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Characters of the Negative Level Highest-Weight Modules for Affine Lie Algebras

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0 Introduction

Our main result in this paper is the following theorem conjectured by G. Lusztig [L].

Let $\mathfrak{g}$ be an affine Lie algebra with Cartan subalgebra $\mathfrak{h}$ and simple coroots $\{h_i\}_{i\in I}$. For $\lambda \in \mathfrak{h}^*$ let $M(\lambda)$ (resp. $L(\lambda)$) be the Verma module (resp. the irreducible module) with highest weight λ.

Theorem. For $\lambda \in \mathfrak{h}^*$ such that $(\lambda+\rho)(h_i) \in \mathbb{Z}_{<0}$ for any $i \in I$, we have

$$\operatorname{ch} L\left(w(\lambda+\rho)-\rho\right) = \sum_{y \leqq w} (-1)^{l(w)-l(y)} P_{y,w}(1) \operatorname{ch} M\left(y(\lambda+\rho)-\rho\right) \tag{0.1}$$

for any $w \in W$. Here W is the Weyl group, $\rho \in \mathfrak{h}^*$ is such that $\rho(h_i) = 1$ for any $i \in I$, l is the length function, $\leqq$ is the Bruhat order, $P_{y,w}$ is the Kazhdan-Lusztig polynomial, and ch denotes the character.

This type of character formula was first conjectured by Kazhdan and Lusztig [KL] for finite-dimensional semisimple Lie algebras and was proved by Beilinson-Bernstein [BB] and Brylinski-Kashiwara [BK]. Then its generalization to symmetrizable Kac-Moody Lie algebras concerning the dominant integral weights λ was given by Kashiwara (-Tanisaki) [K2], [KT] and Casian [C1]. This paper is concerned with a different version in the case of the antidominant integral weights for affine Lie algebras.

The schemes of the proofs of those character formulas are all similar. The $\mathfrak{g}$-modules correspond to $\mathcal{D}$-modules on the flag manifold, and the $\mathcal{D}$-modules correspond to perverse sheaves by the Riemann-Hilbert correspondence. Since the perverse sheaf corresponding to a dual Verma module (resp. irreducible highest-weight module) is the zero (resp. minimal) extension of the constant sheaf on a Schubert cell, the proof is

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reduced to the calculation of the local intersection cohomology groups of the Schubert varieties. This last step is now standard by the theory of Weil sheaves (cf. [BBD]) or the theory of Hodge modules (cf. [S]).

However, there are some differences between [K2], [KT] and our case. The natural setting in our case is of the right $\mathcal{D}$-modules supported on the finite-dimensional Schubert varieties, while in [K2], [KT] we used left $\mathcal{D}$-modules supported on the finite-codimensional Schubert varieties. The category of left $\mathcal{D}$-modules and the one of right $\mathcal{D}$-modules are equivalent on finite-dimensional manifolds. The flag manifold in our case is infinite-dimensional, and those two categories are not equivalent. The left $\mathcal{D}$-modules behave well under the pull-back while the right $\mathcal{D}$-modules behave well under the push-forward. This is the reason why we use right $\mathcal{D}$-modules.

For a technical reason, we do not directly treat the right $\mathcal{D}$-modules on the flag manifold itself. Instead, considering the fact that the flag manifold is locally isomorphic to the projective limit of finite-dimensional smooth varieties, we use a "projective limit" of right $\mathcal{D}$-modules on these finite-dimensional varieties as a substitute.

In this paper we will give descriptions of the category of those projective limits of right $\mathcal{D}$-modules, and the functor from this category to the category of $\mathfrak{g}$-modules. Our main result follows from several properties of this functor. Details of the proof will appear elsewhere.

The same result is claimed in the preprint of Casian [C2]. Our method is different, since a functor in the opposite direction is used in [C2].

1 The Kac-Moody Lie algebra

We recall basic facts concerning the Kac-Moody Lie algebra.

