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The invariant holonomic system on a semisimple Lie algebra

R. Hotta[1], and M. Kashiwara[2]

[1] Mathematical Institute, Tohoku University, Sendai 980, Japan
[2] R.I.M.S., Kyoto University, Kyoto 606, Japan

Introduction

Our motivation is to study the geometric theory of Weyl group representations, so called Springer's representations (see [21] and its references), from the view point of holonomic systems on Lie algebras. In this attempt, the system of differential equations defining invariant eigendistributions, which was extensively investigated by Harish-Chandra, occurs quite naturally as the regular holonomic system corresponding to the intersection cohomology complex defining Springer's representations, through the Riemann-Hilbert correspondence. Thus we obtain a decomposition of this holonomic system according to the action of the Weyl group, which is also related to the decomposition according to the monodromies. Through the Fourier transform, this decomposition gives an "analytic" proof of a recent theorem of Borho and MacPherson ([4], [5]), which has been first proved by using a deep theorem of Bernstein-Beilinson-Deligne-Gabber. Secondly, applying this result, we can investigate structures of solutions to this system of differential equations. In particular, we can extend, in a unified way, a recent result of Barbasch and Vogan ([7], [8]) on the Fourier transforms of the nilpotent orbital integrals.

We are going into more details. Let $\mathfrak{g}$ be a complex semisimple Lie algebra with connected group G. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and denote by $\mathfrak{h}^*$ the dual space of $\mathfrak{h}$ which is often identified with $\mathfrak{h}$ through the Killing form. Let $S(\mathfrak{g})^G$ (resp. $\mathbb{C}[\mathfrak{g}]^G$) be the ring of G-invariant symmetric tensors (resp. G-invariant polynomials) on $\mathfrak{g}$. For $\lambda\in\mathfrak{h}^*$, we consider the following systems of differential equations:

$$\mathcal{M}_\lambda:\begin{cases}\langle [A,x],\partial_x\rangle u_\lambda=0 & (A\in\mathfrak{g})\\ (P(x)-P(\lambda))u_\lambda=0 & (P\in\mathbb{C}[\mathfrak{g}]^G)\end{cases}$$

and

$$\mathcal{M}_\lambda^F:\begin{cases}\langle [A,x],\partial_x\rangle u_\lambda^F=0 & (A\in\mathfrak{g})\\ (Q(\partial_x)-Q(\lambda))u_\lambda^F=0 & (Q\in S(\mathfrak{g})^G)\end{cases}$$

where $\langle [A,x], \partial_x \rangle$ denotes the vector field whose tangent vector at x equals $[A,x] \in \mathfrak{g}$. Thus $\mathcal{M}_\lambda^F$ is a defining system of invariant eigendistributions and the Fourier transform of $\mathcal{M}_\lambda$. These systems will be known to be regular holonomic. Our aim is to study their structures as $\mathcal{D}_\mathfrak{g}$-Modules, mainly aspects related to the Weyl group action. This will be done in §6. However, for this, we first consider the total deformation of these $\mathcal{M}_\lambda$ with parameter $\lambda \in \mathfrak{h}^* \simeq \mathfrak{h}$.

In §4, we shall consider the $\mathcal{D}_{\mathfrak{g}\times\mathfrak{h}}$-Module $\mathcal{N}$ defined by the following system on $\mathfrak{g}\times\mathfrak{h}$

$$\mathcal{N}: \begin{cases} \langle [A,x], \partial_x \rangle \tilde{u} = 0 & (A \in \mathfrak{g}) \\ (P(x) - P(t))\tilde{u} = 0 & (P \in \mathbb{C}[\mathfrak{g}]^G) \\ (Q(\partial_x) - Q(-\partial_t))\tilde{u} = 0 & (Q \in S(\mathfrak{g})^G). \end{cases}$$

Here $\mathcal{D}_{\mathfrak{g}\times\mathfrak{h}}$ is the sheaf of linear differential operators with coefficients in rational functions. We remark, in this paper, we mainly consider $\mathcal{D}$-Modules in the algebraic category (see §1). Let $\mathcal{B}$ be the flag manifold consisting of all Borel subalgebras of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ the incidence subvariety of $\mathcal{B}\times\mathfrak{g}$, i.e., $\tilde{\mathfrak{g}} = \{(\mathfrak{b}', x) \in \mathcal{B}\times\mathfrak{g} \mid x \in \mathfrak{b}'\}$. Denote by ρ: $\tilde{\mathfrak{g}} \to \mathfrak{g}$ the projection to the second factor. Fixing a Borel subalgebra $\mathfrak{b}$ containing the Cartan subalgebra $\mathfrak{h}$, we have the smooth map θ: $\tilde{\mathfrak{g}} \to \mathfrak{g}$ defined by $\theta((g\mathfrak{b}, x)) = g^{-1}x \bmod \mathfrak{n}$ $(g \in G)$ where $\mathfrak{n}$ is the nilpotent radical of $\mathfrak{b}$ and $\mathfrak{h}$ is identified with $\mathfrak{b}/\mathfrak{n}$. We thus have the commutative diagram

$$\begin{tikzcd} \tilde{\mathfrak{g}} \arrow[r, "\rho"] \arrow[d, "\theta"'] & \mathfrak{g} \arrow[d, "\kappa"] \\ \mathfrak{h} \arrow[r] & \mathfrak{h}/W \end{tikzcd}$$

where W is the Weyl group for $(\mathfrak{g}, \mathfrak{h})$ and κ is the invariant map. Using these maps, we consider the product map

$$f = \rho \times \theta \colon \tilde{\mathfrak{g}} \to \mathfrak{g} \times \mathfrak{h}$$

where $f(\tilde{\mathfrak{g}}) = \mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h} \subset \mathfrak{g}\times\mathfrak{h}$. As our first main result, we prove

$$\mathcal{N} \simeq \int_f \mathcal{O}_{\tilde{\mathfrak{g}}} \simeq {}^\pi\mathcal{B}_{f(\tilde{\mathfrak{g}})|\mathfrak{g}\times\mathfrak{h}}$$

which implies that $\mathcal{N}$ is a simple completely regular (for definition, see 1.5) holonomic $\mathcal{D}_{\mathfrak{g}\times\mathfrak{h}}$-Module (Theorems 4.1, 4.2 and 5.1). Here the symbol $\int_f$ is the functor integration along fibers, and ${}^\pi\mathcal{B}_{f(\tilde{\mathfrak{g}})|\mathfrak{g}\times\mathfrak{h}}$ is the minimal extension of $\mathcal{B}_{f(\tilde{\mathfrak{g}})_{rs}|\mathfrak{g}_{rs}\times\mathfrak{h}}$ ($\mathfrak{g}_{rs}$ is the set of regular semisimple elements in $\mathfrak{g}$ and $f(\tilde{\mathfrak{g}})_{rs} = f(\tilde{\mathfrak{g}}) \cap \mathfrak{g}_{rs}\times\mathfrak{h}$).

The W-action on $\mathfrak{g}\times\mathfrak{h}$, defined by that on $\mathfrak{h}$, makes $f(\tilde{\mathfrak{g}})$ stable and hence gives rise to the W-action on the $\mathcal{D}_{\mathfrak{g}\times\mathfrak{h}}$-Modules $\mathcal{N} \simeq \int_f \mathcal{O}_{\tilde{\mathfrak{g}}}$. In §5, we shall investigate this W-action. Borho-MacPherson's result [4] will be clarified as follows. Note that the following two isomorphisms (Prop. 4.8.1);

$$\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}} \simeq \int_p \mathcal{N} \quad (p = \mathrm{pr}_1 \colon \mathfrak{g}\times\mathfrak{h} \to \mathfrak{g}),$$

$$\int_{\rho^{-1}(\mathrm{N})\to\mathfrak{g}} \mathcal{O}_{\rho^{-1}(\mathrm{N})} = j^* \mathcal{N} \quad (j \colon \mathfrak{g} = \mathfrak{g}\times\{0\} \hookrightarrow \mathfrak{g}\times\mathfrak{h}),$$

where the both are single $\mathscr{D}_{\mathfrak{g}}$-Modules ($\mathbf{N}$ is the nilpotent variety of $\mathfrak{g}$). Considering the Fourier transform $(*)^F$ on $\mathfrak{g}$, we have

$$(\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}})^F \simeq (\int_{\rho^{-1}(\mathbf{N})\to\mathfrak{g}} \mathcal{O}_{\rho^{-1}(\mathbf{N})}) \otimes \mathrm{sgn}$$

as W-$\mathscr{D}_{\mathfrak{g}}$-Modules, where sgn is the sign representation of W (Theorem 5.2). Since

$$\mathrm{DR}(\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}}) = \mathbb{R}\rho_* \mathbb{C}_{\tilde{\mathfrak{g}}} \quad \text{and} \quad \mathrm{DR}(\int_{\rho^{-1}(\mathbf{N})\to\mathfrak{g}} \mathcal{O}_{\rho^{-1}(\mathbf{N})}) = \mathbb{R}\rho_* \mathbb{C}_{\tilde{\mathfrak{g}}}|_{\mathbf{N}}[-\operatorname{rank}\mathfrak{g}]$$

through the Riemann-Hilbert correspondence DR, we have the natural decompositions into simple constituents of the both objects (Theorem 5.3), which recovers Borho-MacPherson's result. We also see why the sign representation is involved in various situations in Springer's representations.

In § 6, we shall relate the previous $\mathscr{D}_{\mathfrak{g}}$-Modules $\mathscr{M}_\lambda$ and $\mathscr{M}_\lambda^F$ with $\mathscr{N}$ using harmonic polynomials. Let $\mathfrak{h}(\lambda)$ be the span of the coroots orthogonal to λ in $\mathfrak{h}$ and $W(\lambda) = \{w \in W \mid w\lambda = \lambda\}$. Let $\mathscr{H}(\mathfrak{h}(\lambda))$ be the space of harmonic polynomials on $\mathfrak{h}(\lambda)$ with respect to the small Weyl group $W(\lambda)$. We can then define a natural isomorphism

$$\Phi_\lambda \colon \mathscr{M}_\lambda \xrightarrow{\sim} \mathrm{Hom}_{W(\lambda)}(\mathscr{H}(\mathfrak{h}(\lambda)), \mathscr{N}|_{t=\lambda})$$

(Theorem 6.1). At this point, we use Harish-Chandra's ancient result on $\mathscr{M}_\lambda^F$ (Theorem 6.7.2) by which he derived the famous regularity theorem on invariant eigendistributions. We, however, know some method which gives our results without relying on his result, at least, in the classical case (see 6.6).

We now consider the particular case $\lambda = 0$. Using the earlier decomposition of

$$\mathscr{N}|_{t=0} \simeq \int_{\rho^{-1}(\mathbf{N})\to\mathfrak{g}} \mathcal{O}_{\rho^{-1}(\mathbf{N})}, \qquad \text{we have } \mathscr{M}_0 \simeq \bigoplus_{\chi\in\hat{W}} \mathscr{M}_\chi^{d(\chi)}$$

where $\hat{W}$ is the set of the equivalence classes of irreducible representations of W and $d(\chi)$ is the degree of $\chi \in \hat{W}$. Here $\mathscr{M}_\chi$ is a simple $\mathscr{D}_{\mathfrak{g}}$-Module. We now take the Fourier transform

$$\mathscr{M}_0^F \simeq \bigoplus_{\chi\in\hat{W}} (\mathscr{M}_\chi^F)^{d(\chi)}.$$

Then $\mathscr{M}_0^F$ is a regular holonomic system which satisfies the asymptotics of distribution characters ([6]). In the above, $\mathscr{M}_\chi^F$ is a system with monodromy representation χ. Using this result, in §§ 7 and 8, we shall generalize some results of Barbasch-Vogan ([7], [8]) and King ([17]). For instance, let $\mathbf{O}$ be a nilpotent orbit of $\mathfrak{g}$. The nilpotent orbital integral

$$\mu_{\mathbf{O}}(f) = \int_{\mathbf{O}} f(x)\, d\mu_{\mathbf{O}}(x)$$

then defines the invariant measure $\mu_{\mathbf{O}}$ on $\mathfrak{g}$. Then the Fourier transform $\hat{\mu}_{\mathbf{O}}$ is explicitly given (up to constant multiplication) in terms of a harmonic polynomials on $\mathfrak{h} \times \mathfrak{h}$ using Springer's correspondence (Theorem 8.2). For "special" $\mathbf{O}$ this result has been proved in [7], [8]. We shall also deduce some results on real Lie algebras (§ 8).

We are grateful to J. Sekiguchi who has treated related problems in other aspects and discussion with whom has helped us in some point (6.6).

§ 1. Holonomic systems on algebraic varieties

1.1. Let $(X, \mathcal{O}_X)$ be a smooth algebraic variety defined over $\mathbb{C}$. We denote by $(X_{an}, \mathcal{O}_{X_{an}})$ the underlying complex manifold. Let ι_X denote the morphism of $\mathbb{C}$-ringed spaces

$$\iota = \iota_X: (X_{an}, \mathcal{O}_{X_{an}}) \to (X, \mathcal{O}_X). \tag{1.1}$$

The sheaf $\mathcal{D}_X$ of differential operators on X and the sheaf $\mathcal{D}_{X_{an}}$ of differential operators on X_{an} are related by

$$\mathcal{D}_{X_{an}} = \mathcal{O}_{X_{an}} \otimes_{\iota^{-1}\mathcal{O}_X} \iota^{-1}\mathcal{D}_X = \iota^{-1}\mathcal{D}_X \otimes_{\iota^{-1}\mathcal{O}_X} \mathcal{O}_{X_{an}}. \tag{1.2}$$

For a left $\mathcal{D}_X$-Module $\mathcal{M}$, we shall write $\mathcal{M}_{an}$ for

$$\mathcal{D}_{X_{an}} \otimes_{\iota^{-1}\mathcal{D}_X} \iota^{-1}\mathcal{M} = \mathcal{O}_{X_{an}} \otimes_{\iota^{-1}\mathcal{O}_X} \iota^{-1}\mathcal{M}.$$

A coherent $\mathcal{D}_X$-Module $\mathcal{M}$ is called holonomic (resp. regular holonomic, cf. [16]) if so is $\mathcal{M}_{an}$. As in the case of coherent $\mathcal{D}_{X_{an}}$-Modules, we can define the characteristic variety of a coherent $\mathcal{D}_X$-Module $\mathcal{M}$ as a Zariski closed subset of the cotangent bundle T^*X of X, which we shall denote by $\mathrm{Ch}(\mathcal{M})$. The characteristic variety of $\mathcal{M}_{an}$ coincides with $\iota_{T^*X}^{-1}(\mathrm{Ch}(\mathcal{M}))$. The following proposition immediately follows from GAGA [20].

Theorem 1.1. *Assume X to be smooth and proper over $\mathbb{C}$. Then we have*
(1) *For coherent $\mathcal{D}_X$-Modules $\mathcal{M}$ and $\mathcal{N}$, we have*

$$\mathbb{R}\Gamma(X; \mathbb{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N})) \cong \mathbb{R}\Gamma(X_{an}; \mathbb{R}\,\mathcal{H}om_{\mathcal{D}_{X_{an}}}(\mathcal{M}_{an}, \mathcal{N}_{an})).$$

(2) *The category of coherent $\mathcal{D}_X$-Modules and that of coherent $\mathcal{D}_{X_{an}}$-Modules generated by a coherent sub-$\mathcal{O}_{X_{an}}$-Module are equivalent.*

1.2. For a quasi-coherent $\mathcal{O}_X$-Module $\mathcal{F}$ and a closed subset Y of X, we have (see [9], [13]).

$$\mathbb{R}\Gamma_{[Y_{an}]}(\mathcal{F}_{an}) = \mathbb{R}\Gamma_Y(\mathcal{F})_{an} \tag{1.2.1}$$

and

$$\mathbb{R}\Gamma_{[X_{an}|Y_{an}]}(\mathcal{F}_{an}) = \mathbb{R}\Gamma_{X|Y}(\mathcal{F})_{an}. \tag{1.2.2}$$

In particular, for a holonomic (resp. regular holonomic) $\mathcal{D}_X$-Module $\mathcal{M}$, $\mathcal{H}^j_Y(\mathcal{M})$ and $\mathcal{H}^j_{X|Y}(\mathcal{M})$ are also holonomic (resp. regular holonomic).

1.3. Let $f: X \to Y$ be a morphism of smooth algebraic varieties X and Y defined over $\mathbb{C}$. We define the sheaves $\mathcal{D}_{X\to Y}$ and $\mathcal{D}_{Y\leftarrow X}$ on X by

$$\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y, \quad \mathcal{D}_{Y\leftarrow X} = f^{-1}(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}) \otimes_{f^{-1}\mathcal{O}_Y} \Omega_X, \tag{1.3.1}$$

where Ω_Y and Ω_X are the invertible sheaves of the highest degree forms on Y and X, respectively. As in the analytic case, $\mathcal{D}_{X\to Y}$ is a $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-bi-Module and $\mathcal{D}_{Y\leftarrow X}$ is an $(f^{-1}\mathcal{D}_Y, \mathcal{D}_X)$-bi-Module. For a $\mathcal{D}_Y$-Module $\mathcal{N}$, we set

$$\mathbb{L}f^* \mathscr{N} = \mathscr{D}_{X\to Y} \otimes^{\mathbb{L}}_{f^{-1}\mathscr{D}_Y} f^{-1}\mathscr{N} = \mathscr{O}_X \otimes^{\mathbb{L}}_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{N}$$

and for a $\mathscr{D}_X$-Module $\mathscr{M}$, we set

$$\textstyle\int_f \mathscr{M} = \mathbb{R}f_*(\mathscr{D}_{Y\leftarrow X} \otimes^{\mathbb{L}}_{\mathscr{D}_X} \mathscr{M}),$$

where $\otimes^{\mathbb{L}}_{\mathscr{D}_X}$ denotes the left derived functor of $\otimes_{\mathscr{D}_X}$.

