RIMS-674

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December 1989

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§0. Introduction.

The purpose of this report is to determine the structure of some cohomology groups associated with the theta-zerovalue $\vartheta(t)$ of the Riemann theta function, i.e.,

$$\vartheta(t) = \sum_{\nu \in \mathbb{Z}^n} \exp(\pi i \langle t\nu, \nu \rangle)$$

with t being a symmetric complex matrix of size $n \times n$. The cohomology groups are determined via a complex of linear differential operators of infinite order which the theta-zerovalue solves. [A more precise definition shall be given in §1.] When $n = 1$, its structure is explicitly given in [K2], and the result combined with the so-called reconstruction theorem ([KK], Theorem 1.4.9, [SKK2], Theorem 1.5) entails a quite intriguing result to the effect that the sheaf of microfunctions is the $\mathcal{E}^{\mathbb{R}}$-module generated by $\vartheta(t)$. (See [SKK2], p.286 for the precise statement.) When $n > 1$, we encounter some algebraic complexity concerning the commutation relations of operators which are used to determine the complex. (See [K1] for example. See also [KT].) This complexity makes the direct computation of the cohomology groups very hard, if not impossible. However, the microlocal structure of the complex in question is a straightforward generalization of the complex considered for $n = 1$. (See [S] and [K1].) Using this fact, we find the structure of the cohomology groups explicitly.

In this report, we restrict our consideration to the case where $n = 2$ so that the presentation may become simplified; other cases shall be discussed elsewhere.

§1. Definition of the complex.

Let X be the space of symmetric 2×2 complex matrices $(t_{jk})_{1\le j,k\le 2}$ and let ∂_{jk} denote the vector field $\partial/\partial t_{jk}+\partial/\partial t_{kj}$ on X so that $\partial_{jk}t_{lm}=\delta_{jl}\delta_{km}+\delta_{jm}\delta_{kl}$ $(1\le j,k,l,m\le 2)$ holds with δ_{jk} being Kronecker's delta. Let us introduce several sections in $M(3\times 3;\mathcal{D}_X)$, i.e., 3×3 matrices whose entries are linear differential operators in t.

Definition 1.1.

$$P_1=\begin{pmatrix} 0 & t_{11} & t_{12}\\ 2\pi i(1+t_{11}\partial_{11}+t_{12}\partial_{12}) & 0 & 0\\ 2\pi i(t_{11}\partial_{12}+t_{12}\partial_{22}) & 0 & 0\end{pmatrix}\tag{1.1}$$

$$P_2=\begin{pmatrix} 0 & t_{12} & t_{22}\\ 2\pi i(t_{12}\partial_{11}+t_{22}\partial_{12}) & 0 & 0\\ 2\pi i(1+t_{12}\partial_{12}+t_{22}\partial_{22}) & 0 & 0\end{pmatrix}\tag{1.2}$$

$$Q_1=\begin{pmatrix} 0 & 1 & 0\\ 2\pi i\partial_{11} & 0 & 0\\ 2\pi i\partial_{12} & 0 & 0\end{pmatrix}\tag{1.3}$$

$$Q_2=\begin{pmatrix} 0 & 0 & 1\\ 2\pi i\partial_{12} & 0 & 0\\ 2\pi i\partial_{22} & 0 & 0\end{pmatrix}\tag{1.4}$$

$$R_0\underset{\text{def}}{=}[Q_1,Q_2]=\begin{pmatrix} 0 & 0 & 0\\ 0 & -2\pi i\partial_{12} & 2\pi i\partial_{11}\\ 0 & -2\pi i\partial_{22} & 2\pi i\partial_{12}\end{pmatrix}\tag{1.5}$$

$$R_1\underset{\text{def}}{=}[Q_1,R_0]=\begin{pmatrix} 0 & -2\pi i\partial_{12} & 2\pi i\partial_{11}\\ 0 & 0 & 0\\ (2\pi i)^2(\partial_{11}\partial_{22}-\partial_{12}^2) & 0 & 0\end{pmatrix}\tag{1.6}$$

$$(1.7) \quad R_2 \underset{\text{def}}{=} [Q_2, R_0] = \begin{pmatrix} 0 & -2\pi i\partial_{22} & 2\pi i\partial_{12} \\ -(2\pi i)^2(\partial_{11}\partial_{22} - \partial_{12}^2) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Remark 1.2. The relations (1.6) and (1.7) entail

$$(1.8) \qquad R_1 = 2\pi i\partial_{11}Q_2 - 2\pi i\partial_{12}Q_1$$

$$(1.9) \qquad R_2 = 2\pi i\partial_{12}Q_2 - 2\pi i\partial_{22}Q_1.$$

Hence $[Q_j, R_k]$ $(1 \leq j, k \leq 2)$ belongs to $M(3 \times 3; \mathcal{D}_X)R_0$.

Now it is easy to verify the following relations (when $\operatorname{Im} t$ is positive definite).

$$(1.10) \qquad (\exp P_j - I)\begin{pmatrix} \vartheta(t) \\ 0 \\ 0 \end{pmatrix} = 0, \quad j = 1, 2,$$

$$(1.11) \qquad (\exp Q_j - I)\begin{pmatrix} \vartheta(t) \\ 0 \\ 0 \end{pmatrix} = 0, \quad j = 1, 2.$$

Here I denotes the 3×3 identity matrix.

These simultaneous equations (1.10) and (1.11) look quite similar to the equations considered in the case where $n = 1$. (Cf. [K2]) However, $\exp P_j$'s and $\exp Q_k$'s do not commute in our case; they commute only when they are acting upon $\mathcal{N}^\infty \underset{\text{def}}{=} \mathcal{D}_X^\infty \underset{\mathcal{D}_X}{\otimes} \mathcal{N}$, where $\mathcal{D}_X^\infty$ denotes the sheaf of linear differential operators of infinite order in t and $\mathcal{N}$ denotes the following $\mathcal{D}_X$-module:

$$(1.12) \qquad \mathcal{N} = \mathcal{D}_X u_0 \oplus (\mathcal{D}_X u_1 + \mathcal{D}_X u_2),$$

with the following fundamental relations:

(1.12.ai) $$(\partial_{11}\partial_{22} - \partial_{12}^2)u_0 = 0$$

(1.12.ii) $$\partial_{12}u_1 = \partial_{11}u_2$$

(1.12.iii) $$\partial_{22}u_1 = \partial_{12}u_2.$$

In fact, (P_1, P_2, Q_1, Q_2) determines a Jacobi structure in the sense of Definition 1.3 below and this is the reason why $\exp P_j$'s and $\exp Q_k$'s do commute as endomorphisms of $\mathcal{N}^\infty$. See [SKK1] for the details.