Let $\mathfrak{h}$ be a finite-dimensional vector space over $\mathbb{C}$, and let $\{\alpha_i\}_{i\in I}$, $\{h_i\}_{i\in I}$ be linearly independent vectors of $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively, such that $(\langle h_i, \alpha_j\rangle)_{i,j\in I}$ is a symmetrizable generalized Cartan matrix. The Kac-Moody Lie algebra associated to $(\mathfrak{h}, \{\alpha_i\}, \{h_i\})$ is the Lie algebra $\mathfrak{g}$ generated by the vector space $\mathfrak{h}$ and the elements e_i, f_i $(i \in I)$ satisfying the following fundamental relations:

$$[h, h'] = 0 \qquad \text{for } h, h' \in \mathfrak{h}, \tag{1.1}$$

$$[h, e_i] = \alpha_i(h)e_i \qquad \text{for } h \in \mathfrak{h}, \quad i \in I, \tag{1.2}$$

$$[h, f_i] = -\alpha_i(h)f_i \qquad \text{for } h \in \mathfrak{h}, \quad i \in I, \tag{1.3}$$

$$[e_i, f_j] = \delta_{ij}h_i \qquad \text{for } i, j \in I, \tag{1.4}$$

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$$\mathrm{ad}(e_i)^{1-\alpha_j(h_i)}(e_j) = 0 \qquad \text{for } i, j \in I \text{ with } i \neq j, \tag{1.5}$$

$$\mathrm{ad}(f_i)^{1-\alpha_j(h_i)}(f_j) = 0 \qquad \text{for } i, j \in I \text{ with } i \neq j. \tag{1.6}$$

For $i \in I$, let s_i be the linear automorphism of $\mathfrak{h}^*$ given by

$$s_i(\lambda) = \lambda - \lambda(h_i)\alpha_i \qquad \text{for } \lambda \in \mathfrak{h}^*. \tag{1.7}$$

The Weyl group W is by definition the subgroup of $GL(\mathfrak{h}^*)$ generated by $\{s_i\}_{i\in I}$. Then W is a Coxeter group with a canonical system of generators $\{s_i\}_{i\in I}$. We denote the length function by l and the Bruhat order of W by $\leqq$.

Set

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for } h \in \mathfrak{h}\} \qquad \text{for } \alpha \in \mathfrak{h}^*, \tag{1.8}$$

$$\Delta = \left\{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq \{0\}\right\} - \{0\}, \tag{1.9}$$

$$\Delta^+ = \Delta \cap \sum_{i\in I} \mathbb{Z}_{\geqq 0}\alpha_i, \tag{1.10}$$

$$\Delta_{re} = \bigcup_{i\in I} W(\alpha_i), \quad \Delta^+_{re} = \Delta^+ \cap \Delta_{re}, \tag{1.11}$$

and let $\mathfrak{n}$, $\mathfrak{n}^-$, $\mathfrak{b}$, $\mathfrak{b}^-$ be the subalgebras of $\mathfrak{g}$ generated by $\{e_i\}_{i\in I}$, $\{f_i\}_{i\in I}$, $\{e_i\}_{i\in I} \cup \mathfrak{h}$, $\{f_i\}_{i\in I} \cup \mathfrak{h}$, respectively. We then have

$$\Delta = \Delta^+ \cup (-\Delta^+), \quad W(\Delta) = \Delta, \tag{1.12}$$

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-, \tag{1.13}$$

$$\mathfrak{n} = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha}, \tag{1.14}$$

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+, \quad \mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-. \tag{1.15}$$

For $\lambda \in \mathfrak{h}^*$ let $M(\lambda)$, $N(\lambda)$ be the $\mathfrak{g}$-modules defined by

$$M(\lambda) = U(\mathfrak{g}) \Big/ \left(\sum_{h\in\mathfrak{h}} U(\mathfrak{g})\left(h - \lambda(h)\right) + \sum_{i\in I} U(\mathfrak{g})e_i\right), \tag{1.16}$$

$$N(\lambda) = U(\mathfrak{g}) \Big/ \left(\sum_{h\in\mathfrak{h}} U(\mathfrak{g})\left(h - \lambda(h)\right) + \sum_{i\in I} U(\mathfrak{g})f_i\right), \tag{1.17}$$

where $U(\mathfrak{g})$ denotes the enveloping algebra of $\mathfrak{g}$. Let $M^*(\lambda)$ be the $\mathfrak{g}$-module consisting of $\mathfrak{h}$-finite vectors in $\mathrm{Hom}_{\mathbb{C}}\left(N(-\lambda), \mathbb{C}\right)$. The irreducible $\mathfrak{g}$-module $L(\lambda)$ with highest weight λ is naturally isomorphic to the image of the unique nonzero homomorphism $M(\lambda) \to M^*(\lambda)$.

