DUALITY OF D-MODULES ON FLAG MANIFOLDS

DANIEL BARLET AND MASAKI KASHIWARA

ABSTRACT. We shall give the duality of D-modules on the flag manifold which corresponds to the duality of Harish-Chandra modules. This duality is not algebraic but analytic. As an application, we show that the characteristic variety of the dual of an arbitrary Harish-Chandra module is the complex conjugate of the original one.

1. INTRODUCTION

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with an involution θ and $\mathfrak{k}$ the Lie subalgebra of the fixed points of θ. Let K be a connected reductive algebraic group with Lie algebra $\mathfrak{k}$ and assume that the adjoint action of $\mathfrak{k}$ on $\mathfrak{g}$ lifts to an action of K on $\mathfrak{g}$.

Recall that a $(\mathfrak{g}, K)$-module is a module endowed with a $\mathfrak{g}$-module structure and a K-module structure such that the two induced actions of $\mathfrak{k}$ coincide. A $(\mathfrak{g}, K)$-module is called a Harish-Chandra module if any irreducible representation of K appears in it only finitely many times. It is known that any Harish-Chandra module is finitely generated over $U(\mathfrak{g})$.

For a Harish-Chandra module M, its dual M^* is defined as the subspace of $\mathfrak{k}$-finite vectors of $\mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$. It has a canonical structure of a Harish-Chandra module. Then $M \mapsto M^*$ is a contravariant functor and we have a functorial isomorphism

$$M \xrightarrow{\sim} M^{**}. \tag{1.1}$$

For a finitely generated $U(\mathfrak{g})$-module M, we can define its characteristic variety (sometimes called associated variety) as follows. Let

Key words and phrases. representation of Lie group, flag manifold, Harish-Chandra module.

The first named author thanks Research Institute for Mathematical Sciences, Kyoto University for the hospitality which gives the opportunity of this joint work. The second named author benefits by a "Chaire Internationale de Recherche Blaise Pascal de l'Etat et de la Région d'Ile-de-France, gérée par la Fondation de l'Ecole Normale Supérieure".

$F(U(\mathfrak{g}))$ be the increasing filtration of $U(\mathfrak{g})$ defined by

$$(1.2)\quad F_n(U(\mathfrak{g})) = \begin{cases} 0 & \text{for } n < 0, \\ \mathbb{C} & \text{for } n = 0, \\ F_{n-1}(U(\mathfrak{g})) + \mathfrak{g}F_{n-1}(U(\mathfrak{g})) & \text{for } n > 0. \end{cases}$$

Then $\mathrm{Gr}^F(U(\mathfrak{g}))$ is canonically isomorphic to the symmetric algebra $S(\mathfrak{g})$. Let us take a filtration $F(M)$ of M compatible with $F(U(\mathfrak{g}))$ and finitely generated. Then $\mathrm{Gr}^F(M)$ is a $\mathrm{Gr}^F(U(\mathfrak{g}))$-module and its support is regarded as a subvariety of $\mathfrak{g}^* = \mathrm{Spec}(S(\mathfrak{g}))$. This does not depend on the choice of such filtrations F of M, We call it the characteristic variety of M, and denote by $\mathrm{Ch}(M)$.

In [6], the second author asked what is the characteristic variety of the dual of a Harish-Chandra module. In this paper we shall give an answer to this question. Let us choose a compact form K_{u} of K and let $\mathfrak{k}_{\mathrm{u}}$ be its Lie algebra. Let us choose a θ-stable compact form $\mathfrak{g}_{\mathrm{u}}$ of $\mathfrak{g}$ containing $\mathfrak{k}_{\mathrm{u}}$. These choices are unique up to the conjugacy by K. Let c_{u} be the complex conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_{\mathrm{u}}$. Then this complex conjugation commutes with θ. Let us denote by the same letter c_{u} the induced complex conjugation on $\mathfrak{g}^*$.

Theorem 1.1. *For any Harish-Chandra module M, the characteristic variety of M^* is the complex conjugate of* $\mathrm{Ch}(M)$.

Remark 1.2. The complex conjugation c_{u} depends on the choice of a compact form $\mathfrak{g}_{\mathrm{u}}$, but it is unique up to the conjugation by K. Since the characteristic variety is invariant by the action of K, its complex conjugate does not depend on the choice of a compact form.

The main novelty in our proof of this theorem lies in the use of the conjugation functor for regular holonomic D-modules introduced by the second author ([5]). It is straightforward to generalize it to the case of twisted regular holonomic D-modules on the flag manifold relative to the complex conjugation induced by a compact real form $\mathfrak{g}_{\mathrm{u}}$ of $\mathfrak{g}$. It allows us to show that for a weight λ satisfying certain conditions and for a coherent $(\mathcal{D}_{X,\lambda}, K)$-module $\mathcal{M}$, the dual of the Harish-Chandra module $\Gamma(X;\mathcal{M})$ is given by $\Gamma(X; C_\lambda(\mathcal{M}))$, where $C_\lambda(\mathcal{M})$ is the conjugate of $\mathcal{M}$ in the previous sense. Then the generalization of a result of Borho-Brylinski [2] is used to conclude the proof of the main theorem.

