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Inventiones mathematicae

Kazhdan-Lusztig Conjecture and Holonomic Systems

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In [7], D. Kazhdan and G. Lusztig gave a conjecture on the multiplicity of simple modules which appear in a Jordan-Hölder series of the Verma modules. This multiplicity is described in the terms of Coxeter groups and also by the geometry of Schubert cells in the flag manifold (see [8]). The purpose of this paper is to give the proof of their conjecture.

The method employed here is to associate holonomic systems of linear differential equations with R.S. on the flag manifold with Verma modules and to use the correspondance of holonomic systems and constructible sheaves.

Let G be a semi-simple Lie group defined over $\mathbb{C}$ and $\mathfrak{g}$ its Lie algebra. We take a pair (B, B^-) of opposed Borel subgroups of G and let $T=B\cap B^-$ be a maximal torus and W the Weyl group. Let $\mathfrak{b}$, $\mathfrak{b}^-$ and $\mathfrak{k}$ the corresponding Lie algebras and $\mathfrak{N}$ the nilpotent radical of $\mathfrak{b}$. Let us denote by $\mathscr{M}$ the category of holonomic systems with R.S. on $X=G/B$ whose characteristic varieties are contained in the union of the conormal bundles of $X_w=BwB/B$ $(w\in W)$. On the other hand, let $\tilde{\mathcal{O}}$ denote the category of finitely-generated $U(\mathfrak{g})$-modules which are $\mathfrak{N}$-finite. By $\mathcal{O}_{\mathrm{triv}}$ we denote the category of the modules in $\tilde{\mathcal{O}}$ with the trivial central character.

We shall prove that $\mathscr{M}$ and $\tilde{\mathcal{O}}_{\mathrm{triv}}$ are equivalent by the correspondances $\mathfrak{M} \mapsto \Gamma(X;\mathfrak{M})$ and $M\mapsto \mathscr{D}\otimes_{U(\mathfrak{g})}M$. Here $\mathscr{D}$ is the sheaf of differential operators on X. Let us denote by M_w the Verma module with highest weight $-w(\rho)-\rho$ and let $\mathfrak{M}_w$ be the dual $\mathscr{D}$-module of $\mathscr{H}^{\operatorname{codim} X_w}_{[X_w]}(\mathcal{O}_X)$. Then, $\mathfrak{M}_w$ and M_w correspond by the above correspondence. For any $\mathfrak{M}\in\mathscr{M}$, we can calculate the character of $\Gamma(X;\mathfrak{M})$ by the formula

$$ch(\Gamma(X;\mathfrak{M}))=\sum_{w\in W}(-1)^{\operatorname{codim} X_w} X_w(\mathfrak{M})\, ch(M_w)$$

where

$$X_w(\mathfrak{M})=\sum(-1)^j \dim_{\mathbb{C}} \mathscr{E}xt^j_{\mathscr{D}}(\mathcal{O}_X,\mathfrak{M}).$$

This formula can be proved by reduction to the case $\mathfrak{M}=\mathfrak{M}_w$. Let L_w be the simple module with highest weight $-w(\rho)-\rho$. By the formula above, $ch(L_w)$ is calculated if we know $\mathbb{R}\operatorname{Hom}_{\mathscr{D}}(\mathcal{O},\mathscr{D}\otimes L_w)$. We shall show this complex coincides with $\pi_{\bar{X}_w}[-\operatorname{codim} X_w]$, where $\pi_{\bar{X}_w}$ is the complex introduced by Deligne [2].

This paper is divided into three parts. In the first part we give a review of holonomic systems with R.S. In the second part we establish the equivalence of $\mathscr{M}$ and $\tilde{\mathcal{O}}_{\mathrm{triv}}$. After these preparations, we shall give the proof of the conjecture of Kazhdan-Lusztig in the third part.

It should be mentioned that the idea of ontroducing sheaves of modules over $\mathscr{D}$ in an apparently unrelated problem on $\mathfrak{g}$-modules, was arrived at after a careful study of Kempf's work [9], where he interprets the Bernstein-Gelfand-Gelfand resolution of a finite dimensional $\mathfrak{g}$-module, as being dual to the Cousin resolution for the associated invertible sheaf on X, with respect to the stratification $X = \coprod_{w \in W} X_w$. It was already glaringly apparent there that the Verma module M_w was "corresponding" to the Bruhat cell X_w or, put otherwise, to the constructible sheaf $\mathbb{C}_{X_w}$ on X which has fibre $\mathbb{C}$ over X_w and 0 over $X - X_w$. But some time was needed to realize that holonomic $\mathscr{D}$-modules could serve as a bridge between constructible sheaves and Verma modules.

We wish to thank Michel Demazure for conversations on the geometry of X as related to Kempf's paper, Patrick Delorme for various interesting information on the category $\mathcal{O}$, and Jean-Louis Verdier for pointing out to the first author the possibly use of a theorem of Macpherson giving a characterization of the complex π_Y, for any singular variety Y.[1]

§ 1. Holonomic Systems With Regular Singularities

1.1. In this section, we shall summarize the results on holonomic system of linear differential equations with R.S. (abbreviation of regular singularities).

For the details and proofs, we refer the reader to [6, 15–17].

1.2. Throughout this section, we shall denote by X a complex manifold, $\mathcal{O} = \mathcal{O}_X$ the sheaf of holomorphic functions on X, $\Omega = \Omega_X^{\dim X}$ the sheaf of holomorphic $\dim X$-forms and $\mathscr{D}_X$ (resp. $\mathscr{D}_X^\infty$) the sheaf of differential operators of finite order (resp. infinite order). In the sequel a $\mathscr{D}_X$-module means a left $\mathscr{D}_X$-module if not otherwise mentioned.

Let $\mathscr{D}_X(m)$ denote the sheaf of differential operators of degree at most m. Then Specan $(\bigotimes_j \mathscr{D}_X(m)/\mathscr{D}_X(m-1))$ coincides with the cotangent bundle T^*X of X. For a coherent $\mathscr{D}_X$-module $\mathfrak{M}$, an increasing sequence $\{\mathfrak{M}_j\}_{j \in \mathbb{Z}}$ of coherent sub-$\mathcal{O}_X$-modules of $\mathfrak{M}$ is called a *good filtration* if it satisfies

(1.2.1) $$\mathscr{D}(m)\mathfrak{M}_j \subset \mathfrak{M}_{j+m},$$

(1.2.2) $$\mathscr{D}(m)\mathfrak{M}_j = \mathfrak{M}_{j+m} \quad \text{for } j \gg 0 \text{ locally on } X,$$

(1.2.3) $$\mathfrak{M} = U\mathfrak{M}_j.$$

[1] After this article was written, we learnt that Beilinson and Bernstein also solved the Kazhdan-Lusztig conjecture by using methods similar to ours

The support of the coherent sheaf on T^*X associated with $\bigoplus_{j\geqq 0}(\mathfrak{M}_j/\mathfrak{M}_{j-1})$ is called the *characteristic variety* of $\mathfrak{M}$, which will be denoted by $Ch(\mathfrak{M})$; this does not depend on the choice of a good filtration. The characteristic variety is a closed homogeneous involutory subvariety of T^*X. If the dimension of the characteristic variety of $\mathfrak{M}$ is as minimal as possible, i.e. $\dim X$, then we call $\mathfrak{M}$ *holonomic*. We say that a holonomic $\mathscr{D}_X$-module $\mathfrak{M}$ has R.S. if $\mathfrak{M}$ has a good filtration $\{\mathfrak{M}_j\}$ of $\mathfrak{M}$ satisfying the condition:

(1.2.4) For any open set U and any differential operator $P\in\mathscr{D}_X(m)(U)$, if its principal symbol $\sigma_m(P)$ vanishes on $Ch(\mathfrak{M})$, then $P\mathfrak{M}_j\subset\mathfrak{M}_{j+m-1}$ for any j.

1.3. For a holonomic $\mathscr{D}_X$-module we have:

(1.3.1) $$\mathscr{E}xt^j_{\mathscr{D}_X}(\mathfrak{M},\mathscr{D}_X)=0 \qquad \text{for } j\neq n=\dim X$$

and $\mathscr{E}xt^n_{\mathscr{D}_X}(\mathfrak{M},\mathscr{D}_X)$ is a coherent right $\mathscr{D}_X$-module. Hence

$$\mathfrak{M}^* \underset{\text{def}}{=} \mathscr{E}xt^n_{\mathscr{D}_X}(\mathfrak{M},\mathscr{D}_X)\underset{\mathscr{O}_X}{\otimes}\Omega^{\otimes -1}$$

has a structure of left $\mathscr{D}_X$-module. We call $\mathfrak{M}^*$ the dual of $\mathfrak{M}$.

Proposition 1.1. (1) $*$ *is an exact contravariant functor from the category of holonomic $\mathscr{D}_X$-modules in itself.*

(2) $(\mathfrak{M}^*)^*\cong\mathfrak{M}$ *for a holonomic $\mathscr{D}_X$-module $\mathfrak{M}$.*

(3) $Ch(\mathfrak{M}^*)=Ch(\mathfrak{M})$ *for a holonomic $\mathscr{D}_X$-module $\mathfrak{M}$.*

(4) *If $\mathfrak{M}$ is a holonomic $\mathscr{D}_X$-module with R.S., then so is $\mathfrak{M}^*$* (4).

For two holonomic $\mathscr{D}_X$-modules $\mathfrak{M}$ and $\mathfrak{M}'$ we have $\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathfrak{M}',\mathfrak{M})\cong\mathbb{R}\,\mathscr{H}om_{\mathscr{D}}(\mathfrak{M}^*,\mathfrak{M}'^*)$.

1.4. For a closed analytic subset Y of X and an $\mathscr{O}_X$-module $\mathscr{F}$, we shall denote by $\Gamma_{[Y]}(\mathscr{F})$ the $\mathscr{O}_X$-module $\varinjlim_m\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{J}_Y^m,\mathscr{F})$ and by $\Gamma_{[X|Y]}(\mathscr{F})$ the $\mathscr{O}_X$-module $\varinjlim_m\mathscr{H}om_{\mathscr{O}_X}(\mathscr{J}_Y^m,\mathscr{F})$ where $\mathscr{J}_Y=\{f\in\mathscr{O}_X;\ f|Y=0\}$. If Y is a locally closed subset of X such that $\bar Y$ and $\bar Y-Y$ are analytic, we set

$$\Gamma_{[Y]}(\mathscr{F})=\Gamma_{[\bar Y]}\Gamma_{[X|\bar Y-Y]}(\mathscr{F}).$$

We denote by $\mathscr{H}^j_{[Y]}(\mathscr{F})$ its j-th derived functor. If $\mathscr{F}$ is a $\mathscr{D}_X$-module, $\mathscr{H}^j_{[Y]}(\mathscr{F})$ has a structure of $\mathscr{D}_X$-module.

Suppose that X and Y are algebraic. Let us denote by $(X_{\text{alg}},\mathscr{O}_{X_{\text{alg}}})$ the corresponding algebraic variety over $\mathbb{C}$ and let j be the morphism of ringed spaces $(X,\mathscr{O}_X)\to(X_{\text{alg}},\mathscr{O}_{X_{\text{alg}}})$. Then, for any quasi-coherent $\mathscr{O}_{X_{\text{alg}}}$-module $\mathscr{F}$ we have

(1.4.1) $$\mathbb{R}\,\Gamma_{[Y]}(j^*\mathscr{F})=j^*\mathbb{R}\,\Gamma_{Y_{\text{alg}}}(\mathscr{F})$$

1.5. For a closed submanifold Y of codimension l, we set

$$\mathscr{B}_{Y|X}=\mathscr{H}^l_{[Y]}(\mathscr{O}_X)$$

We have:

Proposition 1.2. (1) $\mathscr{B}_{Y|X}$ is a holonomic $\mathscr{D}_X$-module with R.S., and the characteristic variety of $\mathscr{B}_{Y|X}$ coincides with the conormal bundle $T_Y^* X$ of Y.

(2) For a coherent $\mathscr{D}_X$-module $\mathfrak{M}$ such that $Ch(\mathfrak{M}) \subset T_Y^* X$, the sheaf $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{B}_{Y|X}, \mathfrak{M})$ is a locally constant sheaf of finite rank on Y and we have an isomorphism

$$\mathscr{B}_{Y|X} \underset{\mathbb{C}}{\otimes} \mathscr{H}om_{\mathscr{D}_X}(\mathscr{B}_{Y|X}, \mathfrak{M}) \xrightarrow{\sim} \mathfrak{M}.$$

(3) In particular, if Y is connected, then any coherent $\mathscr{D}_X$-sub-module of $\mathscr{B}_{Y|X}$ is either $\mathscr{O}$ or $\mathscr{B}_{Y|X}$.

