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The Universal Verma Module and the b-Function

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§ 0. Introduction

In this paper, we study the universal Verma module and apply this to the determination of the b-functions of the invariants on the flag manifold.

Let $\mathfrak{g}$ be a semi-simple Lie algebra over $\mathbf{C}$, $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{g}$, $\mathfrak{n}$ the nilpotent radical of $\mathfrak{b}$ and $\mathfrak{h}$ a Cartan subalgebra in $\mathfrak{b}$. Let V be a finite-dimensional irreducible representation of $\mathfrak{g}$ and let v be a lowest weight vector of V. Then there exists $f \in U(\mathfrak{h})$ and a commutative diagram

$$
(0.1)\qquad
\begin{array}{ccc}
U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbf{C} & \longrightarrow & U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} V \\
 & \searrow^{f} & \downarrow{\scriptstyle g} \\
 & & U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbf{C}
\end{array}
$$

where g is given by the $\mathfrak{n}$-linear morphism from V to $\mathbf{C}$ sending u to 1. Note that $\mathrm{End}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbf{C}) \cong U(\mathfrak{h})$.

The first problem is to determine the minimal f with such a property. In order to state the answer to this problem, we shall introduce further notations. Let Δ be the root system for $(\mathfrak{g}, \mathfrak{h})$. For $\alpha \in \Delta$, let h_α be the coroot of α. Let Δ^+ be the set of positive roots given by $\mathfrak{b}$ and ρ the half-sum of positive roots. Let $-\mu$ be the lowest weight of V.

Theorem. *There exists a commutative diagram* (0.1), *with*

$$f = \prod_{\alpha \in \Delta^+} (h_\alpha + h_\alpha(\rho) + 1,\ h_\alpha(\mu))$$

where $(x, n) = x(x+1)\cdots(x+n-1)$. *Conversely for any commutative diagram* (0.1), *f is a multiple of* $\prod_{\alpha \in \Delta^+}(h_\alpha + h_\alpha(\rho) + 1,\ h_\alpha(\mu))$.

By using this theorem, we can calculate the b-functions on the flag manifold. Let G be a simply connected algebraic group with Lie algebra $\mathfrak{g}$, and let B and N be the subgroup of G with Lie algebras $\mathfrak{b}$ and $\mathfrak{n}$, respectively, and let B_- be the opposite Borel subgroup.

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Then the semi-group of $B_- \times B$-semi-invariants f on G, i.e. regular functions f on G which satisfies $f(b'gb)=\chi'(b')\chi(b)f(g)$ for $b' \in B_-$, $g \in G$, $b \in B$ with characters χ' and χ of B_- and B, is parametrized by the set P_+ of dominant integral weights. More precisely, for $\lambda \in P_+$, let V_λ be a finite-dimensional irreducible representation of G with highest weight λ, v_λ a highest weight vector of V_λ and $v_{-\lambda}$ a lowest weight vector of the dual V_λ^* of V_λ. We normalize them such that $\langle v_\lambda, v_{-\lambda}\rangle = 1$. Then, the regular function f^λ given by

$$f^\lambda(g) = \langle g v_\lambda, v_{-\lambda}\rangle$$

is a semi-invariant, and any semi-invariant is a constant multiple of some f^λ. We have

$$f^{\lambda+\lambda'}(g) = f^\lambda(g) f^{\lambda'}(g).$$

Theorem. *For any dominant integral weight μ, we can find a differential operator P_μ on G such that*

$$P_\mu f^{\lambda+\mu} = b_\mu(\lambda) f^\lambda \qquad \textit{for any } \lambda.$$

Here

$$b_\mu(\lambda) = \prod_{\alpha \in \Delta+} (h_\alpha(\lambda+\rho), h_\alpha(\mu)).$$

Notations

$\mathbf{Z}_+$: the set of non-negative integers.
$\mathbf{Z}_{++}$: the set of positive integers.
$\mathfrak{g}$: a semi-simple Lie algebra over $\mathbf{C}$.
$\mathfrak{b}$: a Borel subalgebra of $\mathfrak{g}$.
$\mathfrak{n}$: $[\mathfrak{b}, \mathfrak{b}]$
$\mathfrak{h}$: a Cartan subalgebra of $\mathfrak{b}$.
$\mathfrak{b}_-$: the opposite Borel subalgebra of $\mathfrak{b}$ such that $\mathfrak{b}_- \cap \mathfrak{b} = \mathfrak{h}$.
$\mathfrak{n}_-$: $[\mathfrak{b}_-, \mathfrak{b}_-]$
Δ : the root system of $(\mathfrak{g}, \mathfrak{h})$.
Δ^+ : the set of positive roots given by $\mathfrak{b}$
h_α : the coroot of $\alpha \in \Delta$
s_α : the reflection $\lambda \mapsto \lambda - h_\alpha(\lambda)\alpha$.
W : the Weyl group of $(\Delta, \mathfrak{h}^*)$
$Q_+(\Delta)$: $\sum_{\alpha \in \Delta+} \mathbf{Z}_+ \alpha$
$Q(\Delta)$: $\sum_{\alpha \in \Delta} \mathbf{Z}\alpha$
P_+ : $\{\lambda \in h^*;\ h_\alpha(\lambda) \in \mathbf{Z}_+ \text{ for any } \alpha \in \Delta^+\}$.
ρ : $(\sum_{\alpha \in \Delta+} \alpha)/2$