Remark that the use of conjugation functor which is transcendent does not allow us to get an algebraic proof of this result.

2. Twisted sheaves and ring of twisted differential operators on the flag manifold

Let us take a semisimple algebraic group G with $\mathfrak{g}$ as its Lie algebra. Let X be the flag manifold of G.

Let us take a Borel subgroup B_0 of G and a Cartan subgroup T_0 of B_0. We take a maximal compact subgroup G_{u} of G such that $(T_0)_{\mathrm{u}} := T_0 \cap G_{\mathrm{u}}$ is a maximal compact subgroup of T_0. Let $x_0 \in X$ be the point corresponding to $\mathfrak{b}_0$. Let U_0 be the unipotent part of B_0. Let $\mathfrak{b}_0$, $\mathfrak{t}_0$ and $\mathfrak{n}_0$ be the Lie algebra of B_0, T_0 and U_0, respectively. Let Δ be the root system $\Delta(\mathfrak{g}, \mathfrak{t}_0)$, and let Δ^+ be the positive root system consisting of the roots appearing in $\mathfrak{n}_0$.

For any point x of X, let $\mathfrak{b}(x)$ be the corresponding Borel subalgebra, and let $\mathfrak{n}(x)$ be the nilpotent radical of $\mathfrak{b}(x)$. Then there is a canonical isomorphism $\mathfrak{b}(x)/\mathfrak{n}(x) \xrightarrow{\sim} \mathfrak{t}_0$. Let us denote by $\varphi_x \colon \mathfrak{b}(x) \to \mathfrak{t}_0$ the canonical projection. Then, the following diagram commutes for any $g \in G$ and $x \in X$.

$$\begin{array}{ccc} \mathfrak{b}(x) & \xrightarrow{\mathrm{Ad}(g)} & \mathfrak{b}(gx) \\ & {\scriptstyle \varphi_x}\searrow & \downarrow {\scriptstyle \varphi_{gx}} \\ & & \mathfrak{t}_0 \end{array} \tag{2.1}$$

In this reason, we call $(\mathfrak{t}_0, \Delta^+)$ the universal Cartan subalgebra.

Let c_{u} be the complex conjugation of $\mathfrak{g}$ with respect to the Lie algebra $\mathfrak{g}_{\mathrm{u}}$ of G_{u}. We have $\mathfrak{b}_0 \cap c_{\mathrm{u}}(\mathfrak{b}_0) = \mathfrak{t}_0$. Let us define the automorphism c_X (over $\mathbb{R}$) of X by

$$\mathfrak{b}(c_X(x)) = c_{\mathrm{u}}(\mathfrak{b}(x)).$$

We sometimes denote by $-$ instead of c_X or c_{u}. Hence, $\bar{B}_0$ and $\bar{U}_0$ are the complex conjugates of B_0 and U_0, and $\bar{\mathfrak{b}}_0$, $\bar{\mathfrak{n}}_0$ are their Lie algebras.

Note that we have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{t}_0 & \hookrightarrow & \bar{\mathfrak{b}}_0 \\ {\scriptstyle w_0\lambda}\downarrow & & {\scriptstyle \varphi_{\bar{x}_0}}\downarrow \\ \mathbb{C} & \xleftarrow{\lambda} & \mathfrak{t}_0 \end{array} \tag{2.2}$$

Here w_0 is the longest element of W.

Let us review twisted sheaves and rings of twisted differential operators in our context (see [4] for the details). For an algebraic variety Y, let us denote by Y_{an} the underlying complex variety. For $\lambda \in \mathfrak{t}_0^*$, let $\mathfrak{L}_\lambda$ be the holonomic $\mathcal{D}_{B_0}$-module generated by w_λ with the defining relation

$$R_A w_\lambda = \lambda(A) w_\lambda \quad \text{for } A \in \mathfrak{b}_0. \tag{2.3}$$

Here, we regard λ as a linear form on $\mathfrak{b}_0$ by $\mathfrak{b}_0 \to \mathfrak{t}_0 \xrightarrow{\lambda} \mathbb{C}$, and for $A \in \mathfrak{b}_0$, R_A denotes the vector field on B_0 induced by the right action of B_0 on B_0. Then

$$\begin{aligned} L_\lambda &:= \mathscr{H}om_{\mathcal{D}_{B_0}}(\mathfrak{L}_\lambda, \mathcal{O}_{B_{0\mathrm{an}}}) \\ &= \{\varphi \in \mathcal{O}_{B_{0\mathrm{an}}}\,;\, R_A\varphi = \lambda(A)\varphi \text{ for any } A \in \mathfrak{b}_0\} \end{aligned}$$

is a locally constant sheaf on $B_{0\mathrm{an}}$ generated by a (multi-valued) function b^λ, where $b^\lambda = \mathrm{e}^{\lambda(A)}$ for $b = \mathrm{e}^A$ ($A \in \mathfrak{b}_0$). Hence L_λ has the monodromy $\mathrm{e}^{2\pi\sqrt{-1}\lambda}$. Let us recall that $X \simeq G/B_0$ is the flag variety. A twisted sheaf with twist λ on an open set U of X_{an} is by definition a sheaf F on $p^{-1}U \subset G_{\mathrm{an}}$ satisfying (with a rough language)

$$F_{gb^{-1}} \simeq F_g \otimes (L_\lambda)_b \quad \text{for any } g \in p^{-1}(U) \text{ and } b \in B_0.$$

We denote by $\mathrm{Mod}^\lambda(\mathbb{C}_U)$ the abelian category of twisted sheaves on U with twist λ.

