Invent. math. 77, 185–198 (1984)

Inventiones mathematicae

The characteristic cycles of holonomic systems on a flag manifold

related to the Weyl group algebra

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1. Introduction

1.1. Joseph [10] considered the characteristic varieties of the highest weight modules of a complex semisimple Lie algebra, and Hotta [7] gave a proof of a conjecture of Joseph in [10] concerning the so-called Springer representations of the Weyl group. These results give a relation between the characteristic varieties of the simple highest weight modules and the representations of the Weyl group (see Sect. 6).

On the other hand the so-called Kazhdan-Lusztig conjecture [15] concerning the characters of simple highest weight modules was proved in Brylinski-Kashiwara [6] and Beilinson-Bernstein [3] using the holonomic systems on the flag manifold associated to the simple highest weight modules (see Sect. 2).

The purpose of this paper is to give a relation between the characteristic cycles of these holonomic systems and the representations of the Weyl group.

1.2. Let G be a connected, simply-connected, complex semisimple algebraic group and B a Borel subgroup of G. We denote the half of the sum of the positive roots by ρ and the Weyl group by W. For $w \in W$ let M_w be the Verma module with highest weight $-w(\rho)-\rho$ and L_w its simple quotient. We denote by $\mathfrak{M}_w$ and $\mathfrak{L}_w$ the regular holonomic systems on the flag manifold $\mathscr{B}=G/B$ corresponding to M_w and L_w, respectively (see Sect. 2). The characteristic variety $\mathrm{Ch}(L_w)$ of L_w is a subset of the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G, and the characteristic variety $\mathrm{Ch}(\mathfrak{L}_w)$ of $\mathfrak{L}_w$ is a subset of the cotangent bundle $T^*\mathscr{B}$. Under the natural map $T^*\mathscr{B} \xrightarrow{\gamma} \mathfrak{g}^*$ Brylinski showed that $\gamma(\mathrm{Ch}(\mathfrak{L}_w)) = \mathrm{Ch}(L_w)$ (see Proposition 2). Thus $\mathrm{Ch}(\mathfrak{L}_w)$ has more information than $\mathrm{Ch}(L_w)$. Each irreducible component of $\mathrm{Ch}(\mathfrak{L}_w)$ is the closure of the conormal bundle $T^*_{\mathscr{B}_y}\mathscr{B}$ of a Schubert cell $\mathscr{B}_y = ByB/B$ for some $y \in W$. Let $\tilde{\mathscr{M}}$ be the abelian category consisting of the regular holonomic systems on $\mathscr{B}$ whose characteristic varieties are contained in $\coprod_{w \in W} T^*_{\mathscr{B}_w}\mathscr{B}$. Then the Grothendieck group $K(\tilde{\mathscr{M}})$ is decomposed into the direct sums:

$$K(\tilde{\mathscr{M}})=\bigoplus_{w\in W}\mathbb{Z}[\mathfrak{M}_w]=\bigoplus_{w\in W}\mathbb{Z}[\mathfrak{L}_w].$$

We consider not only the characteristic variety of $\mathfrak{M}\in\tilde{\mathscr{M}}$ but also the multiplicities $m_w(\mathfrak{M})$ along the irreducible components $\overline{T^*_{\mathscr{B}_w}\mathscr{B}}$, that is the algebraic cycle:

$$\mathbf{Ch}(\mathfrak{M})=\sum_{w\in W} m_w(\mathfrak{M})\,[\overline{T^*_{\mathscr{B}_w}\mathscr{B}}],$$

which is called the characteristic cycle of $\mathfrak{M}$. Then $\mathbf{Ch}$ defines an additive map from $K(\tilde{\mathscr{M}})$ into the group of the algebraic cycles of $T^*\mathscr{B}$. Let h be the $\mathbb{Z}$-linear isomorphism from $K(\tilde{\mathscr{M}})$ onto the group ring $\mathbb{Z}[W]$ determined by $h([\mathfrak{M}_w])=w$. There exists a unique basis $\{\mathbf{c}(w)\}_{w\in W}$ of $\mathbb{Z}[W]$ so that $h([\mathfrak{M}])=\sum_{w\in W} m_w(\mathfrak{M})\,\mathbf{c}(w)$ for any $\mathfrak{M}\in\tilde{\mathscr{M}}$, and hence $\mathbf{Ch}(h^{-1}(\mathbf{c}(w))=[\overline{T^*_{\mathscr{B}_w}\mathscr{B}}]$.

In the studies of the Springer representations of the Weyl group, Kazhdan-Lusztig [17] defined a basis of $\mathbb{Z}[W]$ which is also parametrized by W. Our main theorem is that this basis of Kazhdan-Lusztig coincides with our $\{\mathbf{c}(w)\}_{w\in W}$ (Theorem 6). This theorem was conjectured by the second-named author in [24].

1.3. Set $\mathbf{a}(w)=h([\mathfrak{L}_w])=\sum_{y\in W} m_y(\mathfrak{L}_w)\,\mathbf{c}(y)$. The basis $\{\mathbf{a}(w)\}_{w\in W}$ is related to the left cell representations of the Weyl group, which arose from the studies of the primitive ideals of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ (see Sect. 6). Hence our theorem gives a relation between the two bases $\{\mathbf{a}(w)\}_{w\in W}$ and $\{\mathbf{c}(w)\}_{w\in W}$ of $\mathbb{Z}[W]$, which are related to the left cell representations and the Springer representations of W respectively, in view of the holonomic systems on the flag manifold associated to the simple highest weight modules. As an application to our theorem we give some relations between the left cell representations and the Springer representations using the results of Joseph and Hotta.

We remark that the results of Joseph and Hotta together with an unpublished result of Borho-Brylinski give a unified proof of the irreducibility of the associated varieties of the primitive ideals of $U(\mathfrak{g})$, which was proved by Borho-Brylinski [4] using case-by-case computations (see the remark at the end of Sect. 6). We learned that Joseph also gave the similar proof.

It is desirable to find an algorithm for computing the multiplicities $m_y(\mathfrak{L}_w)$, or at least $\mathrm{Ch}(\mathfrak{L}_w)$. Since the Weyl group as a Coxeter group does not determine them (see the example in 5.4), we must take the root system into account.

1.4. The authors express their hearty thanks to Professor R. Hotta for leading them to this problem and making many valuable suggestions in private communications. They also thank Professor J.-L. Brylinski for useful advices. The second-named author thanks Professor N. Iwahori for kind encouragements.

2. Review on the results of Brylinski-Kashiwara [6] and Beilinson-Bernstein [3]

2.1 Let G be a connected, simply-connected, semisimple algebraic group over the complex number field $\mathbb{C}$. We fix a Borel subgroup B of G and a maximal torus H of G contained in B. We denote the Lie algebras of G, B and H by $\mathfrak{g}$, $\mathfrak{b}$ and $\mathfrak{h}$, respectively. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and $\mathscr{D}_{\mathscr{B}}$ the sheaf of the algebraic differential operators on the flag manifold $\mathscr{B}=G/B$. $U(\mathfrak{g})$ and $\mathscr{D}_{\mathscr{B}}$ have natural increasing filtrations:

$$0\subset U_0(\mathfrak{g})\subset U_1(\mathfrak{g})\subset\ldots\subset U(\mathfrak{g})\qquad (U(\mathfrak{g})=\bigcup_i U_i(\mathfrak{g})),$$

$$0\subset \mathscr{D}_{\mathscr{B},0}\subset \mathscr{D}_{\mathscr{B},1}\subset\ldots\subset \mathscr{D}_{\mathscr{B}}\qquad (\mathscr{D}_{\mathscr{B}}=\bigcup_i \mathscr{D}_{\mathscr{B},i}).$$

The natural action of G on $\mathscr{B}$ induces an algebra homomorphism $U(\mathfrak{g})\xrightarrow{D}\Gamma(\mathscr{B},\mathscr{D}_{\mathscr{B}})$ and a linear map $U_i(\mathfrak{g})\xrightarrow{D_i}\Gamma(\mathscr{B},\mathscr{D}_{\mathscr{B},i})$. Let $\mathfrak{z}$ be the center of $U(\mathfrak{g})$.