1.6. We shall give the properties of holonomic $\mathscr{D}$-modules with R.S. In the statements, $\mathfrak{M}^\infty$ stands for $\mathscr{D}_X^\infty \underset{\mathscr{D}_X}{\otimes} \mathfrak{M}$. Note that $\mathscr{D}_X^\infty$ is fairthfully flat over $\mathscr{D}_X$.

Proposition 1.3. (1) *For any holonomic $\mathscr{D}$-module with R.S., its coherent sub-$\mathscr{D}$-modules, its coherent quotients are also with R.S.*

(2) *Any holonomic $\mathscr{D}$-module with R.S. has globally a good filtration which satisfies the condition* (1.2.4).

(3) *If $\mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}''$ is an exact sequence of coherent $\mathscr{D}$-modules and if $\mathfrak{M}'$ and $\mathfrak{M}''$ are holonomic $\mathscr{D}$-modules with R.S., then so is $\mathfrak{M}$.*

(4) *If $\mathfrak{M}$ and $\mathfrak{M}'$ are holonomic $\mathscr{D}$-modules with R.S. then*

$$\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathfrak{M}, \mathfrak{M}') = \mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathfrak{M}, \mathfrak{M}'^{\infty}).$$

If $X = \cup X_\alpha$ is a stratification of Whitney such that $Ch(\mathfrak{M}) \cup Ch(\mathfrak{M}') \subset \cup T_{X_\alpha}^* X$, then $\mathscr{E}xt_{\mathscr{D}}^j(\mathfrak{M}, \mathfrak{M}')|X_\alpha$ is a locally constant sheaf of finite rank.

(5) *For any holonomic $\mathscr{D}$-module $\mathfrak{M}$, there exists a unique sub-$\mathscr{D}$-module $\mathfrak{M}_{\mathrm{reg}}$ of $\mathfrak{M}^\infty$ such that $\mathfrak{M}_{\mathrm{reg}}^\infty \xrightarrow{\sim} \mathfrak{M}^\infty$ and that $\mathfrak{M}_{\mathrm{reg}}$ is a holonomic $\mathscr{D}$-module with R.S.*

(6) *For any holonomic $\mathscr{D}$-module $\mathfrak{M}$, we have*

$$\mathbb{R}\mathscr{H}om_{\mathbb{C}}(\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathfrak{M}, \mathscr{O}), \mathscr{O}) = \mathfrak{M}^\infty$$

(7) *For any difference Y of closed analytic subsets of X and any holonomic $\mathscr{D}$-module $\mathfrak{M}$ with R.S., the $\mathscr{H}_{[Y]}^j(\mathfrak{M})$ are also holonomic $\mathscr{D}$-modules with R.S., and we have*

$$\mathscr{H}_{[Y]}^j(\mathfrak{M})^\infty \cong \mathscr{H}_Y^j(\mathfrak{M}^\infty).$$

(8) *For a holonomic $\mathscr{D}$-module $\mathfrak{M}$, we have*

$$\begin{aligned}\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathfrak{M}, \mathscr{O}_X) &= \mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}_X, \mathfrak{M}^*)\\ &= \mathbb{R}\mathscr{H}om_{\mathbb{C}}(\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathfrak{M}^*, \mathscr{O}_X), \mathbb{C}_X)\\ &= \mathbb{R}\mathscr{H}om_{\mathbb{C}}(\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}_X, \mathfrak{M}), \mathbb{C}_X).\end{aligned}$$

(9) *For two holonomic $\mathscr{D}$-modules $\mathfrak{M}$ and $\mathfrak{M}'$, we have*

$$\begin{aligned}\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathfrak{M}, \mathfrak{M}'^{\infty}) &= \mathbb{R}\mathscr{H}om_{\mathbb{C}}(\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}_X, \mathfrak{M}), \mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}_X, \mathfrak{M}'))\\ &= \mathbb{R}\mathscr{H}om_{\mathbb{C}}(\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathfrak{M}', \mathscr{O}_X), \mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathfrak{M}, \mathscr{O}_X)).\end{aligned}$$

(10) *Let $\mathfrak{M}$ be a holonomic $\mathscr{D}$-module with R.S., and Y and a difference of closed analytic subsets. If $X=\coprod X_\alpha$ is a Whitney stratification satisfying*

(a) $$Ch(\mathfrak{M})\subset U\, T^*_X X \quad \text{and}$$

(b) $$Y=U\{X_\alpha; X_\alpha\subset Y\},$$

then we have

$$Ch(\mathscr{H}^j_{[Y]}(\mathfrak{M}))\subset \bigcup_{X_\alpha\subset Y} T^*_{X_\alpha}X.$$

By (4) and (7) of the preceding proposition, we have:

Proposition 1.4. *For two holonomic $\mathscr{D}$-modules with R.S. $\mathfrak{M}$ and $\mathfrak{M}'$, we have*

$$\mathbb{R}\Gamma_Y(\mathbb{R}\,\mathscr{H}om_{\mathscr{D}}(\mathfrak{M},\mathfrak{M}'))=R\,\mathscr{H}om_{\mathscr{D}}(\mathfrak{M},R\Gamma_{[Y]}(\mathfrak{M}'),$$

By (5) and (6) of Proposition 1.3, we have:

Proposition 1.5. *Let $\mathfrak{M}$ and $\mathfrak{M}'$ be two holonomic $\mathscr{D}_X$-modules with R.S. If $\mathbb{R}\,\mathscr{H}om_{\mathscr{D}}(\mathfrak{M},\mathscr{O})\cong\mathbb{R}\,\mathscr{H}om_{\mathscr{D}}(\mathfrak{M}',\mathscr{O})$, then $\mathfrak{M}\cong\mathfrak{M}'$.*

We shall give here one of the characterizations of R.S.

Proposition 1.6. *Let $\mathfrak{M}$ be a holonomic $\mathscr{D}_X$-module. Then, $\mathfrak{M}$ has R.S. if and only if*

$$\mathscr{E}xt^j_{\mathscr{D}}(\mathfrak{M},\mathscr{O}_x)\to\mathscr{E}xt^j_{\mathscr{D}}(\mathfrak{M},\hat{\mathscr{O}}_x)$$

is an isomorphism for any $x\in X$ and any j. Here $\hat{\mathscr{O}}_x$ is the Krull completion of the local ring $\mathscr{O}_x$.

Proposition 1.7. *If Y is a submanifold of X and if $\mathfrak{M}$ is a holonomic $\mathscr{D}_X$-module with R.S., then*

$$\mathbb{R}\,\mathscr{H}om_{\mathscr{D}_X}(\mathfrak{M},\mathscr{B}_{Y|X})=\mathbb{R}\,\mathscr{H}om_{\mathbb{C}}(\mathbb{R}\,\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_X,\mathfrak{M})|_Y,\mathbb{C}_Y)\,[\text{-codim } Y]$$

Proof. We have $\mathscr{B}_{Y|X}=\mathbb{R}\Gamma_{[Y]}(\mathscr{O}_X)$ [codim Y]. Therefore, Proposition 1.4 implies

$$\mathbb{R}\,\mathscr{H}om_{\mathscr{D}_X}(\mathfrak{M},\mathscr{B}_{Y|X})=\mathbb{R}\Gamma_{[Y]}(\mathbb{R}\,\mathscr{H}om_{\mathscr{D}_X}(\mathfrak{M},\mathscr{O}_X))\,[\text{codim } Y]$$

By Proposition 1.3 (8), we have

$$\begin{aligned}\mathbb{R}\Gamma_Y(\mathbb{R}\,\mathscr{H}om_{\mathscr{D}_X}(\mathfrak{M},\mathscr{O}_X)&=\mathbb{R}\,\Gamma_Y\,\mathbb{R}\,\mathscr{H}om_{\mathbb{C}}(\mathbb{R}\,\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X,\mathfrak{M}),\mathbb{C}_X)\\&=\mathbb{R}\,\mathscr{H}om_{\mathbb{C}}(\mathbb{R}\,\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X,\mathfrak{M}),\mathbb{R}\Gamma_Y(\mathbb{C}_X)).\end{aligned}$$

Proposition follows from $\mathbb{R}\,\Gamma_Y(\mathbb{C}_X)=\mathbb{C}_Y\,[-2\,\text{codim } Y]$. Q.E.D.

§ 2. The Category $\tilde{\mathscr{O}}$

2.1. Let $\mathfrak{g}$ be a semi-simple Lie algebra defined over $\mathbb{C}$, $\mathfrak{k}$ a Cartan subalgebra of $\mathfrak{g}$ and let Δ be the corresponding root system. We fix an ordering of Δ and let Δ^+ and Δ^- be the set of positive and negative roots, respectively.

For $\alpha \in \Delta$, we take a non zero X_α in $\mathfrak{g}$ whose weight is α. We set $\mathfrak{N} = \sum_{\alpha \in \Delta^+} \mathbb{C} X_\alpha$, $\mathfrak{N}^- = \sum_{\alpha \in \Delta^-} \mathbb{C}_{X_\alpha}$, $\mathfrak{b} = \mathfrak{k} \oplus \mathfrak{N}$ and $\mathfrak{b}^- = \mathfrak{k} \oplus \mathfrak{N}^-$. Let W be the Weyl group and w_0 the longest element of W. For $\alpha \in \Delta$, let s_α be the corresponding reflection.

Let $U = U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$ and $\mathfrak{z}$ its center. We shall denote by I the ideal of U generated by $\mathfrak{z} \cap U(\mathfrak{g})\mathfrak{g}$ and by R the quotient ring U/I.

Set $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$ and let M_w be the Verma module with highest weight $-w(\rho) - \rho$; i.e.

$$M_w = U/U\mathfrak{N} + \sum_{H \in \mathfrak{k}} U(H + \langle w(\rho) + \rho, H \rangle).$$

Let L_w be the simple U-module with highest weight $-w(\rho) - \rho$.

Let us denote by $\tilde{\mathcal{O}}$ the category of finitely generated U-modules M such that any $u \in M$ satisfies $\dim_{\mathbb{C}} U(\mathfrak{b}) u < \infty$.

The following lemma is immediate.

Lemma 2.1. (1) *Any submodule and any quotient of a module in $\tilde{\mathcal{O}}$ belong to $\tilde{\mathcal{O}}$.*

(2) *If $M' \to M \to M''$ is an exact sequence of U-modules and if M' and M'' belong to $\tilde{\mathcal{O}}$, then so does M.*

Remark that the property (2) does not hold for the category introduced by Bernstein-Gelfand-Gelfand [1] where they assumed the action of $\mathfrak{t}$ is semi-simple. However, a module in $\tilde{\mathcal{O}}$ is not necessarily semi-simple as a $\mathfrak{t}$-module.

2.2. For any $\lambda \in \mathfrak{k}^*$ and $M \in \tilde{\mathcal{O}}$, we set $M^\lambda = \{u \in M;$ there exists $r > 0$ such that $(H - \langle \lambda, H \rangle)^r u = 0$ for any $H \in \mathfrak{k}\}$.

We say that λ is a weight of M if $M^\lambda \neq 0$. It is easy to see

$$M = \bigoplus M^\lambda \quad \text{and} \quad \dim_{\mathbb{C}} M^\lambda < \infty.$$

We set

$$ch(M) = \sum (\dim M^\lambda) e^\lambda$$

and call this the character of M.

Let $\tilde{\mathcal{O}}_{\text{triv}}$ be the category of $M \in \tilde{\mathcal{O}}$ such that $IM = 0$. It is known that $M(w)$ and $L(w)$ belong to $\tilde{\mathcal{O}}_{\text{triv}}$, and that any highest weight of $\mathfrak{g}$-module in $\tilde{\mathcal{O}}_{\text{triv}}$ has the form $-w(\rho) - \rho$ for some $w \in W$. For any $M \in \tilde{\mathcal{O}}_{\text{triv}}$ and $w \in W$, we shall denote by $[M; L(w)]$ the number of times of appearance of $L(w)$ in a Jordan-Hölder series of M. Then we have the trivial formula:

$$ch(M) = \sum [M; L(w)]\, ch(L(w)).$$

2.3. There exists a unique automorphism τ of $\mathfrak{g}$, which normalizes $\mathfrak{k}$, induces -1 on $\mathfrak{k}$ and sends X_α to $X_{-\alpha}$. For any $U(\mathfrak{g})$-module M, we provide $\mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ with a structure of $\mathfrak{g}$-module by the formula

$$\langle Zf, n \rangle = -\langle f, \tau(Z) n \rangle \tag{2.3.1}$$

for $f \in \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$, $n \in M$ and $Z \in \mathfrak{g}$.