Then $U \mapsto \mathrm{Mod}^\lambda(\mathbb{C}_U)$ is a stack (a sheaf of categories) on X_{an}. It is locally equivalent to the stack of sheaves on X_{an}. Let us denote by $\mathcal{O}_{X_{\mathrm{an}}}(\lambda)$ the sheaf on G_{an} of holomorphic functions $f(g)$ such that $f(gb) = b^{-\lambda}f(g)$ for $b \in B_{0\mathrm{an}}$ sufficiently near to the identity. Then $\mathcal{O}_{X_{\mathrm{an}}}(\lambda)$ is an object of $\mathrm{Mod}^\lambda(\mathbb{C}_{X_{\mathrm{an}}})$. For an open set of X_{an}, if there is a continuous section s of $p^{-1}U \to U$, then we have an equivalence of categories

$$\mathrm{Mod}^\lambda(\mathbb{C}_U) \to \mathrm{Mod}(\mathbb{C}_U)$$

given by $F \mapsto s^{-1}F$. If moreover s is holomorphic, $\mathcal{O}_{X_{\mathrm{an}}}(\lambda)$ corresponds to $\mathcal{O}_{X_{\mathrm{an}}}$ by this equivalence .

In fact, $\mathrm{Mod}^\lambda(\mathbb{C}_X)$ depends on the choice of a Borel subgroup B_0. Let B_1 be another Borel subgroup and let x_1 be the corresponding point of X. Let $p_1 : G \to X$ be the morphism $g \mapsto gx_1$. Let us choose $g \in G$ such that $x_1 = gx_0$. Then we have the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{R_g} & G \\ {\scriptstyle p_1}\downarrow & & {\scriptstyle p}\downarrow \\ X & \xrightarrow{\mathrm{id}_X} & X \end{array}$$

Here $R_g : G \to G$ is the right multiplication map on G by g from the right. Let us denote by ${}^1\mathrm{Mod}^\lambda(\mathbb{C}_X)$ be the category of twisted sheaves defined by B_1 instead of B_0.

Then $\Psi_g : F \mapsto (R_g)^*F$ defines an equivalence of categories

$$\Psi_g : \mathrm{Mod}^\lambda(\mathbb{C}_X) \xrightarrow{\sim} {}^1\mathrm{Mod}^\lambda(\mathbb{C}_X).$$

Note that Ψ_g depends on the choice of g, but up to scalar. Namely if $g_1 = gb$ for $b \in B_0$, then there is a functorial isomorphism in F:

$$\Psi_{g_1}(F) \simeq (L_\lambda)_b \otimes_{\mathbb{C}} \Psi_g(F).$$

We denote by $\mathcal{D}_{X_{\mathrm{an}},\lambda}$ the subring of the endomorphism ring of $\mathcal{O}_{X_{\mathrm{an}}}(\lambda)$ consisting of endomorphisms locally expressed by finite-order differential operators. Hence $\mathcal{D}_{X_{\mathrm{an}},\lambda}$ is a sheaf of rings on X_{an}, locally isomorphic to $\mathcal{D}_{X_{\mathrm{an}}}$. It is algebraically defined. Namely, there is a ring $\mathcal{D}_{X,\lambda}$ on X containing $\mathcal{O}_X$ as a subring such that its pull back to X_{an} is isomorphic to $\mathcal{D}_{X_{\mathrm{an}},\lambda}$.

We call a coherent $\mathcal{D}_{X_{\mathrm{an}},\lambda}$-module $\mathfrak{M}$ a good $\mathcal{D}_{X_{\mathrm{an}},\lambda}$-module, if $\mathfrak{M}$ is a sum of coherent $\mathcal{O}_{X_{\mathrm{an}}}$-submodules. Then, by the theorem of GAGA, the category $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_{X,\lambda})$ of coherent $\mathcal{D}_{X,\lambda}$-modules is equivalent to the category $\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_{X_{\mathrm{an}},\lambda})$ of good $\mathcal{D}_{X_{\mathrm{an}},\lambda}$-modules by the functor

$$\mathfrak{M} \mapsto \mathfrak{M}^{\mathrm{an}} := \mathcal{D}_{X_{\mathrm{an}},\lambda} \otimes_{\mathcal{D}_{X,\lambda}} \mathfrak{M}.$$

There is a canonical ring homomorphism $U(\mathfrak{g}) \to \Gamma(X; \mathcal{D}_{X,\lambda})$.