Definition 1.3. Let $\mathcal{N}$ be a coherent left $\mathcal{D}$-module and let P_j $(1 \le j \le 2n)$ be $\mathcal{D}$-endomorphisms of $\mathcal{N}$. Suppose that $\{P_j\}_j$ and $\mathcal{N}$ satisfy the following conditions:

(1.13) P_j acts on $\mathcal{N}$ from the right.

(1.14) There exists a rational number $\lambda < 1$ and there locally exists a good filtration $\{\mathcal{N}_k\}_{k \in \mathbf{Q}}$ of $\mathcal{N}$ for which

$$\mathcal{N}_k P_j \subset \mathcal{N}_{k+\lambda}$$

holds for each k and j.

(1.15) There exists an integral matrix $(e_{jk})_{1 \le j,k \le 2n}$ with non-vanishing determinant so that

$$[P_j, P_k] = -2\pi i e_{jk} \quad (1 \le j, k \le 2n)$$

holds in $\mathcal{E}nd_{\mathcal{D}}(\mathcal{N})$.

It is known ([SKK1] §2) that, if $\{P_j\}_j$ is a Jacobi structure with respect to $\mathcal{N}$, then $\exp P_j$'s are $\mathcal{D}^\infty$-endomorphisms of $\mathcal{N}^\infty$ commuting mutually.

The fact that (P_1, P_2, Q_1, Q_2) determines a Jacobi structure is guaranteed by the following proposition.

Proposition 1.4.

(1.16) $[P_1, P_2] = (t_{11}t_{22} - t_{12}^2)R_0$

(1.17) $[Q_1, Q_2] = R_0$

(1.18) $[P_j, Q_k] = -2\pi i \delta_{jk} + h_{jk}R_0 \quad (1 \leq j, k \leq 2)$, where

$$h_{11} = -t_{12}, \quad h_{12} = t_{11}, \quad h_{21} = -t_{22} \quad \text{and} \quad h_{22} = t_{12}.$$

(1.19) $[P_j, R_0] = t_{j1}R_1 + t_{j2}R_2 \quad$ for $j = 1, 2$.

(1.20) $[P_j, R_k] = 2\pi i(\delta_{jk} + T_{jk} + 2\varepsilon_{jk})R_0 \quad (1 \leq j, k \leq 2)$, where δ_{jk} is Kronecker's delta, T_{jk} is a scalar operator given by $\sum_{l=1}^{2} t_{jl}\partial_{lk}$ and ε_{jk} is a constant matrix given as follows:

$$\varepsilon_{11} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varepsilon_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\varepsilon_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad \varepsilon_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let us denote $\exp P_j - I$ (resp., $\exp Q_k - I$) by Φ_j (resp., Ψ_k) for $j, k = 1, 2$. Since they commute on $\mathcal{N}^\infty$, we can construct the following Koszul complex K:

(1.21)

$$K : 0 \leftarrow \mathcal{N}^\infty \xleftarrow{(\Phi_1, \Phi_2, \Psi_1, \Psi_2)} (\mathcal{N}^\infty)^4 \leftarrow (\mathcal{N}^\infty)^6 \leftarrow (\mathcal{N}^\infty)^4 \leftarrow \mathcal{N}^\infty \leftarrow 0.$$

As $\mathcal{N}$ admits a free resolution [actually a rather trivial resolution in our case where $n = 2$], Lemma 1.4 below guarantees that K is locally quasi-isomorphic to a finite complex of free $\mathcal{D}^{\infty}$-modules of finite rank, and hence K is a good complex in the sense of [SKK2]; thus we can talk about its characteristic set $\mathrm{Ch}(K)$. A concrete description of $\mathrm{Ch}(K)$ shall be given in §2.

Lemma 1.5. Let A be a sheaf of rings and let M be a finite complex of left A-modules each of whose components admits a finite resolution by finitely generated free A-modules, that is,

$$0 \leftarrow M_j \leftarrow L_{j,0} \leftarrow L_{j,1} \leftarrow \cdots \leftarrow L_{j,l_j} \leftarrow 0 \tag{1.22}$$

with $L_{j,k}$ being a finitely generated free A-modules. Then M is locally quasi-isomorphic to a finite complex of finitely generated free A-modules.

As a proof of this lemma is a straightforward one, we omit it here.

§2. The characteristic set of K.

The purpose of this section is to describe the characteristic set $\mathrm{Ch}(K)$ of the complex K introduced in §1. In what follows, we introduce the fiber coordinates on T^*X by setting $\tau_{jk} = \sigma_1(\partial_{jk})$, the principal symbol of the operator ∂_{jk}, and we set $\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix}$. Then the fundamental 1-form on T^*X is given by $\mathrm{tr}(\tau dt)/2$.

Theorem 2.1. The characteristic set $\mathrm{Ch}(K)$ is contained in $\{(t;\tau) \in T^*X; (t;\tau)$ satisfies the following conditions (2.1), (2.2) and (2.3)$\}$.

(2.1) $\det \tau = 0.$

(2.2) $\mathrm{Re}\, \tau = 0$ and $\mathrm{Im}\, \tau$ is positive semi-definite.