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2 The flag manifold

We fix a $\mathbb{Z}$-lattice P of $\mathfrak{h}^*$ satisfying

$$\alpha_i \in P, \quad \langle h_i, P\rangle \in \mathbb{Z} \qquad \text{for } i \in I. \tag{2.1}$$

For $\alpha = \sum_{i\in I} m_i\alpha_i \in \Delta^+$ set $|\alpha| = \sum_{i\in I} m_i$, and let

$$\mathfrak{n}_k = \bigoplus_{\substack{\alpha\in\Delta^+\\ |\alpha|\geqq k}} \mathfrak{g}_\alpha, \quad \mathfrak{n}_k^- = \bigoplus_{\substack{\alpha\in\Delta^+\\ |\alpha|\geqq k}} \mathfrak{g}_{-\alpha} \qquad \text{for } k \in \mathbb{Z}_{\geqq 0}. \tag{2.2}$$

Define group schemes by

$$T = \mathrm{Spec}(\mathbb{C}[P]), \tag{2.3}$$

$$U = \varprojlim_k \exp(\mathfrak{n}/\mathfrak{n}_k), \tag{2.4}$$

$$U^- = \varprojlim_k \exp(\mathfrak{n}^-/\mathfrak{n}_k^-), \tag{2.5}$$

$$B = (\text{the semidirect product of } T \text{ and } U), \tag{2.6}$$

$$B^- = (\text{the semidirect product of } T \text{ and } U^-). \tag{2.7}$$

Here, for a finite-dimensional nilpotent Lie algebra $\mathfrak{a}$, $\exp(\mathfrak{a})$ denotes the unipotent algebraic group with $\mathfrak{a}$ as its Lie algebra. We have natural isomorphisms

$$\exp : \prod_{\alpha\in\Delta^+} \mathfrak{g}_\alpha \longrightarrow U, \quad \exp : \prod_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha} \longrightarrow U^-. \tag{2.8}$$

In [K1] the first-named author constructed the flag variety of $(\mathfrak{g}, \mathfrak{h}, P)$ as the quotient

$$X = G/B, \tag{2.9}$$

where G is a scheme with locally free right action of B.

Let $\lambda \in P$. We denote the composite of the homomorphisms

$$B \longrightarrow T \xrightarrow{\lambda} \mathbb{G}_m$$

by $b \mapsto b^\lambda$. Let $\pi : G \to X$ be the canonical morphism. An invertible $\mathcal{O}_X$-module $\mathcal{O}_X(\lambda)$ is

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defined by

$$\Gamma\left(V; \mathcal{O}_X(\lambda)\right) = \left\{\varphi \in \Gamma(\pi^{-1}V; \mathcal{O}_G) \mid \varphi(gb) = \varphi(g)b^{-\lambda} \text{ for } (g, b) \in \pi^{-1}V \times B\right\} \tag{2.10}$$

for any open subset V of X.

We have natural left actions of the group scheme B^- and the braid group W' of W on X. Let Y be a T-stable subset of X. For $w' \in W'$, the set $w'Y$ depends only on the image $w \in W$ of w' under the canonical homomorphism $W' \to W$, and hence we simply denote it by wY.