Let $\mathfrak{z}(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. For $\lambda \in \mathfrak{t}_0^*$, $\chi_\lambda \colon \mathfrak{z}(\mathfrak{g}) \to \mathbb{C}$ is the infinitesimal character associated with the Verma module with lowest weight λ. Hence we have for any w in the Weyl group W,

$$\chi_{w\circ\lambda} = \chi_\lambda,$$

where $w \circ \lambda = w(\lambda - \rho) + \rho$.

Let $U(\mathfrak{g})_\lambda$ denote the ring $U(\mathfrak{g})/\big(U(\mathfrak{g})\,\mathrm{Ker}(\chi_\lambda)\big)$.

Theorem 2.1 (Beilinson-Bernstein[1]). (i) $\Gamma(X; \mathcal{D}_{X,\lambda}) \simeq U(\mathfrak{g})_\lambda$.

(ii) *Assume that $\lambda-\rho$ is integrally anti-dominant (i.e. $\langle \lambda-\rho, \alpha^\vee\rangle \notin \mathbb{Z}_{>0}$ for any $\alpha \in \Delta^+$).*

(a) *For any coherent $\mathcal{D}_{X,\lambda}$-module $\mathfrak{M}$, we have*

$$H^i(X; \mathfrak{M}) = 0 \quad \textit{for any } i \neq 0.$$

(b) *For any $U(\mathfrak{g})_\lambda$-module M,*

$$M \xrightarrow{\sim} \Gamma(X; \mathcal{D}_{X,\lambda} \otimes_{U(\mathfrak{g})_\lambda} M).$$

(iii) *Assume that $\lambda - \rho$ is regular and integrally anti-dominant (i.e. $\langle \lambda - \rho, \alpha^\vee\rangle \notin \mathbb{Z}_{\geq 0}$ for any $\alpha \in \Delta^+$). Then the abelian category* $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_{X,\lambda})$ *of coherent $\mathcal{D}_{X,\lambda}$-modules is equivalent to the abelian category* $\mathrm{Mod}_{\mathrm{fg}}(U(\mathfrak{g})_\lambda)$ *of finitely generated $U(\mathfrak{g})_\lambda$-modules by the functor* $\Gamma(X;\,\bullet\,)$.

When λ is not regular, we have the following result instead of (iii). For an integrally anti-dominant $\lambda \in \mathfrak{t}_0^*$, let us denote by $\mathcal{N}_\lambda$ the full subcategory of $\mathrm{Mod}^\lambda_{\mathrm{coh}}(\mathcal{D}_{X,\lambda})$ consisting of objects $\mathfrak{M}$ such that $\Gamma(X; \mathfrak{M}) =$

0. Then $\mathcal{N}_\lambda$ is invariant by extensions, subobjects and quotients. We have the equivalence of categories:

$$\Gamma(X;\ \cdot\ ) : \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_{X,\lambda})/\mathcal{N}_\lambda \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{fg}}(U(\mathfrak{g})_\lambda).$$

3. De Rham functor and Complex conjugation

Let X be a complex variety. Let us denote by $\mathrm{Reg}(\mathcal{D}_X)$ the abelian category of regular holonomic $\mathcal{D}_X$-modules. Let us denote by $\mathrm{Perv}(\mathbb{C}_X)$ the abelian category of perverse sheaves on X. Then the de Rham functor $DR(\mathfrak{M}) = \mathbb{R}\mathscr{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathfrak{M})$ gives an equivalence of categories:

$$DR_X : \mathrm{Reg}(\mathcal{D}_X) \xrightarrow{\sim} \mathrm{Perv}(\mathbb{C}_X).$$

Let X^c be the complex conjugate of X. Then the two abelian categories $\mathrm{Perv}(\mathbb{C}_X)$ and $\mathrm{Perv}(\mathbb{C}_{X^c})$ are canonically equivalent. Let us denote by $\mathcal{D}b_X$ the sheaf of distributions on X regarded as a real variety. Then $\mathcal{D}b_X$ is a $\mathcal{D}_X \otimes \mathcal{D}_{X^c}$-module. Hence for any $\mathcal{D}_X$-module $\mathfrak{M}$, $C(\mathfrak{M}) := \mathscr{H}om_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{D}b_X)$ has a structure of $\mathcal{D}_{X^c}$-module.

Theorem 3.1 ([5]). (i) *For any* $\mathfrak{M} \in \mathrm{Reg}(\mathcal{D}_X)$, *we have*

$$\mathcal{E}xt^i_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{D}b_X) = 0 \quad \text{for } i \neq 0.$$

(ii) *The functor* $\mathfrak{M} \mapsto C(\mathfrak{M})$ *gives an equivalence of categories:*

$$C : \mathrm{Reg}(\mathcal{D}_X)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Reg}(\mathcal{D}_{X^c}).$$

(iii) *For any* $\mathfrak{M} \in \mathrm{Reg}(\mathcal{D}_X)$,

$$DR_{X^c}(C(\mathfrak{M})) \xrightarrow{\sim} R\mathscr{H}om_{\mathbb{C}_X}(DR_X(\mathfrak{M}), \mathbb{C}_X).$$