For any M in $\tilde{\mathcal{O}}$ set

$$M^* = \{f \in \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C}); f(M^\lambda) = 0 \text{ except for finitely many } \lambda\}$$
$$= \{f \in \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C}; \dim_{\mathbb{C}} U(\mathfrak{b}) f < \infty\}$$

then it is easy to see

(2.3.2) If M belongs to $\tilde{\mathcal{O}}$ (resp. $\tilde{\mathcal{O}}_{\mathrm{triv}}$) then so does M^*

(2.3.3) One has $(M^*)^* \cong M$, $(M^*)^\lambda = (M^\lambda)^*$ for M in $\tilde{\mathcal{O}}$ and $\lambda \in \mathfrak{t}^*$

(2.3.4) $$ch(M_w^*) = ch(M_w)$$

§ 3. Flag Manifold

3.1. Let G be a connected simply-connected Lie group whose Lie algebra is $\mathfrak{g}$, and let B, B^-, N, N^- and T be the subgroups of G with $\mathfrak{b}, \mathfrak{b}^-$, $\mathfrak{N}, \mathfrak{N}^-$ and $\mathfrak{t}$ as Lie algebras, respectively. We set $X = G/B$ and we shall identify W as the subset of X by $w \mapsto WB/B$ $(w \in W)$. Define the Bruhat cell X_w to be:

$$X_w = Bw = Nw \subset X.$$

Then X is a disjoint union of X_w's. The following lemma follows immediately from the fact that the set of points of X_w where $X_{w'}$ does not satisfy the Whitney condition is nowhere dense in X_w.

Lemma 3.1. *$\{X_w\}$ is a Whitney stratification of X*

We set $n = \dim \mathfrak{N} = \dim X = \# \Delta^+$,

$$l(w) = \dim X_w = \text{length of } w = \# (\Delta^+ \cap w\Delta^-)$$

Proposition 3.2. *For any $w, w' \in W$ the following conditions are equivalent*

(a) $\bar{X}_w \supset \bar{X}_{w'}$

(b) $\mathrm{Hom}_U(M_{w'}, M_w) \neq 0$

(c) *there exist an integer $N \geqq 1$, $\alpha_j \in \Delta$ $(j = 2, \ldots, N)$ and $w_j \in W$ $(j = 1, \ldots, N)$ such that $w_1 = w'$, $w_j = s_{\alpha_j} \cdot w_{j-1}$ $(j = 2, \ldots, N)$, $w_N = w$ and $l(w_{j-1}) < l(w_j)$ $(j = 2, \ldots, N)$.*

If they are satisfied, one says that w' is smaller than w for the Bruhat order, and one writes $w' \leqq w$.

The proof of (a) $\Leftrightarrow$ (c) goes back to Chevalley (unpublished). One may refer to [3], p. 75. The proof that (c) implies (b) is given in [13] where Verma also conjectures the converse implication, which is proven in [1], §8.

In particular these conditions imply $w'(\rho) \geqq w(\rho)$. Here, $\lambda \geqq \mu$ $(\lambda, \mu \in \mathfrak{t}^*)$ signifies that $\lambda - \mu$ is a non-negative coefficient linear combination of $\alpha \in \Delta^+$.

For any $w \in W$, we shall denote by

(3.2.1) $$Y(w) = \bigcup \{X_{w'}; w'(\rho) \geqq w(\rho)\}.$$

By the proposition above, $Y(w)$ is a closed analytic subset of X containing $\bar{X}_w$. In general, they do not coincide.

We say that a subset Z of X is admissible if $Z \cap X_w \neq \emptyset$ implies $Z \supset Y(w)$. It is equivalent to say that Z is a union of $Y(w)$'s.

Lemma 3.2. *If Z is admissible and if $w \in W$ is such that $w(\rho)$ is minimal in the set $\{w(\rho);\ X_w \subset Z\}$ (i.e. $Z \supset X_{w'}$, $w'(\rho) \leqq w(\rho)$ implies $w = w'$) then X_w is open in Z.*

Proof. If not, $\overline{Z - X_w} \supset X_w$. Hence there is $w' \neq w$ such that $Z \supset X_{w'}$ and $\overline{X_{w'}} \supset X_w$. This implies that $w'(\rho) \leqq w(\rho)$ which is a contradiction.

§ 4. The Category $\mathcal{M}$

4.1. We view $\mathfrak{g}$ as the Lie algebra of right invariant vector fields on X. Denote p the projection $p\colon G \to G/B$. For any $\xi \in \mathfrak{g}$, there exists a unique vector field $\bar{\xi}$ on G/B such that $dp_x(\xi) = \bar{\xi}_{p(x)}$ for any $x \in G$. There exists as Lie algebra homomorphism:

$$\varphi\colon \mathfrak{g} \to \mathscr{D}_X \qquad \text{such that } \varphi(\xi) = \bar{\xi} \text{ for all } \xi \in \mathfrak{g}.$$

Actually, φ is easily seen to be independent of the choice of a base point on X (i.e. of an identification of X with $G|B$.

One extends φ to a ring homomorphism $\varphi\colon U \to \mathscr{D}_X$. It is known that $\varphi(I) = 0$, so one gets a factorization

$$U \to R \to \mathscr{D}_X$$

Hence, for any coherent $\mathscr{D}_X$-module $\mathfrak{M}$, R operates on $H^j(X, \mathfrak{M})$.

4.2. We shall denote by $\mathcal{M}$ the category of holonomic $\mathscr{D}_X$-modules with R.S., whose characteristic varieties are contained in $\bigcup_{w \in W} T^*_{X_w}X$. Here $T^*_{X_w}X$ denotes the conormal bundle of X_w in X.

Theorem 4.1. (1) *For any $M \in \tilde{\mathcal{O}}_{\mathrm{triv}}$, $\mathscr{D} \underset{R}{\otimes} M$ belongs to $\mathcal{M}$.*

(2) *For any $\mathfrak{M} \in \mathcal{M}$, $\Gamma(X; \mathfrak{M})$ belongs to $\tilde{\mathcal{O}}_{\mathrm{triv}}$.*

(3) *For any $M \in \tilde{\mathcal{O}}_{\mathrm{triv}}$, $\mathscr{T}or_j^R(\mathscr{D}, M) = 0$ for $j \neq 0$*

(4) *For any $\mathfrak{M} \in \mathcal{M}$, $H^j(X; \mathfrak{M}) = 0$ for $j \neq 0$*

(5) *For any $M \in \tilde{\mathcal{O}}_{\mathrm{triv}}$, $M \to \Gamma(X; \mathscr{D} \underset{U}{\otimes} M)$ is bijective.*

(6) *For any $\mathfrak{M} \in \mathcal{M}$, $\mathscr{D} \underset{U}{\otimes} \Gamma(X; \mathfrak{M}) \to \mathfrak{M}$ is an isomorphism*

(7) *For any $\mathfrak{M} \in \mathcal{M}$*

$$ch(\Gamma(X; \mathfrak{M}^*)) = ch(\Gamma(X; \mathfrak{M}))$$

Before entering into the proof of this theorem, we shall give here a sketch of the proof.

First we establish (1) by using the fact; if u satisfies

$$\sum_{j=1}^{n} a_j x_j \frac{\partial}{\partial x_j} u = cu \qquad (a_j > 0,\ c \in \mathbb{C})$$

and u satisfies a holonomic system with R.S. outside the origin, then u satisfies a system with R.S.. Next we consider the module $\mathfrak{M}_w^* = \mathscr{H}_{[X_w]}^{n-l(w)}(\mathcal{O}_X)$. For this module we have

(4) $$H^j(X; \mathfrak{M}_w^*) = 0 \quad \text{for } j \neq 0 \text{ and } H^0(X; \mathfrak{M}_w^*) = M_w^*.$$

Moreover this satisfies $\mathscr{D} \otimes \Gamma(X; \mathfrak{M}_w^*) \twoheadrightarrow \mathfrak{M}_w^*$. At the next step, we shall prove (3) for $M = M_w$, and establish the injectivity of $M_w \to \Gamma(X; \mathscr{D} \otimes M_w)$ By looking at $\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{D} \otimes M_w, \mathcal{O})$, we also show that $\mathscr{D} \otimes M_w \cong \mathfrak{M}_w$. By using these, theorem will be proved by induction on $\dim \operatorname{Supp} \mathfrak{M}$.

4.3. Proof of (1) *of Theorem* 1. First we shall show that $Ch(\mathscr{D} \underset{U}{\otimes} M)$ is contained in $\bigcup_w T_{X_w}^* X$. Since $\mathscr{D} \underset{U}{\otimes} M$ is a quotient of a direct sum of copies of $\mathscr{D}/\mathscr{D} \cdot \varphi(\mathfrak{N}^k)$, its characteristic variety is contained in $\{p \in T^*X; \sigma_1(\varphi(Y)(p)) = 0$ for any $Y \in \mathfrak{N}\}$. For a point q in X_w, the vectors $\varphi(Y)$ $(Y \in \mathfrak{N})$ generates $T_q X_w$. If $p \in T_q^* X$ satisfies $\sigma_1(\varphi(Y))(p) = 0$ for $Y \in n$, p is orthogonal to $T_q X_w$ and hence p belongs to $T_{X_w}^* X$. Thus we have proved the statement for $Ch(\mathscr{D} \underset{U}{\otimes} M)$. We shall prove next that $\mathscr{D} \underset{U}{\otimes} M$ is with R.S. First remark the following lemma.

Lemma 4.2. *Let $\mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}''$ be an exact sequence of holonomic $\mathscr{D}$-modules. If $\mathfrak{M}'$ and $\mathfrak{M}''$ have R.S. then so does $\mathfrak{M}$.*

By applying this lemma, one can easily reduce (1) to the case where $M = M_w$. Then

$$\mathscr{D} \underset{U}{\otimes} M = \mathscr{D} / \sum_{Y \in \mathfrak{N}} \mathscr{D} \varphi(Y) + \sum_{H \in \mathfrak{t}} \mathscr{D}(\varphi(H) + \langle w(\rho) + \rho, H \rangle).$$

By using the induction procedure, it is enough to show the following statement.

4.3.1. Let w, w' be elements of W. If $\mathscr{D} \otimes M_w$ has R.S. on $U - X_{w'}$ for an open neighborhood U of $X_{w'}$, then $\mathscr{D} \otimes M_w$ has R.S. on U.

Now, remark that $\varphi(Y)$ $(Y \in \mathfrak{N})$ are tangent to $X_{w'}$ and generate $TX_{w'}$. Moreover, for $H \in \mathfrak{t}$, $\varphi(H)$ is tangent to $X_{w'}$, vanishes at w', and the eigenvalues of the isotropy action of H in $T_{w'}X / T_{w'}X_{w'}$ are $-\alpha(H)$ for $\Delta^+ \cap (w')^{-1}\Delta^+$. Therefore (4.3.1) is a consequence of the following more general proposition. The proof of this proposition will be given in the Appendix.

Proposition 4.3. *Let X be a complex manifold, Y a connected submanifold of X, y a point of Y and $\mathfrak{M}$ a holonomic $\mathscr{D}_X$-module generated by a section u.*

Assume that

(a) *$\mathfrak{M}$ has R.S. on $X - Y$*

(b) *There exist vector fields $V_1, \ldots, V_N$ such that $V_j u \in \mathcal{O}_X u$ and that $\{V_j\}$ generates TY.*

(c) *There exists a vector field V_0 such that $V_0 u \in \mathcal{O}_X u$, V_0 vanishes at y, V_0 is tangent to Y and the eigenvalues of the isotropy action of V_0 on $T_y X / T_y Y$ are strictly positive.*

Then $\mathfrak{M}$ has R.S. on X.

§ 5. The Sheaf $\mathfrak{M}_w^*$

5.1. The following proposition is proved in [10].

Proposition 5.1. (1) $H^j_{[X_w]}(X; \mathcal{O}_X) = 0$ for $j \neq n - l(w)$,

(2) $\mathcal{H}^j_{[X_w]}(\mathcal{O}_X) = 0$ for $j \neq n - l(w)$,

(3) On $H^{n-l(w)}_{[X_w]}(X; \mathcal{O}_X)$, $\mathfrak{k}$ acts semi simply and we have

$$ch(H^{n-l(w)}_{[X_w]}(X; \mathcal{O}_X)) = ch(M_w).$$

The third properly implies that $H^{n-l(w)}_{[X_w]}(X; \mathcal{O}_X)$ belongs to $\tilde{\mathcal{O}}_{\mathrm{triv}}$.