$S(\Delta^+)$: the set of simple roots of Δ^+.
$U(*)$: the universal enveloping algebra
$U_j(\mathfrak{g})$: $U_0(\mathfrak{g})=\mathbf{C}$, $U_j(\mathfrak{g})=U_{j-1}(\mathfrak{g})\mathfrak{g}+U_{j-1}(\mathfrak{g})$
R : $S(\mathfrak{h})=U(\mathfrak{h})$
c : the canonical homomorphism $\mathfrak{h}\to R$
$U_R(*)$: $R\otimes_{\mathbf{C}} U(*)$
$R_{c+\mu}$: for $\mu\in\mathfrak{h}^*$, the $U_R(\mathfrak{b})$-module $U_R(\mathfrak{b})/(U_R(\mathfrak{b})\mathfrak{n}+\sum_{h\in\mathfrak{h}}U_R(\mathfrak{b})(h-c(h)-\mu(h)))$
$1_{c+\mu}$: the canonical generator of $R_{c+\mu}$
$\mathbf{C}_\lambda$: for $\lambda\in\mathfrak{h}^*$, the $U(\mathfrak{b})$-module $U(\mathfrak{b})/(U(\mathfrak{b})\mathfrak{n}+\sum_{h\in\mathfrak{h}}U(\mathfrak{b})(h-\lambda(h)))$
$\mathscr{Z}(\mathfrak{g})$: the center of $U(\mathfrak{g})$
χ_λ : the central character $\mathscr{Z}(\mathfrak{g})\to\mathbf{C}$ of $U(\mathfrak{g})\otimes_{U(\mathfrak{b})}\mathbf{C}_{\lambda-\rho}$; $\chi_\lambda=\chi_{w\lambda}$ for $w\in W$
V_λ : for $\lambda\in P_+$, a finite dimensional irreducible representation of $\mathfrak{g}$ with highest weight λ
v_λ : a highest weight vector of V_λ
$v_{-\lambda}$: a lowest weight vector of V_λ^*
(x, m) : $x(x+1)\cdots(x+m-1)$
G, B, N, B_-, N_-, T: the group with $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{n}$, $\mathfrak{b}_-$, $\mathfrak{n}_-$ and $\mathfrak{h}$ as their Lie algebras.

§1. The universal Verma module

For a ring R and a Lie algebra $\mathfrak{a}$ over $\mathbf{C}$, we write $U_R(\mathfrak{a})$ for $R\otimes_{\mathbf{C}}U(\mathfrak{a})=U(R\otimes_{\mathbf{C}}\mathfrak{a})$. Hereafter we take $S(\mathfrak{h})=U(\mathfrak{h})$ for R, where $\mathfrak{h}$ is a Cartan subalgebra of a semi-simple Lie algebra $\mathfrak{g}$. Let c be the canonical injection from $\mathfrak{h}$ into R. We define R_c by $R_c=U_R(\mathfrak{b})/U_R(\mathfrak{b})\mathfrak{n}+\sum_{h\in\mathfrak{h}}U_R(\mathfrak{b})(h-c(h))$. Then R_c is isomorphic to R as R-module. We write 1_c for the canonical generator of R_c.

Definition 1.1. *We call $U_R(\mathfrak{g})\otimes_{U_R(\mathfrak{b})}R_c$ the universal Verma module.*

As a $\mathfrak{g}$-module, $U_R(\mathfrak{g})\otimes_{U_R(\mathfrak{b})}R_c$ is isomorphic to $U(\mathfrak{g})\otimes_{U(\mathfrak{n})}\mathbf{C}$. For $\lambda\in\mathfrak{h}^*$, let $\mathbf{C}_\lambda$ be the $U(\mathfrak{b})$-module given by $U(\mathfrak{b})/(U(\mathfrak{b})\mathfrak{n}+\sum_{h\in\mathfrak{h}}U(\mathfrak{b})(h-\lambda(h)))$. We regard $\mathbf{C}_\lambda$ also as an R-module by $R\hookrightarrow U(\mathfrak{b})$. Then $\mathbf{C}_\lambda\otimes_R(U_R(\mathfrak{g})\otimes_{U_R(\mathfrak{b})}R_c)$ is nothing but the Verma module with highest weight λ. Note that the universal Verma module is, as an R-module, isomorphic to $R\otimes_{\mathbf{C}}U(\mathfrak{n}_-)$, and in particular it is a free R-module.

For $\mu\in\mathfrak{h}^*$, we write $R_{c+\mu}$ for the $U_R(\mathfrak{b})$-module $\mathbf{C}_\mu\otimes_{\mathbf{C}}R_c$.

The following lemma is almost obvious.

Lemma 1.2. $\mathrm{End}_{U_R(\mathfrak{g})}(U_R(\mathfrak{g})\otimes_{U_R(\mathfrak{b})}R_c)=R$.

Now, we choose a non-degenerate W-invariant symmetric bilinear form (,) on $\mathfrak{h}^*$.

Lemma 1.3. *For $\mu \in \mathfrak{h}^*$, let f_μ be the function on $\mathfrak{h}^*$ given by*

$$f_\mu(\lambda)=(\lambda+\mu+\rho, \lambda+\mu+\rho)-(\lambda+\rho, \lambda+\rho)$$
$$=2(\mu, \lambda+\rho)+(\mu, \mu).$$

and regard this as an element of R.
Then we have

$$f_\mu \operatorname{Ext}^j_{U_R(\mathfrak{g})}(U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} R_c, U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} R_{c+\mu})=0 \qquad \textit{for any } j.$$

Proof. The Laplacian $\Delta \in \mathscr{Z}(\mathfrak{g})$ acts on $U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{b})} R_c$ by the multiplication of $(\lambda+\rho, \lambda+\rho)$ and on $U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{b})} R_{c+\mu}$ by $(\lambda+\mu+\rho, \lambda+\mu+\rho)$. Hence $(\lambda+\mu+\rho, \lambda+\mu+\rho)-(\lambda+\rho, \lambda+\rho)$ annihilates Ext^j.
Q.E.D.

Now, let F be a finite-dimensional $\mathfrak{b}$-module generated by a weight vector u of a weight $\lambda_0 \in \mathfrak{h}^*$. Hence $\mathfrak{h}$ acts semisimply on F. We shall choose a decreasing finite filtration $\{F^j\}$ of F by $\mathfrak{b}$-modules such that

$$F^0=F \tag{1.1}$$

$$F^j/F^{j+1} \text{ has a unique weight } \lambda_j. \tag{1.2}$$

$$\lambda_j \neq \lambda_{j'} \qquad \text{for } j \neq j'. \tag{1.3}$$

Therefore, we have $F^1=\mathfrak{n}F$ and $F^0/F^1 \cong \mathbf{C}_{\lambda_0}$. Hence there exists an isomorphism

$$\varphi_1: U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} R_{c+\lambda_0} \xrightarrow{\sim} U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} (R_c \underset{\mathbf{C}}{\otimes} F^0/F^1).$$

Now, we shall construct a commutative diagram

$$(1.4)_j: \quad \begin{array}{ccc} U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} R_{c+\lambda_0} & \xrightarrow{\varphi_j} & U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} (R_c \underset{\mathbf{C}}{\otimes} F^0/F^j) \\ \downarrow{\scriptstyle f_j} & & \downarrow \\ U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} R_{c+\lambda_0} & \underset{\varphi_1}{\xrightarrow{\sim}} & U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} (R_c \underset{\mathbf{C}}{\otimes} F^0/F^1) \end{array}$$

with $f_j \in R$, by the induction on j.