Let us return to the original situation where X is the flag variety. A coherent $\mathcal{D}_{X,\lambda}$-module is called regular holonomic, if it is a regular holonomic $\mathcal{D}_X$-module by any local isomorphism between $\mathcal{D}_{X,\lambda}$ and $\mathcal{D}_X$. Let us denote by $\mathrm{Reg}(\mathcal{D}_{X,\lambda})$ the abelian category of regular holonomic $\mathcal{D}_{X,\lambda}$-modules. Similarly we can define the notion of twisted perverse sheaves, and let us denote by $\mathrm{Perv}^\lambda(\mathbb{C}_{X_{\mathrm{an}}})$ the category of twisted perverse sheaves on X_{an} with twist λ. Then the functor $DR^\lambda_X(\mathfrak{M}) := \mathbb{R}\mathscr{H}om_{\mathcal{D}_{X_{\mathrm{an}},\lambda}}(\mathcal{O}_{X_{\mathrm{an}}}(\lambda), \mathfrak{M}^{\mathrm{an}})$ gives an equivalence of categories:

$$DR^\lambda_X : \mathrm{Reg}(\mathcal{D}_{X,\lambda}) \xrightarrow{\sim} \mathrm{Perv}^{-\lambda}(\mathbb{C}_{X_{\mathrm{an}}}).$$

4. Duality on the flag manifolds

Let $p_i\colon X \times X \to X$ be the i-th projection ($i = 1, 2$). There is a unique open orbit of $X \times X$ with respect to the diagonal action of G, which we shall denote by Z. Then we have

$$Z = \{(x, y) \in X \times X; \mathfrak{g} = \mathfrak{b}(x) + \mathfrak{b}(y)\}.$$

The open orbit Z contains $(x_0, \bar{x}_0)$, and the isotropy subgroup of G at this point is $T_0 = B_0 \cap \bar{B}_0$.

Let us set

$$Z_{\mathbb{R}} = \{(x, y) \in X \times X; x = \bar{y}\}.$$

Then $Z_{\mathbb{R}}$ is a compact real analytic submanifold of Z, and Z is a complexification of $Z_{\mathbb{R}}$. By the first and the second projection, $Z_{\mathbb{R}}$ is isomorphic to X:

$$\begin{array}{ccc} Z_{\mathbb{R}} & \xrightarrow{p_1} & X \\ {\scriptstyle p_2}\downarrow & \swarrow{\scriptstyle c_X} & \\ X. & & \end{array}$$

Proposition 4.1. *For any $\lambda \in \mathfrak{t}_0^*$, there exists a unique holomorphic function $f_\lambda(g_1, g_2)$ defined on an open neighborhood of*

$$\{(g_1, g_2) \in G_{\mathrm{an}} \times G_{\mathrm{an}}; g_1^{-1} g_2 \in U_{0\,\mathrm{an}} \cdot \bar{U}_{0\,\mathrm{an}}\}$$

in $G_{\mathrm{an}} \times G_{\mathrm{an}}$ such that

$$f_\lambda(g, g) = 1 \quad \text{for } g \in G,$$
$$f_\lambda(g_1 b_1, g_2 b_2) = b_1^{-\lambda} b_2^{\lambda} f_\lambda(g_1, g_2)$$
$$\text{for } g_1,\ g_2 \in G_{\mathrm{an}}, \text{ and } b_1 \in B_{0\,\mathrm{an}},\ b_2 \in \bar{B}_{0\,\mathrm{an}} \text{ close to the identity.}$$

Moreover f_λ can be continued as a multi-valued holomorphic function on the open subset $B_{0\,\mathrm{an}} \cdot \bar{B}_{0\,\mathrm{an}}$ of $G_{\mathrm{an}} \times G_{\mathrm{an}}$.

Corollary 4.2. $\mathrm{Mod}^{(\lambda, -w_0\lambda)}(\mathbb{C}_{Z_{\mathrm{an}}})$ *is equivalent to* $\mathrm{Mod}(\mathbb{C}_{Z_{\mathrm{an}}})$.

Remark 4.3. The shift $w_0\lambda$ arises because of (2.2).

Hence, $\mathrm{Mod}^{(\lambda,\mu)}(\mathbb{C}_{Z_{\mathrm{an}}})$ is equivalent to $\mathrm{Mod}(\mathbb{C}_{Z_{\mathrm{an}}})$ provided that $\lambda + w_0\mu \in P$. Let $\mathcal{C}^\infty_{Z_{\mathbb{R}}}$ and $\mathcal{D}b_{Z_{\mathbb{R}}}$ be the sheaf of C^∞-functions and distributions on $Z_{\mathbb{R}}$, respectively. When $\lambda + w_0\mu \in P$,