Let B_0 be the subgroup of B generated by T and the elements of the form $\exp(x)$ with $x \in \mathfrak{g}_\alpha$, $\alpha \in \Delta^+_{re}$. Note that B_0 is not a scheme but an inductive limit of schemes. We have a natural left action of B_0 on X compatible with those of B^- and W'. Let $x_0 \in X$ be 1 mod B. Then x_0 is a unique B_0-fixed point. We set

$$X^w = B^- w x_0, \quad X_w = B_0 w x_0 \qquad \text{for } w \in W. \tag{2.11}$$

Proposition 1 [K1]. (i) X^w is a locally closed subscheme of X with codimension $l(w)$, and if $\mathfrak{g}$ is not of finite type, it is isomorphic to the infinite-dimensional affine space $\mathbb{A}^\infty$.

(ii) $X = \sqcup_{w \in W} X^w$.

(iii) $\overline{X^w} = \sqcup_{y \geqq w} X^y$.

Moreover we have the following results.

Proposition 2. (i) X_w is a locally closed subscheme of X isomorphic to the affine space $\mathbb{A}^{l(w)}$.

(ii) $\sqcup_{w \in W} X_w$ is naturally isomorphic to the "flag manifold" treated in [KP], [T], and others.

(iii) $\overline{X_w} = \sqcup_{y \leqq w} X_y$.

3 $\mathcal{D}$-modules

For a finite subset F of W set

$$X^F = \bigcup_{w \in F} X^w, \quad X_F = \bigcup_{w \in F} X_w. \tag{3.1}$$

By Propositions 1 and 2, the set X^F (resp. X_F) is open (resp. closed) if and only if F satisfies

$$w \in F, \quad y \in W, \qquad y \leqq w \Longrightarrow y \in F. \tag{3.2}$$

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A subset Y (resp. Z) of X is called an admissible open (resp. closed) subset if $Y = X^{\mathsf{F}}$ (resp. $Z = X_{\mathsf{F}}$) for a finite subset F of W satisfying (3.2).

Let Z be an admissible closed subset of X. In this case Z is projective. Take an admissible open subset Y of X containing Z. We can take such Y since we have $X^{\mathsf{F}} \supset X_{\mathsf{F}}$ if F satisfies (3.2). For $k \geqq 0$ set $U_k^- = \exp(\prod_{\substack{\alpha \in \Delta^+ \\ |\alpha| \geqq k}} \mathfrak{g}_{-\alpha}) \subset U^-$. If k is sufficiently large, then U_k^- acts on Y locally freely, and hence the quotient $U_k^- \backslash Y$ is a finite-dimensional smooth variety. We do not know if $U_k^- \backslash Y$ is separated or not. If k is large enough, the natural morphism $Z \to U_k^- \backslash Y$ is a closed immersion. Fix such Z, Y, and k. For $l \geqq k$ set

$$Y_l = U_l^- \backslash Y \tag{3.3}$$

and let

$$\pi_l : Y \longrightarrow Y_l \tag{3.4}$$

$$p_l : Y_{l+1} \longrightarrow Y_l \tag{3.5}$$

$$i_l : Z \hookrightarrow Y_l \tag{3.6}$$

be the natural morphisms. For $\lambda \in P$, we can naturally define an invertible $\mathcal{O}_{Y_l}$-module $\mathcal{O}_{Y_l}(\lambda)$ satisfying

$$\pi_l^* \mathcal{O}_{Y_l}(\lambda) \cong \mathcal{O}_X(\lambda) \mid Y \quad \text{and} \tag{3.7}$$

$$p_l^* \mathcal{O}_{Y_l}(\lambda) \cong \mathcal{O}_{Y_{l+1}}(\lambda). \tag{3.8}$$

Let $\mathcal{D}_{Y_l}$ be the sheaf of differential operators on Y_l, and set

$$\mathcal{D}_{Y_{l+1} \to Y_l} = \mathcal{O}_{Y_{l+1}} \otimes_{p_l^{-1}\mathcal{O}_{Y_l}} p_l^{-1} \mathcal{D}_{Y_l}, \tag{3.9}$$

$$\mathcal{D}_{Y_l}(\lambda) = \mathcal{O}_{Y_l}(-\lambda) \otimes_{\mathcal{O}_{Y_l}} \mathcal{D}_{Y_l} \otimes_{\mathcal{O}_{Y_l}} \mathcal{O}_{Y_l}(\lambda), \tag{3.10}$$

$$\mathcal{D}_{Y_{l+1} \to Y_l}(\lambda) = \mathcal{O}_{Y_{l+1}}(-\lambda) \otimes_{\mathcal{O}_{Y_{l+1}}} \mathcal{D}_{Y_{l+1} \to Y_l} \otimes_{p_l^{-1}\mathcal{O}_{Y_l}} p_l^{-1} \mathcal{O}_{Y_l}(\lambda). \tag{3.11}$$