Proposition 1 (Beilinson-Bernstein [3]). *D_i is surjective with* $\operatorname{Ker} D_i=U_i(\mathfrak{g})\cap U(\mathfrak{g})(\mathfrak{g}U(\mathfrak{g})\cap\mathfrak{z})$. *Thus D is surjective with* $\operatorname{Ker} D=U(\mathfrak{g})(\mathfrak{g}U(\mathfrak{g})\cap\mathfrak{z})$.

We set $R=U(\mathfrak{g})/\operatorname{Ker} D=\Gamma(\mathscr{B},\mathscr{D}_{\mathscr{B}})$. For a finitely generated R-module M, we have a coherent $\mathscr{D}_{\mathscr{B}}$-Module $\mathscr{D}_{\mathscr{B}}\otimes_R M$.

Theorem 1 (Beilinson-Bernstein [3]). (i) *Let $\mathfrak{M}$ be a coherent $\mathscr{D}_{\mathscr{B}}$-Module. Then $\mathfrak{M}$ is generated by its global sections* (i.e. $\mathscr{D}_{\mathscr{B}}\otimes_R\Gamma(\mathscr{B},\mathfrak{M})\cong\mathfrak{M}$) *and* $H^j(\mathscr{B},\mathfrak{M})=0$ *for* $j>0$.

(ii) $\mathscr{D}_{\mathscr{B}}\otimes_R(\cdot)$ *gives an equivalence between the abelian category of finitely generated R-modules and that of coherent $\mathscr{D}_{\mathscr{B}}$-Modules.*

2.2. Let Δ be the root system of $(\mathfrak{g},\mathfrak{h})$. For a root $\alpha\in\Delta$ we denote by $\mathfrak{g}_\alpha$ the root space. We choose a positive root system Δ^+ so that $\mathfrak{b}=\mathfrak{h}+\sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$ and set $\rho=(\sum_{\alpha\in\Delta^+}\alpha)/2\in\mathfrak{h}^*$. Let W be the Weyl group of $(\mathfrak{g},\mathfrak{h})$. For $w\in W$ we denote by M_w and L_w the Verma module with highest weight $-w(\rho)-\rho$ and its simple quotient, respectively. Let $\tilde{\mathcal{O}}_0$ be the abelian category of finitely generated R-modules which are locally finite as $U(\mathfrak{b})$-modules. The Grothendieck group $K(\tilde{\mathcal{O}}_0)$ is decomposed into the direct sums:

$$K(\tilde{\mathcal{O}}_0)=\bigoplus_{w\in W}\mathbb{Z}[M_w]=\bigoplus_{w\in W}\mathbb{Z}[L_w].$$

We set $\mathscr{B}_w=BwB/B$. Then $\mathscr{B}_w$ is isomorphic to the $l(w)$-dimensional affine space where $l(w)$ is the length of w with respect to the simple reflections, and $\mathscr{B}=\coprod_{w\in W}\mathscr{B}_w$ gives a cellular decomposition of $\mathscr{B}$. $\mathscr{B}_w$ is called a Schubert cell and its closure $\overline{\mathscr{B}_w}$ is called a Schubert variety. We denote by $\tilde{\mathscr{M}}$ the abelian category consisting of regular holonomic $\mathscr{D}_{\mathscr{B}}$-Modules $\mathfrak{M}$ whose characteristic varieties $\operatorname{Ch}(\mathfrak{M})$ are contained in $\coprod_{w\in W} T^*_{\mathscr{B}_w}\mathscr{B}$, where $T^*_{\mathscr{B}_w}\mathscr{B}$ is the conormal bundle of $\mathscr{B}_w$ in $\mathscr{B}$. Set $d=\dim\mathscr{B}$.

Theorem 2 (Brylinski-Kashiwara [6], Beilinson-Bernstein [3]). (i) *For* $M \in \tilde{\mathcal{O}}_0$ $\mathcal{D}_{\mathcal{B}} \otimes_R M$ *belongs to* $\tilde{\mathcal{M}}$ *and this gives an equivalence between* $\tilde{\mathcal{O}}_0$ *and* $\tilde{\mathcal{M}}$.

(ii) $\mathfrak{M}_w := \mathcal{D}_{\mathcal{B}} \otimes_R M_w$ *is isomorphic to* $(\mathcal{H}^{d-l(w)}_{\mathcal{B}_w}(\mathcal{O}_{\mathcal{B}}))^*$, *where* $\mathcal{O}_{\mathcal{B}}$ *is the structure sheaf of the algebraic variety* $\mathcal{B}$.

(iii) $\mathfrak{L}_w := \mathcal{D}_{\mathcal{B}} \otimes_R L_w$ *is the minimal extension of* $\mathfrak{M}_w | \mathcal{B} - \partial\mathcal{B}_w$ *to* $\mathcal{B}$ *as a regular holonomic system, where* $\partial\mathcal{B}_w = \overline{\mathcal{B}_w} - \mathcal{B}_w$.

Let $\mathcal{O}_{\mathcal{B}_{\mathrm{an}}}$ be the sheaf of the holomorphic functions on $\mathcal{B}$. For $\mathfrak{M} \in \tilde{\mathcal{M}}$ we set $\mathcal{DR}(\mathfrak{M}) = \mathbb{R}\mathcal{H}om_{\mathcal{D}_{\mathcal{B}_{\mathrm{an}}}}(\mathcal{O}_{\mathcal{B}_{\mathrm{an}}}, \mathfrak{M}_{\mathrm{an}})$, where $\mathcal{D}_{\mathcal{B}_{\mathrm{an}}} = \mathcal{O}_{\mathcal{B}_{\mathrm{an}}} \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{D}_{\mathcal{B}}$ and $\mathfrak{M}_{\mathrm{an}} = \mathcal{D}_{\mathcal{B}_{\mathrm{an}}} \otimes_{\mathcal{D}_{\mathcal{B}}} \mathfrak{M}$ (see Kashiwara [14]). $\mathcal{DR}(\mathfrak{M})$ belongs to the abelian category $\mathcal{F}$ consisting of the perverse complexes on $\mathcal{B}$ whose cohomology sheaves are constant on each $\mathcal{B}_w$.

Theorem 3 (Brylinski-Kashiwara [6], Beilinson-Bernstein [3]). (i) $\mathcal{DR}$ *gives a category equivalence between* $\tilde{\mathcal{M}}$ *and* $\mathcal{F}$.