Theorem 1.2 ([19], [16]). (1) *If $\mathscr{N}$ is a holonomic (resp. regular holonomic) $\mathscr{D}_Y$-Module, then $\mathscr{H}^j(\mathbb{L}f^*\mathscr{N})$ is a holonomic (resp. regular holonomic) $\mathscr{D}_X$-Module for any j.*
(2) *If f is proper and if $\mathscr{M}$ is a coherent (resp. holonomic, regular holonomic) $\mathscr{D}_X$-Module, then any cohomology group of $\int_f \mathscr{M}$ is a coherent (resp. holonomic, resp. regular holonomic) $\mathscr{D}_Y$-Module. Moreover we have $\mathrm{Ch}(\mathscr{H}^j(\int_f \mathscr{M})) \subset \varpi\rho^{-1}(\mathrm{Ch}(\mathscr{M}))$, where ϖ is the projection $X\times_Y T^*Y \to T^*Y$ and ρ is the morphism $X\times_Y T^*Y\to T^*X$.*

1.4. For a holonomic $\mathscr{D}_X$-Module $\mathscr{M}$ and for an irreducible component Λ of the characteristic variety of $\mathscr{M}$, we can define the *multiplicity*, denoted by $\mathrm{mult}_\Lambda(\mathscr{M})$, of $\mathscr{M}$ along Λ. (See [14]). We shall call the algebraic cycle $\sum_\Lambda \mathrm{mult}_\Lambda(\mathscr{M})\Lambda$ the *characteristic cycle* of $\mathscr{M}$ and we shall denote it by $\mathbf{Ch}(\mathscr{M})$.
If $0\to\mathscr{M}'\to\mathscr{M}\to\mathscr{M}''\to 0$ is an exact sequence of holonomic $\mathscr{D}_X$-Modules, then we have

$$\mathbf{Ch}(\mathscr{M}) = \mathbf{Ch}(\mathscr{M}') + \mathbf{Ch}(\mathscr{M}''). \tag{1.4.1}$$

As shown in [14], we have

Theorem 1.3. *Let $f\colon X\to Y$ be a morphism of smooth algebraic varieties X and Y, and let*

$$\varpi\colon X\times_Y T^*Y\to T^*Y \quad \text{and} \quad \rho\colon X\times_Y T^*Y\to T^*X$$

be the canonical morphisms.
(1) *Let $\mathscr{N}$ be a holonomic $\mathscr{D}_Y$-Module. If $\varpi^{-1}(\mathrm{Ch}(\mathscr{N}))$ is finite over T^*X, then we have $\mathscr{H}^j(\mathbb{L}f^*\mathscr{N})=0$ for $j\neq 0$ and $\mathbf{Ch}(f^*\mathscr{N})=\rho\varpi^{-1}(\mathbf{Ch}(\mathscr{N}))$.*
(2) *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-Module. If $\rho^{-1}(\mathrm{Ch}(\mathscr{M}))$ is finite over T^*Y, then we have*

$$\mathscr{H}^j(\textstyle\int_f\mathscr{M})=0 \ \text{ for } j\neq 0 \quad \text{and} \quad \mathbf{Ch}(\mathscr{H}^0(\int_f\mathscr{M}))=\varpi\rho^{-1}(\mathbf{Ch}(\mathscr{M})).$$

1.5. Let $j\colon X\hookrightarrow \bar{X}$ be a compactification of a smooth algebraic variety X; i.e. j is an open embedding and $\bar{X}$ is proper and smooth over $\mathbb{C}$. A holonomic $\mathscr{D}_X$-Module $\mathscr{M}$ is called *completely regular* if $j_*\mathscr{M}$ is a regular holonomic $\mathscr{D}_{\bar{X}}$-Module. This definition does not depend on a choice of a compactification j. In order to see this, for another compactification $j'\colon X\hookrightarrow\bar{X}'$, we take a third compactification $j''\colon X\hookrightarrow\bar{X}''$ and morphisms $f\colon\bar{X}''\to\bar{X}$ and $f'\colon\bar{X}''\to\bar{X}'$ such that $j=f\circ j''$ and $j'=f'\circ j''$. If $j_*\mathscr{M}$ is regular holonomic, $f^*j_*\mathscr{M}$ is regular holonomic. Therefore

$$\mathscr{M}'=\mathscr{H}^0(\textstyle\int_{f'} f^*j_*\mathscr{M}) \quad \text{and} \quad j'_*\mathscr{M}=\mathscr{H}^0_{\bar{X}'|\bar{X}'-X}(\mathscr{M}')$$

are also regular holonomic.

Theorem 1.4. (1) *If $\mathcal{M}$ and $\mathcal{N}$ are completely regular holonomic $\mathcal{D}_X$-Modules, then*

$$\mathbb{R}\iota_* \mathbb{R}\mathcal{H}om_{\mathcal{D}_{X_{an}}}(\mathcal{M}_{an}, \mathcal{N}_{an}) = \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}).$$

(2) *The category of completely regular holonomic $\mathcal{D}_X$-Modules is equivalent to that of regular holonomic $\mathcal{D}_{X_{an}}$-Modules whose characteristic variety is Zariski closed.*

Proof. In order to prove (1), we have to show

$$\mathbb{R}\Gamma(\iota^{-1}U; \mathbb{R}\mathcal{H}om_{\mathcal{D}_{X_{an}}}(\mathcal{M}_{an}, \mathcal{N}_{an})) = \mathbb{R}\Gamma(U; \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}))$$

for any open subset U of X. By replacing U by X we may assume from the beginning $U = X$. Let j: $X \hookrightarrow \bar{X}$ be a compactification of X and suppose that $\bar{X} - X$ is a divisor. Set $\tilde{\mathcal{N}} = j_* \mathcal{N}$ and $\tilde{\mathcal{M}} = j_* \mathcal{M}$. By the definition they are regular holonomic. We have

$$\text{(1.5.1)} \qquad \mathbb{R}\Gamma(\bar{X}; \mathbb{R}\mathcal{H}om_{\mathcal{D}_{\bar{X}}}(\tilde{\mathcal{M}}, \tilde{\mathcal{N}})) \xrightarrow{\sim} \mathbb{R}\Gamma(X; \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}))$$

and

$$\text{(1.5.2)} \quad \mathbb{R}\Gamma(\bar{X}_{an}; \mathbb{R}\mathcal{H}om_{\mathcal{D}_{\bar{X}_{an}}}(\tilde{\mathcal{M}}_{an}, \tilde{\mathcal{N}}_{an})) \xrightarrow{\sim} \mathbb{R}\Gamma(X_{an}; \mathbb{R}\mathcal{H}om_{\mathcal{D}_{X_{an}}}(\mathcal{M}_{an}, \mathcal{N}_{an})).$$

The first is trivial and the second follows from Theorem 5.4.1 and Theorem 6.4.1 in [16]. On the other hand, Serre's GAGA [20] implies that the left hand sides of (1.5.1) and (1.5.2) are isomorphic. This shows (1).

The statement (2) follows from Theorem 1.1, Theorem 2.1 and (1).

Theorem 1.5. *Let f: $X \to Y$ be a morphism of smooth algebraic varieties X and Y.*
(1) *For a completely regular holonomic $\mathcal{D}_Y$-Module $\mathcal{N}$, $\mathcal{H}^j(\mathbb{L}f^* \mathcal{N})$ is also a completely regular $\mathcal{D}_X$-Module for any j.*
(2) *For a completely regular $\mathcal{D}_X$-Module $\mathcal{M}$, any cohomology group of*

$$\int_f \mathcal{M} = \mathbb{R}f_*(\mathcal{D}_{Y \leftarrow X} \otimes^{\mathbb{L}}_{\mathcal{D}_X} \mathcal{M})$$

is completely regular.

Proof. Let us embed f into $\bar{f}$: $\bar{X} \to \bar{Y}$ where $\bar{X}$ and $\bar{Y}$ are compactifications of X and Y, respectively. Then (1) follows from Theorem 1.2. Let j denote the embedding $X \hookrightarrow \bar{X}$. Then

$$\int_f \mathcal{M} = \mathbb{R}\bar{f}_*(\mathcal{D}_{\bar{Y} \leftarrow \bar{X}} \otimes^{\mathbb{L}}_{\mathcal{D}_{\bar{X}}} \mathbb{R}j_* \mathcal{M})|_Y.$$

Since any cohomology group of $\mathbb{R}j_* \mathcal{M}$ is regular holonomic, (2) follows from Theorem 1.2.

§ 2. Correspondence of holonomic systems and constructible sheaves

2.1. Let X be a complex manifold. We denote by $\mathrm{Mod}(\mathcal{D}_X)$ the category of $\mathcal{D}_X$-Modules and $\mathrm{RH}(X)$ the full subcategory of $\mathrm{Mod}(\mathcal{D}_X)$ consisting of regular holonomic $\mathcal{D}_X$-Modules. Let $D(\mathcal{D}_X)$ denote the derived category of $\mathrm{Mod}(\mathcal{D}_X)$ and let $D^b_{rh}(\mathcal{D}_X)$ the full subcategory of $D(\mathcal{D}_X)$ consisting of bounded complexes whose cohomology groups are regular holonomic.

Let $\mathrm{Mod}(X)$ denote the category of sheaves of $\mathbb{C}$-vector spaces on X and $D(X)$ its derived category.

We denote by $D_c^b(X)$ the full subcategory of $D(X)$ consisting of bounded complexes $\mathscr{F}^\bullet$ satisfying

(2.1.1) $\{\mathscr{H}^n(\mathscr{F}^\bullet)\}_n$ is constructible, i.e., there exists a decreasing sequence closed analytic subsets $X = X_0 \supset X_1 \supset \ldots$ such that $\cap X_i = \emptyset$ and that $\mathscr{H}^n(\mathscr{F}^\bullet)|_{X_j - X_{j+1}}$ is locally constant of finite rank for any j and n.

Let $\mathrm{Perv}(X)$ denote the full subcategory of $D_c^b(X)$ consisting of $\mathscr{F}^\bullet$ satisfying

(2.1.2) $$\operatorname{codim} \operatorname{Supp} \mathscr{H}^j(\mathscr{F}^\bullet) \geqq j$$

and

(2.1.3) $$\operatorname{codim} \operatorname{Supp} \mathscr{E}xt^j_{\mathbb{C}}(\mathscr{F}^\bullet, \mathbb{C}_X) \geqq j$$

where $\mathbb{C}_X$ is the constant sheaf on X with $\mathbb{C}$ as its stalk.

Then we have

Theorem 2.1. (1) $D^b_{\mathrm{rh}}(\mathscr{D}_X) \xrightarrow{\sim} D^b_c(X)$ *and* $\mathrm{RH}(X) \xrightarrow{\sim} \mathrm{Perv}(X)$ *by*

$$\mathrm{DR}_X\colon \mathscr{M}^\bullet \mapsto \mathbb{R}\,\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{M}^\bullet).$$

(2) *We denote by* $\mathrm{Sol}_X(\mathscr{M}) = \mathbb{R}\,\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$. *Then* Sol_X *and* DR_X *are related by*

$$\mathrm{Sol}_X(\mathscr{M}^\bullet) = \mathbb{R}\,\mathscr{H}om_{\mathbb{C}_X}(\mathrm{DR}_X(\mathscr{M}^\bullet), \mathbb{C}_X) \qquad \text{for } \mathscr{M}^\bullet \in D^b_{\mathrm{rh}}(\mathscr{D}_X).$$

Now, let X be a smooth algebraic variety. We denote by $\mathrm{RH}(X)$ the category of *completely regular* holonomic $\mathscr{D}_X$-Modules. We denote by $\mathrm{Perv}(X)$ the full subcategory of $\mathrm{Perv}(X_{\mathrm{an}})$ obtained by replacing (2.1.1) with the following condition.

(2.1.1′) $\{\mathscr{H}^n(\mathscr{F}^\bullet)\}_n$ is algebraically constructible i.e. (2.1.1) holds by choosing X_j to be Zariski closed.

Then we have

Theorem 2.2. $\mathrm{RH}(X) \xrightarrow{\sim} \mathrm{Perv}(X)$ *by* $\mathrm{DR}_X\colon \mathscr{M} \mapsto \mathrm{DR}_{X_{\mathrm{an}}}(\mathscr{M}_{\mathrm{an}})$.

2.2. Let X be a complex manifold.

For $\mathscr{M}^\bullet \in D^b_{\mathrm{rh}}(\mathscr{D}_X)$ we set

(2.2.1) $$\mathscr{M}^{\bullet *} = \mathbb{R}\,\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}^\bullet, \mathscr{D}_X) \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1}[\dim X].$$

Then $\mathscr{M}^\bullet \mapsto \mathscr{M}^{\bullet *}$ is a contravariant functor from $D^b_{\mathrm{rh}}(\mathscr{D}_X)$ into itself.

Proposition 2.2.1. (1) $\mathscr{M}^{\bullet **} = \mathscr{M}^\bullet$ *for* $\mathscr{M}^\bullet \in D^b_{\mathrm{rh}}(\mathscr{D}_X)$.
(2) $\mathrm{DR}_X(\mathscr{M}^{\bullet *}) = \mathrm{Sol}_X(\mathscr{M}^\bullet)$ *for* $\mathscr{M}^\bullet \in D^b_{\mathrm{rh}}(\mathscr{D}_X)$.
(3) $\mathscr{M} \mapsto \mathscr{M}^*$ *is an exact contravariant functor from* $\mathrm{RH}(X)$ *into itself.*
(4) $\mathbf{Ch}(\mathscr{M}) = \mathbf{Ch}(\mathscr{M}^*)$ *for any* $\mathscr{M} \in \mathrm{RH}(X)$.
(5) $*$ *commutes with integration; i.e., for a proper morphism* $f\colon X \to Y$ *and* $\mathscr{M}^\bullet \in D^b_{\mathrm{rh}}(\mathscr{D}_X)$, $\int_f \mathscr{M}^{\bullet *} = (\int_f \mathscr{M}^\bullet)^*$.

(6) * *commutes with non-characteristic pull-back; i.e. under the situation of* (1) *in Theorem* 1.3, $(f^* \mathcal{N})^* = f^*(\mathcal{N}^*)$.

2.3. Let X be a smooth complex manifold, and Y a closed analytic subset of X. Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_{X-Y}$-Module. Assume that $\mathcal{M}$ is extendable; i.e. there exists a coherent $\mathcal{D}_X$-Module $\tilde{\mathcal{M}}$ on X whose restriction to $X-Y$ is isomorphic to $\mathcal{M}$. This is equivalent to saying that $j_! \mathrm{DR}_{X-Y}(\mathcal{M})$ is contructible, where j is the embedding $X-Y \hookrightarrow X$.

Theorem 2.3. *Under the above assumption, there exists a regular holonomic $\mathcal{D}_X$-Module $\mathcal{M}'$ which satisfies the following conditions*

$$(2.3.1) \qquad \mathcal{M}'|_{X-Y} \simeq \mathcal{M}$$

$$(2.3.2) \qquad \mathcal{H}_Y^0(\mathcal{M}')=0 \quad \text{and} \quad \mathcal{H}_Y^0(\mathcal{M}'^*)=0.$$

Moreover, such an $\mathcal{M}'$ is unique up to isomorphism. (See [3].)

Definition 2.4. We call $\mathcal{M}'$ the *minimal extension* of $\mathcal{M}$ and denote it by ${}^{\pi}\mathcal{M}$.

It is easy to see that $\mathcal{M} \mapsto {}^{\pi}\mathcal{M}$ is a functor from the category of extendable regular holonomic $\mathcal{D}_{X-Y}$-Modules to $\mathrm{RH}(X)$.

Theorem 2.5. *Let $\mathcal{M}$ be an extendable regular holonomic $\mathcal{D}_{X-Y}$-Module and ${}^{\pi}\mathcal{M}$ its minimal extension. Set $\mathcal{F}^{\boldsymbol{\cdot}} = \mathrm{DR}_{X-Y}(\mathcal{M})$ and $\mathcal{F}'^{\boldsymbol{\cdot}} = \mathrm{DR}_X({}^{\pi}\mathcal{M})$. Then we have*

$$(2.3.2) \qquad \mathcal{F}'^{\boldsymbol{\cdot}}|_{X-Y} \cong \mathcal{F}^{\boldsymbol{\cdot}}$$

$$(2.3.2) \qquad \begin{aligned} &\operatorname{codim}_X \operatorname{Supp} \mathcal{H}^j(\mathcal{F}'^{\boldsymbol{\cdot}}) \cap Y > j \\ &\operatorname{codim}_X \operatorname{Supp} \mathcal{E}xt^j_{\mathbb{C}}(\mathcal{F}'^{\boldsymbol{\cdot}}, \mathbb{C}_X) \cap Y > j. \end{aligned}$$

Conversely, if $\mathcal{F}'^{\boldsymbol{\cdot}}$ satisfies these conditions, then $\mathcal{F}'^{\boldsymbol{\cdot}} \simeq \mathrm{DR}_X({}^{\pi}\mathcal{M})$.

Definition 2.6. We call $\mathcal{F}'^{\boldsymbol{\cdot}}$ the *minimal extension* of $\mathcal{F}^{\boldsymbol{\cdot}}$.

Proof of Theorem 2.5.

Lemma 2.3.1. *Let $\mathcal{M}^{\boldsymbol{\cdot}} \in D^b_{\mathrm{rh}}(\mathcal{D}_X)$ and Z a smooth subvariety of X. Then*

$$\mathbb{R}\, \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^{\boldsymbol{\cdot}}, \mathcal{B}_{Z|X})|_Z = \mathbb{R}\, \mathcal{H}om_{\mathbb{C}_Z}(\mathrm{DR}_X(\mathcal{M}^{\boldsymbol{\cdot}})|_Z, \mathbb{C}_Z)[-\operatorname{codim} Z].$$

Here $\mathcal{B}_{Z|X}$ denotes $\mathcal{H}^{\operatorname{codim} Z}_{[Z]}(\mathcal{O}_X)$.

Proof. We may assume Z is closed. Then we have

$$\mathbb{R}\, \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^{\boldsymbol{\cdot}}, \mathcal{B}_{Z|X}) = \mathbb{R}\Gamma_Z\, \mathbb{R}\, \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^{\boldsymbol{\cdot}}, \mathcal{O}_X)[\operatorname{codim} Z].$$

On the other hand, $\mathbb{R}\, \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^{\boldsymbol{\cdot}}, \mathcal{O}_X) = \mathbb{R}\, \mathcal{H}om_{\mathbb{C}_X}(\mathrm{DR}_X(\mathcal{M}^{\boldsymbol{\cdot}}), \mathbb{C}_X)$ implies

$$\begin{aligned} \mathbb{R}\Gamma_Z\, \mathbb{R}\, \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^{\boldsymbol{\cdot}}, \mathcal{O}_X) &= \mathbb{R}\, \mathcal{H}om_{\mathbb{C}_X}(\mathrm{DR}_X(\mathcal{M}^{\boldsymbol{\cdot}})|_Z, \mathbb{C}_X) \\ &= \mathbb{R}\, \mathcal{H}om_{\mathbb{C}_Z}(\mathrm{DR}_X(\mathcal{M}^{\boldsymbol{\cdot}})|_Z, \mathbb{C}_Z)[-2\operatorname{codim} Z] \quad \text{Q.E.D.} \end{aligned}$$

We shall apply this to $\mathcal{F}'^{\boldsymbol{\cdot}}$ and ${}^{\pi}\mathcal{M}$. Then if $\mathcal{F}'^{\boldsymbol{\cdot}}|_Z$ is locally constant and $Z \subset Y$,

$$\mathcal{E}xt^j_{\mathcal{D}_X}({}^{\pi}\mathcal{M}, \mathcal{B}_{Z|X}) = \mathcal{H}om_{\mathbb{C}_Z}(\mathcal{H}^{\operatorname{codim} Z - j}(\mathcal{F}'^{\boldsymbol{\cdot}})|_Z, \mathbb{C}_Z).$$

Hence $\mathscr{H}om_{\mathscr{D}_X}({}^{\pi}\mathscr{M}, \mathscr{B}_{Z|X})=0$ implies $\mathscr{H}^j(\mathscr{F}^{\cdot})|_Z=0$ for $j \geqq \operatorname{codim} Z$. Hence we obtain $\operatorname{codim}_X \operatorname{Supp} \mathscr{H}^j(\mathscr{F}^{\cdot}) \cap Y > j$. Since $({}^{\pi}\mathscr{M})^*$ is the minimal extension of $\mathscr{M}^*$, the same discussion implies

$$\operatorname{codim}_X \operatorname{Supp} \mathscr{E}xt^j(\mathscr{F}^{\cdot}, \mathbb{C}) \cap Y > j.$$

The converse is also proved by the similar argument. Q.E.D.