Then $\mathcal{D}_{Y_l}(\lambda)$ is a ring acting on $\mathcal{O}_{Y_l}(-\lambda)$ from the left, and on $\Omega_{Y_l} \otimes \mathcal{O}_{Y_l}(\lambda)$ from the right. Here Ω_{Y_l} is the sheaf of differential forms of degree $\dim Y_l$. We have a natural $(\mathcal{D}_{Y_{l+1}}(\lambda), p_l^{-1}\mathcal{D}_{Y_l}(\lambda))$-bimodule structure on $\mathcal{D}_{Y_{l+1} \to Y_l}(\lambda)$.

Let $\mathbb{H}_l$ be the abelian category of right holonomic $\mathcal{D}_{Y_l}(\lambda)$-modules supported in Z. For $\mathcal{M} \in \mathbb{H}_l$ we define $\mathcal{M}^* \in \mathbb{H}_l$ by

$$\mathcal{M}^* = \Omega_{Y_l} \otimes \mathcal{O}_{Y_l}(2\lambda) \otimes \mathcal{E}xt^{\dim Y_l}_{\mathcal{D}_{Y_l}(\lambda)}\left(\mathcal{M}, \mathcal{D}_{Y_l}(\lambda)\right). \tag{3.12}$$

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This defines a contravariant exact functor $* : \mathbb{H}_l \to \mathbb{H}_l$ such that $** = \mathrm{id}$. For $\mathcal{M} \in \mathbb{H}_{l+1}$ we define $\int_{p_l} \mathcal{M} \in \mathbb{H}_l$ by

$$\int_{p_l} \mathcal{M} = (p_l)_* \left(\mathcal{M} \otimes_{\mathcal{D}_{Y_{l+1}(\lambda)}} \mathcal{D}_{Y_{l+1} \to Y_l}(\lambda) \right). \tag{3.13}$$

Since $i_{l+1}(Z) \xrightarrow{\sim} i_l(Z)$, $\int_{p_l}$ induces an equivalence $\mathbb{H}_{l+1} \xrightarrow{\sim} \mathbb{H}_l$.

We define an abelian category $\mathbb{H} = \mathbb{H}(Z, \lambda, Y, k)$ as follows. An object is a family $\mathcal{M} = (\mathcal{M}_l)_{l \geqq k} \in \prod_{l \geqq k} \mathrm{Ob}(\mathbb{H}_l)$ together with isomorphisms

$$r_l : \int_{p_l} \mathcal{M}_{l+1} \xrightarrow{\sim} \mathcal{M}_l, \tag{3.14}$$

and morphisms are given by

$$\begin{aligned} &\mathrm{Hom}_{\mathbb{H}}(\mathcal{M}, \mathcal{N}) \\ &= \left\{ (\varphi_l) \in \prod_{l \geqq k} \mathrm{Hom}_{\mathbb{H}_l}(\mathcal{M}_l, \mathcal{N}_l) \;\middle|\; \varphi_l \circ r_l = r_l \circ \left(\int_{p_l} \varphi_{l+1} \right) \text{ for } l \geqq k \right\}. \end{aligned} \tag{3.15}$$

Note that $\mathbb{H}$ is equivalent to $\mathbb{H}_l$. The duality functor $* : \mathbb{H} \to \mathbb{H}$ is defined by

$$\mathcal{M}^* = (\mathcal{M}_l^*)_{l \geqq k}. \tag{3.16}$$

Since $\int_{p_l}$ commutes with the duality, this is well defined. It is a contravariant exact functor such that $** = \mathrm{id}$. Note that the category $\mathbb{H}(Z, \lambda, Y, k)$ does not depend on Y and k. Moreover for $\lambda, \lambda' \in P$, $\mathbb{H}(Z, \lambda, Y, k)$ and $\mathbb{H}(Z, \lambda', Y, k)$ are equivalent by $\mathcal{M}_l \mapsto \mathcal{M}_l \otimes \mathcal{O}_{Y_l}(\lambda' - \lambda)$.