Assuming that $(1.4)_j$ has been already constructed ($j \geqq 1$), we shall construct $(1.4)_{j+1}$. We have an exact sequence

$$0 \longrightarrow U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} (R_c \otimes F^j/F^{j+1}) \longrightarrow U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} (R_c \otimes F^0/F^{j+1}) \longrightarrow$$

$$\longrightarrow U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} (R_c \otimes F^0/F^j) \longrightarrow 0.$$

This gives an exact sequence

$$\begin{aligned}
&\mathrm{Hom}_{U_R(\mathfrak{g})}\,(U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} R_{c+\lambda_0},\ U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} (R_c \otimes F^0/F^{j+1}))\\
\longrightarrow\ &\mathrm{Hom}_{U_R(\mathfrak{g})}\,(U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} R_{c+\lambda_0},\ U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} (R_c \otimes F^0/F^{j}))\\
\overset{\delta}{\longrightarrow}\ &\mathrm{Ext}^1_{U_R(\mathfrak{g})}\,(U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} R_{c+\lambda_0},\ \ U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} (R_c \otimes F^j/F^{j+1})).
\end{aligned}$$

On the other hand, F^j/F^{j+1} is a direct sum of copies of $R_{c+\lambda_j}$. Therefore, by Lemma 1.3, we have

$$g_j\ \mathrm{Ext}^1_{U_R(\mathfrak{g})}\,(U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} R_{c+\lambda_0},\ U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} (R_c \otimes F^j/F^{j+1}))=0$$

where $g_j \in R$ is given by $g_j(\lambda)=(\lambda+\lambda_j+\rho, \lambda+\lambda_j+\rho)-(\lambda+\lambda_0+\rho, \lambda+\lambda_0+\rho)$. Hence $g_j\delta(\varphi_j)=0$, which shows that $g_j\varphi_j$ lifts to ψ: $U_R(\mathfrak{g})\otimes_{U_R(\mathfrak{b})} R_{c+\lambda_0} \to U_R(\mathfrak{g})\otimes_{U_R(\mathfrak{b})}(R_c\otimes F^0/F^{j+1})$.

If ψ is divisible by g_j, then φ_j itself lifts and we obtain $(1.4)_{j+1}$ with $f_{j+1}=f_j$.

Assume that ψ is not divisible by g_j. For $\lambda \in \mathfrak{h}^*$, let us denote by $\psi(\lambda)$ the specialization of ψ, i.e. $\mathbf{C}_\lambda \otimes_R \psi$. Then, for a generic point λ of $g_j^{-1}(0)$, $\psi(\lambda)\neq 0$. Hence we obtain a diagram

$$(1.5)\qquad \begin{array}{ccc}
 & \overset{h}{\nearrow} & U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} (\mathbf{C}_\lambda \otimes F^j/F^{j+1})\\
 & & \downarrow\\
U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} \mathbf{C}_{\lambda+\lambda_0} & \xrightarrow{\ \psi(\lambda)\ } & U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} (\mathbf{C}_\lambda \otimes F^0/F^{j+1})\\
 & {\scriptstyle g_j(\lambda)\varphi_j(\lambda)} \searrow & \downarrow\\
 & & U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} (\mathbf{C}_\lambda \otimes F^0/F^{j})
\end{array}$$

Since $g_j(\lambda)=0$, we obtain a nonzero homomorphism h: $U(\mathfrak{g})\otimes_{U(\mathfrak{b})} \mathbf{C}_{\lambda+\lambda_0} \to U(\mathfrak{g})\otimes_{U(\mathfrak{b})} (\mathbf{C}_\lambda\otimes F^j/F^{j+1})$. Since $U(\mathfrak{g})\otimes_{U(\mathfrak{b})} (\mathbf{C}_\lambda\otimes F^j/F^{j+1})$ is a direct sum of copies of $U(\mathfrak{g})\otimes_{U(\mathfrak{b})} \mathbf{C}_{\lambda+\lambda_j}$, the central character of $U(\mathfrak{g})\otimes_{U(\mathfrak{b})} \mathbf{C}_{\lambda+\lambda_0}$ and that of $U(\mathfrak{g})\otimes_{U(\mathfrak{b})} \mathbf{C}_{\lambda+\lambda_j}$ must coincide. Hence there exists $w \in W$ such that $w(\lambda+\lambda_0+\rho)=\lambda+\lambda_j+\rho$. This shows that $w(\lambda+\lambda_0+\rho)=\lambda+\lambda_j+\rho$ holds for any $\lambda \in g_j^{-1}(0)$. Since $\lambda_j\neq\lambda_0$, $w\neq 1$. Since w fixes the hyperplane $(\lambda, \lambda_j-\lambda_0)=0$, w must be the reflection s_α for some $\alpha \in \Delta^+$. Hence we obtain

$$0=\lambda+\lambda_j+\rho-s_\alpha(\lambda+\lambda_0+\rho)=\lambda_j-\lambda_0+h_\alpha(\lambda+\lambda_0+\rho)\alpha.$$

This implies that $\lambda_j=\lambda_0+k\alpha$ for some $k\in\mathbf{C}$. Since $\lambda_j-\lambda_0\in Q_+(\Delta)\backslash\{0\}$, k is a strictly positive integer. Moreover $h_\alpha(\lambda+\lambda_0+\rho)+k=0$ holds on $g_j^{-1}(0)$. Hence g_j is a constant multiple of $h_\alpha(\lambda+\lambda_0+\rho)+k$.