(2.3) $\mathrm{Re}\, t\tau = 0.$

Proof. When $(t;\tau)$ belongs to the zero-section $T^*_X X$, there is nothing to prove. Hence we may assume $\tau \neq 0$. The relation (2.1) is a defining equation of the characteristic variety of $\mathcal{N}$, and hence any point in $\mathrm{Ch}(K)$ should satisfy it. To obtain relations (2.2) and (2.3) outside the zero-section $T^*_X X$, we use the following.

Lemma 2.2. Let $A = (a_{jk})_{1\le j,k\le N}$ be an $N \times N$ matrix of micro-differential operators of finite order. Let $\{r_j\}_{1\le j\le N}$ be a set of rational numbers and let ρ be a real number strictly smaller than 1. Let us consider conditions (2.4) and (2.5) below. Here $\sigma_\lambda(a_{jk})$ denotes the principal symbol of a_{jk} and $\sigma_\lambda(a_{jk})(x^*)$ denotes its value at a point x^* of the cotangent bundle.

$$\text{(2.4)} \qquad \text{ord } a_{jk} \leq r_j - r_k + \rho$$

(2.5) every eigenvalue of the matrix

$$(\sigma_{r_j - r_k + \rho}(a_{jk}(x^*)))_{1 \leq j,k \leq N}$$

is contained in $\mathbf{C} \backslash i\mathbf{R}$.

Then under condition (2.4) each component of $\exp A$ is a well-defined section of the sheaf $\mathcal{E}^{\mathbf{R}}$ of holomorphic microlocal operators, and, if (2.5) is further satisfied, $\exp A - I$ is invertible in $M(N \times N; \mathcal{E}^{\mathbf{R}}_{x^*})$.

The proof of this lemma is essentially the same as the proof of [AKK], Theorem 2, and hence we omit it here.

As the matrix of symbols determined by P_1 etc. has an eigenvalue that is identically 0, Lemma 2.2 cannot be applied to P_1 etc. To avoid this technical trouble, we replace P_1 etc. by $\widetilde{P}_1$ etc. given below so that $P_1 = \widetilde{P}_1$ etc. hold in $\mathcal{E}nd(\mathcal{E}^{\mathbf{R}} \otimes \mathcal{N})$. Before writing down $\widetilde{P}_1$ etc., let us note that (2.1) combined with the assumption $\tau \neq 0$ entails either τ_{11} or τ_{22} is different from 0. Hence, in what follows, we assume without loss of generality that $\tau_{11} \neq 0$.

$$\text{(2.6)} \qquad \widetilde{P}_1 = P_1 - \begin{bmatrix} 0 & 0 & 0 \\ 0 & -t_{12}/\sqrt{2\pi i \partial_{11}} & 0 \\ 0 & t_{11}/\sqrt{2\pi i \partial_{11}} & 0 \end{bmatrix} R_0$$

$$\text{(2.7)} \qquad \widetilde{P}_2 = P_2 - \begin{bmatrix} 0 & 0 & 0 \\ 0 & -t_{22}/\sqrt{2\pi i \partial_{11}} & 0 \\ 0 & t_{12}/\sqrt{2\pi i \partial_{11}} & 0 \end{bmatrix} R_0$$

$$(2.8)\qquad \widetilde{Q}_1 = Q_1 - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/\sqrt{2\pi i\partial_{11}} & 0 \end{bmatrix} R_0$$

$$(2.9)\qquad \widetilde{Q}_2 = Q_2 - \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1/\sqrt{2\pi i\partial_{11}} & 0 \\ 0 & 0 & 0 \end{bmatrix} R_0$$

In calculating the eigenvalues of the matrices $\sigma(\widetilde{P}_1)$ etc. of symbols associated with the operators $\widetilde{P}_1$ etc., we may replace τ_{22} by τ_{12}^2/τ_{11} in $\sigma(\widetilde{P}_1)$ etc., as $\tau_{22} = \tau_{12}^2/\tau_{11}$ holds on $\mathrm{Supp}(\mathcal{E}^{\mathrm{R}} \otimes \mathcal{N})$. After this replacement we find the following:

(2.10) The eigenvalues of $\sigma(\widetilde{P}_1)$ are :

$$\lambda_1 \underset{\mathrm{def}}{=} 2\pi i(t_{11}\tau_{12} + t_{12}\tau_{21})/\sqrt{2\pi i\tau_{11}},\ -\lambda_1,\ -\lambda_1.$$

(2.11) The eigenvalues of $\sigma(\widetilde{P}_2)$ are :

$$\lambda_2 \underset{\mathrm{def}}{=} 2\pi i(t_{21}\tau_{11} + t_{22}\tau_{21})/\sqrt{2\pi i\tau_{11}},\ -\lambda_2,\ -\lambda_2.$$

(2.12) The eigenvalues of $\sigma(\widetilde{Q}_1)$ are :

$$\sqrt{2\pi i\tau_{11}},\ -\sqrt{2\pi i\tau_{11}},\ -\sqrt{2\pi i\tau_{11}}.$$

(2.13) The eigenvalues of $\sigma(\widetilde{Q}_2)$ are :

$$2\pi i\tau_{12}/\sqrt{2\pi i\tau_{11}},\ -2\pi i\tau_{12}/\sqrt{2\pi i\tau_{11}},\ -2\pi i\tau_{12}/\sqrt{2\pi i\tau_{11}}.$$

It then follows from Lemma 2.2 and (2.12) that for $(t;\tau)$ in $\mathrm{Ch}(K)$

$$(2.14)\qquad \tau_{11} = i\alpha \quad \text{with } \alpha > 0$$

holds. Similarly we find by (2.13) that

(2.15) τ_{12} is a purely imaginary number.

Then (2.1), (2.14) and (2.15) imply that τ_{22} is also purely imaginary. Thus we obtain (2.2) by (2.1).

Let us now verify (2.3). It follows from Lemma 2.2, (2.2) and (2.10) (resp.,(2.11)) that

(2.16) $t_{11}\tau_{11} + t_{12}\tau_{12}$ (resp., $t_{21}\tau_{11} + t_{22}\tau_{21}$) is purely imaginary.