The proof of these properties is based on the fact that X_w has an affine neighborhood $(wB^-w^{-1})w$ and the pair $((wB^-w^{-1})w, X_w)$ is isomorphic to $(wN^-w^{-1}, N \cap wN^-w^{-1}) \cong (\mathrm{Ad}(w)\mathfrak{N}^-, \mathfrak{N} \cap \mathrm{Ad}(w)\mathfrak{N}^-)$ as the pair of spaces on which T acts.

We define $\mathfrak{M}_w$ to be the dual of $\mathcal{H}^{n-l(w)}_{[X_w]}(\mathcal{O}_X)$.

Since $X = \cup X_w$ is a Whitney stratification $\mathfrak{M}_w$ and $\mathfrak{M}_w^*$ belong to $\mathcal{M}$ (Proposition 1.1 and Proposition 1.3(10)).

Moreover, we have

(5.1.1) $$\mathrm{Supp}\,\mathfrak{M}_w = \mathrm{Supp}\,\mathfrak{M}_w^* = \overline{X_w},$$

(5.1.2) $$H^j(X; \mathfrak{M}_w^*) = 0 \quad \text{for } j \neq 0,$$

(5.1.3) $$ch(\Gamma(X; \mathfrak{M}_w^*)) = ch(M_w),$$

(5.1.4) $$\mathfrak{M}_w^*|_{X - \partial X_w} \cong \mathcal{B}_{X_w|X - \partial X_w},$$

(5.1.5) $$\mathcal{H}^j_{[\partial X_w]}(\mathfrak{M}_w^*) = 0 \quad \text{for any } j.$$

The last property implies that for any $\mathfrak{M} \in \mathcal{M}$

(5.1.6) $$\mathbb{R}\Gamma(X; \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \mathfrak{M}_w^*)) \xrightarrow{\sim} \mathbb{R}\Gamma(X - \partial X_w; \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \mathfrak{M}^*{}_w))$$

by Proposition 1.4.

This implies in particular any homomorphism from $\mathfrak{M}$ into $\mathfrak{M}_w^*$ defined on $X - \partial X_w$ can be uniquely prolonged to a homomorphism defied on X.

Since $\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \mathfrak{M}_w^*)|_{X - \partial X_w}$ is a complex of sheaves whose cohomology sheaves are locally constant sheaves on X_w and since X_w is isomorphic to $\mathbb{C}^{l(w)}$, we can conclude

$$\mathbb{R}\Gamma(X - \partial X_w; \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \mathfrak{M}_w^*)) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \mathfrak{M}_w^*)_w.$$

Thus we obtain the following

Proposition 5.3. For any $\mathfrak{M} \in \mathcal{M}$,

$$\mathbb{R}\Gamma(X; \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \mathfrak{M}_w^*)) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \mathfrak{M}_w^*)_w.$$

The following proposition is also necessary to prove Theorem 4.1, in spite that this is a very special case of that theorem.

Proposition 5.4. $\mathcal{D} \underset{U}{\otimes} \Gamma(X;\ \mathfrak{M}_w^*) \to \mathfrak{M}_w^*$ *is surjective.*

The proposition is a corollary of the following lemma which can be easily proved.

Lemma 5.5. *Let S be a separated scheme, U an affine open subset of S and let j be the inclusion $U \hookrightarrow S$. Then, for any quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}$, the homomorphism $\mathcal{O}_S \underset{\mathbb{Z}}{\otimes} \Gamma(U;\mathcal{F}) \to j_*\mathcal{F}$ is surjective.*

In order to obtain Proposition 5.4, it is enough to apply this lemma for $S = X$, $U = (wB^- w^{-1})w$ and $\mathcal{F} = \mathfrak{M}_w^*$ and use Serre's GAGA in order to show $\Gamma(X;\mathfrak{M}_w^*)$ equals its algebro-geometric counterpart. We remark also (1.4.1).

5.2. In order to calculate $\Gamma(X,\mathfrak{M}_w^*)$, we shall Remark (5.2.1)

$$\mathrm{Hom}_R(M_w, \Gamma(X;\mathfrak{M}_w^*)) = \begin{cases} \mathbb{C} & \text{if} \quad w = w' \\ 0 & \text{if} \quad w(\rho) \not\geqq w'(\rho) \end{cases}$$

which immediately follows from

$$ch(\Gamma(X;\mathfrak{M}_w^*)) = ch(M_w).$$

Proposition 5.3 implies that one has a natural isomorphism for $M \in \tilde{\mathcal{O}}_{\mathrm{triv}}$

$$\mathcal{H}om_{\mathcal{D}}(\mathcal{D} \underset{R}{\otimes} M, \mathfrak{M}_w^*)_w \xrightarrow{\approx} \mathrm{Hom}_R(M, \Gamma(X, \mathfrak{M}_w^*)) \tag{5.2.2}$$

where one uses the R-morphism $M \to \Gamma(X, \mathcal{D} \underset{R}{\otimes} M)$ and the $\mathcal{D}$-morphism $\mathcal{D} \otimes \Gamma(X, \mathfrak{M}_w^*) \to \mathfrak{M}_w^*$.

Lemma 5.6. Supp$(\mathcal{D} \otimes M_w) \subset Y(w)$.

If the statement is false, there exist $w' \in W$ such that $X_{w'}$ is open in Supp$(\mathcal{D} \otimes M_w)$, and disjoint from $Y(w)$. By Proposition 1.2, one has $\mathrm{Hom}_{\mathcal{D}}(\mathcal{D} \otimes M_w, \mathfrak{M}_w^*)_{w'} \neq 0$ which implies $w(\rho) \leqq w'(\rho)$ by (5.2.1) and (5.2.2). This is a contradiction. Q.E.D.

Corollary 5.7. *For any $M \in \tilde{\mathcal{O}}_{\mathrm{triv}}$, we have*

$$\begin{aligned} \mathrm{Supp}(\mathcal{D} \otimes M) &\subset U\{Y(w); w(\rho) - \rho \textit{ is a weight of } M\} \\ &= U\{Y(w);\ -w(\rho) - \rho \textit{ is a highest weight of } M\}. \end{aligned}$$

Proof. We shall prove this by the induction of $l(M)$. If $-w(\rho) - \rho$ is a highest weight of M, then there exists an exact sequence $M_w \to M \to M' \to 0$ with $l(M') < l(M)$.

Then Corollary follows from the preceding lemma and

$$\mathrm{Supp}(\mathcal{D} \otimes M) \subset \mathrm{Supp}(\mathcal{D} \otimes M') \cup \mathrm{Supp}(\mathcal{D} \otimes M_w).$$

Corollary 5.8. $\Gamma(X;\mathfrak{M}_w^*) = M_w^*$.

Proof. Set $M = \Gamma(X;\mathfrak{M}_w^*)$. Then we have

$$ch(M^*) = Ch(M_w^*) = ch(M_w).$$

Hence there exists a non zero homomorphism $M_w \to M^*$. Taking the dual we obtain

$$0 \to N \to M \xrightarrow{f} M_w^*$$

Any highest weight of N has the form $-w'(\rho)-\rho$ with $w'(\rho) \ngeq w(\rho)$. For such a w', we have $Y(w') \cap X_w = \emptyset$. Therefore, the preceding corollary implies $\mathrm{Supp}(\mathcal{D} \otimes N) \cap X_w = \emptyset$. Hence N is contained in $\Gamma_{\partial X_w}(X; \mathfrak{M}_w^*) = 0$, which implies the injectivity of f. The comparison of the characters conclude the bijectivity of f. Q.E.D.

One may refer to [10] for a different proof of Corollary 5.8.

§ 6. The Sheaf $\mathcal{D} \otimes M_w$

6.1. In this section we shall study the properties of $\mathcal{D} \otimes M_w$.

Proposition 6.1. (1) $\mathcal{T}\mathrm{or}_j^{U(\mathfrak{N})}(\mathcal{D}, \mathbb{C}) = 0$ *for* $j \neq 0$.

(2) $\mathrm{Tor}_j^{U(\mathfrak{N})}(R, \mathbb{C}) = 0$ *for* $j \neq 0$

Proof. It is known that $\mathbb{C}$ has a free resolution

$$0 \leftarrow \mathbb{C} \leftarrow U(\mathfrak{N}) \leftarrow U(\mathfrak{N}) \underset{\mathbb{C}}{\otimes} \mathfrak{n} \leftarrow U(\mathfrak{N}) \underset{\mathbb{C}}{\otimes} \Lambda^2 \mathfrak{N} \leftarrow \ldots \leftarrow U(\mathfrak{N}) \otimes \Lambda^n \mathfrak{N} \leftarrow 0.$$

Hence in order to prove (1), we have to show the following sequence is exact;

$$\mathcal{D} \leftarrow \mathcal{D} \otimes \mathfrak{N} \leftarrow \mathcal{D} \otimes \Lambda^2 \mathfrak{N} \leftarrow \ldots \leftarrow \mathcal{D} \otimes \Lambda^n \mathfrak{N} \leftarrow 0. \tag{6.1.1}$$

Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{N}$. Then the graduation of (6.1.1)

$$\mathrm{gr}\, \mathcal{D} \leftarrow \mathrm{gr}\, \mathcal{D} \otimes \mathfrak{N} \leftarrow \ldots \leftarrow \mathrm{gr}\, \mathcal{D} \otimes \Lambda^n \mathfrak{N} \leftarrow 0 \tag{6.1.2}$$

is nothing but the Koszul complex of $\mathrm{gr}\, \mathcal{D}$ with respect to $(\sigma_1(\varphi(X_1)), \ldots, \sigma_1(\varphi(X_n)))$. Since the common zero of $\sigma_1(\varphi(X_1), \ldots, \sigma_1(\varphi(X_n))$ is $UT^*_{X_w} X$, this has codimension n. Therefore $(\sigma_1(\varphi(X_1)), \ldots, \sigma_1(\varphi(X_n)))$ is a regular system, which implies the exactitude of (6.1.1). The property (2) is also proved by the same argument. Q.E.D.

Note that for any $H \in \mathfrak{k}$ we have

$$\coprod_{w \in W} (H + \langle w(\rho) + \rho, H \rangle) \in I + U\mathfrak{N}$$

Hence we obtain

$$R/R\mathfrak{N} = \bigoplus_{w \in W} M_w \tag{6.1.3}$$

On the other hand, Proposition 6.1 implies

$$\mathcal{T}\mathrm{or}_j^R(\mathcal{D}, R/R\mathfrak{N}) = 0 \quad \text{for } j \neq 0. \tag{6.1.4}$$

Thus we obtain

Proposition 6.2. $\mathcal{T}\mathrm{or}_j^R(\mathcal{D}, M_w) = 0$ for $j \neq 0$.

6.2. Now, we shall calculate $\mathcal{D} \otimes M_w$. We have already seen in Proposition 5.3

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{D}/\mathcal{D}\mathfrak{N}, \mathfrak{M}_w^*)_w \cong \mathbb{R}\Gamma(X; \mathbb{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{D}/\mathcal{D}\mathfrak{N}; \mathfrak{M}_w^*))$$

We have

$$\mathbb{R}\Gamma(X;\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{D}/\mathscr{D}\mathfrak{N},\mathfrak{M}_w^*))=\mathbb{R}\mathscr{H}om_{U(\mathfrak{N})}(\mathbb{C},\mathbb{R}\Gamma(X;\mathfrak{M}_w^*))$$
$$=\mathbb{R}\mathscr{H}om_{U(\mathfrak{N})}(\mathbb{C},M_w^*)$$

(the last equality using Corollary 5.8).