Summing up, we obtain

Lemma 1.4. (i) *If λ_j is not of the form $\lambda_0+k\alpha$ with $\alpha\in\Delta_+$, $k\in\mathbf{Z}_{++}$, then φ_j lifts to φ_{j+1}: $U_R(\mathfrak{g})\otimes_{U_R(\mathfrak{b})}R_{c+\lambda_0}\to U_R(\mathfrak{g})\otimes_{U_R(\mathfrak{b})}(R_c\otimes F^0/F^{j+1})$*
(ii) *If $\lambda_j=\lambda_0+k\alpha$ for some $\alpha\in\Delta^+$ and $k\in\mathbf{Z}_{++}$, then $(c(h_\alpha)+h_\alpha(\lambda_0+\rho)+k)\varphi_j$ lifts to φ_{j+1}.*

Repeating this procedure we obtain

Theorem 1.5. *There exists a commutative diagram*

$$\begin{array}{ccc} U_R(\mathfrak{g})\underset{U_R(\mathfrak{b})}{\otimes}R_{c+\lambda_0} & \xrightarrow{\varphi} & U_R(\mathfrak{g})\underset{U_R(\mathfrak{b})}{\otimes}(R_c\otimes F) \\ \Big\downarrow f & & \Big\downarrow \\ U_R(\mathfrak{g})\underset{U_R(\mathfrak{b})}{\otimes}R_{c+\lambda_0} & \xrightarrow{\sim} & U_R(\mathfrak{g})\underset{U_R(\mathfrak{b})}{\otimes}(R_c\otimes F^0/F^1). \end{array} \tag{1.6}$$

Here $f=\prod_{(\alpha,k)\in\mathfrak{S}(F)}(h_\alpha+h_\alpha(\lambda_0+\rho)+k)$ and $\mathfrak{S}(F)$ is the set of pairs (α,k) of positive root α and a positive integer k such that $\lambda_0+k\alpha$ is a weight of F.

Example 1.6. We set $F_k=U(\mathfrak{b})/(U(\mathfrak{b})\mathfrak{h}+U(\mathfrak{b})\mathfrak{n}^k)$. Let K be the quotient field of R. Then for any k, there exists a unique

$$\varphi_k:\ U_K(\mathfrak{g})\underset{U_R(\mathfrak{b})}{\otimes}R_c\to U_K(\mathfrak{g})\underset{U_R(\mathfrak{b})}{\otimes}(R_c\otimes F_k)$$

such that the following diagram commutes

$$\begin{array}{ccc} U_K(\mathfrak{g})\underset{U_R(\mathfrak{b})}{\otimes}R_c & \xrightarrow{\varphi_k} & U_K(\mathfrak{g})\underset{U_R(\mathfrak{b})}{\otimes}(R_c\otimes F_k) \\ & {}_{1}\searrow & \Big\downarrow \\ & & U_K(\mathfrak{g})\underset{U_R(\mathfrak{b})}{\otimes}(R_c\otimes F_0). \end{array}$$

Hence, taking the projective limit, we obtain

$$\hat{\varphi}:\ U_K(\mathfrak{g})\underset{U_R(\mathfrak{b})}{\otimes}R_c\to\varprojlim_k U_K(\mathfrak{g})\underset{U_R(\mathfrak{b})}{\otimes}(R_c\otimes F_k).$$

When $\mathfrak{g}=sl_2$, we shall calculate $\hat{\varphi}$. Let us take the generator X_+, X_-, h such that $[h,X_\pm]=\pm2X_\pm$, $[X_+,X_-]=h$. Set $\lambda=c(h)$. We can write $P=\hat{\varphi}(1)$ in the following form

$$P=\sum_{j=0}^{\infty}a_jX_-^j\otimes X_+^j(1_c\otimes1)$$

with $a_0=1$. Then

$$\begin{aligned}X_+P&=\sum a_jX_+X_-^j\otimes X_+^j(1_c\otimes 1)\\&=\sum a_jX_-^j\otimes X_+^{j+1}(1_c\otimes 1)+\sum ja_jX_-^{j-1}(h-j+1)\otimes X_+^j(1_c\otimes 1)\\&=\sum a_jX_-^j\otimes X_+^{j+1}(1_c\otimes 1)+\sum j(\lambda+j+1)a_jX_-^{j-1}\otimes X_+^j(1_c\otimes 1).\end{aligned}$$

Here we have used the relation $[X_+, X_-^j]=jX_-^{j-1}(h-j+1)$.

Hence we obtain the recursion formula

$$a_j=-\frac{1}{j(\lambda+j+1)}a_{j-1}\qquad \text{for } j\geqq 1.$$

Solving this, we obtain

(1.7) $$P=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!(\lambda+2,j)}X_-^j\otimes X_+^j(1_c\otimes 1).$$

Let V_μ^* be a finite-dimensional irreducible representation of $\mathfrak{g}$ with a lowest weight $-\mu$ and $v_{-\mu}$ a lowest weight vector. As well-known, $-\mu+k\alpha$ is a weight of V_μ^* if and only if $0\leqq k\leqq h_\alpha(\mu)$. Hence Theorem 1.5 implies the following Theorem.

Theorem 1.7. *There exists a homomorphism*

$$\varphi_0\colon U_R(\mathfrak{g})\underset{U_R(\mathfrak{b})}{\otimes}R_c\longrightarrow U_R(\mathfrak{g})\underset{U_R(\mathfrak{b})}{\otimes}(R_{c+\mu}\otimes V_\mu^*)$$

such that $g\circ\varphi_0=\prod_{\alpha\in\Delta^+}(h_\alpha+h_\alpha(\rho)+1, h_\alpha(\mu))$, *where* $g\colon U_R(\mathfrak{g})\otimes_{U_R(\mathfrak{b})}(R_{c+\mu}\otimes V_\mu^*)\to U_R(\mathfrak{g})\otimes_{U_R(\mathfrak{b})}R_c$ *is given by* $g(1\otimes 1_{c+\mu}\otimes v_{-\mu})=1\otimes 1_c$.

Now, we shall show the converse.