For $\mathcal{M} \in \mathbb{H}$ the homomorphism $\mathcal{O}_{Y_{l+1}} \to \mathcal{D}_{Y_{l+1} \to Y_l}(\lambda)$ induces a homomorphism $p_{l*}\mathcal{M}_{l+1} \to \mathcal{M}_l$, which gives a homomorphism

$$H^i(Y_{l+1}; \mathcal{M}_{l+1}) \longrightarrow H^i(Y_l; \mathcal{M}_l) \qquad \text{for } i \geqq 0. \tag{3.17}$$

Set

$$H^i(Y; \mathcal{M}) = \varprojlim_l H^i(Y_l; \mathcal{M}_l) \qquad \text{for } i \geqq 0. \tag{3.18}$$

Proposition 3. For $\mathcal{M} \in \mathbb{H}$, $H^i(Y; \mathcal{M})$ carries a natural structure of a $\mathfrak{g}$-module.

There is no homomorphism from $\mathfrak{g}$ to $\mathcal{D}_{Y_l}(\lambda)$ because there is no action of $\mathfrak{g}$ on Y_l. However, we can construct a section of $\mathcal{D}_{Y_{l+m} \to Y_l}(\lambda)$ corresponding to $A \in \mathfrak{g}$ if $m \gg 0$. This induces an action of $\mathfrak{g}$ on $H^i(Y; \mathcal{M})$.

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For $w \in W$ such that $X_w \subset Z$, we define $\mathcal{B}_w, \mathcal{M}_w, \mathcal{L}_w \in \mathrm{Ob}(\mathbb{H})$ by

$$(\mathcal{B}_w)_l = \mathcal{H}^{\mathrm{codim}\, i_l(X_w)}_{i_l(X_w)}\left(\Omega_{Y_l} \otimes_{\mathcal{O}_{Y_l}} \mathcal{O}_{Y_l}(\lambda)\right), \tag{3.19}$$

$$\mathcal{M}_w = \mathcal{B}_w^*, \tag{3.20}$$

$$\mathcal{L}_w = \text{the image of the natural morphism } \mathcal{M}_w \to \mathcal{B}_w. \tag{3.21}$$

Then $\mathcal{L}_w$ is a simple object of $\mathbb{H}$ satisfying $\mathcal{L}_w^* \cong \mathcal{L}_w$.

Let $\mathbb{H}_0 = \mathbb{H}_0(Z, \lambda, Y, k)$ be the full subcategory of $\mathbb{H}$ consisting of $\mathcal{M} \in \mathbb{H}$ whose composition factors are isomorphic to $\mathcal{L}_w$ for some $w \in W$ satisfying $X_w \subset Z$.

4 The main result

We fix $\rho \in \mathfrak{h}^*$ satisfying

$$\langle h_i, \rho \rangle = 1 \qquad \text{for any } i \in I. \tag{4.1}$$

Define a shifted action of the Weyl group W on P by

$$w \circ \lambda = w(\lambda + \rho) - \rho. \tag{4.2}$$

Assume that $\lambda \in P$ satisfies the following:

$$\langle h_i, \lambda \rangle < -1 \qquad \text{for any } i \in I; \tag{4.3}$$

$$\text{the Verma module } M(\lambda) \text{ is an irreducible } \mathfrak{g}\text{-module.} \tag{4.4}$$

We keep the notation of Section 3. Let $F \subset W$ be such that $Z = X_F$. Then we have the following theorem.

Theorem.

