D}f^s u$. This implies that $\mathscr{D}f^s u/\mathscr{D}f^m f^s u$ is a holonomic $\mathscr{D}_X$-Module.

There exists a surjective homomorphism $\mathscr{D}f^s u/\mathscr{D}f^m \cdot f^s u \to \mathscr{D}(f^\lambda u)/\mathscr{D} \cdot (f^m \cdot f^\lambda u)$, which shows that $\mathscr{D}f^\lambda u/\mathscr{D}(f^m \cdot f^\lambda u)$ is holonomic. Since $\mathscr{D}(f^m \cdot f^\lambda u)$ is holonomic, $\mathscr{D}f^\lambda\, u$ is also holonomic. Thus, Theorem 2.5 is proved.

2.4. Since $\mathscr{N}$ has a structure of a $\mathscr{D}[s,t]$-Module, we can define the b-function as in [6]. Recall that the b-function is a generator of the ideal of $\mathbb{C}[s]$ consisting of $b(s)$ such that $b(s)\mathscr{N} \subset t\mathscr{N}$. That is equivalent to saying that there exists $P(s) \in \mathscr{D}[s]$ such that $P(s)f^{s+1}u = b(s)f^s u$. However, we cannot apply [6] directly in order to prove the existence of nonzero b-functions, because $\mathscr{N}$ is not a coherent $\mathscr{D}$-Module in general.

Theorem 2.7. *For any point $x_0 \in f^{-1}(0)$, there exist a nonzero polynomial $b(s)$ of s and $P(s) \in \mathscr{D}[s]_{x_0}$ such that*

$$P(s) f^{s+1} u = b(s) f^s u.$$

Proof. We set $\mathscr{M}' = \mathcal{O}_{\mathbb{C}} \hat{\otimes} \mathscr{M}$. Then $\mathscr{M}'$ is a holonomic $\mathscr{D}_{X'}$-Module on $X' = \mathbb{C} \times X$. We denote by u' the section $1 \otimes u$ of $\mathscr{M}'$. Set $f'(y,x) = yf(x)\,(y \in \mathbb{C},\ x \in X)$. We have

$$\mathscr{D}_{X'}[s] f^s u' = \mathscr{D}_{X'} f'^s u'.$$

In fact, we have

$$\left(y \frac{\partial}{\partial y}\right)^m f'^s u' = s^m f'^s u'.$$

Therefore, $\mathcal{N}' = \mathcal{D}_{X'}[s] f'^s u'$ is subholonomic by Theorem 2.5 and has a structure of $\mathcal{D}_{X'}[s, t]$-Module. Therefore, we can apply [6], There exist a polynomial $b(s)$ and a differential operator $P(y, x, D_y, D_x)$ defined in a neighborhood of $(y, x) = (0, x_0)$ such that

$$(2.3) \quad P(y, x, D_y, D_x) f'^{s+1} u' = b(s) f'^s u'.$$

Let P_0 be the homogeneous component of P of degree -1 with respect to y. Then, comparing the degree of homogeneity of (2.3), we have

$$P_0 f'^{s+1} u' = b(s) f'^s u'.$$

P_0 has the form

$$P_0 = \Sigma A_j(x, D_x)(y D_y)^j D_y.$$

Therefore, we have

$$(s+1) \Sigma s^j A_j(x, D_x) f'^s f u' = b(s) f'^s u',$$

which implies

$$(s+1) \Sigma s^j A_j(x, D_x) f^{s+1} u = b(s) f^s u. \quad \text{Q.E.D.}$$

Now, it is easy to see that the canonical homomorphism

$$\mathcal{N}_{\lambda+1} \to \mathcal{N}_\lambda \qquad (f^{\lambda+1} u \mapsto f \cdot f^\lambda u)$$

is an isomorphism when $b(\lambda) \neq 0$, because we can construct the inverse $f^\lambda u \mapsto b(\lambda)^{-1} P(\lambda) f^{\lambda+1} u$.

Therefore, we get the following

Corollary 2.8. *$\varinjlim_m \mathcal{N}_{\lambda-m}$ is a holonomic $\mathcal{D}_X$-Module.*

Proposition 2.9. *For any coherent $\mathcal{D}_X$-Module $\mathcal{M}$, $\mathcal{M}_f$ is a (coherent) holonomic $\mathcal{D}_X$-Module if $\mathcal{M}$ is holonomic outside $f^{-1}(0)$.*

Proof. Since $\mathcal{M} \mapsto \mathcal{M}_f$ is an exact functor, we may assume without loss of generality that $\mathcal{M}$ is generated by a section u.