On the other hand, using (2.1) we find

$$t_{22}\tau_{22} + t_{12}\tau_{12} = \tau_{12}(t_{22}\tau_{12} + t_{12}\tau_{11})/\tau_{11} \tag{2.17}$$

and

$$t_{12}\tau_{22} + t_{11}\tau_{12} = \tau_{12}(t_{12}\tau_{12} + t_{11}\tau_{11})/\tau_{11}. \tag{2.18}$$

Since τ_{12}/τ_{22} is also real-valued on $\mathrm{Ch}(K)$ by (2.1) and (2.2), condition (2.2) follows from (2.17) and (2.18) combined with (2.16).

This completes the proof of Theorem 2.1.

As an immediate consequence of Theorem 2.1, we obtain the following proposition.

Proposition 2.3. Set $S = \{t \in X; \det \mathrm{Im}\, t = 0\}$. Then

$$\mathrm{Ch}(K) \subset T^*_{X\setminus S}X \sqcup T^*_{S\setminus\{0\}}X \sqcup T^*_{\{0\}}X.$$

§3. Structure of the solution complex $\mathcal{S}$ and its cohomology groups.

The purpose of this section is to clarify the structure of the solution complex $\mathcal{S} = \mathrm{R}\mathcal{H}om_{\mathcal{D}^\infty}(\mathcal{K}, \mathcal{O})$ and its cohomology groups. The results in §2 combined with [SKK2], Theorem 1.5 and [KS] Lemma 8.2.7 imply that $H^j(\mathcal{S})$ is a locally constant sheaf of finite rank on $X \backslash S$, $S \backslash \{0\}$ and $\{0\}$, where $S = \{t \in X; \det \mathrm{Im}\, t = 0\}$. To describe the structure of $H^j(\mathcal{S})$ more concretely, we will make use of the tangential system $\mathcal{L}(x)$ induced from $\mathcal{N}$ onto $Y(x) = \{t \in X; t_{12} = x\}$ $(x \in \mathbf{C})$. The structure of $\mathcal{S}$ itself shall be clarified after determing the structure of $H^j(\mathcal{S})$.

To begin with, let us note that $\mathcal{L}(x)$ is a free $\mathcal{D}_{Y(x)}$-module, which can be expressed as follows:

$$(3.1) \qquad \bigoplus_{j=0}^{3} \mathcal{D}_{Y(x)} v_j,$$

where

$$(3.2) \qquad v_j = u_j \,|_{Y(x)} \quad \text{for} \quad j = 0, 1, 2$$

for the generators u_j $(j = 0, 1, 2)$ of $\mathcal{N}$, and

$$(3.3) \qquad v_3 = (2\pi i \partial_{12} u_0) \,|_{Y(x)} \,.$$

Let $\{\phi_j(x), \psi_k(x)\}_{j,k=1,2}$ denote the Jacobi structure on $\mathcal{L}(x)$ which is induced from the Jacobi structure on $\mathcal{N}$ determined by $\{P_j, Q_k\}_{j,k=1,2}$. Then one can immediately find the following:

$$(3.4) \quad \phi_1(x) = \begin{pmatrix} 0 & t_{11} & 0 & 0 \\ 2\pi i(1 + t_{11}\partial_{11}) & 0 & 0 & 0 \\ 0 & 0 & 0 & t_{11} \\ 0 & 0 & 2\pi i(1 + t_{11}\partial_{11}) & 0 \end{pmatrix}$$

$$+x\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2\pi i\partial_{22} & 0 & 0 & 0 \\ 0 & 2\pi i\partial_{22} & 0 & 0 \end{pmatrix}$$

$$(3.5)\quad \phi_2(x) = \begin{pmatrix} 0 & 0 & t_{22} & 0 \\ 0 & 0 & 0 & t_{22} \\ 2\pi i(1+t_{22}\partial_{22}) & 0 & 0 & 0 \\ 0 & 2\pi i(1+t_{22}\partial_{22}) & 0 & 0 \end{pmatrix}$$

$$+x\begin{pmatrix} 0 & 1 & 0 & 0 \\ 2\pi i\partial_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2\pi i\partial_{11} & 0 \end{pmatrix}$$

$$(3.6)\quad \psi_1(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2\pi i\partial_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2\pi i\partial_{11} & 0 \end{pmatrix}$$

$$(3.7)\quad \psi_2(x) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2\pi i\partial_{22} & 0 & 0 & 0 \\ 0 & 2\pi i\partial_{22} & 0 & 0 \end{pmatrix}.$$

Since $Y(x)$ is non-characteristic with respect to $\mathcal{N}$,

$$(3.8)\qquad \mathrm{R}\mathcal{H}om_{\mathcal{D}_X^\infty}(\mathcal{N}^\infty, \mathcal{O}_X)\,|_{Y(x)} \cong \mathrm{R}\mathcal{H}om_{\mathcal{D}_{Y(x)}^\infty}(\mathcal{D}^\infty_{Y(x)\hookrightarrow X} \underset{\mathcal{D}_X^\infty}{\otimes} \mathcal{N}^\infty, \mathcal{O}_{Y(x)})$$

holds. Hence we obtain

$$(3.9)\qquad \mathrm{R}\mathcal{H}om_{\mathcal{D}_X^\infty}(K, \mathcal{O}_X)\,|_{Y(x)} \cong \mathrm{R}\mathcal{H}om_{\mathcal{D}_{Y(x)}^\infty}(\mathcal{D}^\infty_{Y(x)\hookrightarrow X} \underset{\mathcal{D}_X^\infty}{\otimes} K, \mathcal{O}_{Y(x)}).$$

As $\mathcal{D}^\infty_{Y(x)\hookrightarrow X} \otimes K$ is quasi-isomorphic to the Koszul complex $K(x)$ determined by $(e^{\phi_1(x)} - I_4, e^{\phi_2(x)} - I_4, e^{\psi_1(x)} - I_4, e^{\psi_2(x)} - I_4)$ by the definition of $(\phi_1(x), \phi_2(x), \psi_1(x), \psi_2(x))$, it now suffices to compute $\mathrm{R}\mathcal{H}om_{\mathcal{D}_{Y(x)}^\infty}(K(x), \mathcal{O}_{Y(x)})$.