Proposition 1.8. *For any homomorphism*

$$\varphi\colon U_R(\mathfrak{g})\underset{U_R(\mathfrak{b})}{\otimes}R_c\longrightarrow U_R(\mathfrak{g})\underset{U_R(\mathfrak{b})}{\otimes}(R_{c+\mu}\otimes V_\mu^*),$$

set $f=g\circ\varphi\in R$. *Then* f *is a multiple of* $\prod_{\alpha\in\Delta^+}(h_\alpha+h_\alpha(\rho)+1, h_\alpha(\mu))$.

Proof. Note that $h_\alpha+h_\alpha(\rho)+k=c(h_{\alpha'}+h_{\alpha'}(\rho)+k')$ with $\alpha,\alpha'\in\Delta^+, k, k', c\in\mathbf{C}$ implies, $\alpha=\alpha'$, $k=k'$. Hence we can construct another φ such that $g\circ\varphi$ is the greatest common divisor of f and $\prod(h_\alpha+h_\alpha(\rho)+1, h_\alpha(\mu))$. Therefore, we may assume from the beginning that f is a divisor of $\prod(h_\alpha+\rho(h_\alpha)+1, h_\alpha(\mu))$.

Set $M=U_R(\mathfrak{g})\otimes_{U_R(\mathfrak{b})}(R_{c+\mu}\otimes V_\mu^*)\cong U(\mathfrak{g})\otimes_{U(\mathfrak{n})}V_\mu^*$ and let M_j be the image of $U_j(\mathfrak{g})\otimes V_\mu^*$ in M. Then we can easily show

$$\operatorname{gr} M=\bigoplus M_j/M_{j-1}=(S(\mathfrak{g})/S(\mathfrak{g})\mathfrak{n})\underset{\mathbf{C}}{\otimes} V_\mu^*$$

as an $\mathfrak{n}$-module.

Now, $v=\varphi(1)$ is a non-zero element of M which is $\mathfrak{n}$-invariant. Let j be the smallest integer such that $v\in M_j$ and let $\bar{v}$ be the image of v in M_j/M_{j-1}. Then $\bar{v}$ is also $\mathfrak{n}$-invariant. By the Killing form we identify $\mathfrak{g}$ and $\mathfrak{g}^*$. Then $S(\mathfrak{g})/S(\mathfrak{g})\mathfrak{n}$ is isomorphic to $\mathbf{C}[\mathfrak{b}]$, the polynomial ring of $\mathfrak{b}$. Hence we can regard $\bar{v}$ as a V_μ^*-valued function on $\mathfrak{b}$, and we denote it Ψ. By the assumption, v has the form

$$v=f\otimes v_{-\mu} \bmod U(\mathfrak{b}_-)\mathfrak{n}_-\otimes \mathfrak{n}V_\mu^*.$$

Hence $j\geqq \deg f$ and we have either

(1.8) $$j>\deg f \quad \text{and} \quad \Psi|\mathfrak{h}=0$$

or

(1.9) $$j=\deg f \quad \text{and} \quad \Psi(h)=\bar{f}(h)v_{-\mu} \quad \text{for} \quad h\in\mathfrak{h}.$$

Here $\bar{f}$ is the homogeneous part of f. Since $N\mathfrak{h}$ is an open dense subset of $\mathfrak{b}$, $\Psi|\mathfrak{h}=0$ implies $\Psi=0$. Hence the first case (1.8) does not occur and we have (1.9).

Let $S(\Delta^+)$ be the set of simple roots. For $\alpha\in\Delta$, let x_α be a root vector with root α. We normalize as $[x_\alpha, x_{-\alpha}]=h_\alpha$. We set

$$x_+=\sum_{\alpha\in S(\Delta+)} x_\alpha \qquad x_-=\sum_{\alpha\in S(\Delta+)} x_{-\alpha}.$$

We take the element $h_0\in\mathfrak{h}$ such that $h_0(\alpha)=2$ for $\alpha\in S(\Delta^+)$. Then $h_0=\sum_{\alpha\in\Delta+}h_\alpha$. Now, we can show easily $[h_0, x_\pm]=\pm 2x_\pm$, $[x_+, x_-]=h_0$ and hence $\langle h_0, x_+, x_-\rangle_{\mathbf{C}}$ forms a Lie algebra isomorphic to sl_2. We have

$$e^{tx_+}h_0=h_0-2tx_+.$$

Therefore, we obtain

$$\begin{aligned}\Psi(ah_0-2x_+)&=\Psi(ae^{a^{-1}x_+}h_0)=e^{a^{-1}x_+}\Psi(ah_0)\\&=\bar{f}(ah_0)e^{a^{-1}x_+}v_{-\mu}\\&=\sum_{k\geqq 0}\frac{(a^{-1})^k}{k!}\bar{f}(ah_0)x_+^k v_{-\mu}.\end{aligned}$$

The representation theory of sl_2 implies that $x_+^k v_{-\mu}\neq 0$ for $(0\leqq k\leqq h_0(\mu))$ and $x_+^k v_{-\mu}=0$ for $k>h_0(\mu)$. Since $\Psi(ah_0-2x_+)$ is a polynomial in a, $\bar{f}(ah_0)a^{-h_0(\mu)}$ is also a polynomial in a. Moreover $\bar{f}(h_0)\neq 0$ because $\bar{f}$ is a factor of $\prod h_\alpha^{h_\alpha(\mu)}$. This shows that

$$\deg f = \deg \bar{f} \geqq h_0(\mu) = \sum_{\alpha \in \Delta^+} h_\alpha(\mu).$$

Hence f is $\prod (h_\alpha + h_\alpha(\rho) + 1, h_\alpha(\mu))$ up to constant multiple. Q.E.D.

For a $\mathfrak{g}$-module V and a $\mathfrak{b}$-module F, we have a canonical isomorphism

$$U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} (F \otimes V) \longrightarrow V \underset{\mathbf{C}}{\otimes} (U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} F) \tag{1.10}$$

by $1 \otimes (f \otimes v) \mapsto v \otimes (1 \otimes f)$ for $v \in V, f \in F$.