(ii) $\mathcal{DR}(\mathfrak{M}_w) = \mathbb{C}_{\mathcal{B}_w}[-(d-l(w))]$, *where* $\mathbb{C}_{\mathcal{B}_w}$ *is the constructible sheaf whose stalks are* 0 *on* $\mathcal{B} - \mathcal{B}_w$ *and* $\mathbb{C}$ *on* $\mathcal{B}_w$.

(iii) $\mathcal{DR}(\mathfrak{L}_w) = {}^{\pi}\mathbb{C}_{\mathcal{B}_w}[-(d-l(w))]$, *where* ${}^{\pi}\mathbb{C}_{\mathcal{B}_w}$ *is the Deligne-Goresky-MacPherson extension of* $\mathbb{C}_{\mathcal{B}_w}$ *to* $\overline{\mathcal{B}_w}$.

2.3. For $\mathfrak{M} \in \tilde{\mathcal{M}}$ and $w \in W$ we set $\chi_w(\mathfrak{M}) = \sum_i (-1)^i \dim \mathcal{H}^i(\mathcal{DR}(\mathfrak{M}))_{wB}$ and $h(\mathfrak{M}) = \sum_{w \in W} (-1)^{d-l(w)} \chi_w(\mathfrak{M}) w \in \mathbb{Z}[W]$. Then $h(\mathfrak{M}_w) = w$ and h induces an isomorphism $K(\tilde{\mathcal{M}}) \xrightarrow{h} \mathbb{Z}[W]$. Set $\mathbf{a}(w) = h(\mathfrak{L}_w)$. From a theorem of Kazhdan-Lusztig [16] we have:

$$\mathbf{a}(w) = \sum_{y \leqq w} (-1)^{l(w)+l(y)} P_{y,w}(1)\, y,$$

where $\leqq$ is the Bruhat order on W and $P_{y,w}$ is the Kazhdan-Lusztig polynomial defined in [15]. Thus we have the following theorem.

Theorem 4 (Kazhdan-Lusztig conjecture [15], Brylinski-Kashiwara [6], Beilinson-Bernstein [3]). *In the Grothendieck group* $K(\tilde{\mathcal{O}}_0)$ *we have:*

$$[L_w] = \sum_{y \leqq w} (-1)^{l(w)+l(y)} P_{y,w}(1)\, [M_y].$$

2.4. The decomposition of $\mathcal{B} \times \mathcal{B}$ to G-orbits is given by $\mathcal{B} \times \mathcal{B} = \coprod_{w \in W} \mathcal{Y}_w$ with $\mathcal{Y}_w = G \cdot (\{eB\} \times \mathcal{B}_w)$. Let us denote by $\mathcal{M}$ the abelian category consisting of the regular holonomic $\mathcal{D}_{\mathcal{B} \times \mathcal{B}}$-Modules whose characteristic varieties are contained in $\coprod_{w \in W} T^*_{\mathcal{Y}_w}(\mathcal{B} \times \mathcal{B})$. Then the functor $\mathcal{M} \to \tilde{\mathcal{M}}$ $(\mathfrak{M} \mapsto \mathfrak{M} | \{eB\} \times \mathcal{B})$ induces an isomorphism $K(\mathcal{M}) \to K(\tilde{\mathcal{M}})$. The homomorphism $K(\mathcal{M}) \to \mathbb{Z}[W]$ induced by $h: K(\tilde{\mathcal{M}}) \to \mathbb{Z}[W]$ is also denoted by h. We have a multiplicative structure on $K(\mathcal{M})$ by

$$[\mathfrak{M}_1] \circ [\mathfrak{M}_2] = \sum_i (-1)^i \left[\int^i_{p_{13}} ((p^*_{12}\mathfrak{M}_1) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{\mathcal{B} \times \mathcal{B} \times \mathcal{B}}} (p^*_{23}\mathfrak{M}_2)) \right],$$

where p_{ij}: $\mathscr{B}\times\mathscr{B}\times\mathscr{B}\to\mathscr{B}\times\mathscr{B}$ are the obvious projections. This induces a ring structure on $K(\mathscr{M})$ so that h is a ring homomorphism (see Lusztig-Vogan [19] and Springer [22]).

3. Characteristic cycles

3.1. We denote the multiplicity of a holonomic system $\mathfrak{M}\in\tilde{\mathscr{M}}$ along $\overline{T^*_{\mathscr{B}_w}\mathscr{B}}$ by $m_w(\mathfrak{M})$. Then the characteristic cycle $\mathbf{Ch}(\mathfrak{M})$ of $\mathfrak{M}$ is by definition the algebraic cycle $\sum_{w\in W} m_w(\mathfrak{M})[\overline{T^*_{\mathscr{B}_w}\mathscr{B}}]$.

Lemma 1. (i) $\mathbf{Ch}(\mathfrak{M}_w)=\sum_{y\leqq w} d_{y,w}[\overline{T^*_{\mathscr{B}_y}\mathscr{B}}]$. ($d_{y,w}>0$, $d_{w,w}=1$.)

(ii) $\mathbf{Ch}(\mathfrak{L}_w)=\sum_{y\leqq w} e_{y,w}[\overline{T^*_{\mathscr{B}_y}\mathscr{B}}]$. ($e_{y,w}\geqq 0$, $e_{w,w}=1$.)

Proof. Since $\mathrm{Supp}(\mathfrak{M}_w)=\mathrm{Supp}(\mathfrak{L}_w)=\overline{\mathscr{B}_w}$, $\mathbf{Ch}(\mathfrak{M}_w)$ and $\mathbf{Ch}(\mathfrak{L}_w)$ are non-negative $\mathbb{Z}$-linear combinations of $[\overline{T^*_{\mathscr{B}_y}\mathscr{B}}]$ for $y\leqq w$. We have $\mathfrak{M}_w|\mathscr{B}-\partial\mathscr{B}_w=\mathfrak{L}_w|\mathscr{B}-\partial\mathscr{B}_w=\mathscr{H}^{d-l(w)}_{\mathscr{B}_w}(\mathscr{O}_{\mathscr{B}-\partial\mathscr{B}_w})$ because $\mathscr{B}_w$ is a closed submanifold of $\mathscr{B}-\partial\mathscr{B}_w$. Thus $d_{w,w}=e_{w,w}=1$. $d_{y,w}>0$ follows from the fact that $\mathfrak{L}_y$ is a subquotient of $\mathfrak{M}_w$ for $y\leqq w$.

Thus we have an isomorphism $K(\tilde{\mathscr{M}})\xrightarrow{\mathbf{Ch}}\bigoplus_{w\in W}\mathbb{Z}[\overline{T^*_{\mathscr{B}_w}\mathscr{B}}]$. Similarly we can define $K(\mathscr{M})\xrightarrow{\mathbf{Ch}}\bigoplus_{w\in W}\mathbb{Z}[\overline{T^*_{\mathscr{Y}_w}(\mathscr{B}\times\mathscr{B})}]$ and we have a commutative diagram:

$$\begin{array}{ccc} K(\mathscr{M}) & \xrightarrow{\mathbf{Ch}} & \bigoplus_{w\in W}\mathbb{Z}[\overline{T^*_{\mathscr{Y}_w}(\mathscr{B}\times\mathscr{B})}] \\ \downarrow & & \downarrow \\ K(\tilde{\mathscr{M}}) & \xrightarrow{\mathbf{Ch}} & \bigoplus_{w\in W}\mathbb{Z}[\overline{T^*_{\mathscr{B}_w}\mathscr{B}}]. \end{array}$$

Here the right vertical arrow is given by $[\overline{T^*_{\mathscr{Y}_w}(\mathscr{B}\times\mathscr{B})}]\mapsto[\overline{T^*_{\mathscr{B}_w}\mathscr{B}}]$.