Now let (p_j, q_j) denote the Jacobi structure determined by $\vartheta(t_{jj})$ on $Y_j = \{t_{jj} \in \mathbf{C}\}$ for $j = 1, 2$: To be more explicit,

$$p_j = \begin{pmatrix} 0 & t_{jj} \\ 2\pi i(1 + t_{jj}\partial_{jj}) & 0 \end{pmatrix}, \quad j = 1, 2 \tag{3.10}$$

and

$$q_j = \begin{pmatrix} 0 & 1 \\ 2\pi i\partial_{jj} & 0 \end{pmatrix}, \quad j = 1, 2 \tag{3.11}$$

Using the convention that $A \otimes B$ denotes $\begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{21}B \end{pmatrix}$ for 2×2 matrices $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and B, we can rewrite (3.4)~(3.7) as follows:

$$\phi_1(x) = I_2 \otimes p_1 + xq_2 \otimes I_2 \tag{3.12}$$

$$\phi_2(x) = p_2 \otimes I_2 + xI_2 \otimes q_1 \tag{3.13}$$

$$\psi_1(x) = I_2 \otimes q_1 \tag{3.14}$$

$$\psi_2(x) = q_2 \otimes I_2. \tag{3.15}$$

Here I_2 denotes the 2×2 identity matrix.

Using this expression let us calculate $\mathrm{R}\mathcal{H}om_{\mathcal{D}^{\infty}_{Y(0)}}(\mathcal{K}(0), \mathcal{O}_{Y(0)})$. We first note the following lemma.

Lemma 3.1. Let X_j be a complex manifold and let $p_j : X_1 \times X_2 \to X_j$ denote the projection $(j = 1, 2)$. Let $\mathcal{N}_j$ be an R-holonomic complex of $\mathcal{D}^{\infty}_{X_j}$-modules for $j = 1, 2$ and denote by $\mathcal{N}_1 \widehat{\otimes} \mathcal{N}_2$ the complex of $\mathcal{D}^{\infty}_{X_1 \times X_2}$-modules $\mathcal{D}^{\infty}_{X_1 \times X_2} \overset{\mathbf{L}}{\underset{p_1^{-1}\mathcal{D}^{\infty}_{X_1} \underset{\mathbf{C}}{\otimes} p_2^{-1}\mathcal{D}^{\infty}_{X_2}}{\otimes}} (p_1^{-1}\mathcal{N}_1 \underset{\mathbf{C}}{\otimes} p_2^{-1}\mathcal{N}_2)$.

Then we find

$$\begin{aligned} &\mathrm{R}\mathcal{H}om_{\mathcal{D}^{\infty}_{X_1 \times X_2}}(\mathcal{N}_1 \widehat{\otimes} \mathcal{N}_2, \mathcal{O}_{X_1 \times X_2}) \\ &\quad = p_1^{-1}\mathrm{R}\mathcal{H}om_{\mathcal{D}^{\infty}_{X_1}}(\mathcal{N}_1, \mathcal{O}_{X_1}) \underset{\mathbf{C}}{\otimes} p_2^{-1}\mathrm{R}\mathcal{H}om_{\mathcal{D}^{\infty}_{X_2}}(\mathcal{N}_2, \mathcal{O}_{X_2}). \end{aligned} \tag{3.16}$$

The proof of this lemma is similar to the proof of Proposition 1.4.3 of [KK], and we do not repeat it here.

Now, (3.12)~(3.15) imply that $K(0)$ is quasi-isomorphic to $\mathcal{M}_2 \widehat{\otimes} \mathcal{M}_1$, where $\mathcal{M}_j$ denotes the Koszul complex determined by $(e^{p_j} - I_2, e^{q_j} - I_2)\,(j = 1, 2)$. Furthermore the following result is proved in [K2].

Lemma 3.2.

$$\mathrm{R}\mathcal{H}om_{\mathcal{D}^{\infty}_{\mathbf{C}}}(\mathcal{M}_1, \mathcal{O}_{\mathbf{C}}) = \mathbf{C}_U \oplus \mathbf{C}_Z[-1], \tag{3.17}$$

where $U = \{t_{11} \in \mathbf{C}; \mathrm{Im}\, t_{11} > 0\}$ and $Z = \{t_{11} \in \mathbf{C}; \mathrm{Im}\, t_{11} \leq 0\}$.

Combining (3.9), Lemma 3.1 and Lemma 3.2, we obtain the following result.

Proposition 3.3. Let $W_j\,(j = 0, 1, 2)$ denote the following subset of $Y(0)$:

$$W_0 = \{(t_{11}, t_{22}) \in Y(0); \mathrm{Im}\, t_{11}, \mathrm{Im}\, t_{22} > 0\} \tag{3.18}$$

$$W_1 = \{(t_{11}, t_{22}) \in Y(0); \mathrm{Im}\, t_{11} \leq 0, \mathrm{Im}\, t_{22} > 0\} \sqcup \{(t_{11}, t_{22}) \in Y(0); \mathrm{Im}\, t_{11} > 0, \mathrm{Im}\, t_{22} \leq 0\} \tag{3.19}$$

$$W_2 = \{(t_{11}, t_{22}) \in Y(0); \mathrm{Im}\, t_{11}, \mathrm{Im}\, t_{22} \leq 0\}. \tag{3.20}$$

Then we find

$$\mathcal{S}\,|_{Y(0)} \cong \mathbf{C}_{W_0} \oplus \mathbf{C}_{W_1}[-1] \oplus \mathbf{C}_{W_2}[-2]. \tag{3.21}$$

To state Theorem 3.4 below, we introduce the following notations:

(3.22) $X_+ = \{t \in X; \mathrm{Im}\, t \text{ is positive definite }\}$

(3.23) $X_0 = \{t \in X; \mathrm{Im}\, t \text{ has exactly one positive eigenvalue }\}$

(3.24) $X_- = \{t \in X; \mathrm{Im}\, t \text{ is negative semi-definite }\}$.