Since $\mathcal{M}_f$ is the quotient of $\varinjlim_m \mathcal{D} f^{-m} u$, $\mathcal{M}_f$ is holonomic. Q.E.D.

§ 3. Proof of Theorem

Proposition 2.9 implies Theorem 1.4 almost immediately. First note the following proposition.

Proposition 3.1 ([2], [3]). *Let Y_1 and Y_2 be two analytic sets. Then there exists a spectral sequence*

$$\mathcal{E}_2^{pq} = \mathcal{H}^p_{[Y_1]}(\mathcal{H}^q_{[Y_2]}(\mathcal{M})) \Rightarrow \mathcal{H}^{p+q} = \mathcal{H}^{p+q}_{[Y_1 \cap Y_2]}(\mathcal{M}).$$

In particular, if $\mathscr{E}_2^{pq}$ are holonomic, then $\mathscr{H}^{p+q}$ are holonomic. Therefore, if Theorem 1.4 is true for Y_1 and Y_2, then it is so for $Y_1 \cap Y_2$. Since Y is locally a finite intersection of hypersurfaces, we can reduce the theorem to the case in which Y is a hypersurface by induction. This case is nothing but Proposition 2.9.

More generally, we have the following theorem. The author is grateful to J.-M. Kantor for kindly pointing out this result.

Theorem 3.1. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-Module and Y an analytic set of X. Suppose that $\mathscr{M}$ is holonomic on $X-Y$. Then $\mathscr{H}^i_{[X|Y]}(\mathscr{M})$ is coherent and holonomic for any i.*

Proof. Let $\mathscr{M}'$ be the sub-Module of $\mathscr{M}$ consisting of sections u of $\mathscr{M}$ such that $\mathscr{D}_X u$ is holonomic. Then, $\mathscr{M}'$ is a holonomic system. Since $\mathscr{M}=\mathscr{M}'$ outside $X-Y$, the support of $\mathscr{M}/\mathscr{M}'$ is contained in Y. *Therefore, we have* $\mathbb{R}\Gamma_{[X|Y]}(\mathscr{M}/\mathscr{M}')=0$, which implies that

$$\mathbb{R}\Gamma_{[X|Y]}(\mathscr{M})=\mathbb{R}\Gamma_{[X|Y]}(\mathscr{M}').$$

Thus, replacing $\mathscr{M}$ with $\mathscr{M}'$, we may assume that $\mathscr{M}$ is holonomic from the first time. By the exact sequence

$$0\to\mathscr{H}^0_{[Y]}(\mathscr{M})\to\mathscr{M}\to\mathscr{H}^0_{[X|Y]}(\mathscr{M})\to\mathscr{H}^1_{[Y]}(\mathscr{M})\to 0,$$

and by the isomorphisms

$$\mathscr{H}^j_{[X|Y]}(\mathscr{M})=\mathscr{H}^{j+1}_{[Y]}(\mathscr{M}) \qquad (j\geqq 1),$$

this theorem follows from Theorem 1.4. Q.E.D.

§ 4. Restriction of $\mathscr{D}_X$-Modules

4.1. Let X and Y be complex manifolds and f a holomorphic map from Y to X. As in [4], we define the sheaf $\mathscr{D}_{Y\to X}$ (resp. $\mathscr{D}_{X\leftarrow Y}$) by $\mathscr{O}_Y \underset{f^{-1}\mathscr{O}_X}{\otimes} f^{-1}\mathscr{D}_X$ (resp. $f^{-1}(\mathscr{D}_X \underset{\mathscr{O}_X}{\otimes} (\Omega_X^{\dim X})^{\otimes -1}) \underset{f^{-1}\mathscr{O}_X}{\otimes} \Omega_Y^{\dim Y}$), where Ω_X^j signifies the sheaf of the j-forms. The sheaf $\mathscr{D}_{Y\to X}$ has a structure of right $f^{-1}\mathscr{D}_X$-Module by the multiplication from the right. We can endow $\mathscr{D}_{Y\to X}$ with a structure of left $\mathscr{D}_X$-Module as follows. For $v\in\Theta_Y$, $f_*(v)\in\mathscr{O}_Y \underset{f^{-1}\mathscr{O}_X}{\otimes} f^{-1}\Theta_X$ is given $\Sigma a_j\otimes\omega_j$ with $a_j\in\mathscr{O}_Y$ and $\omega_j\in\Theta_X$. Then $v(b\otimes P)=\Sigma a_j b\otimes\omega_j P+v(b)\otimes P$.