It is known that

$$\mathscr{E}xt^j_{U(\mathfrak{N})}(\mathbb{C};M_w^*)=\begin{cases}0 & \text{for } j\neq 0\\ \mathbb{C} & \text{for } j=0\end{cases} \tag{6.2.1}$$

Thus we obtain

$$\mathscr{E}xt^j(\mathscr{D}/\mathscr{D}\mathfrak{N},\mathfrak{M}_w^*)_w=\begin{cases}0 & \text{for } j\neq 0\\ \mathbb{C} & \text{for } j=0\end{cases}$$

Since $\mathscr{D}/\mathscr{D}\mathfrak{N}=\bigoplus\limits_w \mathscr{D}\otimes M_w$, we have

$$\mathscr{E}xt^j_{\mathscr{D}}(\mathscr{D}\otimes M_w,\mathfrak{M}_{w'}^*)_{w'}=0 \quad \text{for } j\neq 0,$$

and

$$\bigoplus_w \mathscr{E}xt^0_{\mathscr{D}}(\mathscr{D}\otimes M_w,\mathfrak{M}_{w'}^*)_{w'}\cong\mathbb{C} \quad \text{for any } w'$$

On the other hand, we know already, by (5.2.2)

$$\mathscr{E}xt^0_{\mathscr{D}}(\mathscr{D}\otimes M_w,\mathfrak{M}_w^*)_w=\mathbb{C}.$$

Therefore we can conclude

$$\mathscr{E}xt^j_{\mathscr{D}}(\mathscr{D}\otimes M_w),\mathfrak{M}_{w'}^*)_{w'}=\begin{cases}\mathbb{C} & \text{if } w=w' \text{ and } j=0\\ 0 & \text{otherwise.}\end{cases} \tag{6.2.2}$$

On the other hand, Proposition 1.7 implies

$$\mathbb{R}\mathscr{H}om(\mathscr{D}\otimes M_w,\mathfrak{M}_{w'}^*)_{w'}=\mathrm{Hom}_{\mathbb{C}}(\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}_X,\mathscr{D}\otimes M_w)_{w'},\mathbb{C})[-\operatorname{codim} X_{w'}].$$

This, together with (6.2.2) implies

$$\mathscr{E}xt^j_{\mathscr{D}}(\mathscr{O}_X,\mathscr{D}\otimes M_w)_{w'}=\begin{cases}\mathbb{C}, & j=\operatorname{codim} X_w,\ w=w'\\ 0 & \text{otherwise}\end{cases}$$

Since $\mathscr{E}xt^j_{\mathscr{D}}(\mathscr{O}_X,\mathscr{D}\otimes M_w)|X_{w'}$ is constant sheaf, we finally calculate $\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}_X,\mathscr{D}\otimes M_w)$.

Proposition 6.3. $\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}_X,\mathscr{D}\otimes M_w)=\mathbb{C}_{X_w}[-\operatorname{codim} X_w]$.

Corollary 6.4. $\mathscr{D}\otimes M_w\cong\mathfrak{M}_w$.

Proof. By the definition of $\mathfrak{M}_{w'}^*$ we have

$$\mathfrak{M}_w^*=\mathbb{R}\Gamma_{[X_w]}(\mathscr{O}_X)[\operatorname{codim} X_w].$$

Hence by Proposition 1.4, we obtain

$$\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}_X,\mathfrak{M}_w^*)=\mathbb{R}\Gamma_{X_w}(\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}_X,\mathscr{O}_X))[\operatorname{codim} X_w].$$

Since $\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathcal{O}_X, \mathcal{O}_X) = \mathbb{C}_X$. We obtain

$$\mathbb{R}\,\mathscr{H}om\,(\mathfrak{M}_w, \mathcal{O}_X) = \mathbb{R}\,\Gamma_{X_w}(\mathbb{C})\ [\operatorname{codim} X_w].$$

On the other hand, by Proposition 1.3 (8), we have

$$\begin{aligned}\mathbb{R}\,\mathscr{H}om\,(\mathscr{D} \otimes M_w, \mathcal{O}_X) &= \mathbb{R}\,\mathscr{H}om_{\mathbb{C}}(\mathbb{R}\,\mathscr{H}om_{\mathscr{D}X}(\mathcal{O}_X, \mathscr{D} \otimes M_w), \mathbb{C}_X)\\ &= \mathbb{R}\,\mathscr{H}om_{\mathbb{C}}(\mathbb{C}_{X_w}[-\operatorname{codim} X_w], \mathbb{C}_X)\\ &= \mathbb{R}\,\Gamma_{X_w}(\mathbb{C}_X)\,[\operatorname{codim} X_w].\end{aligned}$$

Therefore, $\mathbb{R}\,\mathscr{H}om_{\mathscr{D}}(\mathfrak{M}_w, \mathcal{O}_X)$ is isomorphic to $\mathbb{R}\,\mathscr{H}om_{\mathscr{D}}(\mathscr{D} \otimes M_w, \mathcal{O}_X)$. Proposition 1.5 implies that $\mathfrak{M}_w$ is isomorphic to $\mathscr{D} \otimes M_w$. Q.E.D.

6.3. We shall denote by $\mathscr{E}$ the sheaf of micro-differential operators (see [4]).

Proposition 6.5. *$M_w \to \Gamma(X; \mathscr{D} \otimes M_w)$ is injective.*

Proof. Set $\Omega = \{q \in T^*X;\ \sigma_1(\varphi(X_{-\alpha})) \neq 0$ for any simple root $\alpha\}$. In order to prove Proposition 6.5, it is sufficient to show the injectivity of

$$M_w \to \Gamma(\Omega; \mathscr{E} \underset{R}{\otimes} M_w)$$

because this is a composition of $M_w \to \Gamma(X; \mathscr{D} \otimes M_w)$ and

$$\Gamma(X; \mathscr{D} \otimes M_w) \to \Gamma(\Omega; \mathscr{E} \otimes M_w).$$

By Verma [13]. there exists an injective map f: $M_e \to M_w$. Moreover, if we denote by $u(e)$ and $u(w)$ the canonical generators of M_1 and M_w, respectively, we have

$$f(u(e)) = X^{m_1}_{-\alpha_1} \ldots X^{m_N}_{-\alpha_N} u(w)$$

where α_j are simple roots and m_j are positive integers. By using the fact that $\varphi(X_{-\alpha})$ is invertible on Ω, the same argument as Verma shows that

$$1 \otimes f\colon \mathscr{E} \underset{U}{\otimes} M_e \to \mathscr{E} \underset{U}{\otimes} M_w$$

is an isomorphism on Ω.

The preceding corollary implies that

$$\mathscr{E} \underset{U}{\otimes} M(e) = C_{X_e|X} \underset{\text{def}}{=} \mathscr{E} \underset{\mathscr{D}}{\otimes} \mathscr{B}_{X_e|X}$$

Therefore $\mathscr{E} \underset{U}{\otimes} M_w|\Omega$ is isomorphic to $C_{X_e|X}|_\Omega$. For a non zero section v of $C_{X_e|X}$ and $P \in \mathscr{E}$, $Pv = 0$ implies $\sigma(P)|\,T_{X_e^*X} = 0$, because otherwise P is invertible. Any element M_w can be written in a unique way as $Pu(w)$ for $P \in U(\mathfrak{N}^-)$. If P belongs to $U_m(\mathfrak{N}^-) = (\mathbb{C} + \mathfrak{N}^-)^m$, then $\sigma_m(\varphi(P))|T^*_{X_e}X$ is nothing but the modulo class of P in $U_m(\mathfrak{N}^-)/U_{m-1}(\mathfrak{N}^-) = S^m(\mathfrak{N}^-)$, which is regarded as a polynomial on $(\mathfrak{N}^-)^* \cong T^*_{X_e}X$. Hence if $1 \otimes Pu(w) = \varphi(P)(1 \otimes u(w)) = 0$ on Ω, we have $\sigma_m(\varphi(P))|_\Omega = 0$. This implies that $P \in U_{m-1}(\mathfrak{N}^-)$. By continuing this, we can conclude that $P = 0$.

6.4. On X_w, $\mathfrak{M}_w^*$ is isomorphic to $\mathscr{B}_{X_w|X-\partial X_w}$. Hence $\mathfrak{M}_w$ and $\mathfrak{M}_w^*$ are isomorphic on X_w. Let φ_w be a homomorphism from $\mathfrak{M}_w$ into $\mathfrak{M}_w^*$ defined on X which extends this isomorphism on X_w (which exists by (5.1.6)). We shall define $\mathfrak{L}_w$ as the image of φ_w.

Proposition. $\mathfrak{L}_w$ *satisfies the following properties*

(1) Supp $\mathfrak{L}_w = \overline{X_w}$,
(2) Supp $(\mathfrak{M}_w^*/\mathfrak{L}_w) \subset \partial X_w$,
(3) $\mathscr{H}^0_{\partial X_w}(\mathfrak{L}_w)=0$,
(4) $\mathfrak{L}_w \cong \mathfrak{L}_w^*$,
(5) $\mathfrak{L}_w$ *is a simple object in* $\mathscr{M}$.

Proof. The properties (1), (2), (3) are obvious. The property (4) follows from the fact that $\varphi=\varphi^*$. We shall prove (5). Let $\mathscr{F}$ be a coherent sub-$\mathscr{D}_X$-module of $\mathfrak{L}_w$. By Proposition 1.2, $\mathscr{F}=0$ or $\mathfrak{L}_w$ on X_w. If $\mathscr{F}=0$ on X_w, then $\mathscr{F}\subset\mathscr{H}^0_{\partial X_w}(\mathfrak{L}_w)=0$. If $\mathscr{F}=\mathfrak{L}_w$ on X_w, then $(\mathfrak{L}_w/\mathscr{F})^*\subset\mathscr{H}^0_{\partial X_w}(\mathfrak{L}_w^*)=0$. Therefore, $\mathscr{F}$ is $\mathcal{O}$ or $\mathfrak{L}_w$.

§ 7. Proof of Theorem 4.1

7.1. For a subset Z of X, we say that Z is admissible if $Z\cap X_w \neq \emptyset$ implies $Z\supset Y(w)$. For an admissible Z, we denote by $\mathscr{M}_Z$ the category of $\mathfrak{M}\in\mathscr{M}$ with Supp $\mathfrak{M}\subset Z$. Let $\tilde{\mathcal{O}}_Z$ be the category of $M\in\tilde{\mathcal{O}}_{\text{triv}}$ such that any highest weight of M has the form $-w(\rho)-\rho$ for some w with $X_w\subset Z$. This condition is equivalent to say that, for any weight λ of M, there exists w such that $-w(\rho)-\rho\geqq\lambda$ and $X_w\subset Z$.

For an admissible Z, we shall consider the following statements

$(1)_Z$ *For any* $M\in\tilde{\mathcal{O}}_Z$, $\mathscr{D}\otimes M$ *belongs to* $\mathscr{M}_Z$.
$(2)_Z$ *For any* $\mathfrak{M}\in\mathscr{M}$, $\Gamma(X;\mathfrak{M})$ *belongs to* $\tilde{\mathcal{O}}_Z$.
$(3)_Z$ *For any* $M\in\tilde{\mathcal{O}}_Z$, $\mathscr{T}\mathrm{or}_j^R(\mathscr{D},M)=0$ *for* $j\neq 0$.
$(4)_Z$ *For any* $\mathfrak{M}\in\mathscr{M}_Z$, $H^j(X;\mathfrak{M})=0$ *for* $j\neq 0$.
$(5)_Z$ *For any* $M\in\tilde{\mathcal{O}}_Z$, $M\to\Gamma(X;\mathscr{D}\underset{U}{\otimes}M)$ *is an isomorphism.*
$(6)_Z$ *For any* $\mathfrak{M}\in\mathscr{M}_Z$, $\mathscr{D}\otimes\Gamma(X;\mathfrak{M})\to\mathfrak{M}$ *is an isomorphism.*
$(7)_Z$ *For any* $\mathfrak{M}\in\mathscr{M}_Z$, *we have* $ch(\Gamma(X;\mathfrak{M}^*))=ch(\Gamma(X;\mathfrak{M}))$.

The property $(1)_Z$ has already been proved (§4.3 and Corollary 5.7). We shall prove the remaining statements by induction on $\#\{w; X_w\subset Z\}$. If $Z=\emptyset$, there is nothing to prove. Assuming $Z\neq\emptyset$, we shall take a w such that $X_w\subset Z$ and that $w(\rho)$ is minimal (i.e. $X_{w'}\subset Z$ and $w'(\rho)\leqq w(\rho)$ implies $w=w'$). Then X_w is an open subset of Z. Set $Z'=Z-X_w$. Then Z' is also admissible. By the hypothesis of the induction we can assume that $(2)_{Z'},\ldots,(7)_{Z'}$ are true.

7.2. *Proof of* $(2)_Z$, $(4)_Z$ *and* $(6')_Z$

$(6')_Z$: For any $\mathfrak{M}\in\mathscr{M}_Z$, $\mathscr{D}\otimes\Gamma(X;\mathfrak{M})\to\mathfrak{M}$ is surjective.