Let Z be a locally closed subset of X such that $\bar{Z}$ and $\bar{Z}-Z$ are complex analytic. Let us take as Y a closed analytic subset of $\bar{Z}$ such that Y is nowhere dense in $\bar{Z}$ and Y contains the singular locus of Z. The minimal extension of $\mathscr{B}_{Z-Y|X-Y}$ is denoted by ${}^{\pi}\mathscr{B}_{Z|X}$. This does not depend on the choice of Y.

These notions are also generalized to the algebraic case.

§ 3. Fourier transformation

3.1. Let V be a finite-dimensional vector space over $\mathbb{C}$. Then we have, by regarding V as an algebraic variety

$$\Gamma(V, \mathscr{D}_V) = \mathbb{C}[V] \otimes_{\mathbb{C}} \mathbb{C}[V^*] = S(V^*) \otimes_{\mathbb{C}} S(V)$$

where $\mathbb{C}[V] = S(V^*) = \Gamma(V, \mathscr{O}_V)$ and we regard $\mathbb{C}[V^*]$ as the ring of constant coefficient differential operators. Hence $\Gamma(V, \mathscr{D}_V)$ is a $\mathbb{C}$-algebra generated by $V \oplus V^*$ with the fundamental relation

$$(3.1.1) \qquad [v_1, v_2] = [v_1^*, v_2^*] = 0, \qquad [v, v^*] = \langle v, v^* \rangle$$
$$\text{for } v_1, v_2, v \in V \text{ and } v_1^*, v_2^*, v^* \in V^*.$$

Therefore, $\Gamma(V, \mathscr{D}_V)$ is isomorphic to $\Gamma(V^*, \mathscr{D}_{V^*})$ by

$$V \oplus V^* \to V^* \oplus V \quad ((v, v^*) \mapsto (v^*, -v)).$$

On the other hand, the category of coherent $\mathscr{D}_V$-Modules is equivalent to that of finitely generated $\Gamma(V, \mathscr{D}_V)$-modules. Hence we obtain the functor F from the category of coherent $\mathscr{D}_V$-Modules onto the category of coherent $\mathscr{D}_{V^*}$-Modules. For a coherent $\mathscr{D}_V$-Module $\mathscr{M}$, we call the *Fourier transform* of $\mathscr{M}$ the image of $\mathscr{M}$ by F and denote it by $\mathscr{M}^F$.

If we denote by a the isomorphism $v \mapsto -v$ of V, then we have $(\mathscr{M}^F)^F = a^* \mathscr{M}$ for a coherent $\mathscr{D}_V$-Module $\mathscr{M}$.

The Fourier transforms of holonomic $\mathscr{D}_V$-Modules are holonomic $\mathscr{D}_{V^*}$-Modules. In fact, $\mathscr{M}$ is holonomic if and only if $\mathscr{E}xt^j_{\mathscr{D}_V}(\mathscr{M}, \mathscr{D}_V) = 0$ for $j \neq \dim V$.

3.2. Let ϑ denote the element in $V^* \otimes V \subset \Gamma(V, \mathscr{D}_V)$ corresponding to the identity. For a linear coordinate $(x_1, \ldots, x_n)$ of V, ϑ is explicitly given by

$$(3.2.1) \qquad \vartheta = \sum_j x_j \frac{\partial}{\partial x_j},$$

that is, ϑ is the infinitesimal transformation of homotheties.

For a coherent $\mathscr{D}_V$-Module $\mathscr{M}$, we say $\mathscr{M}$ is *homogeneous* if, for any element $v \in \Gamma(V, \mathscr{M})$, $\mathbb{C}[\vartheta]v$ is a finite-dimensional vector space.

It is easy to see that a homogeneous regular holonomic $\mathscr{D}_V$-Module is completely regular.

For a homogeneous $\mathscr{D}_V$-Module $\mathscr{M}$ we can describe the relation between

$$\mathbb{R}\,\mathscr{H}om_{\mathscr{D}_{V_{\mathrm{an}}}}(\mathscr{M}_{\mathrm{an}}, \mathscr{O}_{V_{\mathrm{an}}}) \quad \text{and} \quad \mathbb{R}\,\mathscr{H}om_{\mathscr{D}_{V^*_{\mathrm{an}}}}(\mathscr{M}^F_{\mathrm{an}}, \mathscr{O}_{V^*_{\mathrm{an}}}).$$

Let Z be the closed subset of $V_{\mathrm{an}} \times V^*_{\mathrm{an}}$ given by

$$\{(x, y) \in V_{\mathrm{an}} \times V^*_{\mathrm{an}};\ \mathrm{Re}\langle x, y\rangle \geqq 0\}.$$

Let τ and π be the projections from $V_{\mathrm{an}} \times V^*_{\mathrm{an}}$ onto V_{an} and V^*_{an}, respectively. Then we have

Theorem 3.1. *Let $\mathscr{M}$ be a coherent homogeneous $\mathscr{D}_V$-Module. Then we have*

$$\mathbb{R}\,\mathscr{H}om_{\mathscr{D}_{V^*_{\mathrm{an}}}}(\mathscr{M}^F_{\mathrm{an}}, \mathscr{O}_{V^*_{\mathrm{an}}}) \cong \mathbb{R}\pi_* \mathbb{R}\Gamma_Z(\tau^{-1}\,\mathbb{R}\,\mathscr{H}om_{\mathscr{D}_{V_{\mathrm{an}}}}(\mathscr{M}_{\mathrm{an}}, \mathscr{O}_{V_{\mathrm{an}}})).$$

Theorem 3.2. *Let $\mathscr{M}$ be a homogeneous holonomic $\mathscr{D}_V$-Module. Then* $\mathbf{Ch}(\mathscr{M}) = \mathbf{Ch}(\mathscr{M}^F)$. *Here we identify T^*V with $T^*(V^*)$ (both isomorphic to $V \times V^*$).*

Proof. We may assume that $\mathscr{M}$ is generated by u with $(\vartheta - \lambda)^m u = 0$ for some $\lambda \in \mathbb{C}$ and m. Now, let I be the annihilator of u in $\Gamma(V, \mathscr{D}_V)$. Then I is generated by homogeneous elements. Here, we assign the degree 1 and -1 to the elements of V and V^*, respectively. Let F^1 be the filtration of $\Gamma(V, \mathscr{D}_V)$ by the order, and let F^2 be the filtration of $\Gamma(V^*, \mathscr{D}_{V^*})$ by the order. If we denote by $\Gamma(V, \mathscr{D}_V)_m$ the homogeneous part of degree m, then

$$F^1_k \cap \Gamma(V, \mathscr{D}_V)_m = F^2_{m+k} \cap \Gamma(V, \mathscr{D}_V)_m.$$

Thus we obtain

$$\mathrm{gr}^{F^1}(I) = \mathrm{gr}^{F^2}(I).$$

Since $\mathbf{Ch}(\mathscr{M})$ is the algebraic cycle corresponding to $\mathscr{O}_{V \times V^*}/\mathrm{gr}^{F^1}(I)$ and $\mathbf{Ch}(\mathscr{M}^F)$ is the one corresponding to $\mathscr{O}_{V \times V^*}/\mathrm{gr}^{F^2}(I)$, we obtain the desired result. Q.E.D.

3.3. The Fourier transformation discussed in § 3.1 can be generalized to $\mathscr{D}$-Modules on a vector bundle. In such a general case we have to twist $\mathscr{D}$ and the argument becomes slightly complicated. However, if a vector bundle is trivial, the same argument as in § 3.1 can be applied. So, we shall restrict ourselves to such a case.

Let V be a finite-dimensional vector space and X a smooth algebraic variety. Let f and g be the projections from $X \times V$ and $X \times V^*$ onto X, respectively. Then $f_* \mathscr{D}_{X \times V}$ and $g_* \mathscr{D}_{X \times V^*}$ are isomorphic to

$$\mathscr{D}_X \otimes_{\mathbb{C}} \Gamma(V, \mathscr{D}_V) \quad \text{and} \quad \mathscr{D}_X \otimes_{\mathbb{C}} \Gamma(V^*, \mathscr{D}_{V^*}),$$

respectively. Hence $f_* \mathscr{D}_{X \times V}$ is isomorphic to $g_* \mathscr{D}_{X \times V^*}$ via the isomorphism between $\Gamma(V, \mathscr{D}_V)$ and $\Gamma(V^*, \mathscr{D}_{V^*})$ given in § 3.1. Therefore the category of coherent $\mathscr{D}_{X \times V}$-Modules are equivalent to that of coherent $\mathscr{D}_{X \times V^*}$-Modules. We call this the *partial Fourier transformation* with respect to V.

3.4. Let X be a smooth algebraic variety and f a section of $\mathcal{O}_X$. We define a coherent $\mathcal{D}_X$-Module $\mathcal{M}$ as follows:

$$\mathcal{M} = \mathcal{D}_X / \sum_{v \in \Theta} \mathcal{D}_X (v - v(f))$$

where Θ is the sheaf of vector fields on X. We have $\mathcal{M}_{an} \cong \mathcal{O}_{X_{an}}$ and the isomorphism is given by $1 \mapsto \exp f$. In this reason, we shall denote by $\exp f$ the canonical generator of $\mathcal{M}$. The holonomic Module $\mathcal{D}_X \exp f$ is regular but this is not completely regular except when f is locally constant.

3.5. Let X be an algebraic manifold and let f: $X \to V$ be a morphism from X to a vector space V. Then f defines a section f' of $\mathcal{O}_{X \times V^*}$ by $\langle f(x), v^* \rangle$ for $(x, v^*) \in X \times V^*$. We write $\mathcal{D}_{X \times V^*} \exp f$ for $\mathcal{D}_{X \times V^*} \exp f'$. Let Y be the graph of f. Then it is easy to see

Proposition 3.5.1. *$\mathcal{D}_{X \times V^*} \exp f$ is the partial Fourier transform of $\mathcal{B}_{Y|X \times V}$. Here $\exp f$ and $\delta(v - f(x))$ correspond.*

After writing up the first draft of this paper, we came to know the following references related to the geometric Fourier transformation [1] and [2], which will help the reader's understanding of this section.

§ 4. Some holonomic systems on semisimple Lie algebras

4.1. Let $\mathfrak{g}$ be a semisimple Lie algebra defined over $\mathbb{C}$ and G a connected affine algebraic group with $\mathfrak{g}$ as its Lie algebra. We fix a G-invariant non-degenerate quadratic form on $\mathfrak{g}$ by which we identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$.

We denote by $\mathcal{B}$ the flag manifold, i.e., the set of Borel subgroups of G, or equivalently the set of Borel subalgebras of $\mathfrak{g}$.

We denote by $\tilde{\mathfrak{g}}$ the subvariety of $\mathcal{B} \times \mathfrak{g}$ defined by

$$\tilde{\mathfrak{g}} = \{(\mathfrak{b}', x) \in \mathcal{B} \times \mathfrak{g}; \mathfrak{b}' \ni x\} \tag{4.1.1}$$

and let ρ denote the canonical projection $\tilde{\mathfrak{g}}$ onto $\mathfrak{g}$. The map ρ is proper and we have

$$\dim \rho^{-1}(x) = \dim \mathcal{B} - \tfrac{1}{2} \dim Gx \tag{4.1.2}$$

for any $x \in \mathfrak{g}$.

We shall fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$. Let Δ be the root system of $(\mathfrak{g}, \mathfrak{h})$ and Δ_+ the set of positive roots given by $\mathfrak{b}$. Let W denote the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. If we denote by $\mathfrak{n}$ the nilpotent radical of $\mathfrak{b}$, we have $\mathfrak{b}/\mathfrak{n} \xrightarrow{\sim} \mathfrak{h}$.

We define the map

$$\theta \colon \tilde{\mathfrak{g}} \to \mathfrak{h} \tag{4.1.3}$$

by $(\mathfrak{b}', x) \mapsto gx \bmod \mathfrak{n}$ where g is an element of G satisfying $g\mathfrak{b}' = \mathfrak{b}$. Here the actions of G on $\mathfrak{g}$ and $\mathcal{B}$ are via the adoint action. This θ is a smooth morphism.

Let dx be a nowhere-vanishing global section of the sheaf $\Omega_{\mathfrak{g}}$ of differential forms of the highest degree on $\mathfrak{g}$. The sheaf $\Omega_{\tilde{\mathfrak{g}}}$ has also a unique nowhere-vanishing global section ω up to a constant multiple. We normalize ω and dx so that

$$\rho^*(dx)=\Big(\prod_{\alpha\in\Delta_+}\langle\theta,\alpha\rangle\Big)\omega. \tag{4.1.4}$$

Let $\mathfrak{g}_{rs}$ be the set of regular semisimple elements of $\mathfrak{g}$. Then $\mathfrak{g}_{rs}$ is Zariski open in $\mathfrak{g}$. We denote by $\mathfrak{h}_{rs}=\mathfrak{g}_{rs}\cap\mathfrak{h}$.

Let κ be the canonical morphism from $\mathfrak{g}$ onto $\mathfrak{h}/W$ such that

$$\begin{tikzcd} \tilde{\mathfrak{g}} \arrow[r, "\rho"] \arrow[d, "\theta"'] & \mathfrak{g} \arrow[d, "\kappa"] \\ \mathfrak{h} \arrow[r] & \mathfrak{h}/W \end{tikzcd}$$

is commutative.

We denote by $\mathbf{N}$ the set of nilpotent elements of $\mathfrak{g}$. Then $\mathbf{N}$ is given by

$$\mathbf{N}=\{x\in\mathfrak{g};\ P(x)=P(0)\ \text{for any}\ P\in\mathbb{C}[\mathfrak{g}]^G\} \tag{4.1.5}$$

where $\mathbb{C}[\mathfrak{g}]^G$ denotes the ring of G-invariants on $\mathfrak{g}$.

4.2. Let $f\colon\tilde{\mathfrak{g}}\to\mathfrak{g}\times\mathfrak{h}$ be the morphism given by ρ and θ. Then the image of f is given by

$$f(\tilde{\mathfrak{g}})=\{(x,t)\in\mathfrak{g}\times\mathfrak{h};\ P(x)=P(t)\ \text{for any}\ P\in\mathbb{C}[\mathfrak{g}]^G\}, \tag{4.2.1}$$

and isomorphic to $\mathfrak{g}\times_{\mathfrak{h}/W}\mathfrak{h}$. Here $\mathbb{C}[\mathfrak{g}]^G$ is the ring of G-invariant polynomials on $\mathfrak{g}$. By f, $\tilde{\mathfrak{g}}$ is a desingularization of $f(\tilde{\mathfrak{g}})$.

Now, we shall investigate the property of $\int_f\mathcal{O}_{\tilde{\mathfrak{g}}}$.

Theorem 4.1. (1) *$\mathcal{H}^j(\int_f\mathcal{O}_{\tilde{\mathfrak{g}}})=0$ for $j\neq 0$ and $\mathcal{H}^0(\int_f\mathcal{O}_{\tilde{\mathfrak{g}}})$ is a completely regular holonomic $\mathcal{D}_{\mathfrak{g}\times\mathfrak{h}}$-Module.*

(2) $\mathrm{Ch}(\int_f\mathcal{O}_{\tilde{\mathfrak{g}}})=\{(x,y;t,s)\in\mathfrak{g}\times\mathfrak{g}\times\mathfrak{h}\times\mathfrak{h}=\mathfrak{g}\times\mathfrak{g}^*\times\mathfrak{h}\times\mathfrak{h}^*=T^*(\mathfrak{g}\times\mathfrak{h});\ [x,y]=0$ *and there exists* $\mathfrak{b}'\in\mathcal{B}$ *such that* $\mathfrak{b}'\ni x,y$ *and* $\theta((\mathfrak{b}',x))=t$, $\theta((\mathfrak{b}',y))=-s\}$,

and this is irreducible.

(3) *$\int_f\mathcal{O}_{\tilde{\mathfrak{g}}}$ is a simple Module and isomorphic to ${}^{\pi}\mathcal{B}_{f(\tilde{\mathfrak{g}})|\mathfrak{g}\times\mathfrak{h}}$.*

Proof. Let ϖ_f and ρ_f be the projections from $\tilde{\mathfrak{g}}\times_{\mathfrak{g}\times\mathfrak{h}}T^*(\mathfrak{g}\times\mathfrak{h})$ to $T^*(\mathfrak{g}\times\mathfrak{h})$ and to $T^*\tilde{\mathfrak{g}}$, respectively. Then by Theorem 1.2, $\mathrm{Ch}(\int_f\mathcal{O}_{\tilde{\mathfrak{g}}})$ is contained in $\Lambda=\varpi_f\rho_f^{-1}(T^*_{\tilde{\mathfrak{g}}}\tilde{\mathfrak{g}})$. For

$$(\mathfrak{b}',x)\in\tilde{\mathfrak{g}},\qquad T_{(\mathfrak{b}',x)}\tilde{\mathfrak{g}}\subset T_{(\mathfrak{b}',x)}(\mathcal{B}\times\mathfrak{g})=\mathfrak{g}/\mathfrak{b}'\oplus\mathfrak{g}$$

is given by

$$T_{(\mathfrak{b}',x)}\tilde{\mathfrak{g}}=\{(A,[A,x])\in\mathfrak{g}/\mathfrak{b}'\oplus\mathfrak{g};\ A\in\mathfrak{g}\}+(0\oplus\mathfrak{b}').$$

On the other hand, we have $T_{f(\mathfrak{b}',x)}(\mathfrak{g}\times\mathfrak{h})=\mathfrak{g}\times\mathfrak{h}$ and df: $T_{(\mathfrak{b}',x)}\tilde{\mathfrak{g}}\to T_{f(\mathfrak{b}',x)}(\mathfrak{g}\times\mathfrak{h})$ is given by

$$\mathfrak{g}/\mathfrak{b}'\oplus\mathfrak{g}\ni(A,[A,x])\mapsto[A,x]\oplus 0$$

$$0\oplus\mathfrak{b}'\ni 0\oplus A\mapsto A\oplus(gA \bmod \mathfrak{n}),$$

where $g\in G$ is an element such that $g\mathfrak{b}'=\mathfrak{b}$. Using this, we can easily show that Λ coincides with the set given in (2). Remark that Λ is a Lagrangian variety and hence Λ has pure dimension $\dim\mathfrak{g}+\operatorname{rk}\mathfrak{g}$.

Lemma 4.2.1. ([18]) *$V=\{(x,y)\in\mathfrak{g}\times\mathfrak{g};\ [x,y]=0\}$ is irreducible and has dimension $\dim\mathfrak{g}+\operatorname{rk}\mathfrak{g}$.*

Proof. Since the canonical projection $\Lambda\to V$ is finite and surjective, V has also pure dimenion $\dim\mathfrak{g}+\operatorname{rk}\mathfrak{g}$. We shall stratify $\mathfrak{g}$ by $\mathfrak{g}=\bigcup_{j=0}\mathfrak{g}_j$ so that the fiber dimenion of $V\to\mathfrak{g}$ is constant over $\mathfrak{g}_j$, and that $\mathfrak{g}_j$'s are G-invariant. We take $\mathfrak{g}_{rs}$ as $\mathfrak{g}_0$. Take a generic point x_j of $\mathfrak{g}_j$. Then

$$\dim V\cap(\mathfrak{g}_j\times\mathfrak{g})=\dim\mathfrak{g}_j+\dim\mathfrak{g}_{x_j},\qquad\text{where } \mathfrak{g}_{x_j}=\{y\in\mathfrak{g}\,|\,[y,x_j]=0\}.$$

On the other hand, for $j\neq 0$, κ: $\mathfrak{g}_j\to\mathfrak{h}/W$ has nowhere dense image and the fiber dimension equals $\dim[\mathfrak{g},x_j]$ because the fiber has a finite number of G-orbits. Hence $\dim\mathfrak{g}_j<\operatorname{rank}\mathfrak{g}+\dim[\mathfrak{g},x_j]$. Hence for $j\neq 0$, $\dim V\cap(\mathfrak{g}_j\times\mathfrak{g})<\dim\mathfrak{g}+\operatorname{rank}\mathfrak{g}=\dim V$, which implies that $V\cap(\mathfrak{g}_{rs}\times\mathfrak{g})$ is Zariski dense in V. Since $V\cap(\mathfrak{g}_{rs}\times\mathfrak{g})$ is irreducible, V is also irreducible. Q.E.D.