Set $\mathbf{c}(w)=h\circ\mathbf{Ch}^{-1}([\overline{T^*_{\mathscr{B}_w}\mathscr{B}}])$. The following is obvious.

Lemma 2. *If* $\mathbf{Ch}(\mathfrak{M})=\sum_w n_w[\overline{T^*_{\mathscr{B}_w}\mathscr{B}}]$ *for* $\mathfrak{M}\in\tilde{\mathscr{M}}$, *then* $h(\mathfrak{M})=\sum_w n_w\mathbf{c}(w)$.

Remark. By the index theorem of holonomic systems (see Kashiwara [12], [13]) we have:

$$\mathbf{c}(w)=\sum_{y\leqq w}(-1)^{l(w)+l(y)}c_{yB}(\overline{\mathscr{B}_w})\,y,$$

where $c_{yB}(\overline{\mathscr{B}_w})$ is the local characteristic of $\overline{\mathscr{B}_w}$ at yB defined in [12]. We do not use this fact in the following.

3.2. For a finitely generated $U(\mathfrak{g})$-module M we can define the characteristic variety $\mathrm{Ch}(M)$ $(\subset\mathfrak{g}^*)$ of M just like the characteristic variety of a coherent $\mathscr{D}$-

Module. We identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by the Killing form. Let $\mathcal{N}$ be the subvariety of $\mathfrak{g}$ consisting of the nilpotent elements. Let $\mathfrak{n}$ be the nilpotent radical of $\mathfrak{b}$, that is, $\mathfrak{n}=[\mathfrak{b},\mathfrak{b}]=\sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$. The cotangent bundle $T^*\mathscr{B}$ can be identified with $\{(gB,x)\in\mathscr{B}\times\mathcal{N}\mid g^{-1}x\in\mathfrak{n}\}$ by the Killing form. Then the natural map $T^*\mathscr{B}\xrightarrow{\gamma}\mathcal{N}$ gives a resolution of the singularity of $\mathcal{N}$, which is called the resolution of Grothendieck-Springer.

The following proposition is due to Brylinski. We include here its proof, which is also due to him, because of the absence of this result in the literature.

Proposition 2 (Brylinski). $\mathrm{Ch}(\Gamma(\mathscr{B},\mathfrak{M}))=\gamma(\mathrm{Ch}(\mathfrak{M}))$ for a coherent $\mathscr{D}_{\mathscr{B}}$-Module $\mathfrak{M}$.

Proof. For a good filtration $\{\mathfrak{M}_k\}_{k\in\mathbb{Z}}$ we set $M=\Gamma(\mathscr{B},\mathfrak{M})$ and $M_k=\Gamma(\mathscr{B},\mathfrak{M}_k)$. It is sufficient to show that $\{M_k\}_{k\in\mathbb{Z}}$ is a good filtration of M, that is, $\mathrm{gr}\,M=\bigoplus_{k\in\mathbb{Z}}(M_k/M_{k-1})$ is finitely generated over $\mathrm{gr}\,U(\mathfrak{g})=S(\mathfrak{g})\cong\mathbb{C}[\mathfrak{g}]$.

Since

$$0\to\mathfrak{M}_{k-1}\to\mathfrak{M}_k\to\mathfrak{M}_k/\mathfrak{M}_{k-1}\to 0$$

is exact,

$$0\to M_{k-1}\to M_k\to\Gamma(\mathscr{B},\mathfrak{M}_k/\mathfrak{M}_{k-1})$$

is also exact. Hence we have

$$\mathrm{gr}\,M=\bigoplus_{k\in\mathbb{Z}}M_k/M_{k-1}\subset\bigoplus_{k\in\mathbb{Z}}\Gamma(\mathscr{B},\mathfrak{M}_k/\mathfrak{M}_{k-1})=\Gamma(\mathscr{B},\mathrm{gr}\,\mathfrak{M}).$$

Since $\mathrm{gr}\,\mathfrak{M}$ is a coherent $\mathrm{gr}\,\mathscr{D}_{\mathscr{B}}$-Module, $\tilde{\mathfrak{M}}=\mathscr{O}_{T^*\mathscr{B}}\otimes_{p^{-1}(\mathrm{gr}\,\mathscr{D}_{\mathscr{B}})}p^{-1}(\mathrm{gr}\,\mathfrak{M})$ is a coherent $\mathscr{O}_{T^*\mathscr{B}}$-Module, where $T^*\mathscr{B}\xrightarrow{p}\mathscr{B}$ is the natural projection. Thus $\gamma_*(\tilde{\mathfrak{M}})$ is a coherent $\mathscr{O}_{\mathfrak{g}}$-Module because γ is proper. Hence $\Gamma(\mathscr{B},\mathrm{gr}\,\mathfrak{M})=\Gamma(T^*\mathscr{B},\tilde{\mathfrak{M}})=\Gamma(\mathfrak{g},\gamma^*(\tilde{\mathfrak{M}}))$ and its submodule $\mathrm{gr}\,M$ are finitely generated $\mathbb{C}[\mathfrak{g}]$-modules and we are done.

4. Representations of the Weyl group

4.1. In this section we review Kazhdan-Lusztig's approach [17] to the Springer representations [21] of the Weyl group and a result of Hotta [7] concerning these representations.

Set $Z=\{(gB,g'B,x)\in\mathscr{B}\times\mathscr{B}\times\mathcal{N}\mid g^{-1}x,g'^{-1}x\in\mathfrak{n}\}$ and

$$Z_w=\overline{\{(gB,gwB,x)\in Z\mid g\in G\}}\quad\text{for } w\in W.$$

Then $\dim Z_w=2d$ and $Z=\bigcup_{w\in W}Z_w$ gives the irreducible decomposition of Z. Z can be identified with a subspace of $T^*(\mathscr{B}\times\mathscr{B})$ so that $Z_w=\overline{T^*_{\mathscr{Y}_w}(\mathscr{B}\times\mathscr{B})}$. Kazhdan-Lusztig [17] defined an action of $W\times W$ on the Borel-Moore homology group $H_{4d}(Z)=H_{4d}(Z,\mathbb{C})=\bigoplus_{w\in W}\mathbb{C}[Z_w]$ and showed that there exists an isomorphism $\Psi\colon H_{4d}(Z)\to\mathbb{C}[W]$ as $W\times W$-modules so that $\Psi([Z_e])=e$. Set $\mathbf{b}(w)=\Psi([Z_w])$.

Proposition 3 (Kazhdan-Lusztig [17]). *Let s be a simple reflection of W.*

(i) $\mathbf{b}(w)s=-\mathbf{b}(w)$ *if* $ws<w$.

(ii) $\mathbf{b}(w)s=\mathbf{b}(w)+\sum_{ys<y\leqq ws}\delta_s(y,w)\mathbf{b}(y)$ *if* $ws>w$, *where* $\delta_s(y,w)$ *is a certain non-negative integer and* $\delta_s(ws,w)=1$.