Let $\mathfrak{M}$ be an object in $\mathscr{M}_Z$. Then, by Proposition 1.2 there exists an integer N and an isomorphism f: $\mathfrak{M}\to\mathfrak{M}_w^{*N}$ on a neighborhood of X_w. By the remark after (5.1.6), f extends to a homomorphism defined on X. Thus we obtain an exact sequence

$$0\to\mathscr{F}\to\mathfrak{M}\xrightarrow{f}\mathfrak{M}_w^*\to\mathscr{G}\to 0$$

where $\mathscr{F}$ and $\mathscr{G}$ belong to $\mathscr{M}_{Z'}$. Let $\mathscr{I}$ be the image of f. By (5.1.2) and $(4)_{Z'}$, we have $H^j(X;\mathscr{G})=H^j(X;\mathfrak{M}_w^{*N})=0$ for $j\neq 0$.

Therefore, from the exact sequence $0\to\mathscr{I}\to\mathfrak{M}_w^{*N}\to\mathscr{G}\to 0$ we obtain $H^j(X;\mathscr{I})=0$ for $j\neq 0$, 1 and an exact sequence

$$0\to\Gamma(X;\mathscr{I})\to\Gamma(X;\mathfrak{M}_w^{*N})\to\Gamma(X;\mathscr{G})\to H^1(X;\mathscr{I})\to 0. \tag{7.2.1}$$

Since $\Gamma(X;\mathfrak{M}_w^*)\in\tilde{\mathscr{O}}_Z$ and $\Gamma(X;\mathscr{G})\in\tilde{\mathscr{O}}_{Z'}$, we have $\Gamma(X,\mathscr{I})\in\tilde{\mathscr{O}}_Z$ and $H^1(X;\mathscr{I})\in\tilde{\mathscr{O}}_{Z'}$. Tensoring $\mathscr{D}$ to (7.2.1), we obtain the diagramm

$$\begin{tikzcd}
\mathscr{D}\otimes\Gamma(X;\mathfrak{M}_w^{*N}) \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & \mathscr{D}\otimes\Gamma(X;\mathscr{G}) \arrow[r] \arrow[d, "\beta"] & \mathscr{D}\otimes H^1(X;\mathscr{I}) \arrow[r] \arrow[d] & 0 \\
\mathfrak{M}_w^{*N} \arrow[r] & \mathscr{G} \arrow[r] & 0 &
\end{tikzcd}$$

By $(6)_{Z'}$ and Proposition 5.4, β is an isomorphism and α is surjective. Therefore γ is surjective which implies $\mathscr{D}\otimes H^1(X;\mathscr{I})=0$. Since $H^1(X;\mathscr{I})\in\tilde{\mathscr{O}}_{Z'}$, we can apply $(5)_{Z'}$ to show $H^1(X;\mathscr{I})=0$. Thus we obtain $H^j(X;\mathscr{I})\simeq 0$ for $j\neq 0$. On the other hand, from the exact sequence $0\to\mathscr{F}\to\mathfrak{M}\to\mathscr{I}\to 0$ and $H^j(X;\mathscr{F})=0$ $(j\neq 0)$, we have

$$H^j(X;\mathfrak{M})=H^j(X;\mathscr{I})=0 \qquad \text{for } j\neq 0$$

and an exact sequence

$$0\to\Gamma(X;\mathscr{F})\to\Gamma(X;\mathfrak{M})\to\Gamma(X;\mathscr{I})\to 0.$$

Thus we obtain $(2)_Z$ and $(4)_Z$.

Now, we shall prove $(6')_Z$. By the diagramm

$$\begin{tikzcd}
& \mathscr{D}\otimes\Gamma(X:\mathscr{I}) \arrow[r] \arrow[d] & \mathscr{D}\otimes\Gamma(X;\mathfrak{M}_w^{*N}) \arrow[r] \arrow[d] & \mathscr{D}\otimes\Gamma(X;\mathscr{G}) \arrow[r] \arrow[d, "\wr"'] & 0 \\
0 \arrow[r] & \mathscr{I} \arrow[r] & \mathfrak{M}_w^{*N} \arrow[r] \arrow[d] & \mathscr{G} \arrow[r] & 0 \\
& & 0 & &
\end{tikzcd}$$

we see that $\mathscr{D}\otimes\Gamma(X;\mathscr{I})\to\mathscr{I}$ is surjective. Another diagramm

$$\begin{tikzcd}
& \mathscr{D}\otimes\Gamma(X;\mathscr{F}) \arrow[r] \arrow[d, "\wr"'] & \mathscr{D}\otimes\Gamma(X;\mathfrak{M}) \arrow[r] \arrow[d] & \mathscr{D}\otimes\Gamma(X;\mathscr{I}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F} \arrow[r] & \mathfrak{M} \arrow[r] & \mathscr{I} \arrow[r] \arrow[d] & 0 \\
& & & 0 &
\end{tikzcd}$$

implies the surjectivity of $\mathscr{D}\otimes\Gamma(X;\mathfrak{M})\to\mathfrak{M}$.

7.3. *Proof of* $(7)_Z$. For $\mathfrak{M} \in \mathcal{M}_Z$ we shall consider a homomorphism $f: \mathfrak{M} \to \mathfrak{M}_w^{*N}$ which is an isomorphism on X_w. Set $\mathfrak{M}' = f^{-1}(\mathfrak{L}_w^N)$. Then we obtain exact sequences

$$0 \to \mathscr{F} \to \mathfrak{M}' \to \mathfrak{L}_w^N \to \mathscr{G} \to 0$$
$$0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0$$

where $\mathscr{F}$, $\mathscr{G}$, $\mathfrak{M}''$ belong to $\mathcal{M}_{Z'}$. By $(4)_Z$, we obtain exact sequences

$$0 \to \Gamma(\mathscr{F}) \to \Gamma(\mathfrak{M}') \to \Gamma(\mathfrak{L}_w^N) \to \Gamma(\mathscr{G}) \to 0$$
$$0 \to \Gamma(\mathfrak{M}') \to \Gamma(\mathfrak{M}) \to \Gamma(\mathfrak{M}'') \to 0$$
$$0 \leftarrow \Gamma(\mathscr{F}^*) \leftarrow \Gamma(\mathfrak{M}'^*) \leftarrow \Gamma(\mathfrak{L}_w^{*N}) \leftarrow \Gamma(\mathscr{G}^*) \leftarrow 0$$
$$0 \leftarrow \Gamma(\mathfrak{M}'^*) \leftarrow \Gamma(\mathfrak{M}^*) \leftarrow \Gamma(\mathfrak{M}''^*) \leftarrow 0$$

Since $ch(\Gamma(\mathscr{F}^*)) = ch(\Gamma(\mathscr{F}))$, $ch(\Gamma(\mathscr{G})) = ch(\Gamma(\mathscr{G}^*))$, $ch(\Gamma(\mathfrak{M}'')) = ch(\Gamma(\mathfrak{M}''^*)$, and $\mathfrak{L}_w = \mathfrak{L}_w^*$ we obtain $ch(\Gamma(X; \mathfrak{M})) = ch(\Gamma(X; \mathfrak{M}^*))$.

Corollary 7.1. $M_w \to \Gamma(X; \mathscr{D} \otimes M_w)$ is an isomorphism.

Proof. We have $\mathscr{D} \otimes M_w = \mathfrak{M}_w$. Hence by $(7)_Z$ we obtain

$$\begin{aligned} ch(\Gamma(X; \mathscr{D} \otimes M_w)) &= ch(\Gamma(X; \mathfrak{M}_w^*)) \\ &= ch(M_w). \end{aligned}$$

We know already the homorphism in question is injective. Therefore we obtain its bijectivity.

7.4. *Proof of* $(3)_Z$ *and* $(5)_Z$. This is analogous to the preceding proof of $(2)_Z$, $(4)_Z$ and $(6')_Z$. We shall prove them by the induction on the length $l(M)$ of $M \in \tilde{\mathcal{O}}_Z$. If M does not have a highest weight $-w(\rho) - \rho$, then M belongs to $\tilde{\mathcal{O}}_{Z'}$. Hence $(1)_Z$, $(3)_Z$ and $(5)_Z$ are true for such an M. Now, suppose that M has a highest weight $-w(\rho) - \rho$. Then there exists an exact sequence

$$0 \to N \to M_w \xrightarrow{f} M \to M' \to 0$$

with $l(M') < l(M)$ and $N \in \tilde{\mathcal{O}}_Z$.

Hence $(3)_Z$ and $(5)_Z$ are true for N and M'. Let I be the image of f. By Proposition (6.2) and $(3)_{Z'}$, we have

$$\mathscr{T}\mathrm{or}_j^R(\mathscr{D}, M_w) = \mathscr{T}\mathrm{or}_j^R(\mathscr{D}, N) = 0 \quad \text{for } j \neq 0.$$

Hence the exact sequence $0 \to N \to M_w \to I \to 0$ gives

(7.4.1) $$\mathscr{T}\mathrm{or}_j^R(\mathscr{D}, I) = 0$$

and

(7.4.2) $$0 \to \mathscr{T}\mathrm{or}_1^R(\mathscr{D}, I) \to \mathscr{D} \otimes N \to \mathscr{D} \otimes M_w \to \mathscr{D} \otimes I \to 0$$

On the other hand, $\mathscr{T}\mathrm{or}_j^R(\mathscr{D}, M') = 0$ for $j \neq 0$ by the hypothesis of induction. Hence we obtain an exact sequence

$$(7.4.3)\qquad 0 \to \mathscr{D} \otimes I \to \mathscr{D} \otimes M \to \mathscr{D} \otimes M' \to 0$$

and

$$(7.4.4)\qquad \mathscr{T}\!or_j^R(\mathscr{D}, M) = \mathscr{T}\!or_j^R(\mathscr{D}, I) \quad \text{for } j \neq 0.$$

From (7.4.2), (7.4.3), $(4)_Z$, $(5)_{Z'}$, Corollary 7.1 and the hypothesis of the induction, we obtain a diagramm

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X; \mathscr{T}\!or_1^R(\mathscr{D}, I)) \arrow[r] & \Gamma(X; \mathscr{D} \otimes N) \arrow[r] & \Gamma(X; \mathscr{D} \otimes M_w) \arrow[r] & \Gamma(X; \mathscr{D} \otimes M) \arrow[r] & \Gamma(X; \mathscr{D} \otimes M') \\
& 0 \arrow[r] & N \arrow[r] \arrow[u, "\wr"] & M_w \arrow[r] \arrow[u, "\wr"] & M \arrow[r] \arrow[u] & M' \arrow[r] \arrow[u, "\wr"] & 0
\end{tikzcd}$$

This shows that $M \to \Gamma(X; \mathscr{D} \otimes M)$ is bijective, and $\Gamma(X; \mathscr{T}\!or_1^R(\mathscr{D}, I)) = 0$. Since $\mathscr{T}\!or_1^R(\mathscr{D}, I)$ belongs to $\mathscr{M}_{Z'}$ by (7.4.2), $(6)_{Z'}$ implies $\mathscr{T}\!or_1^R(\mathscr{D}, I) = 0$. Together with (7.4.1) and (7.4.4) this implies $\mathscr{T}\!or_j^R(\mathscr{D}, M) = 0$ for $j \neq 0$.

7.5. *Proof of* $(6)_Z$. For $\mathfrak{M} \in \mathscr{M}_Z$, set $M = \Gamma(X; \mathfrak{M}) \in \tilde{\mathscr{O}}_Z$. Let $\mathscr{N}$ be the kernel of

$$0 \to \mathscr{N} \to \mathscr{D} \otimes M \to \mathfrak{M} \to 0$$

Then we obtain

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X; \mathscr{N}) \arrow[r] & \Gamma(X; \mathscr{D} \otimes M) \arrow[r, "\beta"] & \Gamma(X; \mathfrak{M}) \arrow[r] & 0 \\
& & M \arrow[u, "\alpha"] \arrow[ur, "\sim"'] & &
\end{tikzcd}$$

Since α is bijective by $(5)_Z$, β is also bijective. This implies $\Gamma(X; \mathscr{N}) = 0$. Hence $\mathscr{N} = 0$ by $(6')_Z$. This completes the proof of Theorem 4.1.

7.6. We have already estimated the support of $\mathscr{D} \otimes M$ in Corollary 5.7. However Theorem 4.1 allows us to show the following proposition.

Proposition 7.2. *For* $M \in \tilde{\mathscr{O}}_{\mathrm{triv}}$, *we have*

$$\operatorname{Supp}(\mathscr{D} \otimes M) = U\{\overline{X}_w;\ [M; L_w] \neq 0\}.$$

Proof. Since $\mathscr{D} \otimes *$ is an exact functor on $\tilde{\mathscr{O}}_{\mathrm{triv}}$ we reduce the proposition to the case when $M = L_w$. In this case, the proposition follows from the following

Proposition 7.3. $\mathscr{D} \otimes L_w = \mathfrak{L}_w$.