$$\begin{aligned} \mathcal{C}^\infty_{Z_{\mathbb{R}}}(\lambda, \mu) &:= \mathcal{C}^\infty_{Z_{\mathbb{R}}} \otimes_{\mathcal{O}_{X_{\mathrm{an}} \times X_{\mathrm{an}}}} \mathcal{O}_{X_{\mathrm{an}} \times X_{\mathrm{an}}}(\lambda, \mu) \quad \text{and} \\ \mathcal{D}b_{Z_{\mathbb{R}}}(\lambda, \mu) &:= \mathcal{D}b_{Z_{\mathbb{R}}} \otimes_{\mathcal{O}_{X_{\mathrm{an}} \times X_{\mathrm{an}}}} \mathcal{O}_{X_{\mathrm{an}} \times X_{\mathrm{an}}}(\lambda, \mu) \end{aligned}$$

are well-defined sheaves and they have a $\mathcal{D}_{X \times X, (\lambda,\mu)}$-module structure. The function $f_\lambda \in \mathcal{C}^\infty_{Z_{\mathbb{R}}}(\lambda, -w_0\lambda)$ is a unique section of $\mathcal{D}b_{Z_{\mathbb{R}}}^{(\lambda, -w_0\lambda)}$ invariant by G_{an} up to a constant multiple.

For a regular holonomic $\mathcal{D}_{X,\lambda}$-module $\mathfrak{M}$,

$$p_{2*}\,\mathscr{H}om_{p_1^{-1}\mathcal{D}_{X,\lambda}}(p_1^{-1}\mathfrak{M}, \mathcal{D}b_{Z_\mathbb{R}}(\lambda, -w_0\lambda))$$

is a regular holonomic $\mathcal{D}_{X_{\mathrm{an}},-w_0\lambda}$-module by Theorem 3.1. Let us denote by $C_\lambda(\mathfrak{M})$ the regular holonomic $\mathcal{D}_{X,-w_0\lambda}$-module such that

$$C_\lambda(\mathfrak{M})^{\mathrm{an}} \simeq p_{2*}\,\mathscr{H}om_{p_1^{-1}\mathcal{D}_{X,\lambda}}(p_1^{-1}\mathfrak{M}, \mathcal{D}b_{Z_\mathbb{R}}(\lambda, -w_0\lambda)).$$

By the same theorem, we have an equivalence of categories:

$$C_\lambda : \mathrm{Reg}(\mathcal{D}_{X,\lambda})^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Reg}(\mathcal{D}_{X,-w_0\lambda}). \tag{4.1}$$

and the isomorphism

$$DR_X^{-w_0\lambda}(C_\lambda(\mathfrak{M})) \simeq c_X^{-1}\mathbb{R}\mathscr{H}om_{\mathbb{C}}(DR_X^\lambda(\mathfrak{M}), \mathbb{C}_{X_{\mathrm{an}}}), \tag{4.2}$$

or the following diagram (quasi-) commutes:

$$\begin{array}{ccc} \mathrm{Reg}(\mathcal{D}_{X,\lambda}) & \xrightarrow{C_\lambda} & \mathrm{Reg}(\mathcal{D}_{X,-w_0\lambda}) \\ {\scriptstyle DR_X^\lambda}\big\downarrow & & \big\downarrow{\scriptstyle DR_X^{-w_0\lambda}} \\ \mathrm{Perv}^{-\lambda}(\mathbb{C}_{X_{\mathrm{an}}}) & \xrightarrow{c_X^{-1}} & \mathrm{Perv}^{w_0\lambda}(\mathbb{C}_{X_{\mathrm{an}}}). \end{array}$$

For $\mathfrak{M} \in \mathrm{Mod}(\mathcal{D}_{X,\lambda})$, we have a $\mathfrak{g}$-linear homomorphism

$$\Gamma(X;\mathfrak{M}) \otimes_{\mathbb{C}} \Gamma(X; C_\lambda(\mathfrak{M})) \to \Gamma(Z_\mathbb{R}; \mathcal{D}b_{Z_\mathbb{R}}(\lambda, -w_0\lambda)).$$

Since $\mathcal{O}_X(2\rho) = \Omega_X^{\dim X}$, by multiplying $f_{-\lambda+2\rho}$, we obtain

$$\Gamma(Z_\mathbb{R}; \mathcal{D}b_{Z_\mathbb{R}}(\lambda, -w_0\lambda)) \xrightarrow{f_{-\lambda+2\rho}} \Gamma(Z_\mathbb{R}; \mathcal{D}b_{Z_\mathbb{R}} \otimes_{\mathcal{O}_{X_{\mathrm{an}}\times X_{\mathrm{an}}}} \Omega_{Z_\mathbb{R}}^{\dim_\mathbb{R}(Z_\mathbb{R})}) \xrightarrow{\int} \mathbb{C}.$$

Thus we obtain the $\mathfrak{g}$-invariant coupling

$$\Gamma(X;\mathfrak{M}) \times \Gamma(X; C_\lambda(\mathfrak{M})) \to \mathbb{C}. \tag{4.3}$$

Proposition 4.4. *Assume that $\lambda - \rho$ is integrally anti-dominant and $\mathfrak{M}$ is generated by global sections. Then the homomorphism*

$$\Gamma(X; C_\lambda(\mathfrak{M})) \to \mathrm{Hom}_{\mathbb{C}}\Big(\Gamma(X;\mathfrak{M}), \mathbb{C}\Big)$$

is injective.