$\mathscr{D}_{X\leftarrow Y}$ has evidently the structure of left $f^{-1}\mathscr{D}_X$-Module. The structure of right $\mathscr{D}_X$-Module on $\mathscr{D}_X$ induces the structure of left $\mathscr{D}_X$-Module on $\mathscr{D}_X \underset{\mathscr{O}_X}{\otimes} (\Omega_X^{\dim X})^{\otimes -1}$ and hence

$$\mathscr{O}_Y \underset{f^{-1}\mathscr{O}_X}{\otimes} (\mathscr{D}_X \underset{\mathscr{O}_X}{\otimes} (\Omega_X^{\dim X})^{\otimes -1})=\mathscr{D}_{Y\to X} \underset{f^{-1}\mathscr{D}_X}{\otimes} f^{-1}(\mathscr{D}_X \underset{\mathscr{O}_X}{\otimes} (\Omega_X^{\dim X})^{\otimes -1})$$

has a structure of left $\mathscr{D}_Y$-Module. This defines the structure of right $\mathscr{D}_Y$-Module on $\mathscr{D}_{X\leftarrow Y}$. Thus, $\mathscr{D}_{Y\to X}$ is a $(\mathscr{D}_Y, f^{-1}\mathscr{D}_X)$-bi-Module and $\mathscr{D}_{X\leftarrow Y}$ is an $(f^{-1}\mathscr{D}_X, \mathscr{D}_Y)$-bi-

Module. Note that we have

$$\mathscr{D}_{Y\to X}=\mathscr{H}^{\dim X}_{[Y]}(\mathscr{O}_{Y\times X}\underset{\mathscr{O}_X}{\otimes}\Omega_X^{\dim X})$$

and

$$\mathscr{D}_{X\leftarrow Y}=\mathscr{H}^{\dim X}_{[Y]}(\Omega_Y^{\dim Y}\underset{\mathscr{O}_Y}{\otimes}\mathscr{O}_{Y\times X}).$$

4.2. Suppose that Y is a submanifold of X of codimension l. Then $\mathscr{D}_{Y\to X}$ and $\mathscr{D}_{X\leftarrow Y}$ are coherent $\mathscr{D}_X$-Modules and faithfully flat over $\mathscr{D}_Y$. We define for a left $\mathscr{D}_X$-Module $\mathscr{M}$,

$$\mathscr{M}_Y=\mathscr{D}_{Y\to X}\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathscr{M}=\mathscr{O}_Y\overset{L}{\underset{\mathscr{O}_X}{\otimes}}\mathscr{M}.$$

Theorem 4.1. *If $\mathscr{M}$ is a holonomic $\mathscr{D}_X$-Module, then $\mathscr{T}or_k^{\mathscr{O}_X}(\mathscr{O}_Y,\mathscr{M})=\mathscr{T}or_k^{\mathscr{D}_X}\cdot(\mathscr{D}_{Y\to X},\mathscr{M})$ is a holonomic $\mathscr{D}_Y$-Module for any k.*

This theorem is a consequence of the following propositions.

Proposition 4.2. *If $\mathscr{M}$ is a coherent $\mathscr{D}_X$-Module whose support is contained in Y, then we have*

$$\mathscr{M}=\mathscr{D}_{X\leftarrow Y}\underset{\mathscr{D}_Y}{\otimes}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_{X\leftarrow Y};\mathscr{M}),$$

$\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_{X\leftarrow Y};\mathscr{M})$ is a coherent $\mathscr{D}_Y$-Module and $\mathscr{E}xt^j_{\mathscr{D}_X}(\mathscr{D}_{X\leftarrow Y};\mathscr{M})=0$ for $j\neq 0$. If, moreover, $\mathscr{M}$ is holonomic, so is $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_{X\leftarrow Y};\mathscr{M})$. See [5].