We give a geometric description of $\delta_s(y,w)$ due to Hotta in the following. Let P be the parabolic subgroup corresponding to s, that is, $P=BsB\cup B$. Set $\mathscr{P}=G/P$ and $\mathscr{P}_w=BwP/P$. Then $\mathscr{P}_w=\mathscr{P}_{ws}$ and $\mathscr{P}=\coprod_{ws<w}\mathscr{P}_w$ gives the decomposition of $\mathscr{P}$ into B-orbits. Let $\mathscr{B}\times_{\mathscr{P}}T^*\mathscr{P}\xrightarrow{\rho}T^*\mathscr{B}$ and $\mathscr{B}\times_{\mathscr{P}}T^*\mathscr{P}\xrightarrow{\varpi}T^*\mathscr{P}$ be the natural maps.

Theorem 5 (Hotta [7; Theorem 2], [26]). *For $w\in W$ and a simple reflection s with $ws>w$, we have*

$$\varpi_*(\rho^*([\overline{T^*_{\mathscr{B}_w}\mathscr{B}}]))=\sum_{ys<y\leqq ws}\delta_s(y,w)[\overline{T^*_{\mathscr{P}_y}\mathscr{P}}],$$

as an algebraic cycle (*see Lemma* 3 *below*).

Hotta's result in [7] is not exactly of this form. But the proof is the same.

4.2. We denote the nilpotent orbit containing $x\in\mathscr{N}$ by O_x. Set

$$\mathscr{B}^x=\{gB\in\mathscr{B}\mid g^{-1}x\in\mathfrak{n}\}\quad\text{and}\quad A(x)=Z_G(x)/Z_G(x)^0\quad\text{for } x\in\mathscr{N}.$$

It is known that $\mathscr{B}^x$ has pure dimension $d_x=d-(\dim O_x)/2$ (see Steinberg [23]). We have an action of W on $H_{2d_x}(\mathscr{B}^x)$ commuting with the natural action of $A(x)$ which is called the Springer representation. Let $\widehat{A(x)}$ and $\hat{W}$ be the set of the irreducible representations of $A(x)$ and that of W, respectively. Then as a $W\times A(x)$-module we have $H_{2d_x}(\mathscr{B}^x)=\bigoplus_{\xi\in\widehat{A(x)}}(V_{O_x,\xi}\otimes\xi)$, where $V_{O_x,\xi}$ is 0 or an irreducible W-module. Moreover the mapping from the set

$$\{(O_x,\xi)\mid O_x\in\mathscr{N}/G,\ \xi\in\widehat{A(x)},\ V_{O_x,\xi}\neq 0\}$$

to $\hat{W}$ given by $(O_x,\xi)\mapsto V_{O_x,\xi}$ is bijective.

For $w\in W$ we define a nilpotent orbit $St(w)$ by $\overline{St(w)}=\overline{G(\mathfrak{n}\cap w(\mathfrak{n}))}$. Any nilpotent orbit coincides with some $St(w)$.

Proposition 4 (Kazhdan-Lusztig [17]). *For a nilpotent orbit O, $\bigoplus_{St(w)\subset\bar{O}}\mathbb{C}\mathbf{b}(w)$ is* $W\times W$-invariant and isomorphic to

$$\bigoplus_{O_x\subset\bar{O}}(H_{2d_x}(\mathscr{B}^x)\otimes H_{2d_x}(\mathscr{B}^x))^{A(x)}.$$

5. Main theorem

5.1. Our main theorem is the following.

Theorem 6. $\mathbf{b}(w)=\mathbf{c}(w)$ *for any* $w\in W$.

This means that the diagram

$$\begin{array}{ccc} K(\mathscr{M}) & \xrightarrow{h} & \mathbb{Z}[W] \\ {\scriptstyle \mathbf{Ch}}\downarrow & & \downarrow \\ H_{2d}(Z, \mathbb{C}) & \xrightarrow{\Psi} & \mathbb{C}[W] \end{array}$$

commutes.

5.2. In order to prove Theorem 6 we need some general facts concerning the behaviour of the characteristic cycles of holonomic systems under pull-back or integration along fibers.

Theorem 7. *Let X and Y be projective non-singular algebraic varieties over $\mathbb{C}$ and $f: X \to Y$ be a smooth map. We denote by $\rho_f: X \times_Y T^*Y \to T^*X$ and $\varpi_f: X \times_Y T^*Y \to T^*Y$ the natural maps.*

(i) *Let $\mathfrak{N}$ be a holonomic [resp. regular holonomic] $\mathscr{D}_Y$-Module. Then the pull-back $f^*(\mathfrak{N})$ of $\mathfrak{N}$ is a holonomic [resp. regular holonomic] $\mathscr{D}_X$-Module, and its characteristic cycle is given by*

$$\mathbf{Ch}(f^*(\mathfrak{N})) = \rho_{f*}(\varpi_f^*(\mathbf{Ch}(\mathfrak{N})).$$

(ii) *Let $\mathfrak{M}$ be a holonomic [resp. regular holonomic] $\mathscr{D}_X$-Module with* $\mathbf{Ch}(\mathfrak{M}) = \sum_{\Lambda} n_\Lambda[\Lambda]$, *where Λ runs through the irreducible components of* $\mathrm{Ch}(\mathfrak{M})$.

(a) *The integration $\int_f \mathfrak{M}$ of $\mathfrak{M}$ along f is a bounded complex of $\mathscr{D}_Y$-Modules so that the i-th cohomology sheaf $\int_f^i \mathfrak{M}$ is holonomic [resp. regular holonomic].*

(b) *For an irreducible component Λ of* $\mathrm{Ch}(\mathfrak{M})$ *we denote the set of the irreducible components of $\varpi_f(\rho_f^{-1}(\Lambda))$ by I_Λ. For $\Lambda' \in I_\Lambda$ there exists an integer $m(\Lambda', \Lambda)$ which depends only on Λ and Λ' (and not on $\mathfrak{M}$), so that the following holds:*

$$\mathbf{Ch}(\textstyle\int_f \mathfrak{M}) \left(:= \sum_i (-1)^i \, \mathbf{Ch}(\int_f^i \mathfrak{M})\right) = \sum_\Lambda n_\Lambda \left(\sum_{\Lambda' \in I_\Lambda} m(\Lambda', \Lambda) \, [\Lambda'] \right).$$

(c) *If f is non-characteristic for Λ, that is,* $\dim \rho_f^{-1}(\Lambda) = \dim Y$ *(and hence $\rho_f^{-1}(\Lambda) \to \varpi_f(\rho_f^{-1}(\Lambda))$ is finite at the generic points of $\rho_f^{-1}(\Lambda)$), then*

$$\sum_{\Lambda' \in I_\Lambda} m(\Lambda', \Lambda)[\Lambda'] = \varpi_{f*}(\rho_f^*([\Lambda])).$$

(i) is contained in Kashiwara [13]. (ii) follows from Sabbah [20].

5.3. We go back to our situation. We fix a simple reflection s. Set $P = B \cup BsB$ and $\mathscr{P} = G/P$. Let $\pi: \mathscr{B} \to \mathscr{P}$ be the natural map. We can check the following easily. So we omit the proof.