(i) $H^n(Y; \mathcal{M}) = 0$ for any $\mathcal{M} \in \mathbb{H}_0$ and $n > 0$;

(ii) $H^0(Y; *)$ defines an exact functor from $\mathbb{H}_0$ to the category of $U(\mathfrak{g})$-modules;

(iii) $H^0(Y; \mathcal{B}_w) = \hat{M}^*(w \circ \lambda)$ for any $w \in F$;

(iv) $H^0(Y; \mathcal{M}_w) = \hat{M}(w \circ \lambda)$ for any $w \in F$;

(v) $H^0(Y; \mathcal{L}_w) = \hat{L}(w \circ \lambda)$ for any $w \in F$.

Here $\hat{M}^*(w \circ \lambda)$ is the completion of $M^*(w \circ \lambda)$, etc.

By [KK] the condition (4.4) is satisfied if $\mathfrak{g}$ is an affine Lie algebra and if λ satisfies (4.3). The conjecture of Lusztig is easily derived from this theorem along with the standard arguments.

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The proof of this theorem is accomplished by showing the following statements by induction on the dimension of Z.

$$\text{For } \mathcal{M} \in \mathbb{H}_0,\ \mu \in P, \text{ and } n \in \mathbb{Z}, \text{ the weight space } H^n(Y_l; \mathcal{M}_l)_\mu \text{ is constant for } l \gg k. \tag{4.5}$$

This implies that $H^n(Y;\mathcal{M})$ is isomorphic to $\prod_\mu \left(\varprojlim_l H^n(Y_l;\mathcal{M}_l)_\mu\right)$. We denote $\oplus_\mu\left(\varprojlim_l H^n(Y_l;\mathcal{M}_l)\right)_\mu$ by $\bar{H}^n(Y;\mathcal{M})$. Hence $\bar{H}^n(Y;\mathcal{M})$ is the set of $\mathfrak{h}$-finite vectors of $H^n(Y;\mathcal{M})$.

$$\bar{H}^n(Y;\mathcal{M}) = 0 \qquad \text{for } \mathcal{M} \in \mathbb{H}_0 \text{ and } n > 0. \tag{4.6}$$

$$\mathrm{Hom}_{\mathbb{H}_0}(\mathcal{M},\mathcal{N}) \simeq \mathrm{Hom}_{\mathfrak{g}}\left(\bar{H}^0(Y;\mathcal{M}), \bar{H}^0(Y;\mathcal{N})\right) \qquad \text{for } \mathcal{M},\mathcal{N} \in \mathbb{H}_0. \tag{4.7}$$

$$\bar{H}^0(Y;\mathcal{M}) \otimes \mathcal{D}(\lambda) = \mathcal{M} \qquad \text{for } \mathcal{M} \in \mathbb{H}_0. \tag{4.8}$$

$$\text{Let } \mathcal{M} \in \mathbb{H}_0 \text{ and let } N \text{ be a } U(\mathfrak{g})\text{-submodule of } \bar{H}^0(Y;\mathcal{M}). \text{ If } N \otimes \mathcal{D}(\lambda) = \mathcal{M}, \text{ then } N = \bar{H}^0(Y;\mathcal{M}). \tag{4.9}$$

$$\bar{H}^0(Y;\mathcal{B}_w) = M^*(w\circ\lambda) \qquad \text{for } w \in F. \tag{4.10}$$

$$\bar{H}^0(Y;\mathcal{M}_w) = M(w\circ\lambda) \qquad \text{for } w \in F. \tag{4.11}$$

$$\bar{H}^0(Y;\mathcal{L}_w) = L(w\circ\lambda) \qquad \text{for } w \in F. \tag{4.12}$$

In (4.8) and (4.9), for a $U(\mathfrak{g})$-submodule N of $\bar{H}^0(Y;\mathcal{M})$, $N \otimes \mathcal{D}(\lambda)$ denotes the subobject of $\mathcal{M}$ "generated by" N. We remark that we do not know how to construct the functor $M \mapsto M\otimes D(\lambda)$ from a category of $\mathfrak{g}$-modules to $\mathbb{H}_0$, but there is no problem in constructing $N \otimes \mathcal{D}(\lambda) \subset \mathcal{M}$ for $N \subset \bar{H}^0(Y;\mathcal{M})$.

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