Proposition 4.3. $\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_{X\leftarrow Y};\mathbb{R}\Gamma_{[Y]}(\mathscr{M}))[l]=\mathscr{D}_{Y\to X}\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathscr{M}$ *for any $\mathscr{D}_X$-Module $\mathscr{M}$.*

Proof. Since

$$\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_{X\leftarrow Y};\mathbb{R}\Gamma_{[Y]}(\mathscr{M}))$$
$$=\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_{X\leftarrow Y};\mathbb{R}\Gamma_{[Y]}(\mathscr{D}_X))\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathscr{M},$$

it is enough to show

$$\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_{X\leftarrow Y};\mathbb{R}\Gamma_{[Y]}(\mathscr{D}_X))[l]=\mathscr{D}_{Y\to X}.$$

Set $n=\dim X$. Then, by the definition,

$$\mathscr{D}_{X\leftarrow Y}=\mathscr{D}_X\overset{L}{\underset{\mathscr{O}_X}{\otimes}}(\Omega_Y^{n-l}\otimes(\Omega_X^n)^{\otimes -1}).$$

Hence, we have

$$\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_{X\leftarrow Y};\mathbb{R}\Gamma_{[Y]}(\mathscr{D}_X))[l]=\mathbb{R}\mathscr{H}om_{\mathscr{O}_X}(\Omega_Y^{n-l}\otimes(\Omega_X^n)^{\otimes -1};\mathbb{R}\Gamma_{[Y]}(\mathscr{D}_X))[l].$$

Since $\Omega_Y^{n-l}\otimes(\Omega_X^n)^{\otimes -1}$ is a coherent $\mathscr{O}_X$-Module supported on Y, we have

$$\mathbb{R}\mathscr{H}om_{\mathscr{O}_X}(\Omega_Y^{n-l}\otimes(\Omega_X^n)^{\otimes -1};\mathbb{R}\Gamma_{[Y]}(\mathscr{D}_X))=\mathbb{R}\mathscr{H}om_{\mathscr{O}_X}(\Omega_Y^{n-l}\otimes(\Omega_X^n)^{\otimes -1};\mathscr{D}_X)$$

and the last term equals

$$\mathbb{R}\mathscr{H}om_{\mathcal{O}_X}(\Omega_Y^{n-l}\otimes(\Omega_X^n)^{\otimes-1};\mathcal{O}_X)\overset{L}{\underset{\mathcal{O}_X}{\otimes}}\mathscr{D}_X.$$

Since $\mathscr{E}xt^j_{\mathcal{O}_X}(\mathcal{O}_Y;\mathcal{O}_X)=(\Omega_Y^{n-l})^{\otimes-1}\otimes\Omega_X^n$ for $j=l$ and vanishes for $j\neq l$, we have

$$\mathbb{R}\mathscr{H}om_{\mathcal{O}_X}(\Omega_Y^{n-l}\otimes(\Omega_X^n)^{\otimes-1};\mathcal{O}_X)=\mathcal{O}_Y[-l].$$

Thus we obtain

$$\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_{X\leftarrow Y};\mathbb{R}\Gamma_{[Y]}(\mathscr{D}_X))[l]=\mathcal{O}_Y\overset{L}{\underset{\mathcal{O}_X}{\otimes}}\mathscr{D}_X=\mathscr{D}_{Y\to X}.\quad \text{Q.E.D.}$$

Now, we can prove Theorem 4.1.

By Theorem 1.4, $\mathscr{H}^k_{[Y]}(\mathscr{M})$ are holonomic when $\mathscr{M}$ is holonomic. Then, by Proposition 4.2, $\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_{X\leftarrow Y};\mathbb{R}\Gamma_{[Y]}(\mathscr{M}))$ has holonomic $\mathscr{D}_Y$-Modules as cohomologies, Hence, Theorem 4.1 follows immediately from Proposition 4.3.

4.3. Suppose that $f\colon Y\to X$ is a holomorphic map.

Theorem 4.4. *If $\mathscr{M}$ is a holonomic $\mathscr{D}_X$-Module, then*

$$\mathscr{T}or_k^{f^{-1}\mathcal{O}_X}(\mathcal{O}_Y,f^{-1}\mathscr{M})=\mathscr{T}or_k^{\mathscr{D}_X}(\mathscr{D}_{Y\to X},f^{-1}\mathscr{M})$$

is a holonomic $\mathscr{D}_Y$-Module.