Lemma 3. (i) $\varpi(\rho^{-1}(\overline{T^*_{\mathscr{B}_w}\mathscr{B}})) \subset \bigcup_{ys<y} \overline{T^*_{\mathscr{P}_y}\mathscr{P}}$ *for any $w \in W$, and if $w < ws$, then π is non-characteristic for $\overline{T^*_{\mathscr{B}_w}\mathscr{B}}$ and $\varpi(\rho^{-1}(\overline{T^*_{\mathscr{B}_w}\mathscr{B}})) \subset \bigcup_{ys<y\leqq ws} \overline{T^*_{\mathscr{P}_y}\mathscr{P}}$.*

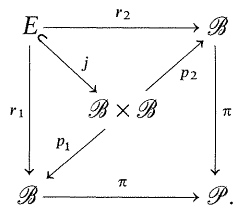
(ii) $\rho_* \circ \varpi^*([\overline{T^*_{\mathscr{P}_w}\mathscr{P}}]) = [\overline{T^*_{\mathscr{B}_w}\mathscr{B}}]$ *if $ws < w$ (in this case $\overline{\mathscr{B}_w} = \pi^{-1}(\overline{\mathscr{P}_w})$).*

We define an additive map $K(\tilde{\mathcal{M}}) \to K(\tilde{\mathcal{M}})$ by

$$[\mathfrak{M}] \mapsto [\mathfrak{M}]^s = \sum_i (-1)^i [\mathscr{H}^i(\pi^* \textstyle\int_\pi \mathfrak{M})] \quad \text{for } \mathfrak{M} \in \tilde{\mathcal{M}}.$$

Proposition 5. *For $\mathfrak{M} \in \tilde{\mathcal{M}}$ we have $h([\mathfrak{M}]^s) = h([\mathfrak{M}])(s-e)$, or equivalently $[\mathfrak{M}]^s = [\mathfrak{M}] \circ [\mathfrak{L}_s]$. Here we endow the ring structure via the isomorphism $K(\mathcal{M}) \to K(\tilde{\mathcal{M}})$.*

Proof. By an exact sequence $0 \to \mathfrak{M}_e \to \mathfrak{M}_s \to \mathfrak{L}_s \to 0$, we have $[\mathfrak{L}_s] = [\mathfrak{M}_s] - [\mathfrak{M}_e]$. Since $h([\mathfrak{M}_w]) = w$, the first assertion is derived from the second (see 2.4). Set $E = \mathscr{B} \times_{\mathscr{P}} \mathscr{B}$ and $\mathfrak{L}'_s = \mathscr{H}^{d-1}_E(\mathscr{B} \times \mathscr{B})$. Since $E = \mathscr{Y}_s$, $[\mathfrak{L}'_s]$ corresponds to $[\mathfrak{L}_s]$ by the isomorphism $K(\mathcal{M}) \to K(\tilde{\mathcal{M}})$. Consider the following diagram:



For $\mathfrak{M} \in \tilde{\mathcal{M}}$ and $\mathfrak{N}' \in \mathcal{M}$ with $\mathfrak{N}' \mid \{eB\} \times \mathscr{B} = \mathfrak{N}$ we have:

$$[\mathfrak{M}] \circ [\mathfrak{N}] = \sum_i (-1)^i [\mathscr{H}^i(\textstyle\int_{p_2} p_1^* \mathfrak{M} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{\mathscr{B} \times \mathscr{B}}} \mathfrak{N}')].$$

Hence $[\mathfrak{M}] \circ [\mathfrak{L}_s] = \sum_i (-1)^i [\mathscr{H}^i(\mathfrak{N})]$ with

$$\mathfrak{N} = \textstyle\int_{p_2} p_1^* \mathfrak{M} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{\mathscr{B} \times \mathscr{B}}} \mathfrak{L}'_s = \int_{p_2} \int_j (\mathbb{L} j^* p_1^* \mathfrak{M}) = \int_{r_2} r_1^* \mathfrak{M} = \pi^* \int_\pi \mathfrak{M},$$

and we are done.

Proof of Theorem 6. We prove this by induction on $l(w)$. If $l(w) = 0$, then $w = e$ and $\mathbf{b}(e) = \mathbf{c}(e) = e$. If $l(w) > 0$, then there exists a simple reflection s so that $ws < w$. By Proposition 3 we have

$$\mathbf{b}(w) = \mathbf{b}(ws)s - \mathbf{b}(ws) - \sum_{ys < y < w} \delta_s(y, ws)\, \mathbf{b}(y).$$

Thus we have only to prove the following:

$$\mathbf{c}(w)s - \mathbf{c}(w) = \sum_{ys < y \leqq ws} \delta_s(y, w)\, \mathbf{c}(y) \quad \text{if } ws > w.$$

We take $\mathfrak{M} \in \tilde{\mathcal{M}}$ and set $\mathbf{Ch}(\mathfrak{M}) = \sum_{w \in W} n_w [\overline{T^*_{\mathscr{B}_w} \mathscr{B}}]$. Then by Theorems 5, Theorem 7 and Lemma 3 we have:

$$\mathbf{Ch}([\mathfrak{M}]^s) = \sum_{w < ws} n_w \Big(\sum_{ys < y \leqq ws} \delta_s(y, w) [\overline{T^*_{\mathscr{B}_y} \mathscr{B}}] \Big) + \sum_{w > ws} n_w \Big(\sum_{ys < y} \gamma_s(y, w) [\overline{T^*_{\mathscr{B}_y} \mathscr{B}}] \Big),$$

where $\delta_s(y, w)$ is the integer given in 4.1 and $\gamma_s(y, w)$ is a certain integer. Thus by Lemma 2 and Proposition 5, we have:

$$h(\mathfrak{M}) = \sum_w n_w \mathbf{c}(w),$$

$$h(\mathfrak{M})(s-e) = \sum_{w<ws} n_w \Big(\sum_{ys<y\leqq ws} \delta_s(y, w)\,\mathbf{c}(y)\Big) + \sum_{w>ws} n_w \Big(\sum_{ys<y} \gamma_s(y, w)\,\mathbf{c}(y)\Big).$$

Since the above formula holds for any $\mathcal{M} \in \tilde{\mathcal{M}}$, we have:

$$\mathbf{c}(w)(s-e) = \sum_{ys<y\leqq ws} \delta_s(y, w)\,\mathbf{c}(y) \qquad \text{if } w<ws.$$

This completes the proof.

5.4. Theorem 6 gives an interpretation of a conjecture of Kazhdan-Lusztig [17] concerning the Springer representations in terms of holonomic systems. That is, the following two conjectures are equivalent.

Conjecture. (Kashdan-Lusztig [17]). *If* $G = \mathrm{SL}_n(\mathbb{C})$, *then*

$$\mathbf{b}(w) = \sum_{y\leqq w} (-1)^{l(w)+l(y)} P_{y,w}(1)\, y,$$

that is, $\mathbf{a}(w) = \mathbf{b}(w)$ *for any* $w \in W$.

Conjecture. *If* $G = \mathrm{SL}_n(\mathbb{C})$, *then* $\mathrm{Ch}(\mathfrak{L}_w) = \overline{T^*_{\mathscr{B}_w}(\mathscr{B})}$ *for any* $w \in W$.

In general $\mathrm{Ch}(\mathfrak{L}_w)$ is not irreducible.