Lemma 4.2.2. *Λ is irreducible.*

Proof. By the preceding lemma, $\Lambda\cap(\mathfrak{g}_{rs}\times\mathfrak{g}\times\mathfrak{h}\times\mathfrak{h})$ is a Zariski dense subset of Λ. Since this is irreducible, Λ is also irreducible.

Now, we shall prove Theorem 4.1. Since f is embedding on $\mathfrak{g}_{rs}\times\mathfrak{h}$, we have

$$\mathscr{H}^j(\textstyle\int_f\mathcal{O}_{\tilde{\mathfrak{g}}})|_{\mathfrak{g}_{rs}\times\mathfrak{h}}=\begin{cases}{}^{\pi}\mathscr{B}_{f(\tilde{\mathfrak{g}})|\mathfrak{g}\times\mathfrak{h}}|_{\mathfrak{g}_{rs}\times\mathfrak{h}} & \text{for } j=0\\ 0 & \text{for } j\neq 0.\end{cases}\tag{4.2.2}$$

Hence $\operatorname{Ch}(\mathscr{H}^j(\int_f\mathcal{O}_{\tilde{\mathfrak{g}}}))$ $(j\neq 0)$ is a nowhere dense subset of Λ. Since the characteristic variety is always involutive, we have

$$\mathscr{H}^j(\textstyle\int_f\mathcal{O}_{\tilde{\mathfrak{g}}})=0\qquad\text{for } j\neq 0.$$

Thus (1) is proved. The property (2) is also evident. The property (3) follows from (4.2.2) and irreducibility of the characteristic variety of $\int_f\mathcal{O}_{\tilde{\mathfrak{g}}}$.

Corollary 4.2.3. *$\mathscr{H}^j(\int_\rho\mathcal{O}_{\tilde{\mathfrak{g}}})=0$ for $j\neq 0$ and $\int_\rho\mathcal{O}_{\tilde{\mathfrak{g}}}$ is the minimal extension of $(\int_\rho\mathcal{O}_{\tilde{\mathfrak{g}}})|_{\mathfrak{g}_{rs}}$.*

Proof. Let p be the projection $\mathfrak{g}\times\mathfrak{h}\to\mathfrak{g}$. Then we have

$$\textstyle\int_\rho\mathcal{O}_{\tilde{\mathfrak{g}}}=\int_p(\int_f\mathcal{O}_{\tilde{\mathfrak{g}}}).\tag{4.2.3}$$

Since $\mathscr{H}^j(\int_f \mathcal{O}_{\tilde{\mathfrak{g}}})=0$ for $j\neq 0$ and $\operatorname{Supp}(\mathscr{H}^0(\int_f \mathcal{O}_{\tilde{\mathfrak{g}}}))\to\mathfrak{g}$ is a finite map, $\mathscr{H}^j(\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}})=0$ $(j\neq 0)$ follows from Theorem 1.3.

Now, let us remark that local cohomology commutes with integration. Hence

$$\mathscr{H}^0_{\mathfrak{g}-\mathfrak{g}_{rs}}(\textstyle\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}})=\int_p \mathscr{H}^0_{p^{-1}(\mathfrak{g}-\mathfrak{g}_{rs})}(\int_f \mathcal{O}_{\tilde{\mathfrak{g}}})=0$$

by (3) of Theorem 4.1. On the other hand, since $\mathcal{O}_{\tilde{\mathfrak{g}}}$ is self-dual, $\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}}$ is self dual. Therefore $\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}}$ has no non-trivial quotient supported in $\mathfrak{g}-\mathfrak{g}_{rs}$. Hence $\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}}$ is the minimal extension of its restriction to $\mathfrak{g}_{rs}$. Q.E.D.

Corollary 4.2.4. (1) $\mathscr{H}^j(\int_{\rho^{-1}(\mathrm{N})\to\mathfrak{g}} \mathcal{O}_{\rho^{-1}(\mathrm{N})})=0$ *for* $j\neq 0$.
(2) $\mathscr{H}^0(\int_{\rho^{-1}(\mathrm{N})\to\mathfrak{g}} \mathcal{O}_{\rho^{-1}(\mathrm{N})})$ *is a completely regular holonomic* $\mathscr{D}_\mathfrak{g}$*-Module.*

Proof. (2) follows from Theorem 1.5. We shall show (1). We have $\rho^{-1}(\mathrm{N}) = f^{-1}(\mathfrak{g}\times\{0\})$ and

$$\mathcal{O}_{\rho^{-1}(\mathrm{N})}=\mathbb{L}i^*\mathcal{O}_{\tilde{\mathfrak{g}}} \qquad \text{where } i\colon \rho^{-1}(\mathrm{N})\hookrightarrow\tilde{\mathfrak{g}}.$$

Therefore, if j denotes the embedding $\mathfrak{g}\hookrightarrow\mathfrak{g}\times\mathfrak{h}$ given by $\{0\}\hookrightarrow\mathfrak{h}$ we have

$$\textstyle\int_{\rho^{-1}(\mathrm{N})\to\mathfrak{g}} \mathcal{O}_{\rho^{-1}(\mathrm{N})}=\mathbb{L}j^*(\int_f \mathcal{O}_{\tilde{\mathfrak{g}}}). \tag{4.2.4}$$

Since $\Lambda\cap T^*_{\mathfrak{g}\times\{0\}}(\mathfrak{g}\times\mathfrak{h})$ is contained in the zero section, Theorem 1.3 implies (1).

4.3. The projection f: $\tilde{\mathfrak{g}}\to\mathfrak{g}\times\mathfrak{h}$ decomposes $\tilde{\mathfrak{g}}\to\tilde{\mathfrak{g}}\times\mathfrak{h}\xrightarrow{\rho\times 1}\mathfrak{g}\times\mathfrak{h}$. Hence the partial Fourier transform of $\int_f \mathcal{O}_{\tilde{\mathfrak{g}}}$ w.r.t. $\mathfrak{h}$ is the integration of the partial Fourier transform of $\mathscr{B}_{\tilde{\mathfrak{g}}|\tilde{\mathfrak{g}}\times\mathfrak{h}}$.

Proposition 4.3.1.
(1) $\mathscr{H}^j(\int_{\rho\times 1}\mathscr{D}_{\tilde{\mathfrak{g}}\times\mathfrak{h}}\exp\theta)=0$ *for* $j\neq 0$ *and* $\mathscr{H}^0(\int_{\rho\times 1}\mathscr{D}_{\tilde{\mathfrak{g}}\times\mathfrak{h}}\exp\theta)$ *is the partial Fourier transform of* $\int_f \mathcal{O}_{\tilde{\mathfrak{g}}}$ *with respect to* $\mathfrak{h}$.
(2) $\int_{\rho\times 1}\mathscr{D}_{\tilde{\mathfrak{g}}\times\mathfrak{h}}\exp\theta$ *is a regular holonomic* $\mathscr{D}_{\mathfrak{g}\times\mathfrak{h}}$*-Module.*
(3) $\operatorname{Ch}(\int_{\rho\times 1}\mathscr{D}_{\tilde{\mathfrak{g}}\times\mathfrak{h}}\exp\theta)\subset\{((x,y),(t,s))\in T^*(\mathfrak{g}\times\mathfrak{h})=\mathfrak{g}\times\mathfrak{g}\times\mathfrak{h}\times\mathfrak{h}^*; s=0, [x,y]=0, y\in\mathrm{N}\}$.

Proof. (1) and (2) follow from the fact that the partial Fourier transform of $\mathscr{B}_{\tilde{\mathfrak{g}}|\tilde{\mathfrak{g}}\times\mathfrak{h}}$ is $\mathscr{D}_{\tilde{\mathfrak{g}}\times\mathfrak{h}}\exp\theta$ (Proposition 3.5.1). (3) follows from $\operatorname{Ch}(\mathscr{D}_{\tilde{\mathfrak{g}}\times\mathfrak{h}}\exp\theta) = T^*_{\tilde{\mathfrak{g}}\times\mathfrak{h}}(\tilde{\mathfrak{g}}\times\mathfrak{h})$ and Theorem 1.2.

4.4. We shall remark that $\int_{\rho\times 1}\mathscr{D}_{\tilde{\mathfrak{g}}\times\mathfrak{h}}\exp\theta$ is not a completely regular holonomic system. This does not contradict (1) of Proposition 4.3.1, because the (partial) Fourier transformation does not preserve completely regular holonomic systems when they are not homogeneous.

For $\lambda\in\mathfrak{h}^*=\mathfrak{h}$, the $\mathscr{D}_\mathfrak{g}$-Module

$$\textstyle\int_{\rho\times 1}\mathscr{D}_{\tilde{\mathfrak{g}}\times\mathfrak{h}}\exp\theta|_{\mathfrak{g}\times\{\lambda\}}=\int_\rho\mathscr{D}_{\tilde{\mathfrak{g}}}\exp\langle\theta,\lambda\rangle$$

is neither completely regular except when $\lambda=0$. However, the corresponding $\mathscr{D}_G$-Module on the group G is completely regular.

4.5. Now, we shall express explicitly the $\mathscr{D}_{\mathfrak{g}\times\mathfrak{h}}$-Module $\int_f \mathscr{O}_{\tilde{\mathfrak{g}}}$. We define a coherent $\mathscr{D}_{\mathfrak{g}\times\mathfrak{h}}$-Module $\mathscr{N}$ by

$$(4.5.1)\qquad \begin{aligned}\mathscr{N} &= \mathscr{D}_{\mathfrak{g}\times\mathfrak{h}}/(\mathscr{D}_{\mathfrak{g}\times\mathfrak{h}}\operatorname{ad}\mathfrak{g} + \sum_{P\in S(\mathfrak{g})^G} \mathscr{D}_{\mathfrak{g}\times\mathfrak{h}}(P(\partial_x) - P(-\partial_t)) \\ &\quad + \sum_{Q\in\mathbb{C}[\mathfrak{g}]^G} \mathscr{D}_{\mathfrak{g}\times\mathfrak{h}}(Q(x) - Q(t))) \\ &= \mathscr{D}_{\mathfrak{g}\times\mathfrak{h}}\tilde{u}.\end{aligned}$$

Here, (x, t) denotes a point of $\mathfrak{g}\times\mathfrak{h}$ and $\operatorname{ad}\mathfrak{g}$ is the vector space of vector fields $\langle [A, x], \partial_x\rangle$ $(A\in\mathfrak{g})$, which denotes $f(x)\mapsto \frac{d}{dt} f(e^{tA}x)|_{t=0}$. By $\tilde{u}$ we denote the canonical generator of $\mathscr{N}$.

Theorem 4.2. $\mathscr{N} \simeq \int_f \mathscr{O}_{\tilde{\mathfrak{g}}}$.

We shall prove this theorem in two steps.

(1) $\mathscr{N}$ is a simple or zero $\mathscr{D}_{\mathfrak{g}\times\mathfrak{h}}$-Module.

(2) To construct a non-zero homomorphism $\mathscr{N}\to\int_f \mathscr{O}_{\tilde{\mathfrak{g}}}$. Since $\int_f \mathscr{O}_{\tilde{\mathfrak{g}}}$ is a simple $\mathscr{D}_{\mathfrak{g}\times\mathfrak{h}}$-Module Theorem 4.2 follows from these two statements.

4.6. *Proof of* (1). By the partial Fourier transformation with respect to $\mathfrak{g}$, it is enough to show that

$$\begin{aligned}\mathscr{N}^F &= \mathscr{D}_{\mathfrak{g}\times\mathfrak{h}}/(\mathscr{D}_{\mathfrak{g}\times\mathfrak{h}}\operatorname{ad}\mathfrak{g} + \sum_{P\in S(\mathfrak{g})^G} \mathscr{D}_{\mathfrak{g}\times\mathfrak{h}}(P(\partial_x) - P(t)) \\ &\quad + \sum_{Q\in\mathbb{C}[\mathfrak{g}]^G} \mathscr{D}_{\mathfrak{g}\times\mathfrak{h}}(Q(x) - Q(\partial_t))) \\ &= \mathscr{D}_{\mathfrak{g}\times\mathfrak{h}}\tilde{u}^F\end{aligned}$$

is a simple $\mathscr{D}_{\mathfrak{g}\times\mathfrak{h}}$-Module or zero.

The characteristic variety of $\mathscr{N}^F$ is clearly contained in

$$\begin{aligned}W &= \{(x, y; t, s)\in T^*\mathfrak{g}\times T^*\mathfrak{h} = \mathfrak{g}\times\mathfrak{g}\times\mathfrak{h}\times\mathfrak{h};\ [x, y] = 0, y\in\mathbf{N}, s = 0\} \\ &\subset T^*\mathfrak{g}\times T^*_{\mathfrak{h}}\mathfrak{h}.\end{aligned}$$

By Theorem 1.3, for a coherent $\mathscr{D}_{\mathfrak{g}\times\mathfrak{h}}$-Module $\mathscr{L}$ such that $\operatorname{Ch}(\mathscr{L})\subset W$, $\mathscr{L}=0$ if and only if $\mathscr{L}|_{\mathfrak{g}\times\{t_0\}}=0$ for some $t_0\in\mathfrak{h}$. Hence it is enough to show that $\mathscr{N}^F|_{\mathfrak{g}\times\{t_0\}}$ is a simple or zero Module for $t_0\in\mathfrak{h}_{\mathrm{rs}}$. By using the partial Fourier transformation with respect to $\mathfrak{g}$ again, (1) follows from the following Lemma.

Lemma 4.6.1. *For* $t_0\in\mathfrak{h}_{\mathrm{rs}}$, $\mathscr{N}|_{\mathfrak{g}\times\{t_0\}}$ *is a simple* $\mathscr{D}_{\mathfrak{g}}$*-Module or zero.*

Proof. First, we shall show that $\mathscr{N}|_{\mathfrak{g}\times\{t_0\}}$ is generated by $1\otimes\tilde{u}$. We have

$$[P(\partial_t), Q(t)]\tilde{u} = -[P(-\partial_x), Q(x)]\tilde{u}$$

for $P\in S(\mathfrak{g})^G$ and $Q\in\mathbb{C}[\mathfrak{g}]^G$. By choosing $P = t_1^2+\ldots+t_l^2$ where $(t_1,\ldots,t_l)$ is an orthonormal basis of $\mathfrak{h}$,

$$2\sum\frac{\partial Q(t)}{\partial t_i}\frac{\partial}{\partial t_i}\tilde{u} = -\left(\sum\frac{\partial^2 Q(t)}{\partial t_i^2} + [P(-\partial_x), Q(x)]\right)\tilde{u}.$$

Since there are $Q_1, \dots, Q_l \in \mathbb{C}[\mathfrak{g}]^G$ such that $dQ_1|_{\mathfrak{h}}, \dots, dQ_l|_{\mathfrak{h}}$ are linearly independent at t_0, we have the relation

$$\frac{\partial}{\partial t_i}\tilde{u} \in \mathscr{D}_{\mathfrak{g}} \mathcal{O}_{\mathfrak{g}\times\mathfrak{h}} \tilde{u}$$

on a neighborhood of t_0. This implies

$$\mathscr{N}|_{\mathfrak{g}\times\{t_0\}} = \mathscr{D}_{\mathfrak{g}}(1 \otimes \tilde{u}).$$

On the other hand $1\otimes\tilde{u}$ satisfies

$$\begin{aligned} &(\operatorname{ad}\mathfrak{g})(1\otimes\tilde{u}) = 0 \\ &(P(x) - P(t_0))(1\otimes\tilde{u}) = 0 \qquad \text{for } P \in \mathbb{C}[\mathfrak{g}]^G. \end{aligned}$$

Since $S = \{x \in \mathfrak{g}; P(x) = P(t_0)\}$ is smooth and consists of a single G-orbit, we have $\mathbf{Ch}(\mathscr{N}|_{\mathfrak{g}\times\{t_0\}}) = 0$ or $T_S^*\mathfrak{g}$. Therefore $\mathscr{N}|_{\mathfrak{g}\times\{t_0\}}$ is a simple $\mathscr{D}_{\mathfrak{g}}$-Module or zero.

4.7. *Proof of* (2). Now, we shall construct a non-zero homomorphism $\mathscr{N} \to \int_f \mathcal{O}_{\mathfrak{g}}$. By taking the partial Fourier transformation, it is enough to construct a non-zero homomorphism

$$\mathscr{N}^F \to \int_{\rho\times 1} \mathscr{D}_{\tilde{\mathfrak{g}}\times\mathfrak{h}} \exp\theta.$$

Let $\tilde{u}_1^F$ be the section of $\int_{\rho\times 1}\mathscr{D}_{\tilde{\mathfrak{g}}\times\mathfrak{h}}\exp\theta$ given by $(dx)^{-1}\otimes\omega\otimes\exp\theta$. We shall show that $u^F \mapsto \tilde{u}_1^F$ gives a $\mathscr{D}_{\mathfrak{g}\times\mathfrak{h}}$-linear homomorphism from $\mathscr{N}^F$ to $\int_{\rho\times 1}\mathscr{D}_{\tilde{\mathfrak{g}}\times\mathfrak{h}}\exp\theta$. In order to see this, we have to show

(4.7.1) $$(\operatorname{ad}\mathfrak{g})\tilde{u}_1^F = 0$$

(4.7.2) $$(P(x) - P(\partial_t))\tilde{u}_1^F = 0 \qquad \text{for } P \in \mathbb{C}[\mathfrak{g}]^G$$

(4.7.3) $$(P(\partial_x) - P(t))\tilde{u}_1^F = 0 \qquad \text{for } P \in S[\mathfrak{g}]^G.$$

The first equality follows from the fact that dx and ω are G-invariant. The second equality follows from $P(\partial_t)\exp\theta = P(x)\exp\theta$.