Proof. Let $\mathscr{N}$ be the holonomic system $\mathcal{O}_Y\hat{\otimes}\mathscr{M}$. Then $\mathscr{N}$ is a holonomic $\mathscr{D}_{Y\times X}$-Module. Identifying Y with the graph of f, we shall prove $\mathscr{T}or_k^{\mathscr{D}_X}(\mathscr{D}_{Y\to X},\mathscr{M})=\mathscr{T}or_k^{\mathscr{D}_{Y\times X}}(\mathscr{D}_{Y\to Y\times X},\mathscr{N})$. This implies immediately the desired result.

Lemma 4.5. $\mathscr{D}_{Y\to Y\times X}\overset{L}{\underset{\mathscr{D}_{Y\times X}}{\otimes}}(\mathcal{O}_Y\underset{\mathbb{C}}{\hat{\otimes}}\mathscr{M})=\mathscr{D}_{Y\to X}\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathscr{M}.$

Proof. Let p_1 and p_2 be the projection from $Y\times X$ onto Y and X, respectively.

$$\mathcal{O}_Y\underset{\mathbb{C}}{\hat{\otimes}}\mathscr{M}=\mathscr{D}_{Y\times X}\underset{p_1^{-1}\mathscr{D}_Y\otimes p_2^{-1}\mathscr{D}_X}{\otimes}(p_1^{-1}\mathcal{O}_Y\otimes p_2^{-1}\mathscr{M}).$$

Thus, we have

$$\begin{aligned}\mathscr{D}_{Y\to Y\times X}\overset{L}{\underset{\mathscr{D}_{Y\times X}}{\otimes}}(\mathcal{O}_Y\underset{\mathbb{C}}{\hat{\otimes}}\mathscr{M})&=\mathscr{D}_{Y\to Y\times X}\overset{L}{\underset{p_1^{-1}\mathscr{D}_Y\otimes p_2^{-1}\mathscr{D}_X}{\otimes}}(p_1^{-1}\mathcal{O}_Y\otimes p_2^{-1}\mathscr{M})\\&=(\mathscr{D}_{Y\to Y\times X}\overset{L}{\underset{p_1^{-1}\mathscr{D}_Y}{\otimes}}p_1^{-1}\mathcal{O}_Y)\overset{L}{\underset{p_2^{-1}\mathscr{D}_X}{\otimes}}p_2^{-1}\mathscr{M}.\end{aligned}$$

Thus, it is enough to show

$$\mathscr{D}_{Y\to Y\times X}\overset{L}{\underset{p_1^{-1}\mathscr{D}_Y}{\otimes}}p_1^{-1}\mathcal{O}_Y=\mathscr{D}_{Y\to X}.$$

It is easy to see

$$\mathscr{D}_{Y\times X}\overset{L}{\underset{p_1^{-1}\mathscr{D}_Y}{\otimes}}p_1^{-1}\mathcal{O}_Y=\mathcal{O}_{Y\times X}\underset{p_2^{-1}\mathcal{O}_X}{\otimes}p_2^{-1}\mathscr{D}_X.$$

We have

$$\mathscr{D}_{Y\to Y\times X} \underset{p_1^{-1}\mathscr{D}_Y}{\overset{L}{\otimes}} p_1^{-1}\mathcal{O}_Y = (\mathcal{O}_Y \underset{\mathcal{O}_{Y\times X}}{\overset{L}{\otimes}} \mathscr{D}_{Y\times X}) \underset{p_1^{-1}\mathscr{D}_Y}{\overset{L}{\otimes}} p_1^{-1}\mathcal{O}_Y$$
$$= \mathcal{O}_Y \underset{\mathcal{O}_{Y\times X}}{\overset{L}{\otimes}} (\mathscr{D}_{Y\times X} \underset{p_1^{-1}\mathscr{D}_Y}{\overset{L}{\otimes}} p_1^{-1}\mathcal{O}_Y)$$
$$= \mathcal{O}_Y \underset{\mathcal{O}_{Y\times X}}{\overset{L}{\otimes}} (\mathcal{O}_{Y\times X} \underset{p_2^{-1}\mathcal{O}_X}{\otimes} p_2^{-1}\mathscr{D}_X) = \mathcal{O}_Y \underset{f^{-1}\mathcal{O}_X}{\otimes} f^{-1}\mathscr{D}_X = \mathscr{D}_{Y\to X}. \quad \text{Q.E.D.}$$

4.4. We shall prove here the tensor products of two holonomic systems are holonomic.

Theorem 4.6. *Let $\mathscr{M}$ and $\mathscr{N}$ be two holonomic $\mathscr{D}_X$-Modules; then $\mathscr{T}or_k^{\mathcal{O}_X}(\mathscr{M}, \mathscr{N})$ is a holonomic $\mathscr{D}_X$-Module for any k.*