We now obtain the following Theorem 3.4 from Proposition 3.3 combined with the local constancy of $H^j(\mathcal{S})$, noting that X_+ and X_- are simply connected.

Theorem 3.4.

(i) $H^0(\mathcal{S}) \cong \mathbf{C}_{X_+}$.

(ii) $H^1(\mathcal{S})$ is a locally constant sheaf of rank 1 on X_0.

(iii) $H^2(\mathcal{S}) \cong \mathbf{C}_{X_-}$.

As an immediate consequence of Theorem 3.4 combined with (3.9) and the fact

$$\begin{aligned}
&\mathrm{Ext}^2(\mathbf{C}_{X_-\cap Y(x)}, \mathbf{C}_{X_0\cap Y(x)}) \\
=&\mathrm{Ext}^2(\mathbf{C}_{X_0\cap Y(x)}, \mathbf{C}_{X_+\cap Y(x)}) \\
=&\mathrm{Ext}^2(\mathbf{C}_{X_-\cap Y(x)}, \mathbf{C}_{X_+\cap Y(x)}) = 0\ ,
\end{aligned}$$

we obtain the following result.

Proposition 3.5.

$\mathrm{R}\mathcal{H}om(K(x), \mathcal{O}_{Y(x)}) \cong \mathbf{C}_{X_+\cap Y(x)} \oplus \mathbf{C}_{X_0\cap Y(x)}[-1] \oplus \mathbf{C}_{X_-\cap Y(x)}[-2]$

holds for any $x \in \mathbf{C}$.

Using Theorem 3.4 we can further determine the structure of the complex $\mathcal{S}$ as in Theorem 3.7 below. In view of Theorem 2.1, we first note that $\mathcal{S}$ has the form $p^{-1}\widetilde{\mathcal{S}}$, where p is the projection from X to the space V of real 2×2 symmetric matrices s by $t \mapsto \operatorname{Im} t$, and $\widetilde{\mathcal{S}}$ is a complex on V. Let $\mathcal{G}$ denote the inverse Fourier-Sato transform of $\widetilde{\mathcal{S}}$ (in the sense of [KS] §5.1). Identifying V^* with the space of symmetric matrices σ via the pairing $\operatorname{tr} \sigma s$, we find the following (3.25) and (3.26) from Theorem 2.1 and [KS], Theorem 10.1.1:

(3.25) $\quad \operatorname{Ch}(\mathcal{G}) \subset T^*_{S^*_+} V^* \sqcup T^*_{\{0\}} V^*,$

where $S^*_+ = \{\sigma \in V^*; \sigma$ has rank 1 and is positive semi-definite $\}$,

(3.26) $\quad \mathcal{G} \mid_{S^*_+}$ has locally constant cohomology groups.

Letting $\mathcal{F}$ denote $\mathcal{G}_{V^* \setminus \{0\}}$, we have the following distinguished triangle:

(3.27)
$$\begin{array}{ccc} & \mathcal{F} & \\ \swarrow & & \nwarrow^{+1} \\ \mathcal{G} & \longrightarrow & \mathcal{G}_0 \otimes \mathbf{C}_{\{0\}} \end{array},$$

where $\mathcal{G}_0$ denotes the germ of $\mathcal{G}$ at the origin 0.

Let C_+ denote $\{s \in V;\ s$ is positive-definite $\}$. Then it follows from Theorem 3.2 that $\widetilde{\mathcal{S}}$ is isomorphic to $\mathbf{C}_{C_+}^{\oplus 2}$ microlocally at a generic point of the conormal bundle of S^*_+. Hence $\mathcal{F}$ is isomorphic to the inverse Fourier-Sato transform of $\mathbf{C}_{C_+}^{\oplus 2}$ microlocally at a generic point of the conormal bundle of S^*_+. Therefore $\mathcal{F} \mid_{S^*_+}$ is a locally constant sheaf of rank 2. Let W be a stalk of $\mathcal{F} \mid_{S^*_+}$ and let M denote the monodromy automorphism of W.

Now we have the following.

Lemma 3.6.

(i) The cohomologies of $\mathcal{F}$ and those of $\mathcal{G}$ are concentrated at the degree 0.

(ii) Letting ι denote the injection: $V^*\backslash\{0\} \hookrightarrow V^*$, we find

$$\mathcal{G} \cong H^0 \iota_* \iota^{-1} \mathcal{F}.$$

(iii) $\mathcal{F}$ has a direct sum decomposition $\mathcal{F}_+ \oplus \mathcal{F}_-$ so that $M|_{\mathcal{F}_\pm} = \pm \mathrm{id}$ and that $\mathcal{F}_\pm$ is a locally constant sheaf of rank 1.

Proof. Besides C_+ defined earlier, let us define C_0 and C_- as follows:

$$C_0 = \{s \in V;\ s \text{ has the signature } (1,1)\} \tag{3.28}$$

$$C_- = \{s \in V;\ s \text{ is negative definite }\}. \tag{3.29}$$

For a point p in C_+ it follows from the definition of the Fourier-Sato transform $\mathcal{F}^\wedge$ that $(\mathcal{F}^\wedge)_p = 0$. It also follows from the definition of $\mathcal{G}$ and Theorem 3.4 that $\mathcal{G}_0 \cong (\mathcal{G}^\wedge)_p \cong \widetilde{\mathcal{S}}_p = \mathbf{C}$, which we denote by W'. In particular, the cohomologies of $\mathcal{G}$ is concentrated at the degree 0. This proves (i).