In order to see (4.7.3), we shall recall the following lemma. We define

(4.7.4) $$\Delta = \prod_{\alpha>0} \alpha.$$

Proposition 4.7.1. ([10]) *Let φ be a $\mathfrak{g}$-invariant function defined on a neighborhood of $t_0 \in \mathfrak{h}_{rs}$. Then for any $P \in \mathbb{C}[\mathfrak{g}]^G$*

$$P(\partial_x)\varphi|_{\mathfrak{h}} = \Delta^{-1} P(\partial_t)(\Delta\varphi|_{\mathfrak{h}}).$$

We shall apply this to show (4.7.3). Since $\rho\times 1$ is a $(\# W)$-sheeted unramified covering of $\mathfrak{g}\times\mathfrak{h}$ over $\mathfrak{g}_{rs}\times\mathfrak{h}$, we have

$$\left(\int_{\rho\times 1}\mathscr{D}_{\tilde{\mathfrak{g}}\times\mathfrak{h}}\exp\theta\right)_{an} \xrightarrow{\sim} (\mathcal{O}_{(\mathfrak{g}\times\mathfrak{h})_{an}})^{\# W}$$

on a neighborhood of $(t_0, t_1) \in \mathfrak{g}_{rs} \times \mathfrak{h}$. Since $dx = \pm \Delta \omega$ at any branch, this isomorphism is given by

$$\tilde{u}_1^F \mapsto \varphi(x, wt), \qquad (w \in W).$$

Here $\varphi(x,t)$ is a $\mathfrak{g}$-invariant function defined on a neighborhood of $t_0 \times \mathfrak{h}_{rs}$ given by $\varphi(x,t)|_{\mathfrak{h} \times \mathfrak{h}} = e^{\langle x,t \rangle}/\Delta(x)$. Hence by applying Proposition 4.7.1, we have

$$\begin{aligned}(P(\partial_x)\varphi)(t',t) &= \Delta(t')^{-1} P(\partial_{t'}) e^{\langle t',t \rangle} \\ &= \Delta(t')^{-1} P(t) e^{\langle t',t \rangle} \\ &= (P(t)\varphi)(t',t).\end{aligned}$$

Hence $(P(\partial_x) - P(t))\varphi = 0$ on $\mathfrak{h}_{rs} \times \mathfrak{h}_{rs}$. Since this is G-invariant, this holds on $\mathfrak{g}_{rs} \times \mathfrak{h}_{rs}$. Thus we obtain

$$(P(\partial_x) - P(t))\tilde{u}_1^F = 0 \quad \text{on} \quad \mathfrak{g}_{rs} \times \mathfrak{h}_{rs}.$$

Since $\int_{\rho \times 1} \mathscr{D}_{\tilde{\mathfrak{g}} \times \mathfrak{h}} \exp \theta$ is a simple Module we have (4.7.3) on $\mathfrak{g} \times \mathfrak{h}$. Thus the proof of (2) is completed.

4.8. As seen in §4.2, $\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}}$ and $\int_{\rho^{-1}(\mathbf{N}) \to \mathfrak{g}} \mathcal{O}_{\rho^{-1}(\mathbf{N})}$ are canonically obtained from $\int_f \mathcal{O}_{\tilde{\mathfrak{g}}}$. Hence we have

Proposition 4.8.1. (1) $\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}} \cong \int_p \mathcal{N}$
(2) $\int_{\rho^{-1}(\mathbf{N}) \to \mathfrak{g}} \mathcal{O}_{\rho^{-1}(\mathbf{N})} = j^* \mathcal{N}$

where $p\colon \mathfrak{g} \times \mathfrak{h} \to \mathfrak{g}$ *and* $j\colon \mathfrak{g} \hookrightarrow \mathfrak{g} \times \mathfrak{h}$ *is given by* $\{0\} \hookrightarrow \mathfrak{h}$. *In particular, we have*

$$\Gamma(\mathfrak{g}, \textstyle\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}}) = \Gamma(\mathfrak{g} \times \mathfrak{h}, \mathcal{N}) / \sum \dfrac{\partial}{\partial t_i} \Gamma(\mathfrak{g} \times \mathfrak{h}, \mathcal{N})$$

$$\Gamma(\mathfrak{g}, \textstyle\int_{\rho^{-1}(\mathbf{N}) \to \mathfrak{g}} \mathcal{O}_{\rho^{-1}(\mathbf{N})}) = \Gamma(\mathfrak{g} \times \mathfrak{h}, \mathcal{N}) / \sum t_i \Gamma(\mathfrak{g} \times \mathfrak{h}, \mathcal{N}).$$

If F' denotes the Fourier transformation with respect to $\mathfrak{g} \times \mathfrak{h}$, we have $\mathcal{N}^{F'} = \mathcal{N}$. Hence

$$\begin{aligned}\Gamma(\mathfrak{g}, (\textstyle\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}})^F) &\cong \Gamma(\mathfrak{g} \times \mathfrak{h}, \mathcal{N}^{F'}) / \sum t_i \Gamma(\mathfrak{g} \times \mathfrak{h}, \mathcal{N}^{F'}) \\ &= \Gamma(\mathfrak{g}, \textstyle\int_{\rho^{-1}(\mathbf{N}) \to \mathfrak{g}} \mathcal{O}_{\rho^{-1}(\mathbf{N})}).\end{aligned}$$

Thus, we obtain

Proposition 4.8.2. $\int_{\rho^{-1}(\mathbf{N}) \to \mathfrak{g}} \mathcal{O}_{\rho^{-1}(\mathbf{N})}$ *is the Fourier transform of* $\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}}$.

Proposition 4.8.3. $\mathbf{Ch}(\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}}) = V \cdot (\mathfrak{g} \times \mathbf{N})$. *Here*

$$V = \{(x, y) \in T^* \mathfrak{g} = \mathfrak{g} \times \mathfrak{g}^* = \mathfrak{g} \times \mathfrak{g};\ [x, y] = 0\}.$$

Proof. By the use of Theorem 1.3 and the relation $\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}} = \int_p \mathcal{N}$, we shall calculate $\mathbf{Ch}(\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}})$. Here p is the projection $\mathfrak{g} \times \mathfrak{h} \to \mathfrak{g}$. We have $\mathbf{Ch}(\mathcal{N}) = \Lambda$; where Λ is the one defined on right hand side of (2) in Theorem 4.1. Hence, we have

$$\mathbf{Ch}(\textstyle\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}}) = \varpi_*(\Lambda \cdot \{s = 0\}).$$

Here ϖ is the projection from the (x,y,t)-space onto the (x,y)-space. Now set, for $w \in W$,

$$\Lambda_w = \{(x,y,t,s);\ (x,y,t,ws) \in \Lambda\}.$$

Then we have $\Lambda_w \cdot \{s=0\} = \Lambda \cdot \{s=0\}$. Hence

$$(\# W) \cdot \mathbf{Ch}(\textstyle\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}}) = \varpi_*((\sum_{w \in W} \Lambda_w) \cdot \{s=0\}).$$

On the other hand, we have

$$\tilde{\Lambda} = \sum \Lambda_w = \{(x,y,t,s);\ [x,y]=0, \kappa(x)=\kappa(t), \kappa(y)=\kappa(-s)\}.$$

Let ϖ_1 be the projection from the (x,y,t,s)-space onto the (x,y,s)-space. Then,

$$\varpi(\tilde{\Lambda} \cdot \{s=0\}) = \varpi_1(\tilde{\Lambda}) \cdot \{s=0\}.$$

On the other hand, it is easy to see that

$$\varpi_1(\tilde{\Lambda}) = (\# W) \cdot S$$

where

$$S = \{(x,y,s);\ [x,y]=0, \kappa(y)=\kappa(-s)\}.$$

Thus, we obtain

$$\mathbf{Ch}(\textstyle\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}}) = S \cdot \{s=0\}.$$

Since $\mathbf{N} = \{y;\ \kappa(y)=\kappa(0)\}$, we obtain the desired result. Q.E.D.

Proposition 4.8.4. $\mathbf{Ch}(\mathcal{N}^F) = \mathbf{Ch}(\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}}) \times T^*_{\mathfrak{h}}\mathfrak{h}$. *In particular,*

$$\mathrm{Ch}(\mathcal{N}^F) = \{(x,y) \in \mathfrak{g} \times \mathfrak{g} = \mathfrak{g} \times \mathfrak{g}^* = T^*\mathfrak{g};\ [x,y]=0, y \in \mathbf{N}\} \times T^*_{\mathfrak{h}}\mathfrak{h}.$$

Proof. We have

$$\mathcal{N}^F = \textstyle\int_{\rho \times \mathrm{id}} \mathcal{D}_{\tilde{\mathfrak{g}} \times \mathfrak{h}} \exp\theta.$$

Since $(\mathcal{D}_{\tilde{\mathfrak{g}} \times \mathfrak{h}} \exp\theta)_{\mathrm{an}} = (\mathcal{O}_{\tilde{\mathfrak{g}} \times \mathfrak{h}})_{\mathrm{an}}$ we have

$$(\mathcal{N}^F)_{\mathrm{an}} = (p^*(\textstyle\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}}))_{\mathrm{an}}$$

where p is the projection $\mathfrak{g} \times \mathfrak{h} \to \mathfrak{g}$. Hence we have

$$\mathbf{Ch}(\mathcal{N}^F) = \mathbf{Ch}(p^* \textstyle\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}}) = \mathbf{Ch}(\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}}) \times T^*_{\mathfrak{h}}\mathfrak{h}. \quad \text{Q.E.D.}$$

Proposition 4.8.5. $\mathbf{Ch}(\int_{\rho^{-1}(\mathbf{N}) \to \mathfrak{g}} \mathcal{O}_{\rho^{-1}(\mathbf{N})}) = V \cdot (\mathbf{N} \times \mathfrak{g}^*)$.

Proof. Since $\int_{\rho^{-1}(\mathbf{N}) \to \mathfrak{g}} \mathcal{O}_{\rho^{-1}(\mathbf{N})}$ is the Fourier transform of $\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}}$ which is homogeneous, this is obtained by Theorem 3.2. This proposition also can be proven in the similar way as Proposition 4.8.3 by using

$$\textstyle\int_{\rho^{-1}(\mathbf{N}) \to \mathfrak{g}} \mathcal{O}_{\rho^{-1}(\mathbf{N})} \cong \int_f \mathcal{O}_{\tilde{\mathfrak{g}}}|_{\mathfrak{g} \times \{0\}}.$$

4.9. The correspondence of RH and Perv implies the following

Proposition 4.9.1.

(1) $$\mathrm{DR}_{\mathfrak{g} \times \mathfrak{h}}(\textstyle\int_f \mathcal{O}_{\tilde{\mathfrak{g}}}) = \mathbb{R}f_* \mathbb{C}_{\tilde{\mathfrak{g}}}[-\operatorname{rank} \mathfrak{g}]$$

(2) $\qquad \mathrm{DR}_{\mathfrak{g}}(\int_{\rho}\mathcal{O}_{\tilde{\mathfrak{g}}})=\mathbb{R}\rho_{*}\mathbb{C}_{\tilde{\mathfrak{g}}}$

(3) $\qquad \mathrm{DR}_{\mathfrak{g}}(\int_{\rho^{-1}(\mathrm{N})\to\mathfrak{g}}\mathcal{O}_{\rho^{-1}(\mathrm{N})})=\mathbb{R}\rho_{*}\mathbb{C}_{\tilde{\mathfrak{g}}}|_{\mathrm{N}}[-\operatorname{rank}\mathfrak{g}]$

(4) $\qquad \mathrm{DR}_{\mathfrak{g}\times\mathfrak{h}}(\int_{\rho\times 1}\mathcal{D}_{\tilde{\mathfrak{g}}\times\mathfrak{h}}\exp\theta)=\mathbb{R}(\rho\times 1)_{*}\mathbb{C}_{\tilde{\mathfrak{g}}\times\mathfrak{h}}$

(5) $\qquad \mathrm{DR}_{\mathfrak{g}}(\mathcal{N}^{F}|_{\mathfrak{g}\times\{\lambda\}})=\mathbb{R}\rho_{*}\mathbb{C}_{\tilde{\mathfrak{g}}}.$

Here we considered them in the usual topology.

§ 5. The action of W on $\mathcal{N}$

5.1. The action of the Weyl group W on $\mathfrak{h}$ induces the action of W on $\mathfrak{g}\times\mathfrak{h}$. The subset $f(\tilde{\mathfrak{g}})$ is clearly invariant by this action. Hence W acts on ${}^{\pi}\mathcal{B}_{f(\tilde{\mathfrak{g}})|\mathfrak{g}\times\mathfrak{h}}\cong\int_{f}\mathcal{O}_{\tilde{\mathfrak{g}}}\cong\mathcal{N}$.

Through Riemann-Hilbert correspondence in Theorem 2.1, this W-action is closely connected with Lusztig's construction of Springer's representations.

5.2. As seen in the preceding section $\tilde{u}^{F}\mapsto\tilde{u}_{1}^{F}$ gives an isomorphism between $\mathcal{N}^{F}$ and $(\int_{f}\mathcal{O}_{\tilde{\mathfrak{g}}})^{F}$. Hence, by taking the partial Fourier transform $\tilde{u}\mapsto\tilde{u}_{1}=(dx\,dt)^{-1}\otimes\omega$ gives an isomorphism between $\mathcal{N}$ and $\int_{f}\mathcal{O}_{\tilde{\mathfrak{g}}}$.

Now, $\tilde{\mathfrak{g}}_{\mathrm{rs}}=\tilde{\mathfrak{g}}\cap f^{-1}(\mathfrak{g}_{\mathrm{rs}}\times\mathfrak{h})$ is isomorphic to $f(\tilde{\mathfrak{g}})\cap\mathfrak{g}_{\mathrm{rs}}\times\mathfrak{h}$. Hence W acts on $\tilde{\mathfrak{g}}_{\mathrm{rs}}$. It is easy to see that $w^{*}(\omega)=(\operatorname{sgn}w)\omega$ for any $w\in W$. Here $\operatorname{sgn}w=\det_{\mathfrak{h}}(w)$. Since $w^{*}(dt)=(\operatorname{sgn}w)dt$, $\tilde{u}_{1}$ is invariant by the action of W.

Proposition 5.2.1. *W acts on $\mathcal{N}$ by $W\ni w$: $\tilde{u}\mapsto\tilde{u}$.*

5.3. The section u can be realized as a section of ${}^{\pi}\mathcal{B}_{f(\tilde{\mathfrak{g}})|\mathfrak{g}\times\mathfrak{h}}$ as follows. Let $\{P_{1}(x),\ldots,P_{l}(x)\}$ be a homogeneous base of $\mathbb{C}[\mathfrak{g}]^{G}$ where $l=\operatorname{rank}\mathfrak{g}$. Then, for a linear coordinate system $(t_{1},\ldots,t_{l})$, the Jacobian of $(P_{1}(t),\ldots,P_{l}(t))$ equals a constant multiple of $\Delta(t)=\prod_{\alpha\in\Delta_{+}}\langle\alpha,t\rangle$.

Fix a point $t_{0}\in\mathfrak{h}_{\mathrm{rs}}$. We shall consider on a neighborhood of $(t_{0},t_{0})\in\mathfrak{g}\times\mathfrak{h}\cap f(\tilde{\mathfrak{g}})$. Then, there exists a function $\varphi_{1}(x),\ldots,\varphi_{l}(x)$ so that $f(\tilde{\mathfrak{g}})$ is defined by $t_{j}=\varphi_{j}(x)$ $(j=1,\ldots,l)$ and also $P_{j}(t)=P_{j}(x)$ $(j=1,\ldots,l)$. Hence u corresponds to

$$\begin{aligned}&\delta(t_{1}-\varphi_{1}(x))\delta(t_{2}-\varphi_{2}(x))\ldots\delta(t_{l}-\varphi_{l}(x))dx^{-1}\otimes\omega\\&=\delta(t_{1}-\varphi_{1}(x))\ldots\delta(t_{l}-\varphi_{l}(x))\Delta(t)^{-1}.\end{aligned}$$

On the other hand we have

$$\begin{aligned}&\delta(P_{1}(t)-P_{1}(x))\ldots\delta(P_{l}(t)-P_{l}(x))\\&\quad=(\text{Jacobian of }(P_{1}(t),\ldots,P_{l}(t)))^{-1}\,\delta(t_{1}-\varphi_{1}(x))\ldots\delta(t_{l}-\varphi_{l}(x)).\end{aligned}$$

Hence, we obtain

Theorem 5.1. *$\mathcal{N}\xrightarrow{\sim}{}^{\pi}\mathcal{B}_{f(\tilde{\mathfrak{g}})|\mathfrak{g}\times\mathfrak{h}}$ is given by*

$$\tilde{u}\mapsto\delta(P_{1}(t)-P_{1}(x))\ldots\delta(P_{l}(t)-P_{l}(x)).$$

Note that Proposition 5.2.1 follows also from this theorem.

5.4. The action of W on $\int_f \mathcal{O}_{\tilde{\mathfrak{g}}}$ induces the action of W on $\int_{\rho^{-1}(\mathrm{N})\to\mathfrak{g}} \mathcal{O}_{\rho^{-1}(\mathrm{N})}$ and $\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}}$ because they are canonically obtained from $\int_f \mathcal{O}_{\tilde{\mathfrak{g}}}$ (see §4.2). On the other hand, we see $\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}}$ is the Fourier transform of $\int_{\rho^{-1}(\mathrm{N})\to\mathfrak{g}} \mathcal{O}_{\rho^{-1}(\mathrm{N})}$. More precisely we have the following theorem.

Theorem 5.2. *Together with the actions of W, we have*

$$(\textstyle\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}})^F = (\int_{\rho^{-1}(\mathrm{N})\to\mathfrak{g}} \mathcal{O}_{\rho^{-1}(\mathrm{N})}) \otimes \mathrm{sgn}.$$

Proof. As seen in §4.8, we have

$$\Gamma(\mathfrak{g}, \textstyle\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}}) = \Gamma(\mathfrak{g}\times\mathfrak{h}; \mathcal{N}) \Big/ \sum_i \frac{\partial}{\partial t_i} \Gamma(\mathfrak{g}\times\mathfrak{h}; \mathcal{N})$$

and

$$\textstyle\int_{\rho^{-1}(\mathrm{N})\to\mathfrak{g}} \mathcal{O}_{\rho^{-1}(\mathrm{N})} = \Gamma(\mathfrak{g}\times\mathfrak{h}; \mathcal{N}) / \sum_i t_i \Gamma(\mathfrak{g}\times\mathfrak{h}; \mathcal{N}).$$

Since $a(t)\tilde{u} \bmod \sum \frac{\partial}{\partial t_i} \Gamma(\mathfrak{g}\times\mathfrak{h}; \mathcal{N})$ corresponds to $a(t)\tilde{u}\,dt$ in $\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}}$ (here $dt = dt_1 \dots dt_l$ and $(t_1, \dots, t_l)$ is a linear coordinate of $\mathfrak{h}$) $w \in W$ acts by $a(t)\tilde{u} \mapsto \mathrm{sgn}\, w a(w^{-1}t)\tilde{u}$.

On the other hand $w \in W$ acts on $a(\partial_t)\tilde{u} \bmod \sum t_i \Gamma(\mathfrak{g}\times\mathfrak{h}; \mathcal{N})$ by $a(\partial_t)\tilde{u} \mapsto a(w^{-1}\partial_t)\tilde{u}$. Thus we obtain the desired result. Q.E.D.

Now, we decompose, according to the W-action,

$$\textstyle\int_{\rho^{-1}(\mathrm{N})\to\mathfrak{g}} \mathcal{O}_{\rho^{-1}(\mathrm{N})} \cong \bigoplus_{\chi\in\hat{W}} V_\chi \otimes \mathcal{M}_\chi \tag{5.4.1}$$

where $\hat{W}$ is the set of irreducible representations of W and V_χ is a representation space of χ.