In fact $\mathfrak{L}_w$ is the image of the non trivial homomorphism $\mathfrak{M}_w \to \mathfrak{M}_w^*$ and L_w is the image of the non trivial homomorphism $M_w \to M_w^*$.

§ 8. Conjecture of Kazhdan-Lusztig

8.1. In order to state the conjecture of Kazhdan-Lusztig, we shall describe the complex of sheaves π given by Deligne, Goresky and MacPherson [2]. Let Y

be an analytic space of pure dimension n. Take a filtration

$$Y = Y_n \supset Y_{n-1} \supset \ldots \supset Y_0 \supset Y_{-1} = \emptyset$$

where Y_i is a closed analytic subset, $Y_i - Y_{i-1}$ is non singular of pure dimension i and $Y_i - Y_{i-1}$ satisfies the Whitney's condition along $Y_j - Y_{j+1}$. Set $U_i = Y - Y^{n-i}$ and ley j_i: $U_i \hookrightarrow U_{i+1}$ be the inclusion. Then π_Y is defined by

$$\pi_Y = \tau_{\leqq n-1} \mathbb{R}_{j_n *} \ldots \tau_{\leqq 0} \mathbb{R}_{j_1 *}(\mathbb{C}_{U_1}).$$

Here $\tau_{\leqq i}$ is the truncation operator.

The complex π_Y is characterized by the following three properties:

(a) $\mathscr{H}^i(\pi_Y) = 0$ for $i < 0$ and $\dim \operatorname{Supp}(\mathscr{H}^i(\pi_Y)) \leqq n - i - 1$ for $i > 0$.

(b) π_Y is self-dual in the derived category of the category of sheaves on Y.

(c) $\pi_Y / Y_{\mathrm{reg}} \cong \mathbb{C}_{Y_{\mathrm{reg}}}$, where Y_{reg} is the regular part of Y.

This is an unpublished result of Mac Pherson.

8.2. By using π_Y, the conjecture of Kazhdan-Lusztig can be stated as follows (see [8]).

Theorem 8.1. *$ch(L_w) = \sum_{w'} (-1)^{l(w)+l(w')} rk_{w'}(\pi_{X_w}), ch(M_{w'})$ where*

$$\begin{aligned} rk_{w'}(\pi_{X_w}) &= \sum_j (-1)^j \dim \mathscr{H}^j(\pi_{X_w})_{w'} \\ &= \sum_j (-1)^j \dim \mathbb{H}^j_{\{w'\}}(\overline{X}_w, \pi_{X_w}). \end{aligned}$$

The last equality follows from the following general fact.

Proposition 8.2. *Let Y be a complex analytic variety, $K^\cdot$ a complex of sheaves on Y with bounded cohomology, the cohomology sheaves of which are constructible. For any point y of Y, the two integers $rk_y(K^\cdot) = \sum_j (-1)^j \dim \mathscr{H}^j(K^\cdot_y)$ and $\sum_j (-1)^j \dim \mathbb{H}^j_{\{y\}}(Y, K^\cdot)$ are equal.*

Proof. One may obviously reduce oneself to the case Y is non singular and $K^\cdot$ is a single sheaf F. One has to show that, F being a constructible sheaf: $\chi(U, F) = \chi(\mathbb{R}\Gamma_{\{y\}}(U; F))$ for a sufficiently small ball U centered at y, or equivalently $\chi(\mathbb{R}\Gamma(U - \{y\}, F)) = 0$. Furthermore, we may assume that there exists a locally closed subset Z such that $F_{|Z}$ is locally constant, $F_{|X-Z} = 0$ and both $\overline{Z}$ and $\overline{Z} - Z$ are analytic. Then $\chi(\mathbb{R}\Gamma(U - \{y\}); F) = \chi(\mathbb{R}\Gamma(U \cap Z; F)) = (\operatorname{rank} F)\, \chi(U \cap Z)$. But $\chi(U \cap Z) = 0$ by a theorem of Sullivan [14]. Q.E.D.

Proposition 8.3. *For any $\mathfrak{M} \in \mathscr{M}$, we have*

$$ch(\Gamma(X; \mathfrak{M})) = \sum_w (-1)^{n-l(w)} \chi_w(\mathfrak{M}) ch(M_w))$$

where $\chi_w(\mathfrak{M}) = \sum (-1)^j \dim \mathscr{E}xt^j(\mathcal{O}_X, \mathfrak{M})_w$.

Proposition 8.4. $\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathcal{O}_X, \mathfrak{L}_w) = \pi_{X_w}[-\operatorname{codim} X_w]$.

Leaving the proof of Proposition 8.3 in the later sections, we shall prove Proposition 8.2. Since both sides of Proposition 8.2 are additive in $\mathfrak{M}$ we can reduce this to the case when $\mathfrak{M}=\mathfrak{M}_w=\mathscr{D}\otimes M_w$. In this case, the assertion follows from Proposition 6.3.

8.3. In fact, Proposition 8.3 can be generalized to an arbitrary complex manifold X. In order to see this let us recall the properties of $\mathfrak{L}_w$. As already seen, $\mathfrak{L}_w$ enjoys the following

$$\mathscr{H}^0_{\partial X_w}(\mathfrak{L}_w)=\mathscr{H}^0_{\partial X_w}(\mathfrak{L}^*_w)=0, \tag{8.3.1}$$

$$\mathfrak{L}_w|_{X_w}\cong\mathscr{B}_{X_w}|_{X-\partial X_w}. \tag{8.3.2}$$

Proposition 8.5. *Let X be a complex manifold, Y a closed analytic subset of pure codimension l, and Z a nowhere dense closed analytic subset of Y containing the singular locus of Y. Then there exists a unique holonomic $\mathscr{D}_X$-module with R.S. which satisfies*

$$\mathfrak{L}|_{X-Z}\cong\mathscr{B}_{Y-Z}|_{X-Z}, \tag{8.3.3}$$

$$\mathscr{H}^0_Z(\mathfrak{L})=\mathscr{H}^0_Z(\mathfrak{L}^*)=0. \tag{8.3.4}$$

For such an $\mathfrak{L}$, we have $\mathfrak{L}\cong\mathfrak{L}^*$.

Proof. We shall prove first the existence of $\mathfrak{L}$. Set $\mathfrak{M}=\mathscr{H}^l_{[Y-Z]}(\mathscr{O}_X)$. Then $\mathfrak{M}$ is a holonomic $\mathscr{D}_X$-module with R.S. which satisfies $\mathscr{H}^0_Z(\mathfrak{M})=0$. Moreover, we have

$$\mathfrak{M}|_{X-Z}\cong\mathscr{B}_{Y-Z}|_{X-Z}.$$

Set $\mathfrak{L}=\mathfrak{M}^*/\mathscr{H}^0_Z(\mathfrak{M})$. Then $\mathfrak{L}$ is a holonomic $\mathscr{D}_X$-module with R.S. On $X-Z$, we have $\mathfrak{L}\cong\mathfrak{M}^*\cong\mathscr{B}_{Y-Z}|_{X-Z}{}^*\cong\mathscr{B}_{Y-Z}|_{X-Z}$. It is evident that $\mathscr{H}^0_Z(\mathfrak{L})=0$. Since $\mathfrak{L}^*$ is a sub-module of $(\mathfrak{M}^*)^*=\mathfrak{M}$, $\mathscr{H}^0_Z(\mathfrak{L}^*)$ also vanishes. Now, let us prove the uniqueness. Let $\mathfrak{L}$ and $\mathfrak{L}'$ be two holonomic $\mathscr{D}_X$-modules with R.S. satisfying (8.3.3) and (8.3.4). Set $\mathscr{N}=\mathscr{H}^0_{[X-Z]}(\mathfrak{L})$. Then the isomorphism

$$\mathfrak{L}'|_{X-Z}\xrightarrow{\sim}\mathfrak{L}|_{X-Z}\cong\mathscr{N}|_{X-Z}$$

extends to a homomorphism $\mathfrak{L}'\to\mathscr{N}$ defined on X (Proposition 1.4). Since $\mathscr{H}^0_Z(\mathfrak{L})=\mathscr{H}^0_Z(\mathfrak{L}')=0$ we can regard $\mathfrak{L}$ and $\mathfrak{L}'$ as sub-Modules of $\mathscr{N}$ which coincide on $X-Z$. The vanishing of $\mathscr{H}^0_Z(\mathfrak{L}^*)$ implies that any quotient of $\mathfrak{L}$ supported in Z must be 0. By applying this to $\mathfrak{L}/\mathfrak{L}\cap\mathfrak{L}'$, we obtain $\mathfrak{L}\subset\mathfrak{L}'$. Similarly we obtain $\mathfrak{L}'\subset\mathfrak{L}$. The property $\mathfrak{L}\cong\mathfrak{L}^*$ follows from the uniqueness. Q.E.D.

Definition 8.6. We shall denote $\mathfrak{L}$ in Proposition 8.4 by $\mathfrak{L}(Y,X)$.

8.4. By the property (8.3.1) and (8.3.2), we can conclude $\mathfrak{L}_w=\mathfrak{L}(\overline{X_w},X)$. Therefore Proposition 8.3 is a corollary of the following theorem.

Theorem 8.6. *Let Y be a closed analytic subset of pure codimension l of a complex manifold X. Then, we have*

$$\mathbb{R}\,\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X,\mathfrak{L}(Y,X))=\pi_Y[-l].$$

Proof. We shall denote $\mathfrak{L}$ for $\mathfrak{L}(Y,X)$. Let $Y=\coprod X_\alpha$ be a Whitney stratification of Y such that $Ch(\mathfrak{B})\subset U T^*_{X_\alpha}X$. Set $d_\alpha=\operatorname{codim}_X X_\alpha$, Set $Y_j=U\{Y_\alpha; d_\alpha\leqq j\}$ and let $j_i\colon X\colon X-Y_i\hookrightarrow X-Y_{i+1}$ be the inclusion. Then we have by the construction of π_Y,

$$\pi_Y[-l]=\tau_{<n}\mathbb{R}_{j_n^*}\dots\tau_{<l+1}\mathbb{R}_{j_{l+1}*}(\mathbb{C}_{Y-Y_{l+1}}[-l]).$$

Hence we have

$$\pi_Y[-l]|_{X-Y_{i+1}}=\tau_{<i}\mathbb{R}j_{i*}(\pi_Y[-l]|X-Y_i) \qquad i=l+1,\dots,n$$

and $\pi_Y[-l]|_{X-Y_{l+1}}\cong\mathbb{C}_{Y-Y_{l+1}}[-l]$.

Set $F^\cdot=\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X,\mathfrak{B})$. We have $F^\cdot|_{X-Y_{l+1}}\cong\mathbb{C}_{Y-Y_{l+1}}[-l]$ because $\mathfrak{B}|_{X-Y_{l+1}}\cong\mathfrak{B}_{Y-Y_{l+1}}|_{X-Y_{l+1}}$. Hence, in order to show $\pi_Y[-l]\cong F^\cdot$, it is enough to show

$$F^\cdot|_{X-Y_{i+1}}\cong\tau_{<i}\mathbb{R}j_{i*}(F^\cdot|_{X-Y_i}) \qquad \text{for } i=l+1,\dots,n.$$

This isomorphism follows from the following

(8.4.1) $\quad \tau_{<i}(F^\cdot|_{X-Y_{i+1}})\xrightarrow{\sim}F^\cdot|_{X-Y_{i+1}} \qquad$ for $i=l+1,\dots,n$,

(8.4.2) $\quad \tau_{<i}(F^\cdot|_{X-Y_{i+1}})\xrightarrow{\sim}\tau_{<i}(\mathbb{R}j_{i*}(F^\cdot|_{X-Y_i})) \qquad$ for $i=l+1,\dots,n$.

By using induction on i, we may assume that (8.4.1) is true on $X-Y_i$. Hence (8.4.1) follows from

(8.4.3) $\quad H^k(F^\cdot)|_{Y_i-Y_{i+1}}=0 \qquad$ for $k\geqq i\geqq l+1$.

The property (8.4.2) follows from

(8.4.4) $\quad \mathscr{H}^k_{Y_i-Y_{i+1}}(F^\cdot)|_{X-Y_{i+1}}=0 \qquad$ for $k\leqq i>l+1$.

Since Y_i-Y_{i+1} is the disjoint union of X_α's with $d_\alpha=i$, (8.4.3) and (8.4.4) are equivalent to the following conditions (8.4.5) and (8.4.6), respectively

(8.4.5) $\quad \mathscr{H}^k(F^\cdot)|_{X_\alpha}=0 \qquad$ for $k\geqq d_\alpha=\operatorname{codim}X_\alpha>l$,

(8.4.6) $\quad \mathscr{H}^k_{X_\alpha}(F^\cdot)|_{X_\alpha}=0 \qquad$ for $k\leqq d_\alpha=\operatorname{codim}X_\alpha>l$.