Similarly, for p in C_0, we find

$$(\mathcal{F}^\wedge)_p \cong W[-1] \tag{3.30}$$

and

$$(\mathcal{G}^\wedge)_p \cong (\mathcal{S}^\wedge)_p \cong \mathbf{C}[-1]. \tag{3.31}$$

Hence we obtain the following exact sequence:

$$0 \to \mathcal{G}_0 \to W \to \mathbf{C} \to 0. \tag{3.32}$$

For p in C_-, we obtain

(3.33) $(\mathcal{F}^\wedge)_p = \underset{1}{W} \xrightarrow{M-\mathrm{id}} \underset{2}{W}$, where the indices 1 and 2 denote the degree of the complex,

(3.34) $(\mathcal{G}^\wedge)_p = \mathbf{C}[-2]$,

and hence

(3.35) $0 \to W' \to W \xrightarrow{M-\mathrm{id}} W \to \mathbf{C} \to 0.$

The above exact sequence (3.35) implies that

(3.36) $\mathcal{G}_0$ is isomorphic to $(\iota_* \iota^{-1} \mathcal{F})_0$,

and hence

(3.37) $\mathcal{G} \cong H^0 \iota_* \iota^{-1} \mathcal{F}$,

proving (ii).

To prove (iii) we investigate the monodromy of

$$H^0(\mathcal{H}om(K, \mathcal{H}^1_{X \setminus p^{-1}(C_+)}(\mathcal{O}_X))) \mid_{p^{-1}(\partial C_+ \setminus \{0\})},$$

where p is the morphism Im : $X \to V$ and ∂C_+ denotes the boundary of C_+.

We first parametrize a loop in $\partial C_+ \setminus \{0\}$ by

(3.38) $$s(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} (\cos\theta \ , \quad \sin\theta) \qquad (0 \leq \theta < \pi).$$

To perform the explicit computation of the monodromy, let us introduce the following auxiliary functions $h_{\mu,\nu}(\mu, \nu \in \mathbf{Z})$ following [K1]:

(3.39) $$h_{\mu,\nu}(t) = \begin{bmatrix} \exp(\pi i(\mu^2 t_{11} + 2\mu\nu t_{12} + \nu^2 t_{22}) \\ 2\pi i \mu \exp(\pi i(\mu^2 t_{11} + 2\mu\nu t_{12} + \nu^2 t_{22}) \\ 2\pi i \nu \exp(\pi i(\mu^2 t_{11} + 2\mu\nu t_{12} + \nu^2 t_{22}) \end{bmatrix}.$$

Then we can easily verify

(3.40) $(\exp Q_j - I)h_{\mu,\nu} = 0$ for $j = 1, 2$ and for any (μ, ν),

(3.41) $R_l h_{\mu,\nu} = 0$ for $l = 0, 1, 2$ and for any (μ, ν),

(3.42) $(\exp P_1)h_{\mu,\nu} = h_{\mu+1,\nu}$ and $(\exp P_2)h_{\mu,\nu} = h_{\mu,\nu+1}$.

(Cf. [K2])

Let us further introduce the following subsets of Z^2:

(3.43) $H(\theta) = \{(\mu, \nu) \in Z^2; -\mu \sin\theta + \nu \cos\theta \geq 0\}$

(3.44) $H_1(\theta) = \{(\mu + 1, \nu) \in Z^2; (\mu, \nu) \in H(\theta)\}$

(3.45) $H_2(\theta) = \{(\mu, \nu + 1) \in Z^2; (\mu, \nu) \in H(\theta)\}$.

We let $\vec{\vartheta}(t)$ and $\varphi_\theta(t)$ respectively denote $\sum_{(\mu,\nu)\in Z^2} h_{\mu,\nu}(t)$ and $\sum_{(\mu,\nu)\in H(\theta)} h_{\mu,\nu}$. Both $\vec{\vartheta}(t)$ and $\varphi_\theta(t)$ are well-defined and holomorphic on $p^{-1}(C_+)$. Furthermore (3.42) entails

(3.46) $$(\exp P_1 - I)\varphi_\theta(t) = \sum_{(\mu,\nu)\in H_1(\theta)\setminus H(\theta)} h_{\mu,\nu}(t)$$

and

(3.47) $$(\exp P_2 - I)\varphi_\theta(t) = \sum_{(\mu,\nu)\in H_2(\theta)\setminus H(\theta)} h_{\mu,\nu}(t) - \sum_{(\mu,\nu)\in H(\theta)\setminus H_2(\theta)} h_{\mu,\nu}(t).$$

We can readily verify also

(3.48) $$(\exp Q_j - I)\varphi_\theta(t) = 0, \quad j = 1, 2$$

and

(3.49) $$R_l \varphi_\theta(t) = 0, \quad l = 0, 1, 2.$$

Since there exists a constant $C > 0$ for which

(3.50) $$(\mu \cos\theta + \nu \sin\theta)^2 \geq C(\mu^2 + \nu^2)$$

holds for $(\mu,\nu) \in (H_1(\theta)\backslash H(\theta)) \cup (H_2(\theta)\backslash H(\theta)) \cup (H(\theta)\backslash H_2(\theta))$, $(\exp P_j - I)\varphi_\theta(t)$ is holomorphic on a neighborhood of the point $p(\theta) = is(\theta)$. Hence $\varphi_\theta(t)$ and $\vec{\vartheta}(t)$ respectively determine elements in $\mathcal{H}^1_{X\backslash p^{-1}(C_+)}(\mathcal{S})_{p(\theta)}$, and $\varphi_\theta(t)$ is locally constant with respect to θ. Since the components of $\vec{\vartheta}(t)$ other than the first one are zero and those of $\varphi_\theta(t)$ are singular at $p(\theta)$ (cf.[K3]), they are linearly independent in $\mathcal{F}_{p(\theta)}$. Similarly $\varphi_0(t) + \varphi_\pi(t) = \vec{\vartheta}(t)$ in $\mathcal{H}^1_{X\backslash p^{-1}(C_+)}(\mathcal{O})_{p(0)}$. Thus the monodromy on $\mathcal{H}^1_{V\backslash C_+}(\widetilde{\mathcal{S}})$ has eigenvalues 1 and -1, and hence the monodromy M of $\mathcal{F}\mid_{S^*_+}$ has eigenvalue -1.

This completes the proof of Lemma 3.6.