Theorem 5.3. (1) *$\mathcal{M}_\chi$ is a simple $\mathcal{D}_\mathfrak{g}$-Module.*
(2) *The $\mathcal{M}_\chi$'s are not isomorphic to each other.*
(3) *The support of $\mathcal{M}_\chi$ is the closure of a nilpotent G-orbit.*

Proof. Note that (3) follows from (1). By Theorem 5.2, we have

$$\textstyle\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}} \cong \bigoplus_{\chi\in\hat{W}} V_\chi \otimes \mathcal{M}^F_{\chi'}, \qquad \chi' = \chi \otimes \mathrm{sgn}. \tag{5.4.2}$$

In order to prove this proposition it is enough to show that the $\mathcal{M}^F_{\chi'}$'s are simple $\mathcal{D}_\mathfrak{g}$-Modules and not isomorphic to each other. Since $\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}}$ is a minimal extension of its restriction to $\mathfrak{g}_{\mathrm{rs}}$, so is $\mathcal{M}^F_{\chi'}$. Hence it is sufficient to show them for $\mathcal{M}^F_{\chi'}|_{\mathfrak{g}_{rs}}$. Now $\rho^{-1}(\mathfrak{g}_{\mathrm{rs}}) \to \mathfrak{g}_{\mathrm{rs}}$ is a principal W-bundle and hence W acts on $\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}}|_{\mathfrak{g}_{rs}}$. This action coincides with the action already defined. Now fix a point $x_0 \in \mathfrak{h}_{\mathrm{rs}}$. Then, $\mathrm{DR}(\int_\rho \mathcal{O}_{\tilde{\mathfrak{g}}})_{x_0} \cong H^0(\rho^{-1}(x_0), \mathbb{C})$. The action of W via the structure of principal W-bundle and the action via the monodromy endow $H^0(\rho^{-1}(x_0), \mathbb{C})$ with the structure of bi-W-module, and this is isomorphic to $\mathbb{C}[W]$. Since $\mathbb{C}[W] \cong \bigoplus_\chi V_\chi \otimes V_\chi^*$ as W-bi-modules, the monodromy action to $\mathrm{DR}(\mathcal{M}^F_{\chi'})_{x_0}$ is isomorphic to V_χ^*. Since the V_χ^*'s are irreducible and not isomorphic to each other, we obtain the desired result. Q.E.D.

By the argument discussed here we obtain

Corollary 5.4.1. *The multiplicity of* $\mathcal{M}_\chi$ *along* $T^*_{\{0\}}\mathfrak{g}$ *equals* $\dim V_\chi$.

We shall denote by $\mathbf{O}(\chi)$ the nilpotent G-orbit whose closure is the support of $\mathcal{M}_\chi$. For example $\mathbf{O}$(trivial) is the orbit of regular nilpotent elements and $\mathbf{O}$(sgn) is the origin.

§ 6. The invariant holonomic system

6.1. For $\lambda\in\mathfrak{h}^*$, we define a $\mathcal{D}_\mathfrak{g}$-module $\mathcal{M}_\lambda$ by

(6.1.1)
$$\begin{aligned}\mathcal{M}_\lambda &= \mathcal{D}_\mathfrak{g}/(\mathcal{D}_\mathfrak{g}\operatorname{ad}\mathfrak{g} + \sum_{P\in\mathbb{C}[\mathfrak{g}]^G}\mathcal{D}_\mathfrak{g}(P(x)-P(\lambda)))\\ &= \mathcal{D}_\mathfrak{g}u_\lambda.\end{aligned}$$

Hence its Fourier transform is given by

(6.1.2)
$$\begin{aligned}\mathcal{M}^F_\lambda &= \mathcal{D}_\mathfrak{g}/(\mathcal{D}_\mathfrak{g}\operatorname{ad}\mathfrak{g} + \sum_{P\in S(\mathfrak{g})^G}\mathcal{D}_\mathfrak{g}(P(\partial_x)-P(\lambda)))\\ &= \mathcal{D}_\mathfrak{g}u^F_\lambda.\end{aligned}$$

This is nothing but the system of linear differential equations for invariant eigendistributions. We call $\mathcal{M}^F_\lambda$ the *invariant holonomic system* on $\mathfrak{g}$. In this section, we shall study the relation between $\mathcal{M}_\lambda$ and $\mathcal{N}$.

6.2. For $\lambda\in\mathfrak{h}^*$, we define

$$\Delta(\lambda)=\{\alpha\in\Delta;(\alpha,\lambda)=0\}$$

and

$$W(\lambda)=\{w\in W;\ w\lambda=\lambda\}.$$

If we denote by $\mathfrak{h}(\lambda)$ the vector space generated by $\Delta(\lambda)$, then $(\mathfrak{h}(\lambda),\Delta(\lambda))$ is a root system and $W(\lambda)$ is its Weyl group. Let us denote by $\mathcal{H}(\mathfrak{h}(\lambda))$ the space of harmonic polynomials on $\mathfrak{h}(\lambda)$. By definition

$$\begin{aligned}\mathcal{H}(\mathfrak{h}(\lambda)) &= \{f\in\mathbb{C}[\mathfrak{h}(\lambda)];\ (\#W(\lambda))f(\mu)=\sum_{w\in W(\lambda)}f(\mu+w\mu')\text{ for any }\mu,\mu'\in\mathfrak{h}(\lambda)\}\\ &= \{f\in\mathbb{C}[\mathfrak{h}(\lambda)];\ P(\partial_\mu)f(\mu)=P(0)f(\mu)\text{ for any }P\in S(\mathfrak{h}(\lambda))^{W(\lambda)}\}.\end{aligned}$$

As is well-known, $\mathcal{H}(\mathfrak{h}(\lambda))$ is isomorphic to $\mathbb{C}[W(\lambda)]$ as representations of $W(\lambda)$.

6.3. As was seen in § 4, we have

(6.3.1)
$$\begin{aligned}\mathcal{N}^F &= \mathcal{D}_{\mathfrak{g}\times\mathfrak{h}}/(\mathcal{D}_{\mathfrak{g}\times\mathfrak{h}}\operatorname{ad}\mathfrak{g} + \sum_{P\in\mathbb{C}[\mathfrak{g}]^G}\mathcal{D}_{\mathfrak{g}\times\mathfrak{h}}(P(\partial_x)-P(t))\\ &\quad + \sum_{Q\in S(\mathfrak{g})^G}\mathcal{D}_{\mathfrak{g}\times\mathfrak{h}}(Q(x)-Q(\partial_t))),\\ &= \mathcal{D}_{\mathfrak{g}\times\mathfrak{h}}\tilde{u}^F.\end{aligned}$$

We shall define a homomorphism

$$\Psi_\lambda\colon \mathscr{H}(\mathfrak{h}(\lambda))\otimes_{\mathbb{C}}\mathscr{M}_\lambda^F \to \mathscr{N}^F|_{t=\lambda} \tag{6.3.2}$$

$$\text{by} \quad P\otimes u_\lambda^F \mapsto P(\partial_t)\tilde{u}^F|_{t=\lambda}.$$

Here, we identify $\mathfrak{h}(\lambda)$ with a vector subspace of $\mathfrak{h}^*$ by the invariant quadratic form.

Proposition 6.3.1. *Ψ_λ is a well-defined $\mathscr{D}_{\mathfrak{g}}$-linear homomorphism.*

In order to see this, we have to show for any $P\in\mathscr{H}(\mathfrak{h}(\lambda))$

(1) $(\mathrm{ad}\,\mathfrak{g})\,P(\partial_t)\tilde{u}^F|_{t=\lambda}=0$

(2) $(Q(\partial_x)-Q(\lambda))\,P(\partial_t)\tilde{u}^F|_{t=\lambda}=0$ for any $Q\in\mathbb{C}[\mathfrak{g}^*]^G$.

The first statement is obvious.

We shall show (2). We have

$$(Q(\partial_x)-Q(\lambda))\,P(\partial_t)\tilde{u}^F=P(\partial_t)(Q(t)-Q(\lambda))\tilde{u}^F.$$

Hence it is enough to show

$$P(\partial_t)(Q(t)-Q(\lambda))\in I_\lambda\mathscr{D}_{\mathfrak{h}}.$$

Here I_λ is the ideal of $\mathscr{O}_{\mathfrak{h}}$ consisting of the functions vanishing at λ.

Let $\mathfrak{h}=\mathfrak{h}(\lambda)\oplus\mathfrak{h}(\lambda)^\perp$ be the orthogonal decomposition of $\mathfrak{h}$. Then $Q\in\mathbb{C}[\mathfrak{h}]^W$ can be written in the form $Q=\sum Q_jR_j$ where $Q_j\in\mathbb{C}[\mathfrak{h}(\lambda)]^{W(\lambda)}$ and $R_j\in\mathbb{C}[\mathfrak{h}(\lambda)^\perp]$ because $W(\lambda)$ acts trivially on $\mathfrak{h}(\lambda)^\perp$,

Since $\lambda\in\mathfrak{h}(\lambda)^\perp$, we have

$$\begin{aligned} Q(t)-Q(\lambda)&=\sum(Q_j(t)R_j(t)-Q_j(0)R_j(\lambda))\\ &=\sum(R_j(t)-R_j(\lambda))Q_j(t)+\sum R_j(\lambda)(Q_j(t)-Q_j(0)). \end{aligned}$$

Since $R_j(t)$ commutes with $P(\partial_t)$, we have

$$P(\partial_t)\sum(R_j(t)-R_j(\lambda))Q_j(t)\in I_\lambda\mathscr{D}_{\mathfrak{h}}.$$

Hence (2) is reduced to the following Lemma.

Lemma 6.3.2. *For $P\in\mathscr{H}(\mathfrak{h}(\lambda))$ and $Q\in\mathbb{C}[\mathfrak{h}(\lambda)]^{W(\lambda)}$ with $Q(0)=0$, we have*

$$P(\partial_t)Q(t)\in I\mathscr{D}_{\mathfrak{h}(\lambda)}.$$

Here I is the maximal ideal of $\mathscr{O}_{\mathfrak{h}(\lambda)}$ consisting of the functions vanishing at the origin.

Proof. By taking the Fourier transformation, it is enough to show

$$P(t)Q(-\partial_t)\in\sum_j\frac{\partial}{\partial t_j}\mathscr{D}_{\mathfrak{h}(\lambda)}.$$

Here $(t_1, \ldots, t_N)$ is a linear coordinate system of $\mathfrak{h}(\lambda)$. By taking the formal adjoint, this is equivalent to

$$Q(\partial_t)P(t) \in \sum_j \mathscr{D}_{\mathfrak{h}(\lambda)} \frac{\partial}{\partial t_j}.$$

This follows from the definition of $\mathscr{H}(\mathfrak{h}(\lambda))$. Q.E.D.

In the preceding section, we have defined the homomorphism

$$\Psi_\lambda^F\colon \mathscr{H}(\mathfrak{h}(\lambda)) \otimes_{\mathbb{C}} \mathscr{M}_\lambda \to \mathscr{N}|_{t=\lambda}.$$

On the other hand, since λ is a fixed point of $W(\lambda)$, $W(\lambda)$ acts on $\mathscr{N}|_{t=\lambda}$, and it is obvious that Ψ_λ^F is $W(\lambda)$-linear. Thus we can define

$$\Phi_\lambda\colon \mathscr{M}_\lambda \to \mathrm{Hom}_{W(\lambda)}(\mathscr{H}(\mathfrak{h}(\lambda)), \mathscr{N}|_{t=\lambda}).$$

Theorem 6.1. *Φ_λ is an isomorphism.*

The proof is devided into three steps.
(1) Φ_λ^F is an isomorphism on $\mathfrak{g}_{rs}$.
(2) $\operatorname{codim} \kappa(S) \geqq 2$. Here S is the support of the kernel of Φ_λ^F and κ is the canonical morphism from $\mathfrak{g}$ onto $\mathfrak{h}/W$.
(3) Φ_λ^F is an isomorphism.

6.5. *Proof of* (1). As was seen in Propositions 4.8.3 and 4.8.4 we have

$$\mathbf{Ch}(\mathscr{N}^F|_{t=\lambda}) = \{(x,y); [x,y]=0\} \cdot (\mathfrak{g} \times \mathbf{N}). \tag{6.5.1}$$

Hence the multiplicity of $\mathscr{N}^F|_{t=\lambda}$ along $T_{\mathfrak{g}}^*\mathfrak{g}$ equals $\# W$. Hence in order to show (1), it is enough to prove that

$$\begin{aligned}\mathscr{H}om_{\mathscr{D}}(\Phi_\lambda^F, \mathscr{O}_{\mathfrak{g}_{\mathrm{an}}})_{x_0}\colon\ & \mathscr{H}om_{\mathscr{D}_{\mathfrak{g}_{\mathrm{an}}}}(\mathrm{Hom}_{W(\lambda)}(\mathscr{H}(\mathfrak{h}(\lambda)), \mathscr{N}^F|_{t=\lambda})_{\mathrm{an}}, \mathscr{O}_{\mathfrak{g}_{\mathrm{an}}})_{x_0}\\ &\to \mathscr{H}om_{\mathscr{D}_{\mathfrak{g}_{\mathrm{an}}}}((\mathscr{M}_\lambda^F)_{\mathrm{an}}, \mathscr{O}_{\mathfrak{g}_{\mathrm{an}}})_{x_0}\end{aligned}$$

is injective for any $x_0 \in \mathfrak{g}_{rs}$.

By the adjoint action, we may assume $x_0 \in \mathfrak{h}_{rs}$. By the well-posedness of Cauchy's problem

$$\mathscr{H}om_{\mathscr{D}_{\mathfrak{g}_{\mathrm{an}}}}(\mathscr{N}^F_{\mathrm{an}}|_{t=\lambda}, \mathscr{O}_{\mathfrak{g}_{\mathrm{an}}})_{x_0} \cong \mathscr{H}om_{\mathscr{D}_{(\mathfrak{g}\times\mathfrak{h})_{\mathrm{an}}}}(\mathscr{N}^F_{\mathrm{an}}, \mathscr{O}_{(\mathfrak{g}\times\mathfrak{h})_{\mathrm{an}}})_{(x_0,\lambda)}.$$

For $w \in W$, let $\varphi_w(x,t)$ be a $\mathfrak{g}$-invariant holomorphic function defined on a neighborhood of $\{x_0\} \times \mathfrak{h}$ given by

$$\varphi_w(x,t)|_{\mathfrak{h}\times\mathfrak{h}} = e^{\langle wx,t\rangle} / \textstyle\prod_{\alpha>0} \langle \alpha, x \rangle.$$

Then, $\mathbb{C}[W] \xrightarrow{\sim} \mathscr{H}om_{\mathscr{D}_{(\mathfrak{g}\times\mathfrak{h})_{\mathrm{an}}}}(\mathscr{N}^F_{\mathrm{an}}, \mathscr{O}_{(\mathfrak{g}\times\mathfrak{h})_{\mathrm{an}}})_{(x_0,\lambda)}$ by $w \mapsto (\tilde{u}^F \mapsto \varphi_w(x,t))$. The action of $W(\lambda)$ on the right hand side is the left multiplication on $\mathbb{C}[W]$. We shall choose $P \in W(\lambda)$ such that $\mathscr{H}(\mathfrak{h}(\lambda)) \xleftarrow{\sim} \mathbb{C}[W(\lambda)]P$. Then, we have

$$\mathbb{C}[W] \simeq \mathscr{H}om_{\mathscr{D}_{\mathfrak{g}_{\mathrm{an}}}}(\mathrm{Hom}_{W(\lambda)}(\mathscr{H}(\mathfrak{h}(\lambda)), \mathscr{N}^F|_{t=\lambda})_{\mathrm{an}}, \mathscr{O}_{\mathfrak{g}_{\mathrm{an}}})_{x_0}.$$

Then $\mathcal{H}om_{\mathcal{D}}(\Phi_\lambda, \mathcal{O}_{\mathfrak{g}})_{x_0}$ is given by

$$w \longmapsto (u_\lambda^F \mapsto (P(\partial_t)\varphi_w(x,t))|_{t=\lambda}).$$

On the other hand $P(\partial_t)\varphi_w = P(wx)\varphi_w(x,\lambda)$. Therefore it is enough to show the homomorphism

$$\mathbb{C}[W] \to \mathcal{O}_{\mathfrak{h},x_0}$$

given by $w \mapsto P(wx)e^{\langle x, w^{-1}\lambda\rangle}$ is injective. This easily follows from the fact that $\sum_j \mathbb{C}[x]e^{\langle x,\mu_j\rangle}$ forms a direct sum for mutually different μ_j's.

This shows (1).

6.6. *Proof of* (2). Since $\mathcal{N}^F|_{t=\lambda}$ is the minimal extension of its restriction to $\mathfrak{g}_{rs}$, the same thing holds for

$$\mathrm{Hom}_{\mathbb{C}[W(\lambda)]}(\mathcal{H}(\mathfrak{h}(\lambda)), \mathcal{N}^F|_{t=\lambda}).$$

Hence the surjectivity of $\Phi_\lambda^F|\mathfrak{g}_{rs}$ implies

(1′) Φ_λ^F is surjective.

Let $\mathcal{L}$ be the kernel of Φ_λ^F. Then we have

$$\mathbf{Ch}(\mathcal{M}_\lambda^F) = \mathbf{Ch}(\mathcal{L}) + \mathbf{Ch}(\mathcal{N}^F|_{t=\lambda}). \tag{6.6.1}$$

On the other hand, we have

$$\mathrm{Ch}(\mathcal{M}_\lambda^F) \subset \{(x,y)\in\mathfrak{g}\times\mathbf{N};\, [x,y]=0\}.$$

Hence if Λ is an irreducible component of the right hand side, we have

$$\mathrm{mult}_\Lambda \mathcal{M}_\lambda^F \geqq \mathrm{mult}_\Lambda \mathcal{N}^F|_{t=\lambda}. \tag{6.6.2}$$

Moreover, if

$$\mathrm{mult}_\Lambda \mathcal{M}_\lambda^F \leqq \mathrm{mult}_\Lambda \mathcal{N}^F|_{t=\lambda} \tag{6.6.3}$$

holds, then $\mathrm{Ch}(\mathcal{L})$ does not contain Λ.

Now, we know that

$$\mathbf{Ch}(\mathcal{N}^F|_{t=\lambda}) = \{(x,y)\in\mathfrak{g}\times\mathfrak{g};\, [x,y]=0\}\cdot\mathfrak{g}\times\mathbf{N}.$$

Assume that at some point $(x_0, y_0)\in\Lambda$ we have

$$\dim \mathfrak{g}_{x_0}\cap\mathfrak{g}_{y_0} = \mathrm{rank}\,\mathfrak{g}. \tag{6.6.4}$$

Then by the Jacobian criterion $\{(x,y);\, [x,y]=0\}$ is non-singular on a neighborhood of (x_0, y_0).

Let $\{P_1, \dots, P_l\}$ be a system of homogeneous generators of $\mathbb{C}[\mathfrak{g}]^G$ ($l = \mathrm{rank}\,\mathfrak{g}$). Then $\mathrm{mult}_\Lambda \mathcal{N}^F|_{t=\lambda}$ is the multiplicity of $\mathcal{O}_{\mathfrak{g}\times\mathfrak{g}}/I$ along Λ where I is the ideal generated by $[x,y]$ and $P_j(y)$. On the other hand, the symbol ideal of u_λ contains I. Hence $\mathrm{mult}_\Lambda \mathcal{M}_\lambda^F$ is equal to or less than the multiplicity of $\mathcal{O}_{\mathfrak{g}\times\mathfrak{g}}/I$ along Λ. Thus (6.6.3) holds and hence $\mathrm{Ch}(\mathcal{L}) \not\supset \Lambda$.

Remark that in the case when $\mathfrak{g}$ is a classical semisimple Lie algebra, (6.6.4) holds for any Λ. Therefore we have $\mathrm{Ch}(\mathscr{L})=\emptyset$ and hence Φ_λ is an isomorphism. However in the exceptional Lie algebra case (e.g. F_4), there exists some Λ where (6.6.4) does not hold (communicated by J. Sekiguchi).