Similarly, we have

$$U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} (R_{c+\mu} \otimes V_\mu^*) \xrightarrow{\sim} V_\mu^* \underset{\mathbf{C}}{\otimes} (U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} R_{c+\mu}). \tag{1.11}$$

Therefore, we have

$$\begin{aligned} &\mathrm{Hom}_{U_R(\mathfrak{g})} (U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} R_c,\ U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} (R_{c+\mu} \otimes V_\mu^*)) \\ &= \mathrm{Hom}_{U_R(\mathfrak{g})} (U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} R_c,\ V_\mu^* \otimes (U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} R_{c+\mu})) \\ &= \mathrm{Hom}_{U_R(\mathfrak{g})} (V_\mu \otimes (U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} R_c),\ U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} R_{c+\mu}) \\ &= \mathrm{Hom}_{U_R(\mathfrak{g})} (U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} (R_c \otimes V_\mu),\ U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} R_{c+\mu}). \end{aligned} \tag{1.12}$$

We choose a lowest weight vector $v_{-\mu}$ of V_μ^* and a highest weight vector v_μ of V_μ, normalized by $\langle v_\mu, v_{-\mu} \rangle = 1$. We define g: $U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{b})} (R_{c+\mu} \otimes V_\mu^*) \to U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{b})} R_c$ and h: $U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{b})} R_{c+\mu} \to U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{b})} (R_c \otimes V_\mu)$ by $g(1 \otimes 1_{c+\mu} \otimes v_{-\mu}) = 1 \otimes 1_c$ and $h(1 \otimes 1_{c+\mu}) = 1 \otimes 1_c \otimes v_\mu$

Theorem 1.9. *Assume that*

$$\varphi \in \mathrm{Hom}_{U_R(\mathfrak{g})} (U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} R_c,\ U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} (R_{c+\mu} \otimes V_\mu^*))$$

and

$$\psi \in \mathrm{Hom}_{U_R(\mathfrak{g})} (U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} (R_c \otimes V_\mu),\ U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} R_{c+\mu})$$

correspond by the isomorphism (1.12). *Set $f = g \circ \varphi \in R$ and $f' = \psi \circ h \in R$. Then, we have*

$$f' = \prod_{\alpha \in \Delta^+} \frac{h_\alpha + h_\alpha(\rho)}{h_\alpha + h_\alpha(\rho + \mu)} f \tag{1.13}$$

Proof. For $\lambda \in \mathfrak{h}^*$, we shall denote by $\varphi(\lambda)$, $\psi(\lambda)$, $h(\lambda)$ and $g(\lambda)$ their specializations at λ. Identifying $V_\mu^* \otimes (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbf{C}_{\lambda+\mu})$ with $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (\mathbf{C}_{\lambda+\mu} \otimes V_\mu^*)$, etc., we have commutative diagrams

$$\begin{array}{ccc} U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} \mathbf{C}_\lambda & \xrightarrow{\varphi(\lambda)} & V_\mu^* \otimes (U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} \mathbf{C}_{\lambda+\mu}) \\ & \searrow^{f(\lambda)} & \downarrow g(\lambda) \\ & & U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} \mathbf{C}_\lambda \end{array}$$

and

$$\begin{array}{ccc} U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} \mathbf{C}_{\lambda+\mu} & & \\ \downarrow h(\lambda) & \searrow^{f'(\lambda)} & \\ V_\mu \otimes (U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} \mathbf{C}_\lambda) & \xrightarrow[\psi(\lambda)]{} & U(\mathfrak{g}) \otimes \mathbf{C}_{\lambda+\mu}. \end{array}$$

Letting λ be a dominant integral weight and employing the homomorphism $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbf{C}_\lambda \to V_\lambda$, etc. we obtain

$$\begin{array}{ccc} V_\lambda & \xrightarrow{\bar{\varphi}} & V_\mu^* \otimes V_{\lambda+\mu} \\ & \searrow^{f(\lambda)} & \downarrow \bar{g} \\ & & V_\lambda \end{array} \tag{1.14}$$

and

$$\begin{array}{ccc} V_{\lambda+\mu} & & \\ \downarrow \bar{h} & \searrow^{f'(\lambda)} & \\ V_\mu \otimes V_\lambda & \xrightarrow[\bar{\psi}]{} & V_{\lambda+\mu} \end{array} \tag{1.15}$$

Here $\bar{g}$ and $\bar{h}$ are characterized by $\bar{g}(v_{-\mu} \otimes v_{\lambda+\mu}) = v_\lambda$ and $\bar{h}(v_{\lambda+\mu}) = v_\mu \otimes v_\lambda$. Moreover, $\bar{\varphi}$ and $\bar{\psi}$ are related by

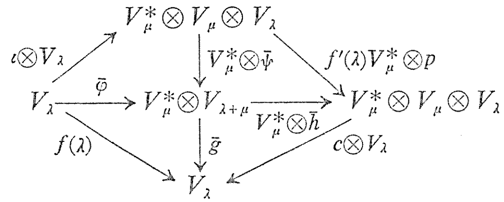
$$(c \otimes \mathrm{id}_{V_{\lambda+\mu}})(w \otimes \bar{\varphi}(v)) = \bar{\psi}(w \otimes v) \quad \text{for } v \in V_\lambda \quad \text{and } w \in V_\mu,$$

where c is the contraction $V_\mu \otimes V_\mu^* \to \mathbf{C}$.

Now, $V_\mu \otimes V_\lambda$ contains $V_{\lambda+\mu}$ with multiplicity 1. Let us denote by p the projector form $V_\mu \otimes V_\lambda$ onto $\bar{h}(V_{\lambda+\mu})$, and regard this as an endomorphism of $V_\mu \otimes V_\lambda$. Then by (1.15), we have

$$\bar{h} \circ \bar{\psi} = f'(\lambda) p.$$

On the other hand, we have a commutative diagram



where ι: $\mathbf{C} \to V_\mu^* \otimes V_\mu$ is the canonical injection. Therefore we have

$$f(\lambda)\,\mathrm{id}_{V_\lambda} = f'(\lambda)(c \otimes V_\lambda) \circ (V_\mu^* \otimes p) \circ (\iota \otimes V_\lambda).$$

Taking the trace, we have

$$(1.16) \qquad f(\lambda) \dim V_\lambda = f'(\lambda)\,\mathrm{tr}_{V_\lambda}(c \otimes V_\lambda) \circ (V_\mu^* \otimes p) \circ (\iota \otimes V_\lambda).$$