Now we shall prove first (8.4.5).

By Proposition 1.6, we have

$$\mathscr{E}xt^k_{\mathscr{D}_X}(\mathfrak{L},\mathscr{B}_{X_\alpha|X})|_{X_\alpha}=\mathscr{H}om_{\mathbb{C}}(\mathscr{E}xt^{d_\alpha-k}_{\mathscr{D}_X}(\mathscr{O}_X,\mathfrak{L})|_{X_\alpha};\mathbb{C}_{X_\alpha}).$$

Hence in order to show (5) it is sufficient to show

$$\mathscr{E}xt^k_{\mathscr{D}_X}(\mathfrak{L},\mathscr{B}_{X_\alpha|X})|_{X_\alpha}=0 \qquad \text{for } k\leqq 0.$$

This is evident for $k<0$, and this is also true for $k=0$ because

$$\mathscr{H}om(\mathfrak{L},\mathscr{B}_{X_\alpha|X})|_{X_\alpha}\cong\mathscr{H}om(\mathscr{B}_{X_\alpha|X},\mathfrak{L}^*)\subset\mathscr{H}om(\mathscr{B}^*_{X_\alpha|X},\mathscr{H}^0_{X_\alpha}(\mathfrak{L}^*))|_{X_\alpha}=0.$$

Finally we shall prove (8.4.6). We have by Proposition 1.4 and Proposition 1.1.

$$\begin{aligned}\mathbb{R}\Gamma_{X_\alpha}(F^{\cdot})|_{X_\alpha} &= \mathbb{R}\Gamma_{X_\alpha}\mathbb{R}\,\mathscr{H}om(\mathfrak{L}^*, \mathcal{O}_X)|_{X_\alpha}\\ &= \mathbb{R}\,\mathscr{H}om(\mathfrak{L}^*; \mathbb{R}\Gamma_{[X_\alpha]}(\mathcal{O}_X))|_{X_\alpha}\\ &= \mathbb{R}\,\mathscr{H}om(\mathfrak{L}^*, \mathscr{B}_{X_\alpha|X})|_{X_\alpha}[-\operatorname{codim} X_\alpha]\\ &= \mathbb{R}\,\mathscr{H}om(\mathscr{B}_{X_\alpha|X}, \mathfrak{L})|_{X_\alpha}[-d_\alpha].\end{aligned}$$

Thus we obtain

$$\mathscr{H}^k_{X_\alpha}(F^{\cdot})|_{X_\alpha} = \mathscr{E}xt^{k-d_\alpha}(\mathscr{B}_{X_\alpha|X}, \mathfrak{L})|_{X_\alpha},$$

Since $\mathscr{H}^0_{X_\alpha}(\mathfrak{L})=0$, the same argument as above shows $\mathscr{E}xt^{k-d_\alpha}(\mathscr{B}_{X_\alpha|X}, \mathfrak{L}|_{X_\alpha}=0$ for $k-d_\alpha \leqq 0$. Thus we obtain (8.4.6). Q.E.D.

Appendix

In this appendix, we shall give the proof of Proposition 4.3.

Proposition 4.3. *Let Y be a connected submanifold of a complex manifold X and y a point of Y. Let $\mathfrak{M}$ be a holonomic $\mathscr{D}_X$-module generated by a section u. Assume the following conditions*

(a) *$\mathfrak{M}$ has R.S. outside Y.*

(b) *There exist vector fields $V_1, \ldots, V_N$ such that $V_j u \in \mathcal{O}_X u$ and $V_1, \ldots, V_N$ generate TY.*

(c) *There exists a vector field V_0 satisfying*

(c_1) *$V_0 u \in \mathcal{O}_X u$,*

(c_2) *V_0 vanishes at y,*

(c_3) *V_0 is tangent to Y,*

(c_4) *The isotropy action of V_0 on T_yX/T_yY has strictly positive eigenvalues.*

Then $\mathfrak{M}$ has R.S.

Proof. By (a), $\pi^{-1}(Y)\cap Ch(\mathfrak{M})\subset T_Y^*X$, where π is the projection from T^*X onto X. Hence it is enough to show that $\mathfrak{M}$ has R.S. on T_Y^*X. Since the set of points in T_Y^*X where $\mathfrak{M}$ has R.S. is open and closed, it is enough to show the regularity of $\mathfrak{M}$ on a neighborhood of y. By replacing u with $f(x)u$ ($f(x)\in\mathcal{O}_X^X$) if necessary, we may assume that

$$X\subset\mathbb{C}^n, \qquad y=0, \qquad Y=\{x\in X; x_1=\ldots=x_l=0\}$$

and $\partial u/\partial x_j=0$ for $j=l+1, \ldots, n$.

Now V_0 has the form

$$V_0=\sum_{j\leqq l} f_j(x)\frac{\partial}{\partial x_j}+\sum_{j>l} f_j(x)\frac{\partial}{\partial x_j} \quad \text{and} \quad V_0 u=h(x)u \qquad \text{for } h\in\mathcal{O}.$$

By the condition on V_0 we have $f_j|_Y=0$ ($1\leqq j\leqq l$) and

(1) the eigenvalues of $(\partial f_j(0)/\partial x_k)$ $1\leqq j, k\leqq l$ are strictly positive.

Since $\partial u/\partial x_j=0$ for $j=l+1, \ldots, n$ we may assume $f_{l+1}=\ldots=f_n=0$ and $f_1, \ldots, f_l$ do not depend on $x_{l+1}, \ldots, x_n$.

By Proposition 1.6, in order to prove Proposition 4.3, it is sufficient to show

$$\mathscr{E}xt^j_{\mathscr{D}}(\mathfrak{M}, \hat{\mathcal{O}}_z/\mathcal{O}_z)=0 \qquad \text{for any } j \text{ and any } z\in Y.$$

Thus Proposition 2.4 is a corollary of the following proposition.

Proposition A.1. Let P be a vector field on $\mathbb{C}^n$ defined on a neighborhood of 0 such that

$$P=\sum_{j=1}^{n} f_j(x)\frac{\partial}{\partial x_j}, \qquad f_j(0)=0 \tag{2}$$

and the eigenvalues of $(\partial f_j(0)/\partial x_k)$ $1\leqq j, k\leqq n$ are strictly positive.

Let $\mathfrak{M}$ be a coherent $\mathscr{D}_{\mathbb{C}^n}$-module defined on a neighborhood of 0 generated by sections $u_1, \dots, u_N$. If $Pu_j\subset \sum_{k=1}^{N} \mathcal{O}u_k$ for any j, then we have

$$\mathscr{E}xt^j_{\mathscr{D}}(\mathfrak{M}, \hat{\mathcal{O}}_0/\mathcal{O}_0)=0 \qquad \text{for any } j.$$

Proof. Write $Pu_j=\sum_k f_{jk}u_k$ with $f_{jk}\in\mathcal{O}$ and let F denote the matrix (f_{jk}) and let u denote the column vector with $u_1, \dots, u_N$ as components.

By [11], Proposition A.1 is true for

$$\mathfrak{M}'=\mathscr{D}^N/\mathscr{D}^N(P-F).$$

Now, we shall prove Proposition (A.1) by descending induction on j. If $j>n$ this is true because the projective dimension of $\mathfrak{M}$ is at most n. We shall prove

$$\mathscr{E}xt^r_{\mathscr{D}}(\mathfrak{M}, \hat{\mathcal{O}}_0/\mathcal{O}_0)=0$$

by assuming $\mathscr{E}xt^{r+1}_{\mathscr{D}}(\mathfrak{M}, \hat{\mathcal{O}}_0/\mathcal{O}_0)=0$ for any $\mathfrak{M}$ satisfying the condition described above. We have an exact sequence

$$0\leftarrow\mathfrak{M}\leftarrow\mathfrak{M}'\leftarrow\mathscr{N}\leftarrow 0.$$

If $\mathscr{I}$ denote the $\mathscr{D}$-module $\{Q\in\mathscr{D}^N; Qu=0\}$, then $\mathscr{N}=\mathscr{I}/\mathscr{D}^N(P-F)$. Let us take m such that $\mathscr{I}=\mathscr{D}(\mathscr{I}\cap\mathscr{D}(m)^N)$ and let $R_1, \dots, R_S$ be a system of generators of $\mathscr{I}\cap\mathscr{D}(m)^N$ as an $\mathcal{O}$-module. Then we have

$$0=PR_ju=[P, R_j]u+R_jFu.$$

Hence $[P, R_j]+R_jF$ belongs to $\mathscr{I}\cap\mathscr{D}(m)^N$.

Therefore there are $g_{jk}(1\leqq j, k\leqq a)$ such that

$$[P, R_j]+R_jF=\sum_k g_{jk}R_k.$$

This shows that

$$PR_j\equiv\sum_k g_{jk}R_k \bmod \mathscr{D}^N(P-F).$$

Hence $\mathscr{N}$ satisfies the same condition as $\mathfrak{M}$, which implies $\mathscr{E}xt_{\mathscr{D}}^{r+1}(\mathscr{N}, \hat{\mathscr{O}}_0/\mathscr{O}_0) = 0$ by the hypothesis of induction. The exact sequence

$$\mathscr{E}xt_{\mathscr{D}}^{r+1}(\mathscr{N}, \hat{\mathscr{O}}_0/\mathscr{O}_0) \to \mathscr{E}xt_{\mathscr{D}}^{r}(\mathfrak{M}, \hat{\mathscr{O}}_0/\mathscr{O}_0) \to \mathscr{E}xt_{\mathscr{D}}^{r}(\mathfrak{M}', \hat{\mathscr{O}}_0/\mathscr{O}_0)$$

implies $\mathscr{E}xt_{\mathscr{D}}^{r}(\mathfrak{M}, \hat{\mathscr{O}}_0/\mathscr{O}_0) = 0$. Q.E.D.

Bibliography

1. Bernstein-Gelfand-Gelfand: Differential operators on the base affine space and a study of $\mathscr{G}$-module, In: Lie groups and their representations, Budapest 1971. Ed. par I.M. Gelfand, Wiley, New York, pp. 39–64, 1975
2. Deligne, P.: Letter to D. Kazhdan and G. Lusztig, Bures-sur-Yvette, April 20, 1979
3. Demazure, M.: Désingularisation des variétés de Schubert généralisées. Ann. Sci. École Norm. Sup. **7**, 53–88 (1974)
4. Kashiwara, M.: Systèmes d'équations micro-différentielles. Cours rédigé par Teresa Monteiro-Fernandes, Prépublications mathématiques de l'Université de Paris-Nord (1977)
5. Kashiwara, M.: Faisceaux constructibles et systèmes holonomes d'équations aux dérivées partielles linéaires à points singuliers réguliers, Exposé au Séminaire Goulaouic-Schwartz 1979–80, Ecole Polytechnique, Palaiseau
6. Kashiwara, M., Kawai, T.: On the holonomic systems of linear differential equations I, II, III. To appear
7. Kazhdan, D., Lusztig, G.: Representations of Coxeter groups and Hecke algebras. Inventiones math. **53**, 165–184 (1979)
8. Kazhdan, D., Lusztig, G.: Schubert varieties and Poincaré duality. Proc. Symp. Pure Math. **36**, 185–203 (1980)
9. Kempf, G.: The geometry of homogeneous spaces versus induced representations
10. Kempf, G.: The Grothendieck-Cousin complex of an induced representation. Advances in Math. **29**, 310–396 (1978)
11. Poincaré, H.: Thèse, Oeuvres I, Paris, 1928
12. Verdier, J.L.: Exposé VI au séminaire de Géométrie Analytique de l'E.N.S., 1974–75, Astérisque No. 36–37
13. Verma, D.N.: Structure of certain induced representations of complex semi-simple Lie algebras. Bull. A.M.S. **74**, 160–166 (1968)
14. Sullivan, D.: Combinatorial invariants of analytic spaces in Proceedings of the Liverpool Singularities Symposium I. Springer Lecture Notes **192**, 165–168 (1971)
15. Mebkhout, Z.: Thèse de doctorat d'Etat. Université de Paris VII (1979)
16. Mebkhout, Z.: Sur le problème de Riemann-Hilbert. Note C.R.A.S., t. 290 (3 Mars 1980)
17. Mebkhout, Z.: Dualité de Poincaré. In: Séminaire sur les singularités. Publications mathématiques de l'Université Paris VII No 7 (1980)

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