Example. We consider the case when G is of type B_2. Let s [resp. t] be the simple reflection of W corresponding to a short [resp. long] root. Then $\mathbf{Ch}(\mathfrak{L}_w)$ is given as follows.

$$\mathbf{Ch}(\mathfrak{L}_w) = [\overline{T^*_{\mathscr{B}_w}\mathscr{B}}] \qquad \text{if } w \neq tst,$$

$$\mathbf{Ch}(\mathfrak{L}_{tst}) = [\overline{T^*_{\mathscr{B}_{tst}}\mathscr{B}}] + [\overline{T^*_{\mathscr{B}_t}\mathscr{B}}].$$

5.5. By Theorem 6 we can describe the ring structure of $\bigoplus_{w\in W} \mathbb{Z}[Z_w]$ derived from that of $\mathbb{Z}[W]$ by Ψ. Let $\mathscr{C}$ be the abelian category of coherent $\mathcal{O}_Z$-Modules and $\mathscr{C}'$ the full subcategory of $\mathscr{C}$ consisting of coherent sheaves whose supports are nowhere dense in Z. Then the Grothendieck group $K(\mathscr{C}/\mathscr{C}')$ is canonically isomorphic to $\bigoplus_{w\in W} \mathbb{Z}[Z_w]$ by $[Z_w] \leftrightarrow [\mathcal{O}_{Z_w}]$. Let us consider the diagram

$$\begin{array}{ccccc} Z & \xleftarrow{\;p_1\;} & Z \times_{T^*\mathscr{B}} Z & \xrightarrow{\;p\;} & Z \\ \downarrow & & \downarrow{\scriptstyle p_2} & & \\ T^*\mathscr{B} & \longleftarrow & Z, & & \end{array}$$

where $Z \times_{T^*_{\mathscr{B}}} Z = \{(g_1 B, g_2 B, g_3 B, x) \in \mathscr{B} \times \mathscr{B} \times \mathscr{B} \times \mathcal{N} \mid x \in g_1 \mathfrak{n} \cap g_2 \mathfrak{n} \cap g_3 \mathfrak{n}\}$ and p_1, p_2, p are given by $(g_1 B,\ g_2 B,\ g_3 B, x) \mapsto (g_1 B,\ g_2 B, x)$, $(g_2 B,\ g_3 B, x)$ and

$(g_1 B, g_3 B, x)$, respectively. Then the multiplication is given by

$$[F_1]\circ[F_2]=\sum_i(-1)^i\,[R^i p_*(\mathbb{L}p_1^* F_1 \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{Z\times_{T^*_{\mathscr{B}}} Z}} \mathbb{L}p_2^* F_2)]$$

for $F_1, F_2 \in \mathscr{C}$.

6. Review on the results of Joseph [10] and Hotta [7]

6.1. We say that a subspace V of $\mathbb{C}[W]$ is $\mathbf{a}$-basal if it is spanned by the $\mathbf{a}(w)$'s contained in it. For $w\in W$ let $\bar{V}^L_w, \bar{V}^R_w$ and $\bar{V}^{LR}_w$ be the smallest $\mathbf{a}$-basal subspaces of $\mathbb{C}[W]$ which contain $\mathbf{a}(w)$ and are invariant under the left action of W, the right action of W and both actions of W, respectively. We define a preorder $\underset{L}{\leqq}$ and an equivalence relation $\underset{L}{\sim}$ on W by

$$w_1 \underset{L}{\leqq} w_2 \quad \text{iff} \quad \bar{V}^L_{w_1} \supset \bar{V}^L_{w_2},$$

$$w_1 \underset{L}{\sim} w_2 \quad \text{iff} \quad \bar{V}^L_{w_1} = \bar{V}^L_{w_2}.$$

We call the equivalence class $\mathscr{C}^L_w=\{w'\in W \mid w' \underset{L}{\sim} w\}$ the left cell containing w. Let K^L_w be the sum of the subspaces $\bar{V}^L_y$ which are properly contained in $\bar{V}^L_w$. Then the W-module $V^L_w := \bar{V}^L_w / K^L_w$ has a natural basis corresponding to the left cell $\mathscr{C}^L_w$. We call the representation of W on V^L_w the left cell representation. We can define $\underset{R}{\leqq}, \underset{LR}{\leqq}, \ldots$, etc. similarly.

Proposition 6 (Joseph [8], Vogan [25]). *The following conditions are equivalent.*

(i) $w_1 \underset{L}{\leqq} w_2$.

(ii) $\mathrm{Ann}_{U(\mathfrak{g})} L_{w_1} \subset \mathrm{Ann}_{U(\mathfrak{g})} L_{w_2}$.

(iii) *There exists a finite dimensional* $U(\mathfrak{g})$*-module* F *so that* $L_{w_2^{-1}}$ *is a subquotient of* $L_{w_1^{-1}} \otimes F$.

Since representations of W are self-dual, the group ring $\mathbb{C}[W]$ is isomorphic to the direct sum $\bigoplus_{\sigma\in\hat{W}} (\sigma\otimes\sigma)$. We define a preorder $\underset{LR}{\leqq}$ and an equivalence relation $\underset{LR}{\sim}$ on $\hat{W}$ as follows. For σ_1 and σ_2 of $\hat{W}$, $\sigma_1 \underset{LR}{\leqq} \sigma_2$ if σ_1 is contained in V^{LR}_w and σ_2 is contained in $\bar{V}^{LR}_w$ for some $w\in W$, and $\sigma_1 \underset{LR}{\sim} \sigma_2$ if they are contained in the same V^{LR}_w.

6.2. For $w\in W$ we define a homogeneous polynomial p_w on $\mathfrak{h}$ by

$$p_w = \sum_{y\leqq w} (-1)^{l(w)+l(y)} P_{y,w}(1)(y^{-1}\rho)^{m_w},$$

where m_w is the least non-negative integer so that the right hand side is non-zero. This is Joseph's Goldie rank polynomial up to some non-zero constant multiple (see Joseph [9]). Let $\sigma(w)$ be the representation of W generated by p_w

in $S^{m_w}(\mathfrak{h}^*)$. Then $\sigma(w)$ is an irreducible representation of W, which is called Goldie rank representation. Furthermore $\sigma(w)$ appears in $S^{m_w}(\mathfrak{h}^*)$ with multiplicity one and does not appear in $S^n(\mathfrak{h}^*)$ for $n<m_w$. It is known that the Goldie rank representations are the special representations in the sense of Lusztig [18] and any special representations occur as Goldie rank representations (see Barbasch-Vogan [1], [2]).

Proposition 7 (Joseph [9]). (i) $w_1 \underset{L}{\sim} w_2$ *if and only if* $\mathbb{C}p_{w_1}=\mathbb{C}p_{w_2}$.

(ii) $\sigma(w)=\bigoplus_{w_1\in\mathscr{C}_w^{LR}/\underset{L}{\sim}} \mathbb{C}p_{w_1}$.