Now we shall prove (2). In order to prove this, it is enough to show that, for Λ with $\operatorname{codim} \kappa\pi(\Lambda)=1$, (6.6.4) holds. Here π is the projection from $T^*\mathfrak{g}$ onto $\mathfrak{g}$.

In this case we can take $h+x_\alpha$ or h as x_0. Here, α is a root, h is a generic point of $\alpha^{-1}(0)$, and x_α is a root vector of α. In the first case we have $\dim \mathfrak{g}_{x_0} = \operatorname{rank} \mathfrak{g}$ and in the second case we have $\dim \mathfrak{g}_{x_0} \cap \mathfrak{g}_{x_\alpha} = \operatorname{rank} \mathfrak{g}$. Thus, (6.6.4) is satisfied in either case, and hence (2) holds.

6.7. *Proof of* (3). In order to prove this we shall show the kernel $\mathscr{L}$ of Φ_λ^F equals zero. Let S be the support of $\mathscr{L}$. By (2), we have

$$\dim \kappa(S) \leqq \operatorname{rank} \mathfrak{g} - 2. \tag{6.7.1}$$

Now, assuming S is non-empty we shall deduce the contradiction. Let x_0 be a generic point of S.

Lemma 6.7.1. $\mathscr{E}xt^1_{\mathscr{D}_{\mathfrak{g}_{\mathrm{an}}}}(\mathscr{N}^F_{\mathrm{an}}|_{t=\lambda}, \mathscr{L}_{\mathrm{an}})_{x_0}=0.$

Proof. On a neighborhood of x_0, $\mathscr{L}_{\mathrm{an}}$ is isomorphic to a direct sum of copies of $\mathscr{B}_{S|\mathfrak{g}_{\mathrm{an}}}$.

Hence it is enough to show that

$$\mathscr{E}xt^1_{\mathscr{D}_{\mathfrak{g}_{\mathrm{an}}}}(\mathscr{N}^F_{\mathrm{an}}|_{t=\lambda}, \mathscr{B}_{S|\mathfrak{g}_{\mathrm{an}}})_{x_0}=0.$$

By Lemma 2.3.1, this is equivalent to

$$\mathscr{H}^{\operatorname{codim} S-1}(\mathrm{DR}_{\mathfrak{g}}(\mathscr{N}^F|_{t=\lambda}))_{x_0}=0.$$

On the other hand, by Proposition 4.9.1,

$$\mathrm{DR}_{\mathfrak{g}}(\mathscr{N}^F|_{t=\lambda})=\mathbb{R}\rho_*\mathbb{C}_{\tilde{\mathfrak{g}}}.$$

Thus it is enough to show

$$H^{\operatorname{codim} S-1}(\rho^{-1}(x_0), \mathbb{C})=0.$$

Thus the lemma is reduced to

$$\operatorname{codim} S - 1 > 2\dim \rho^{-1}(x_0) = \dim \mathfrak{g} - \dim[\mathfrak{g}, x_0] - \operatorname{rank} \mathfrak{g},$$

or

$$\dim S + 1 < \dim[\mathfrak{g}, x_0] + \operatorname{rank} \mathfrak{g}. \tag{6.7.2}$$

Since the fiber of κ is a union of a finite number of G-orbits, we have

$$\dim S = \dim \kappa(S) + \dim[\mathfrak{g}, x_0].$$

Hence (6.7.1) implies (6.7.2).

Now, we shall recall the following famous deep result due to Harish-Chandra ([10, 11]).

Theorem 6.7.2 (Harish-Chandra). *$(\mathscr{M}_\lambda^F)_{\mathrm{an}}$ has no non-zero coherent quotient whose support is contained in $\mathfrak{g}-\mathfrak{g}_{\mathrm{rs}}$.*

What he proved is not exactly of this form, but his main purpose was rather to show the regularity of invariant eigendistributions. In fact, if one inspects his proof, one can apply his method to obtain the above form of expression. Rather, in our point of view, the essence of his regularity theorem lies in the above stated point and the regularity property can be deduced directly from this. (See also unpublished paper of Atiyah-Schmid.)

Now, by Lemma 6.7.1, the exact sequence

$$0 \to \mathscr{L}_{\mathrm{an}} \to (\mathscr{M}_\lambda^F)_{\mathrm{an}} \to (\mathscr{N}^F|_{t=\lambda})_{\mathrm{an}} \to 0$$

splits on a neighborhood of x_0. Hence $\mathscr{L}_{\mathrm{an}}$ is a quotient of $(\mathscr{M}_\lambda^F)_{\mathrm{an}}$ on a neighborhood of x_0. This contradicts Theorem 6.7.2.

This completes the proof of Theorem 6.1.

Corollary 6.7.3. *For λ and $\lambda' \in \mathfrak{h}$, we have*

$$(\mathscr{M}_\lambda^F)_{\mathrm{an}} \cong (\mathscr{M}_{\lambda'}^F)_{\mathrm{an}}.$$

Proof. We have, by Theorem 6.1 and Proposition 4.9.1,

$$\mathrm{DR}(\mathscr{M}_\lambda^F) \cong \mathrm{DR}(\mathscr{M}_{\lambda'}^F) = \mathbb{R}\rho_*(\mathbb{C}_{\tilde{\mathfrak{g}}}).$$

Thus this corollary follows from the Riemann-Hilbert correspondence in Theorem 2.1.

Note that this isomorphism is not canonical. Note also that $\mathscr{M}_\lambda^F$ is not completely regular in general and $\mathscr{M}_\lambda^F$ is not isomorphic to $\mathscr{M}_{\lambda'}^F$.

§ 7. Homogeneous decomposition of $\mathscr{M}_0^F$

7.1. Let $\vartheta_{\mathfrak{g}}$ be the vector field $\sum_j x_j \partial/\partial x_j$ for a linear coordinate system $(x_1, \ldots, x_n)$ of $\mathfrak{g}$. Similarly we define the vector field $\vartheta_{\mathfrak{h}}$ on $\mathfrak{h}$.

We define the $\mathscr{D}_{\mathfrak{g}}$-linear endomorphism ϑ of $\mathscr{M}_0$ by $\vartheta\colon u_0 \mapsto \vartheta_{\mathfrak{g}} u_0$. This is well-defined because the defining ideal of u_0 is homogeneous.

Lemma 7.1.1. $(\vartheta_{\mathfrak{g}} + \vartheta_{\mathfrak{h}} + c)\tilde{u} = 0$ *where* $c = \# \Delta_+ + \operatorname{rank} \mathfrak{g} = \dim \mathfrak{b}$.

Proof. We shall take orthonormal base $(x_1, \ldots, x_n)$ and $(t_1, \ldots, t_l)$ of $\mathfrak{g}$ and $\mathfrak{h}$, where $n = \dim \mathfrak{g}$ and $l = \operatorname{rank} \mathfrak{g}$. Then

$$(x_1^2 + \ldots + x_n^2 - t_1^2 - \ldots - t_l^2)\tilde{u} = 0$$

and

$$\left(\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2} - \frac{\partial^2}{\partial t_1^2} - \ldots - \frac{\partial^2}{\partial t_l^2}\right)\tilde{u} = 0.$$

By taking the commutators, we have

$$\left(4x_1\frac{\partial}{\partial x_1}+\ldots+4x_n\frac{\partial}{\partial x_n}+4t_1\frac{\partial}{\partial t_1}+\ldots+4t_l\frac{\partial}{\partial t_l}+2n+2l\right)\tilde{u}=0.$$

The lemma follows from this and $n=2(\#\Delta_+)+l$.

Proposition 7.1.2. *$\Phi_0(\vartheta u)(P)=\Phi_0(u)((-c-\vartheta_{\mathfrak{h}})(P))$ for any $P\in\mathscr{H}(\mathfrak{h})$ and any $u\in\mathscr{M}_0$.*

Proof. We may assume $u=u_0$. Then we have

$$\Phi_0(\vartheta_{\mathfrak{g}}u_0)(P)=\vartheta_{\mathfrak{g}}\Phi_0(u_0)(P)=\vartheta_{\mathfrak{g}}P(\partial_t)\tilde{u}|_t=0.$$

On the other hand,

$$\vartheta_{\mathfrak{g}}P(\partial_t)\tilde{u}=P(\partial_t)\vartheta_{\mathfrak{g}}\tilde{u}=P(\partial_t)(-c-\vartheta_{\mathfrak{h}})\tilde{u}=(-cP(\partial_t)-[P(\partial_t),\vartheta_{\mathfrak{h}}])\tilde{u}-\vartheta_{\mathfrak{h}}P(\partial_t)\tilde{u}.$$

Since $\vartheta_{\mathfrak{h}}\in\sum t_j\mathscr{D}_{\mathfrak{g}\times\mathfrak{h}}$, and $[P(\partial_t),\vartheta_{\mathfrak{h}}]=(\vartheta_{\mathfrak{h}}P)(\partial_t)$, the statement holds for $u=u_0$. The general case follows from the $\mathscr{D}$-linearity. Q.E.D.

We set $\mathscr{M}(\alpha)=\mathscr{M}_0/(\vartheta-\alpha)\mathscr{M}_0$ for $\alpha\in\mathbb{C}$, and $\mathscr{H}(\alpha)=\{P\in\mathscr{H}(\mathfrak{h});\,\vartheta_{\mathfrak{h}}P=\alpha P\}$. Since $\mathscr{H}(\mathfrak{h})=\bigoplus\mathscr{H}(\alpha)$, we have, by Theorem 6.1 and the preceding lemma

$$\mathscr{M}_0=\bigoplus\mathscr{M}(\alpha) \tag{7.1.1}$$

and

$$\mathscr{M}(\alpha)=\mathscr{H}\!om_W(\mathscr{H}(-c-\alpha),\mathscr{N}|_{t=0}). \tag{7.1.2}$$

Proposition 7.1.3 (Barbasch-Vogan [6]). *For any nilpotent orbit $\mathbf{O}$ of $\mathfrak{g}$, we have*

$$\mathscr{H}\!om_{\mathscr{D}_{\mathfrak{g}_{\mathrm{an}}}}(\mathscr{M}(\alpha)_{\mathrm{an}},\mathscr{D}_{\mathbf{O}|\mathfrak{g}_{\mathrm{an}}})|_{\mathbf{O}}=\begin{cases}0 & \text{for } \alpha>\lambda_{\mathbf{O}}=\frac{1}{2}\dim\mathbf{O}-\dim\mathfrak{g}\\ \mathbb{C}_{\mathbf{O}_{\mathrm{an}}} & \text{for } \alpha=\lambda_{\mathbf{O}}\end{cases}$$

Corollary 7.1.4. (1) *For any nilpotent orbit $\mathbf{O}$, there is $\chi_{\mathbf{O}}\in\hat{W}$ such that $\mathscr{M}_{\chi_{\mathbf{O}}}={}^{\pi}\mathscr{B}_{\mathbf{O}|\mathfrak{g}}$*

$$[\mathscr{H}(\alpha);\chi_{\mathbf{O}}]=\begin{cases}1 & \text{for } \alpha=\#\Delta_+-\frac{1}{2}\dim\mathbf{O}\\ 0 & \text{for } \alpha<\#\Delta_+-\frac{1}{2}\dim\mathbf{O}\end{cases}$$

(2) *The dimension of the support of any non-zero sub-Module of $\mathscr{M}(\alpha)$ is equal to or larger than $2(\alpha+\dim\mathfrak{g})$.*

Corollary 7.1.5. *Let $J_{\mathbf{O}}$ be the defining ideal of $\bar{\mathbf{O}}$ and J the defining ideal of the set $\mathbf{N}$ of nilpotent elements. Then*

$${}^{\pi}\mathscr{B}_{\mathbf{O}|\mathfrak{g}}\cong\mathscr{D}_{\mathfrak{g}}/(\mathscr{D}_{\mathfrak{g}}\operatorname{ad}\mathfrak{g}+\mathscr{D}_{\mathfrak{g}}J_{\mathbf{O}}^k+\mathscr{D}_{\mathfrak{g}}(\vartheta_{\mathfrak{g}}-\lambda_{\mathbf{O}})+\mathscr{D}_{\mathfrak{g}}J)\qquad \textit{for any } k\geqq 1.$$

Proof. Let $\mathscr{L}$ denote the right hand side. Then $\operatorname{Supp}\mathscr{L}\subset\bar{\mathbf{O}}$ and $\mathscr{L}\cong{}^{\pi}\mathscr{B}_{\mathbf{O}|\mathfrak{g}}$ on a neighborhood of $\mathbf{O}$ by Proposition 7.1.3. On the other hand, $\mathscr{L}$ is a quotient of $\mathscr{M}_0$. Since $\mathscr{M}_0$ is completely reducible (i.e. a direct sum of simple $\mathscr{D}_{\mathfrak{g}}$-Modules), so is $\mathscr{L}$. Hence $\mathscr{L}={}^{\pi}\mathscr{B}_{\mathbf{O}|\mathfrak{g}}\oplus\mathscr{L}'$ for some $\mathscr{D}_{\mathfrak{g}}$-Module $\mathscr{L}'$ such that $\operatorname{Supp}\mathscr{L}'\subset\partial\mathbf{O}$. On the other hand the preceding corollary implies $\dim\operatorname{Supp}\mathscr{L}'\geqq\dim\mathbf{O}$, if $\mathscr{L}'\neq 0$. Hence $\mathscr{L}'$ vanishes.

§ 8. Hyperfunction solutions to $\mathcal{N}$ and $\mathcal{M}_0$

8.1. In this section, we shall study hyperfunction solutions to $\mathcal{N}$.

Since the sheaf of hyperfunctions is obtained from the sheaf of holomorphic functions by a cohomological manipulation, the property of hyperfunction solutions is, theoretically derived from that of holomorphic solutions. In our case, the knowledge on $\mathrm{DR}(\mathcal{N})$ permits us to perform this program.

In the sequel, we shall mainly treat the case where $\mathfrak{g}$ is the complexification of a complex semisimple Lie algebra.

Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{g}_{\mathbb{R}}$ a real form of $\mathfrak{g}$. Let us fix a Cartan subalgebra $\mathfrak{h}_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{R}}$ and let $\mathfrak{h}$ be the complexification of $\mathfrak{h}_{\mathbb{R}}$. Let $G_{\mathbb{R}}$ be a connected Lie group with $\mathfrak{g}_{\mathbb{R}}$ as its Lie algebra.

We set

(8.1.1) $\mathfrak{h}_1 = \{t \in \mathfrak{h}_{\mathbb{R}};$ there exists at least two positive roots vanishing at $t\}$.
Then we have

Proposition 8.1.1.

(a) $\mathcal{H}^0_{(\mathfrak{g}_{\mathbb{R}}-\mathfrak{g}_{rs})\times\mathfrak{h}_{\mathbb{R}}}(\mathcal{H}om_{\mathcal{D}_{(\mathfrak{g}\times\mathfrak{h})_{an}}}(\mathcal{N}_{an}, \mathcal{B}_{\mathfrak{g}_{\mathbb{R}}\times\mathfrak{h}_{\mathbb{R}}})) = 0$

(b) $\mathcal{H}^1_{\mathfrak{g}_{\mathbb{R}}\times\mathfrak{h}_1}(\mathcal{H}om_{\mathcal{D}_{(\mathfrak{g}\times\mathfrak{h})_{an}}}(\mathcal{N}_{an}, \mathcal{B}_{\mathfrak{g}_{\mathbb{R}}\times\mathfrak{h}_{\mathbb{R}}})) = 0.$

Here $\mathcal{B}$ denotes the sheaf of hyperfunctions.

Proof. Let us take a subanalytic Whitney stratification

$$\mathfrak{g}_{\mathbb{R}}\times\mathfrak{h}_{\mathbb{R}}\cap f(\tilde{\mathfrak{g}}) = \coprod_i M_i$$

such that

(0) M_i is connected.

(1) Any fiber of $M_i \to \mathfrak{h}_{\mathbb{R}}$ consists of a single $G_{\mathbb{R}}$-orbit (or empty).

(2) $M_i \subset (\mathfrak{g}_{\mathbb{R}} - \mathfrak{g}_{rs}) \times \mathfrak{h}_{\mathbb{R}}$ or $M_i \subset \mathfrak{g}_{rs} \times \mathfrak{h}_{\mathbb{R}}$.

(3) $M_i \subset \mathfrak{g}_{\mathbb{R}} \times (\mathfrak{h}_{\mathbb{R}} - \mathfrak{h}_1)$ or $M_i \subset \mathfrak{g}_{\mathbb{R}} \times \mathfrak{h}_1$.

(4) $M_i \to \mathfrak{h}_{\mathbb{R}}$ has constant rank.

(5) $M_i \ni (x, t) \mapsto \dim[\mathfrak{g}, x]$ is locally constant.

(6) $\mathrm{DR}(\mathcal{N})|_{M_i}$ has locally constant cohomology groups.

The properties (4) and (5) implies

(7) For $(x, t) \in M_i$, $\dim M_i = \dim[\mathfrak{g}, x] + \text{rank of } (M_i \to \mathfrak{h}_{\mathbb{R}})$.