Proof. By the definition, $\Gamma(Z_\mathbb{R}; \mathcal{D}b_{Z_\mathbb{R}}(\lambda, -w_0\lambda))$ is the topological dual of $\Gamma(Z_\mathbb{R}; \mathcal{C}^\infty_{Z_\mathbb{R}}(-\lambda+2\rho, w_0\lambda+2\rho))$. Note that $w_0\lambda+2\rho = -w_0(-\lambda+2\rho)$.

For $\varphi \in \Gamma(X; C_\lambda(\mathfrak{M}))$ such that

$$\int_{Z_\mathbb{R}} \varphi(s) f_{-\lambda+2\rho} = 0 \quad \text{for any } s \in \Gamma(X;\mathfrak{M}),$$

let us show $\varphi(s)=0$. We have for any $P\in U(\mathfrak{g})$

$$\int_{Z_{\mathbb{R}}}\varphi(Ps)f_{-\lambda+2\rho}=\int_{Z_{\mathbb{R}}}(P\otimes 1)\varphi(s)f_{-\lambda+2\rho}=\int_{Z_{\mathbb{R}}}\varphi(s)(P^*\otimes 1)f_{-\lambda+2\rho}.$$

Here $P\mapsto P^*$ is an anti-automorphism of $U(\mathfrak{g})$ sending $A\in\mathfrak{g}$ to $-A$. Hence it is enough to show that $U(\mathfrak{g}\oplus\mathfrak{g}))f_{-\lambda+2\rho}=(U(\mathfrak{g})\otimes 1)f_{-\lambda+2\rho}$ is dense in $\Gamma(Z_{\mathbb{R}};\mathcal{C}^{\infty}_{Z_{\mathbb{R}}}(-\lambda+2\rho,w_0\lambda+2\rho))$. Setting $\mu=-\lambda+2\rho$, the weight $\mu-\rho$ is integrally dominant. Hence this is a consequence of Proposition 4.5 below that we will prove in the next section. □

Proposition 4.5. *Assume that $\lambda-\rho$ is integrally dominant. Then the $U(\mathfrak{g}\oplus\mathfrak{g})$-submodule generated by f_λ is dense in the Fréchet space $\Gamma(Z_{\mathbb{R}};\mathcal{C}^{\infty}_{Z_R}(\lambda,-w_0\lambda))$.*

For a simple root α, let X_α be the partial flag manifold associated with α. Hence there exists a projection $p_\alpha\colon X\to X_\alpha$ whose fiber is isomorphic to $\mathbb{P}^1$.

We have the following theorem.

Theorem 4.6 ([4])**.** *Assume the following conditions.*

(a) *$\lambda-\rho$ is integrally anti-dominant,*
(b) *$\Delta_0(\lambda-\rho):=\{\alpha\in\Delta;\langle\alpha^\vee,\lambda-\rho\rangle=0\}$ is generated by simple roots.*

Then for any coherent $\mathcal{D}_{X,\lambda}$-module $\mathfrak{M}$, $\Gamma(X;\mathfrak{M})=0$ if and only if there exists a finite filtration $0=\mathfrak{M}_{-1}\subset\mathfrak{M}_0\subset\cdots\subset\mathfrak{M}_l=\mathfrak{M}$ of $\mathfrak{M}$ by coherent $\mathcal{D}_{X,\lambda}$-submodules of $\mathfrak{M}$ satisfying the following condition: for each j, there exists a simple root $\alpha_j\in\Delta_0(\lambda-\rho)$ such that $\mathfrak{M}_j/\mathfrak{M}_{j-1}$ is the pull back of a coherent module on X_{α_j} by p_{α_j}.

Proposition 4.7. *Assume that λ satisfies the conditions (a) and (b) in Theorem 4.6. Then for any $\mathfrak{M}\in\mathrm{Reg}(\mathcal{D}_{X,\lambda})$, $\Gamma(X;C_\lambda(\mathfrak{M}))=0$ if and only if $\Gamma(X;\mathfrak{M})=0$.*

Proof. Since any object in $\mathrm{Reg}(\mathcal{D}_{X,\lambda})$ has a finite length, we may assume that $\mathfrak{M}$ is a simple object. By the preceding theorem, $\Gamma(X;\mathfrak{M})=0$ if and only if there exist a simple root $\alpha\in\Delta_0(\lambda-\rho)$ such that $DR^\lambda_X(\mathfrak{M})$ is constant along the fibers of p_α. Let β be the simple root $-w_0\alpha$. Then there is commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{c_X} & X \\ {\scriptstyle p_\beta}\downarrow & & {\scriptstyle p_\alpha}\downarrow \\ X_\beta & \xrightarrow{\sim} & X_\alpha. \end{array}$$

In virtue of this diagram, the condition above is equivalent to the condition that $c_X^{-1} DR_X^{\lambda}(\mathfrak{M})$ is constant along the fibers of p_β. Since $DR_X^{-w_0\lambda}(C_\lambda(\mathfrak{M})) \simeq c_X^{-1} DR_X^{\lambda}(\mathfrak{M})$, we obtain the desired result. □

Proposition 4.8. *Assume the conditions (a) and (b) in Theorem 4.6. Then for any $\mathfrak{M} \in \mathrm{Reg}(\mathcal{D}_{X,\lambda})$, the homomorphism*

$$\Gamma(X; C_\lambda(\mathfrak{M})) \to \mathrm{Hom}_{\mathbb{C}}\Big(\Gamma(X;\mathfrak{M}), \mathbb{C}\Big)$$

is injective.