In order to calculate the right-hand side, we shall take bases $\{w_j\}$ of V_λ, $\{u_k\}$ of V_μ and their dual bases $\{w_j^*\}$ and $\{u_k^*\}$. Then

$$\begin{aligned}
&(c \otimes V_\lambda) \circ (V_\mu^* \otimes p) \circ (\iota \otimes V_\lambda)(w_j) \\
&\quad = \sum_k (c \otimes V_\lambda) \circ (V_\mu^* \otimes p)(u_k^* \otimes u_k \otimes w_j) \\
&\quad = \sum_k (c \otimes V_\lambda)(u_k^* \otimes p(u_k \otimes w_j)).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\mathrm{tr}_{V_\lambda}(c \otimes V_\lambda) \circ (V_\mu^* \otimes p) \circ (\iota \otimes V_\lambda) \\
&\quad = \sum_{j,k} \langle w_j^*, (c \otimes V_\lambda)(u_k^* \otimes p(u_k \otimes w_j))\rangle \\
&\quad = \sum_{j,k} \langle u_k^* \otimes w_j^*, p(u_k \otimes w_j)\rangle \\
&\quad = \mathrm{tr}_{V_\mu \otimes V_\lambda} p = \dim V_{\lambda+\mu}.
\end{aligned}$$

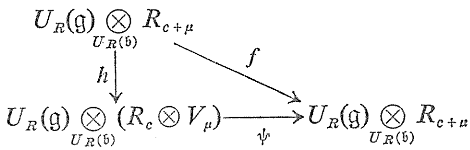
By (1.16), we obtain

$$f(\lambda) \dim V_\lambda = f'(\lambda) \dim V_{\lambda+\mu}.$$

Then the assertion follows from Weyl's dimension formula

$$\dim V_\lambda = \prod_{\alpha \in \Delta+} \frac{h_\alpha(\lambda+\rho)}{h_\alpha(\rho)}. \qquad \text{Q.E.D.}$$

Corollary 1.10. *For a dominant integral weight μ, there exists a commutative diagram*



where $f=\prod_{\alpha\in\Delta^+}(h_\alpha+h_\alpha(\rho),\ h_\alpha(\mu))$ *and* $h(1\otimes 1_{c+\mu})=1\otimes 1_c\otimes v_\mu$.

Remark 1.11. This corollary is also obtained either by a similar argument as the proof of Theorem 1.5 or directly from Theorem 1.7 by the following argument. First note that for any $U_R(\mathfrak{b})$-module F, we have

$$\mathbf{R}\operatorname{Hom}_{U_R(\mathfrak{g})}(U_R(\mathfrak{g})\underset{U_R(\mathfrak{b})}{\otimes}F,\ U_R(\mathfrak{g}))$$
$$=U_R(\mathfrak{g})\underset{U_R(\mathfrak{b})}{\otimes}\mathbf{R}\operatorname{Hom}_{U_R(\mathfrak{b})}(F,\ U_R(\mathfrak{b})).$$

On the other hand, for a finite dimensional $\mathfrak{b}$-module V

$$\mathbf{R}\operatorname{Hom}_{U_R(\mathfrak{b})}(R_c\otimes V,\ U_R(\mathfrak{b}))=R_{-c-2\rho}\otimes V^*[-\dim\mathfrak{b}]$$

where $R_{-c-2\rho}$ is the $U_R(\mathfrak{b})$-module R with weight $-c-2\rho$. Hence the commutative diagram

$$\begin{array}{ccc}
U_R(\mathfrak{g})\underset{U_R(\mathfrak{b})}{\otimes}R_c & \longrightarrow & U_R(\mathfrak{g})\underset{U_R(\mathfrak{b})}{\otimes}(R_{c+\mu}\otimes V_\mu^*) \\
 & {\scriptstyle f'}\searrow & \downarrow \\
 & & U_R(\mathfrak{g})\underset{U_R(\mathfrak{b})}{\otimes}R_c
\end{array}$$

with $f'=\prod_\alpha(h_\alpha+h_\alpha(\rho)+1,\ h_\alpha(\mu))$ gives

$$\begin{array}{ccc}
U_R(\mathfrak{g})\underset{U_R(\mathfrak{b})}{\otimes}R_{-c-2\rho} & \longleftarrow & U_R(\mathfrak{g})\underset{U_R(\mathfrak{b})}{\otimes}(R_{-c-\mu-2\rho}\otimes V_\mu) \\
 & {\scriptstyle f'}\nwarrow & \uparrow \\
 & & U_R(\mathfrak{g})\underset{U_R(\mathfrak{b})}{\otimes}R_{-c-2\rho}\,.
\end{array}$$

Now, the isomorphism $h\mapsto -h-h(2\rho+\mu)$ gives Corollary 1.10.

§ 2. The b-functions of $B_-\times B$-semi-invariants

For a dominant integral weight λ, let V_λ be an irreducible representation of $\mathfrak{g}$ with highest weight λ. Let v_λ be a highest weight vector of V_λ and $v_{-\lambda}$ the lowest weight vector of V_λ^*, normalized by $\langle v_\lambda, v_{-\lambda}\rangle=1$.

Let f^λ be the regular function on G defined by

(2.1) $$f^\lambda(g)=\langle gv_\lambda,\ v_{-\lambda}\rangle.$$

Then f^λ is $B_-\times B$-semi-invariant such that

(2.2) $$f^\lambda(b'gb)=\chi_\lambda^-(b')\chi_\lambda^+(b)f^\lambda(g)\quad\text{for } g\in G,\ b'\in B_-\text{ and } b\in B,$$

where $\chi_\lambda^\pm$ is the character of B and B_- such that

$$\chi_\lambda^\pm(e^h)=e^{\lambda(h)} \qquad \text{for } h\in\mathfrak{h}.$$

Moreover we have

$$f^\lambda(e)=1. \tag{2.3}$$

Note that any $B_-\times B$-semi-invariant with character $\chi_\lambda^-\otimes\chi_\lambda$ is a constant multiple of f^λ and any $B_-\times B$-semi-invariant has a character $\chi_\lambda^-\otimes\chi_\lambda$ for some $\lambda\in P^+$. This follows from the well-known formula

$$\mathcal{O}(G)=\bigoplus_{\lambda\in P+} V_\lambda^*\otimes V_\lambda.$$

In particular, we have

$$f^{\lambda+\lambda'}(g)=f^\lambda(g)f^{\lambda'}(g). \tag{2.4}$$

Theorem 2.1. *For any dominant integral weight μ, there exists a differential operator P_μ such that*

$$P_\mu f^{\lambda+\mu}=b_\mu(\lambda)f^\lambda \qquad \textit{for any } \lambda. \tag{2.5}$$

Here $b_\mu(\lambda)=\prod_{\alpha\in\Delta+}(h_\alpha(\lambda+\rho),\, h_\alpha(\mu))$.