Now we shall remark the following lemma

Lemma 8.1.2. *Let M be a real analytic manifold and N a closed submanifold of M. Let $Y \subset X$ be complexifications of $N \subset M$. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-Module such that all the cohomology groups of $\mathrm{DR}_X(\mathcal{M})|_N$ are locally constant. Then we have*

$$\begin{aligned}&\mathbb{R}\Gamma_N(\mathbb{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M))\\ &\cong \mathbb{R}\,\mathcal{H}om_{\mathbb{C}_N}(\mathrm{DR}_X(\mathcal{M})|_N, \mathbb{C}_N) \otimes_{\mathbb{C}_N} \mathcal{H}^{\operatorname{codim} N}_N(\mathbb{C}_M)[-\operatorname{codim} N].\end{aligned}$$

Proof. We have

$$\begin{aligned}&\mathbb{R}\Gamma_N\, \mathbb{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)\\ &= \mathbb{R}\,\mathcal{H}om_{\mathbb{C}_X}(\mathrm{DR}(\mathcal{M})|_N, \mathbb{C}_X)\\ &= \mathbb{R}\,\mathcal{H}om_{\mathbb{C}_N}(\mathrm{DR}(\mathcal{M})|_N, \mathbb{C}_N) \otimes_{\mathbb{C}_N} \mathcal{H}^{2\dim M - \dim N}_N(\mathbb{C}_X)[\dim N - 2\dim M].\end{aligned}$$

On the other hand, $\mathscr{B}_M = \mathbb{R}\Gamma_M(\mathcal{O}_X)\otimes\mathscr{H}_M^{\dim M}(\mathbb{C}_X)[\dim M]$ implies

$$\begin{aligned}&\mathbb{R}\Gamma_N\,\mathbb{R}\,\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{B}_M)\\&=\mathbb{R}\Gamma_N\,\mathbb{R}\,\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathcal{O}_X)\otimes\mathscr{H}_M^{\dim M}(\mathbb{C}_X)[\dim M]\\&=\mathbb{R}\,\mathscr{H}om_{\mathbb{C}_N}(\mathrm{DR}(\mathscr{M})|_N,\mathbb{C}_N)\otimes\mathscr{H}_N^{2\dim M-\dim N}(\mathbb{C}_X)\otimes\mathscr{H}_M^{\dim M}(\mathbb{C}_X)[-\operatorname{codim} N].\end{aligned}$$

Hence the lemma follows from

$$\mathscr{H}_N^{2\dim M-\dim N}(\mathbb{C}_X)\otimes\mathscr{H}_M^{\dim M}(\mathbb{C}_X)=\mathscr{H}_N^{\operatorname{codim} N}(\mathbb{C}_M).\quad \text{Q.E.D.}$$

We shall resume the proof of Proposition 8.1.1. By using this lemma, we have locally

$$\begin{aligned}&\mathbb{R}\Gamma_{M_i}(\mathbb{R}\,\mathscr{H}om_{\mathscr{D}_{(\mathfrak{g}\times\mathfrak{h})_{an}}}(\mathscr{N}_{an},\mathscr{B}_{\mathfrak{g}_{\mathbb{R}}\times\mathfrak{h}_{\mathbb{R}}}))|_{M_i}\\&=\mathbb{R}\,\mathscr{H}om_{\mathbb{C}_{M_i}}(\mathrm{DR}(\mathscr{N})|_{M_i},\mathbb{C}_{M_i})[-\operatorname{codim} M_i]\\&=\mathbb{R}\,\mathscr{H}om_{\mathbb{C}_{M_i}}(\mathbb{R}f_*(\mathbb{C}_{\tilde{\mathfrak{g}}})|_{M_i},\mathbb{C}_{M_i})[\operatorname{rank}\mathfrak{g}\text{-}\operatorname{codim} M_i].\end{aligned}$$

Hence for $p_i=(x_i,t_i)\in M_i$, we have

$$\begin{aligned}&\mathscr{H}_{M_i}^k(\mathbb{R}\,\mathscr{H}om_{\mathscr{D}_{\mathfrak{g}\times\mathfrak{h}}}(\mathscr{N},\mathscr{B}_{\mathfrak{g}_{\mathbb{R}}\times\mathfrak{h}_{\mathbb{R}}}))_{p_i}\\&\cong\operatorname{Hom}_{\mathbb{C}}(H^{\operatorname{codim} M_i-\operatorname{rank}\mathfrak{g}-k}(f^{-1}(p_i),\mathbb{C}),\mathbb{C}).\end{aligned}$$

Now, we have $2\dim f^{-1}(p_i)=\dim\mathfrak{g}-\operatorname{rank}\mathfrak{g}-\dim[\mathfrak{g},x_i]$. Hence, if

(8.1.2) $$\operatorname{codim} M_i-\operatorname{rank}\mathfrak{g}-k>\dim\mathfrak{g}-\operatorname{rank}\mathfrak{g}-\dim[\mathfrak{g},x_i]$$

holds, then $\mathscr{H}_{M_i}^k(\mathbb{R}\,\mathscr{H}om_{\mathscr{D}_{(\mathfrak{g}\times\mathfrak{h})_{an}}}(\mathscr{N}_{an},\mathscr{B}_{\mathfrak{g}_{\mathbb{R}}\times\mathfrak{h}_{\mathbb{R}}}))|_{M_i}$ vanishes. Now (8.1.2) is equivalent to

(8.1.3) $k<\operatorname{rank}\mathfrak{g}+\dim[\mathfrak{g},x_i]-\dim M_i=\operatorname{rank}\mathfrak{g}-$ the rank of $(M_i\to\mathfrak{h}_{\mathbb{R}})$.

Hence, we obtain

(8.1.4) $$\mathscr{H}_{M_i}^0(\mathbb{R}\,\mathscr{H}om_{\mathscr{D}}(\mathscr{N},\mathscr{B}))=0\qquad\text{for } M_i\subset(\mathfrak{g}-\mathfrak{g}_{rs})\times\mathfrak{h}_{\mathbb{R}}$$

and

(8.1.5) $$\mathscr{H}_{M_i}^1(\mathbb{R}\,\mathscr{H}om_{\mathscr{D}}(\mathscr{N},\mathscr{B}))=0\qquad\text{for } M_i\subset\mathfrak{g}\times\mathfrak{h}_1.$$

This shows immediately (a) and (b).

Proposition 8.1.3.

(1) $\Gamma(\mathfrak{g}_{\mathbb{R}};\mathscr{H}om_{\mathscr{D}_{\mathfrak{g}_{an}}}((\mathscr{M}_\lambda^F)_{an},\mathscr{B}_{\mathfrak{g}_{\mathbb{R}}}))$ *does not depend on* λ.
(2) $\mathscr{H}^0_{(\mathfrak{g}_{\mathbb{R}}-\mathfrak{g}_{rs})}\,\mathscr{H}om_{\mathscr{D}_{\mathfrak{g}_{an}}}((\mathscr{M}_\lambda^F)_{an},\mathscr{B}_{\mathfrak{g}_{\mathbb{R}}})=0.$

Proof. (1) follows immediately from Corollary 9.7.3.

Let us show (2). By Lemma 8.1.2, it is enough to show that for a subvariety N of $\mathfrak{g}_{\mathbb{R}}-\mathfrak{g}_{rs}$ such that $G_{\mathbb{R}}N=N$ and $\mathrm{DR}_{\mathfrak{g}}(\mathscr{M}_\lambda^F)|_N$ is locally constant,

$$\mathscr{H}^{\operatorname{codim} N}(\mathrm{DR}_{\mathfrak{g}}(\mathscr{M}_\lambda^F)|_N)=0.$$

Now, we have $\mathrm{DR}_{\mathfrak{g}}(\mathscr{M}_\lambda^F) = \mathbb{R}\rho_*(\mathbb{C}_{\tilde{\mathfrak{g}}})$. Hence we have, for $x \in N$,

$$\mathscr{H}^j(\mathrm{DR}_{\mathfrak{g}}(\mathscr{M}_\lambda^F))_x = H^j(\rho^{-1}(x), \mathbb{C}) = 0 \qquad \text{for } j > 2\dim \rho^{-1}(x).$$

On the other hand, we have

$$\begin{aligned} 2\dim \rho^{-1}(x) &= \dim \mathfrak{g} - \operatorname{rank} \mathfrak{g} - \dim[\mathfrak{g}, x] \\ &= \dim \mathfrak{g} - \operatorname{rank} \mathfrak{g} - (\dim N - \dim \kappa(N)) \\ &= \operatorname{codim} N - \operatorname{codim} \kappa(N) \\ &< \operatorname{codim} N. \end{aligned}$$

Thus we obtain the desired result.

8.2. In the sequel we shall restrict ourselves to the case where $\mathfrak{g}_{\mathbb{R}}$ is the underlying structure of a complex semisimple Lie algebra $\mathfrak{g}_0$. Hence $\mathfrak{g} = \mathfrak{g}_0 \oplus \overline{\mathfrak{g}_0}$, where $\overline{\mathfrak{g}_0}$ is the complex conjugate of $\mathfrak{g}_0$.

Theorem 8.1. *Under this assumption*

$$\Gamma(\mathfrak{g}_{\mathbb{R}} \times \mathfrak{h}_{\mathbb{R}}, \ \mathscr{H}om_{\mathscr{D}_{(\mathfrak{g}\times\mathfrak{h})_{\mathrm{an}}}}(\mathscr{N}_{\mathrm{an}}, \mathscr{B}_{\mathfrak{g}_{\mathbb{R}} \times \mathfrak{h}_{\mathbb{R}}})) = \mathbb{C}.$$

Proof. Ser $\Omega = f(\tilde{\mathfrak{g}}) \cap (\mathfrak{g}_{\mathbb{R}} \cap \mathfrak{g}_{rs}) \times \mathfrak{h}_{\mathbb{R}}$. Then, the sheaf $\mathscr{H}om_{\mathscr{D}_{\mathrm{an}}}(\mathscr{N}_{\mathrm{an}}, \mathscr{B}_{\mathfrak{g}_{\mathbb{R}} \times \mathfrak{h}_{\mathbb{R}}})|_\Omega$ is isomorphic to $\mathbb{C}_\Omega$. In fact, choosing a base $\{P_1, \ldots, P_l\}$ of homogeneous generators of $\mathbb{C}[\mathfrak{g}_{\mathbb{R}}]^{G_{\mathbb{R}}}$, we set

$$\varphi(x, t) = \textstyle\prod_{j=1}^l \delta(P_j(t) - P_j(x)).$$

Then φ is a well-defined hyperfunction on a neighborhood of Ω and

$$\tilde{u} \longmapsto \varphi$$

defines the isomorphism

$$\mathbb{C}_\Omega \xrightarrow{\sim} \mathscr{H}om_{\mathscr{D}_{\mathrm{an}}}(\mathscr{N}_{\mathrm{an}}, \mathscr{B}_{\mathfrak{g}_{\mathbb{R}} \times \mathfrak{h}_{\mathbb{R}}})|_\Omega.$$

Since we assumed that $\mathfrak{g}_{\mathbb{R}}$ is a complex semisimple Lie algebra,

$$\Omega = f(\tilde{\mathfrak{g}}) \cap \mathfrak{g}_{\mathbb{R}} \times (\mathfrak{h}_{\mathbb{R}} - \mathfrak{h}_1)$$

and Ω is connected. Hence by Proposition 8.1.1, we have

$$\Gamma(\mathfrak{g}_{\mathbb{R}} \times \mathfrak{h}_{\mathbb{R}}, \mathscr{H}om_{\mathscr{D}_{\mathrm{an}}}(\mathscr{N}_{\mathrm{an}}, \mathscr{B}_{\mathfrak{g}_{\mathbb{R}} \times \mathfrak{h}_{\mathbb{R}}})) \xrightarrow{\sim} \Gamma(\Omega, \mathscr{H}om_{\mathscr{D}_{\mathrm{an}}}(\mathscr{N}_{\mathrm{an}}, \mathscr{B}_{\mathfrak{g}_{\mathbb{R}} \times \mathfrak{h}_{\mathbb{R}}})).$$

The last term is isomorphic to $\mathbb{C}$ because Ω is connected. Q.E.D.

By this theorem, the hyperfunction φ is extended to a global hyperfunction solution to $\mathscr{N}$. We shall denote it by $\delta(x, t)$.

8.3. Let W be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ and let $W_{\mathbb{R}}$ be the subgroup $\{w \in W; w\mathfrak{h}_{\mathbb{R}} = \mathfrak{h}_{\mathbb{R}}\}$. Then, $W_{\mathbb{R}}$ is the Weyl group of $\mathfrak{g}_0$, and W is isomorphic to $W_{\mathbb{R}} \times W_{\mathbb{R}}$.

Since the Fourier transform of $\mathscr{N}$ is isomorphic to $\mathscr{N}$ we have

Proposition 8.3.1. *The Fourier transform of δ is a constant multiple of δ.*

Proposition 8.3.2. *For* $P \in \mathscr{H}(\mathfrak{h})$,

(8.3.1) $$P(\partial_t)\delta(x,t)|_{t=0}=0 \quad \textit{if and only if} \quad \sum_{w\in W_{\mathbb{R}}} P(wt)=0.$$

Proof. The Fourier transform of $P(\partial_t)\delta(x,t)|_{t=0}$ is the integration of $P(2\pi i t)$-times the Fourier transform of δ with respect to t. Hence $P(\partial_t)\delta(x,t)|_{t=0}=0$ if and only if

$$\int P(2\pi i t)\delta(x,t)\,dt=0.$$

On the other hand, the Fourier transform of $P(\partial_t)\delta(x,t)|_{t=0}$ satisfies $\mathscr{M}_0^F$, and hence it vanishes if it vanishes on $\mathfrak{g}_{rs}\cap\mathfrak{g}_{\mathbb{R}}$ (Proposition 8.1.3). Since $\operatorname{Supp}\delta\subset\overline{(G_{\mathbb{R}}\mathfrak{h}_{\mathbb{R}})}\times\mathfrak{h}_{\mathbb{R}}$, (8.3.1) is equivalent to

$$\int P(2\pi i t)\delta(x,t)\,dt=0 \qquad \text{for } x\in\mathfrak{h}_{\mathbb{R}}\cap\mathfrak{h}_{rs}.$$

Since

$$\delta(x,t)|_{x\in\mathfrak{h}_{\mathbb{R}}\times\mathfrak{h}_{rs}}=\sum_{w\in W_{\mathbb{R}}}\frac{1}{|\Delta(t)|}\delta(t-wx),$$

we have

$$\int P(2\pi i t)\delta(x,t)\,dt=\sum_{w\in W_{\mathbb{R}}} P(2\pi i w x)/|\Delta(x)|.$$

Thus we obtain the desired result.

8.4. Let $\mathbf{O}$ be a $G_{\mathbb{R}}$-orbit of $\mathfrak{g}_{\mathbb{R}}$ consisting of nilpotent elements. Let $\mathbf{O}_{\mathbb{C}}$ be the G-orbit containing $\mathbf{O}$. Now, the invariant measure on $\mathbf{O}$ extends to a measure on $\mathfrak{g}_{\mathbb{R}}$, which defines a distribution $\mu_{\mathbf{O}}$ on $\mathfrak{g}_{\mathbb{R}}$. This defines a solution to $\mathscr{M}(\lambda_{\mathbf{O}_{\mathbb{C}}})$ on $\mathfrak{g}_{\mathbb{R}}$.

By Proposition 7.1.3, Corollary 7.1.4 and Corollary 7.1.5, we have

Proposition 8.4.1. (1) *$\mu_{\mathbf{O}}$ is a solution to $\mathscr{M}_{\chi_{\mathbf{O}}}$.*
(2) *Any solution to $\mathscr{M}_{\chi_{\mathbf{O}}}$ is a constant multiple of $\mu_{\mathbf{O}}$.*

These results hold for an arbitrary $\mathfrak{g}_{\mathbb{R}}$.

Now assume that $\mathfrak{g}_{\mathbb{R}}$ is the underlying real structure of a complex semi-simple Lie algebra $\mathfrak{g}_0$ and let W and $W_{\mathbb{R}}$ be as in 8.3. Then $\mathfrak{g}=\mathfrak{g}_0\times\bar{\mathfrak{g}}_0$ and $\mathbf{O}_{\mathbb{C}}=\mathbf{O}\times\bar{\mathbf{O}}$, and incidentally $\chi_{\mathbf{O}_{\mathbb{C}}}=\chi_{\mathbf{O}}\otimes\chi_{\mathbf{O}}$, for $\chi_{\mathbf{O}}\in\hat{W}$, where the bar $-$ denotes the complex conjugation.

Theorem 8.2. *There exists*

$$P\in\mathscr{H}(-(\dim\mathfrak{g}+\operatorname{rank}\mathfrak{g})/2-\lambda_{\mathbf{O}_{\mathbb{C}}})$$

which transforms according to $\chi_{\mathbf{O}_{\mathbb{C}}}=\chi_{\mathbf{O}}\otimes\chi_{\mathbf{O}}$ such that

$$P(\partial_t)\delta(x,t)|_{t=0}=\mu_{\mathbf{O}}.$$

Proof. Note that $\mathscr{H}=\mathscr{H}(-(\dim\mathfrak{g}+\operatorname{rank}\mathfrak{g})/2-\lambda_{\mathbf{O}_{\mathbb{C}}})$ contains $\chi_{\mathbf{O}_{\mathbb{C}}}$. Since $\chi_{\mathbf{O}_{\mathbb{C}}}|_{W_{\mathbb{R}}}$ contains the trivial representation there exists $P\in\mathscr{H}$ which transforms according to $\chi_{\mathbf{O}_{\mathbb{C}}}$ such that $\sum_{w\in W_{\mathbb{R}}} P(wt)\neq 0$. Let us take such a P. Then $P(\partial_t)\delta(x,t)|_{t=0}\neq 0$ by Proposition 8.3.2. On the other hand by Theorem 6.1, $P(\partial_t)\delta(x,t)|_{t=0}$ is a solution to $\mathscr{M}_{\chi_{\mathbf{O}_{\mathbb{C}}}}$. Hence it is a non-zero constant multiple of $\mu_{\mathbf{O}}$. Q.E.D.

This theorem is proved by Barbasch-Vogan [7, 8] for special orbits.

References

1. Brylinski, J.-L.: Transformations canoniques, dualité projective, théorie de Lefschetz, transformations de Fourier et sommes trigonométriques. Preprint
2. Brylinski, J.-L., Malgrange, B., Verdier, J.-L.: Transformation de Fourier geometrique, I. C.R. Acad. Sc. Paris **297**, 55–58 (1983)
3. Brylinski, J.-L., Kashiwara, M.: Kazhdan-Lusztig conjecture and holonomic systems. Invent. Math. **64**, 387–410 (1981)
4. Borho, W., MacPherson, R.D.: Représentation des groupes de Weyl et homologie d'intersection pour les variétés de nilpotent. C.R. Acad. Sc. Paris **292**, 707–710 (1981)
5. Borho, W., MacPherson, R.D.: Partial resolutions of nilpotent varieties. Astérisque **101–102**, 23–74 (1983)
6. Barbasch, D., Vogan, D.: The local structure of characters. J. Funct. Anal. **37**, 27–55 (1980)
7. Barbasch, D., Vogan, D.: Primitive ideals and orbital integrals in complex classical groups. Math. Ann. **259**, 153–199 (1982)
8. Barbasch, D., Vogan, D.: Primitive ideals and orbital integrals in complex exceptional groups. J. Algebra **80**, 350–382 (1983)
9. Grothendieck, A.: Cohomologie locale des faisceaux cohérents et Théorèmes de Lefschetz locaux et globaux (SGA 2). North-Holland, Amsterdam-Masson & CIE, Editeur-Paris (1962)
10. Harish-Chandra: Invariant differential operators and distributions on a semisimple Lie algebra. Amer. J. Math. **86**, 534–564 (1964)
11. Harish-Chandra: Invariant eigendistributions on a semisimple Lie algebra. I.H.E.S. Publ. Math. **27**, 5–54 (1965)
12. Kashiwara, M.: B-functions and holonomic systems. Invent. Math. **38**, 33–53 (1976)
13. Kashiwara, M.: On the holonomic systems of linear differential equations II. Invent. Math. **49**, 121–135 (1978a)
14. Kashiwara, M.: Systèmes d'équations micro-différentiels, noted by Monteiro Fernandes. Université Paris-Nord 1978b) (English translation: P.M. 34, Birkhäuser)
15. Kashiwara, M., Kawai, T.: On holonomic systems for $\prod_{l=1}^{N}(f_l+\sqrt{-1}0)^{\lambda_l}$. Publ. R.I.M.S. **15**, 551–575 (1979)
16. Kashiwara, M., Kawai, T.: On holonomic systems of micro-differential equations, III – Systems with regular singularities. ibid **17**, 813–979 (1981)
17. King, D.: The character polynomial of the annihilator of an irreducible Harish-Chandra module. Amer. J. Math. **103**, 1195–1240 (1981)
18. Richardson, R.W.: Commuting varieties of semisimple Lie algebras and algebraic groups. Compositio Math. **38**, 311–322 (1979)
19. Sato, M., Kawai, T., Kashiwara, M.: Microfunctions and pseudo-differential equations. Lecture Notes in Math., vol. 287, pp. 265–529. Berlin-Heidelberg-New York, Springer 1973
20. Serre, J-P.: Géométrie algébrique et géometrie analytique. Ann. Inst. Fourier **6**, 1–42 (1956)
21. Springer, T.A.: Quelques applications de la cohomologie d'intersection. Sém. Bourbaki 34^e année, 1981/82, No. 589

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