Proof. First we shall prove

Proposition 4.7. *For two $\mathscr{D}_X$-Modules $\mathscr{M}$ and $\mathscr{N}$, we have*

$$\mathscr{M} \underset{\mathcal{O}_X}{\overset{L}{\otimes}} \mathscr{N} = \mathscr{D}_{X\to X\times X} \underset{\mathscr{D}_{X\times X}}{\overset{L}{\otimes}} (\mathscr{M} \hat{\otimes} \mathscr{N}),$$

where

$$\mathscr{M} \hat{\otimes} \mathscr{N} = \mathscr{D}_{X\times X} \underset{p_1^{-1}\mathscr{D}_X \otimes p_2^{-1}\mathscr{D}_X}{\otimes} (p_1^{-1}\mathscr{M} \otimes p_2^{-1}\mathscr{N})$$

with the first and the second projections p_1 and p_2 from $X\times X$ onto X.

Proof. Since

$$\mathscr{D}_{X\times X} = \mathcal{O}_{X\times X} \underset{p_1^{-1}\mathcal{O}_X \otimes p_2^{-1}\mathcal{O}_X}{\otimes} (p_1^{-1}\mathscr{D}_X \otimes p_2^{-1}\mathscr{D}_X),$$

we have

$$\mathscr{M} \hat{\otimes} \mathscr{N} = \mathcal{O}_{X\times X} \underset{p_1^{-1}\mathcal{O}_X \otimes p_2^{-1}\mathcal{O}_X}{\otimes} (p_1^{-1}\mathscr{M} \otimes p_2^{-1}\mathscr{N}).$$

Therefore,

$$\mathscr{D}_{X\to X\times X} \underset{\mathscr{D}_{X\times X}}{\overset{L}{\otimes}} (\mathscr{M} \hat{\otimes} \mathscr{N}) = \mathcal{O}_X \underset{\mathcal{O}_{X\times X}}{\overset{L}{\otimes}} (\mathscr{M} \hat{\otimes} \mathscr{N}) = \mathcal{O}_X \underset{p_1^{-1}\mathcal{O}_X \underset{\mathbb{C}}{\otimes} p_2^{-1}\mathcal{O}_X}{\overset{L}{\otimes}} (p_1^{-1}\mathscr{M} \underset{\mathbb{C}}{\otimes} p_2^{-1}\mathscr{N})$$
$$= \mathscr{M} \underset{\mathcal{O}_X}{\overset{L}{\otimes}} \mathscr{N}. \quad \text{Q.E.D.}$$

Theorem 4.6 is a consequence of this proposition and Theorem 4.1.

4.5. We know that $\mathscr{E}xt^j_{\mathscr{D}_X}(\mathscr{M}; \mathscr{D}_X^\infty \otimes_{\mathscr{D}_X} \mathscr{N})$ is a constructible sheaf for any holonomic $\mathscr{D}_X$-Modules $\mathscr{M}$ and $\mathscr{N}$ [5]. Here a sheaf $\mathscr{F}$ is called constructible if there is a stratification of X on each of whose strata $\mathscr{F}$ is locally constant of finite rank. $\mathscr{D}_X^\infty$ is the sheaf of the differential operators of infinite order. Therefore, in particular, $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}; \mathscr{N})$ has a finite-dimensional stalk at each point. Furthermore, by using the previous results, we can prove the following results.

Theorem 4.8. *Let $\mathcal{M}$ and $\mathcal{N}$ be two holonomic $\mathcal{D}_X$-Modules. Then $\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M};\mathcal{N})$ is a constructible sheaf for any j.*

Proof. This is a consequence of Lemma 1.8 because

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M};\mathcal{N})=\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\Omega^n_X;\mathbb{R}\mathcal{H}om(\mathcal{M};\mathcal{D}_X)\overset{L}{\underset{\mathcal{O}_X}{\otimes}}\mathcal{N})[n]$$

and

$$\mathbb{R}\mathcal{H}om(\mathcal{M},\mathcal{D}_X)\overset{L}{\underset{\mathcal{O}_X}{\otimes}}\mathcal{N}$$

has holonomic $\mathcal{D}_X$-Modules as cohomologies. Therefore the theorem follows from the result in [5]. Q.E.D.

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