Since $\widetilde{\mathcal{S}}$ is the Fourier-Sato transform of $\mathcal{G}$, the fact that $\mathcal{S} = p^{-1}\widetilde{\mathcal{S}}$ entails the following Theorem 3.7. There we identify X^* with the space of symmetric 2×2 matrices τ by the pairing $\operatorname{Re}\operatorname{tr} t\tau$.

Theorem 3.7. The complex $\mathcal{S}$ has the form $\mathbf{C}^{\wedge}_{S^*_-} \oplus i_! F[-1]$, where

(3.51) $S^*_- = \{\tau \in X^*; \det \tau = 0, \operatorname{Re}\tau = 0, \operatorname{Im}\tau$ is negative semi-definite $\}$,

(3.52) i is the injection $Z \hookrightarrow X$, with Z being $\{t \in X; \operatorname{Im} t$ has exactly one positive eigenvalue $\}$,

and

(3.53) F is a locally constant sheaf of rank 1 on Z with monodromy -1.

§4. Imaginary transformation.

One crucial point in the results given in the preceding section is that the equation determined by the complex K determines its solution by its local properties. To exemplify this fact concretely, we show how Jacobi's relation between the theta-zerovalue and its imaginary transform follows from the local property of the theta-zerovalue, i.e., the fact that the theta-zerovalue satisfies the equation on X_+.

Let Δ denote $\det t\,(t \in X)$. Then Δ never vanishes on X_+ (see, e.g.[K[3]], p.90) and hence t is invertible there. Furthermore, using the simply-connectedness of X_+, we have a single-valued analytic function $\sqrt{\Delta}$ on X_+ by determining its value at $t_0 \underset{\text{def}}{=} \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}$ to be $\sqrt{-1}$. Let φ denote the automorphism of X_+ given by $t \mapsto -t^{-1}$. Then for P in $M(3 \times 3; \mathcal{D}_{X_+})$ we denote by $\widetilde{P}$ the operator $(\varphi^* P)$ defined so that the relation

$$(4.1) \qquad (\varphi^* P)(f(\varphi(x))) = (Pf)(\varphi(x))$$

may hold for any f, a three-vector of holomorphic functions on X_+.

Now, denoting by T the following matrix

$$\begin{bmatrix} \sqrt{\Delta} & 0 & 0 \\ 0 & \sqrt{\Delta}t_{11} & \sqrt{\Delta}t_{12} \\ 0 & \sqrt{\Delta}t_{12} & \sqrt{\Delta}t_{22} \end{bmatrix},$$

we find the following relations:

$$(4.1) \qquad T^{-1}\widetilde{P}_j T = -Q_j, \quad j = 1, 2,$$

$$(4.2) \qquad T^{-1}\widetilde{Q}_j T = P_j, \quad j = 1, 2,$$

$$(4.3) \qquad T^{-1}\widetilde{R}_0 T = \Delta R_0,$$

(4.4) $$T^{-1}\widetilde{R}_1 T = \Delta(t_{11}R_1 + t_{12}R_2),$$

(4.5) $$T^{-1}\widetilde{R}_2 T = \Delta(t_{12}R_1 + t_{22}R_2).$$

These relations entail that

$$T^{-1}\begin{pmatrix} \vartheta(\varphi(t)) \\ 0 \\ 0 \end{pmatrix}$$

satisfies the same equations as

$$\begin{pmatrix} \vartheta(t) \\ 0 \\ 0 \end{pmatrix}$$

satisfies. Here we have used the fact that $\exp(T^{-1}\widetilde{P}T) = T^{-1}(\exp\widetilde{P})T$ holds for any P in $M(3\times 3; \mathcal{D}_{X+})$.

Hence Theorem 3.4(i) implies that

(4.2) $$T^{-1}\begin{pmatrix} \vartheta(\varphi(t)) \\ 0 \\ 0 \end{pmatrix} = C\begin{pmatrix} \vartheta(t) \\ 0 \\ 0 \end{pmatrix}$$

should hold with some constant C. Since $\varphi(t_0) = t_0$ holds, we evaluate both sides of (4.2) at $t_0 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}$ and find that $C = 1/\sqrt{\Delta(t_0)} = 1/\sqrt{-1}$. Thus we have deduced Jacobi's relation from local properties of the theta-zerovalue.

References

[AKK] Aoki, T., M. Kashiwara and T. Kawai: On a class of linear differential operators of infinite order with finite index, *Adv. in Math.*, **62** (1986), 155-168.

[KK] Kashiwara, M. and T. Kawai: On holonomic systems of micro-differential equations. III, *Publ. RIMS, Kyoto Univ.*, **17** (1981), 813-979.

[K³] Kashiwara, M., T. Kawai and T. Kimura: *Foundations of Algebraic Analysis, Princeton University Press*, Princeton, 1986. (Translated from the Japanese edition (1980).)

[KS] Kashiwara, M. and P. Schapira: Microlocal study of sheaves, *Astérisque* **128** (1985), 1-235.

[K1] Kawai, T: Some remarks on the microlocal analysis of ϑ-functions, *Sûriken-kôkyûroku*, No. 410, *RIMS, Kyoto*, 1980, pp.76-87. (In Japanese.)

[K2] ———: An example of a complex of linear differential operators of infinite order, *Proc. Japan Acad. Ser. A*, **59** (1983), 113-115.

[K3] ———: The Fabry-Ehrenpreis gap theorem and linear differential equations of infinite order, *Amer. J. Math.*, **109** (1987), 57-64.

[KT] Kawai, T. and Y. Takei: "Fundamental Principle" and ϑ-zerovalue, *Sûriken-kôkyûroku*, No. 675, *RIMS, Kyoto*, 1988, pp.79-86. (In Japanese.)

[S] Sato, M.: Pseudo-differential equations and theta functions, *Astérisque*, **2** et **3** (1973), 286-291.

[SKK1] Sato, M., M. Kashiwara and T. Kawai: Linear differential equations of infinite order and theta functions, *Adv. in Math.*, 47 (1983), 300-325.

[SKK2] ———: Microlocal analysis of theta functions, *Advanced Studies in Pure Math.*, 4 (1984), 267-289.