6.3. For a nilpotent orbit O, $O\cap\mathfrak{n}$ has pure-dimension $(\dim O)/2$. We say that a closed subvariety V of $\mathfrak{n}$ is orbital if it is an irreducible component of $\overline{O}\cap\mathfrak{n}$ for some nilpotent orbit O. For an orbital variety V Joseph [10] defined a polynomial p_V on $\mathfrak{h}^*$ as follows. Set $\mathfrak{n}^-=\bigoplus_{\alpha\in\Delta^+}\mathfrak{g}_{-\alpha}$. We identify the coordinate ring $\mathbb{C}[\mathfrak{n}]$ of $\mathfrak{n}$ with the symmetric algebra $S(\mathfrak{n}^-)$ by the Killing form. Let $I(V)$ be the defining ideal of V. Since V is ad$(\mathfrak{h})$-invariant, we have a natural action of $\mathfrak{h}$ on $M=S(\mathfrak{n}^-)/I(V)$. For $h\in\mathfrak{h}^*$ so that $\alpha(h)$ is a negative integer for each $\alpha\in\Delta^+$, set $M_i^h=\{m\in M \mid h\cdot m=im\}$ for a non-negative integer i. Then there exists a unique homogeneous polynomial p_V of degree $d-\dim V$ so that the following holds.

$$\sum_{i=0}^{k}\dim M_i^h=(p_V(h)/\prod_{\alpha\in\Delta^+}\alpha(h))\,k^{\dim V}+O(k^{(\dim V)-1}).$$

Proposition 8 (Gabber, Joseph, see [10]). (i) $\mathrm{Ch}(L_w)$ *is pure-dimensional and its irreducible components are orbital.*

(ii) $p_{w^{-1}}=\sum_V l_V p_V$, *where V runs through the irreducible components of* $\mathrm{Ch}(L_w)$ *and l_V are certain positive integers.*

For a nilpotent orbit O with $O=O_x$ let $V_{O,1}$ be the irreducible representation of W corresponding to the trivial character of $A(x)$ (see Sect. 4). The following is a corollary of a theorem of Borho-MacPherson [5].

Proposition 9 (Borho-MacPherson [5]). $V_{O,1}$ *appears in* $S^{d_x}(\mathfrak{h}^*)$ *with multiplicity one and does not appear in* $S^n(\mathfrak{h}^*)$ *for* $n<d_x$.

Let $Sp(O)$ be the W-submodule of $S^{d_x}(\mathfrak{h}^*)$ which is isomorphic to $V_{O,1}$.

Proposition 10 (Hotta [7]). $Sp(O)=\bigoplus_V\mathbb{C}p_V$, *with V runs through the irreducible components of* $\overline{O}\cap\mathfrak{n}$.

Proposition 11. *For each $w\in W$ there exists a unique nilpotent orbit O_w so that $\sigma(w)=Sp(O_w)$. Any irreducible component of* $\mathrm{Ch}(L_w)$ *is an irreducible component of $\overline{O_w}\cap\mathfrak{n}$. Especially,* $\overline{G\cdot\mathrm{Ch}(L_w)}$ *coincides with* $\overline{O_w}$.

Proof. From the definition of the Kazhdan-Lusztig polynomial $w\underset{LR}{\sim}w^{-1}$ for any $w\in W$, hence $\sigma(w)=\sigma(w^{-1})$ by Proposition 7. Thus our claim follows from Propositions 8, 9 and 10.

Remark. By an unpublished result of Borho-Brylinski, the associated variety of the primitive ideal $\mathrm{Ann}_{U(\mathfrak{g})}L_w$ coincides with $\overline{G\cdot\mathrm{Ch}(L_w)}$. This together with the above Proposition gives a unified proof of the irreducibility of the associated varieties in the integral case, which was proved by Borho-Brylinski [4] using case-by-case computations. We learned that Joseph also gave the similar proof as indicated above.

7. Left cell representations and Springer representations

7.1. The group ring $\mathbb{C}[W]$ has two bases $\{\mathbf{a}(w)\}_{w\in W}$ and $\{\mathbf{b}(w)\}_{w\in W}$. $\{\mathbf{a}(w)\}_{w\in W}$ is related to the left cell representations and $\{\mathbf{b}(w)\}_{w\in W}$ is related to the Springer representations of W. Since $\mathbf{a}(w)=\sum_{y\leqq w} m_y(\mathfrak{L}_w)\,\mathbf{b}(y)$, relations of the left cell representations and the Springer representations will be deduced from the knowledge of the multiplicities $m_y(\mathfrak{L}_w)$. For example if the conjecture in 5.4 is true, or at least if $\mathrm{Ch}(L_w)=\gamma(\overline{T^*_{\mathscr{B}_w}\mathscr{B}})$ $(=\overline{B(\mathfrak{n}\cap w(\mathfrak{n}))})$ for $G=\mathrm{SL}_n(\mathbb{C})$, the left cell representations and the Springer representations of a symmetric group coincide with their natural bases.

7.2. We give below some relations which are deduced from our main theorem and the results of Joseph and Hotta.

Proposition 12. *For each* $w\in W$ $\bar{V}^{LR}_w$ *is contained in* $\bigoplus_{St(y)\subset \overline{O_w}} \mathbb{C}\,\mathbf{b}(y)$.

Proof. For $z\in W$ set $\mathbf{a}(z)=\sum_{y\leqq z} e_{y,z}\,\mathbf{b}(y)$. Then we have $\mathrm{Ch}(\mathfrak{L}_z)=\sum_{y\leqq z} e_{y,z}[\overline{T^*_{\mathscr{B}_y}\mathscr{B}}]$. Thus $\mathrm{Ch}(L_z)=\bigcup_{e_{y,z}\neq 0}\gamma(\overline{T^*_{\mathscr{B}_y}\mathscr{B}})$ by Proposition 2. Since $\gamma(\overline{T^*_{\mathscr{B}_y}\mathscr{B}})=\overline{B(\mathfrak{n}\cap y(\mathfrak{n}))}$, it follows from Proposition 11 that $\overline{O_z}=\bigcup_{e_{y,z}\neq 0}\overline{St(y)}$. So it is sufficient to show that $\overline{O_z}\subset\overline{O_w}$ if $z\underset{LR}{\geqq} w$. Moreover we have only to consider the cases $z\underset{R}{\geqq} w$ and $z\underset{L}{\geqq} w$. If $z\underset{R}{\geqq} w$, then $z^{-1}\underset{L}{\geqq} w^{-1}$ by the properties of the Kazhdan-Lusztig polynomials. Since there exists a finite dimensional $U(\mathfrak{g})$-module F so that L_z is a subquotient of $L_w\otimes F$ by Proposition 7, we have $\mathrm{Ch}(L_z)\subset\mathrm{Ch}(L_w)$ and hence $\overline{O_z}\subset\overline{O_w}$ by Proposition 11. If $z\underset{L}{\geqq} w$, then $z^{-1}\underset{R}{\geqq} w^{-1}$ and hence $\overline{O_z}=\overline{O_{z^{-1}}}\subset\overline{O_{w^{-1}}}$ $=\overline{O_w}$. This proves our assertion.

From Propositions 4 and 12, we have the following.

Corollary. *Let* σ *and* σ' *be the irreducible representations of* W *with* σ *special and* $\sigma\underset{LR}{\leqq}\sigma'$. *Then for nilpotent orbits* O *and* O' *with* $\sigma=V_{O,1}$ *and* $\sigma'=V_{O',\xi}$ *for some* ξ, *we have* $\overline{O'}\subset\overline{O}$.

References

1. Barbasch, D., Vogan, D.: Primitive ideals and orbital integrals in complex classical groups. Math. Ann. **259**, 153–199 (1982)
2. Barbasch, D., Vogan, D.: Primitive ideals and orbital integrals in complex exceptional groups. J. Algebra **80**, 350–382 (1983)