Proof. Let us denote by $\mathcal{D}$ the sheaf of differential operators on G. Then the right-action of G on itself gives a homomorphism R: $U(\mathfrak{g})\to\mathcal{D}(G)$. In particular, $R(U(\mathfrak{g}))$ is the set of left invariant differential operators on G.

By Corollary 1.10, there exists an $\mathfrak{n}$-invariant element P of $V_\mu^*\otimes(U_R(\mathfrak{g})\otimes_{U_R(\mathfrak{b})}R_{c+\mu})$ with weight c, whose coefficient of $v_{-\mu}$ is $\prod_{\alpha\in\Delta+}(c(h_\alpha)+h_\alpha(\rho),\, h_\alpha(\mu))$. Hence P is written in the following form

$$P=\sum_{j=0}^{N} v_j\otimes P_j\otimes 1_{c+\mu}$$

where

$$v_0=v_{-\mu},\quad P_0=\prod_{\alpha\in\Delta+}(h_\alpha+h_\alpha(\rho-\mu),\, h_\alpha(\mu)) \tag{2.6}$$

and

$$v_j\in\mathfrak{n}V_\mu^*,\quad P_j\in U(\mathfrak{b}_-)\mathfrak{n}_- \quad \text{for } j\geqq 1. \tag{2.7}$$

We shall define the differential operator P_μ on G by

$$(P_\mu u)(g)=\sum_j\langle v_\mu.\ gv_j\rangle(R(P_j)u)(g). \tag{2.8}$$

Lemma 2.2. *For any $y \in \mathfrak{n}$, we have*

$$[R(y), P_\mu] \in \mathscr{D}(G)R(\mathfrak{n}).$$

Proof. We have $[R(y), \langle v_\mu, gv_j\rangle] = \langle v_\mu, gyv_j\rangle$. Hence we have

$$([R(y), P_\mu]u)(g) = \sum_j \langle g^{-1}v_\mu, yv_j\rangle (R(P_j)u)(g) + \sum_j \langle g^{-1}v_\mu, v_j\rangle (R([y, P_j])u)(g).$$

Since $\sum v_j \otimes P_j \otimes 1_{c+\mu}$ is $\mathfrak{n}$-invariant, we have

$$\sum_j yv_j \otimes P_j \otimes 1_{c+\mu} + \sum_j v_j \otimes [y, P_j] \otimes 1_{c+\mu} = 0$$

in

$$V_\mu^* \otimes U_R(\mathfrak{g}) \underset{U_R(\mathfrak{b})}{\otimes} R_{c+\mu} = V_\mu^* \otimes (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{n}).$$

Therefore we can write, as the identity in $V_\mu^* \otimes_{\mathbf{C}} U(\mathfrak{g})$,

$$\sum_j yv_j \otimes P_j + \sum v_j \otimes [y, P_j] = \sum w_k \otimes S_k$$

with $w_k \in V_\mu^*$ and $S_k \in U(\mathfrak{g})\mathfrak{n}$. This shows

$$([R(y), P_\mu]u)(g) = \sum_k \langle g^{-1}v_\mu, w_k\rangle (R(S_k)u)(g).$$

Since $R(S_k) \in \mathscr{D}(G)R(\mathfrak{n})$, we have the desired result. Q.E.D.

By this lemma, we have for $y \in \mathfrak{n}$

$$R(y)P_\mu f^{\lambda+\mu} = [R(y), P_\mu]f^{\lambda+\mu} + P_\mu R(y) f^{\lambda+\mu} = 0$$

because $f^{\lambda+\mu}$ is right invariant by N. Therefore $P_\mu f^{\lambda+\mu}$ is also right N-invariant. Since B_-N is an open dense subset of G, it is sufficient to show (2.5) on B_-. Now for $g \in B_-$, we have

$$(P_\mu f^{\lambda+\mu})(g) = \sum_j \langle v_\mu, gv_j\rangle (R(P_j)f^{\lambda+\mu})(g).$$

Note that all P_j belongs to $U(\mathfrak{b}_-)$ and $P_j \in U(\mathfrak{b}_-)\mathfrak{n}_-$ for $j \neq 0$. Since $f^{\lambda+\mu}(n_-h) = f^{\lambda+\mu}(hn_-) = h^{\lambda+\mu}$ for $h \in T$ and $n_- \in N_-$, $f^{\lambda+\mu}|_{B_-}$ is right N_--invariant. This shows $R(P_j)f^{\lambda+\mu}|_{B_-} = 0$ for $j \neq 0$. It is easy to see for $g \in B_-$

$$R(P_0)f^{\lambda+\mu}(g) = \prod_\alpha (h_\alpha(\lambda+\mu) + h_\alpha(\rho-\mu), h_\alpha(\mu)) f^{\lambda+\mu} = b_\mu(\lambda) f^{\lambda+\mu}$$

and $\langle v_\mu, gv_0\rangle = 1/f^\mu$.

This completes the proof of Theorem 2.1.

Remark 2.3. We can show $b_\mu(\lambda)$ in Theorem 2.1 is the best possible one. This follows from the similar argument as Proposition 1.8, or we can use the result in [3]. In fact if w_0 is the longest element of W, then $T^*_{B_-w_0B}G$ is a good Lagrangian variety in the sense in [3], which is equivalent to saying that $\mathfrak{n}$ is a prehomogeneous vector space over $\mathfrak{b}$. Hence we can show the degree of the local b-function is $\sum_{\alpha\in\Delta_+} h_\alpha(\mu)$.

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