Proof. Let $\mathfrak{M}_0$ be the image of $\mathcal{D}_{X,\lambda} \otimes \Gamma(X;\mathfrak{M}) \to \mathfrak{M}$. Then $\mathfrak{M}_0$ is generated by global sections. Set $\mathfrak{M}_1 = \mathfrak{M}/\mathfrak{M}_0$. Then $\Gamma(X;\mathfrak{M}_0) \xrightarrow{\sim} \Gamma(X;\mathfrak{M})$ and $\Gamma(X;\mathfrak{M}_1) = 0$. The preceding proposition implies $\Gamma(X; C_\lambda(\mathfrak{M}_1)) = 0$. Hence $\Gamma(X; C_\lambda(\mathfrak{M})) \to \Gamma(X; C_\lambda(\mathfrak{M}_0))$ is an isomorphism. Hence the assertion follows from Proposition 4.4. □

Remark 4.9. Proposition 4.7 still holds only under the condition that $\lambda - \rho$ is integrally anti-dominant. This can be proved by the braid group action on the category of modules over the rings of twisted differential operators. Hence Proposition 4.8 and Theorem 4.10 below are also true only under the condition of integrally anti-dominance. Since we do not use these facts, we do not give the proof.

Now let us return to the Harish-Chandra module case. Let $\mathfrak{M}$ be a coherent $(\mathcal{D}_{X,\lambda}, K)$-module. Then $\Gamma(X;\mathfrak{M})$ is a Harish-Chandra module. Moreover $C_\lambda(\mathfrak{M})$ is a coherent $(\mathcal{D}_{X,-w_0\lambda}, K)$-module. By (4.3), we have the homomorphism of $(\mathfrak{g}, K)$-modules $\Gamma(X; C_\lambda(\mathfrak{M})) \to \Gamma(X;\mathfrak{M})^*$. Now assume the conditions (a) and (b) in Theorem 4.6. Then, by Proposition 4.8, this homomorphism is injective. By replacing $\mathfrak{M}$ with $C_\lambda(\mathfrak{M})$, the homomorphism $\Gamma(X;\mathfrak{M}) \to \Gamma(X; C_\lambda(\mathfrak{M}))^*$ is also injective. Thus we obtain the following theorem.

Theorem 4.10. *Assume the conditions (a) and (b) in Theorem 4.6. Then, for any coherent $(\mathcal{D}_{X,\lambda}, K)$-module $\mathfrak{M}$, $\Gamma(X; C_\lambda(\mathfrak{M}))$ is the dual Harish-Chandra module of $\Gamma(X;\mathfrak{M})$.*

Let $\mu\colon T^*X \to \mathfrak{g}^*$ denote the moment map. Then we have the following proposition proved by Borho–Brylinski (when $\lambda = 0$).

Proposition 4.11 ([2])**.** *Let $\lambda \in \mathfrak{t}_0^*$, and let $\mathfrak{M}$ be a coherent $\mathcal{D}_{X,\lambda}$-module.*

(i) $\mathrm{Ch}(\Gamma(X;\mathfrak{M})) \subset \mu(\mathrm{Ch}(\mathfrak{M}))$.

(ii) *Assume that that* $\mathfrak{M}$ *is generated by global sections. Then we have*

$$\mathrm{Ch}(\Gamma(X;\mathfrak{M})) = \mu(\mathrm{Ch}(\mathfrak{M})).$$

Proof. Let us take a coherent filtration $F(\mathfrak{M})$ of $\mathfrak{M}$. Then $\mathrm{Gr}^F(\mathfrak{M})$ is a coherent module over $\mathrm{Gr}^F(\mathcal{D}_{X,\lambda}) \simeq S_{\mathcal{O}_X}(\Theta_X)$. Here Θ_X is the sheaf of vector fields on X. By the definition, $\mathrm{Ch}(\mathfrak{M})$ is the support of the coherent $\mathcal{O}_{T^*X}$-module $\mathrm{Gr}^F(\mathfrak{M})^{\sim}$associated with $\mathrm{Gr}^F(\mathfrak{M})$. Set $M = \Gamma(X;\mathfrak{M})$ and let $F_k(M) \subset M$ be the image of $\Gamma(X;F_k(\mathfrak{M}))$. Then we have

$$\mathrm{Gr}^F M \subset \Gamma(X;\mathrm{Gr}^F(\mathfrak{M})).$$

Note that

$$\Gamma\big(X;\mathrm{Gr}^F(\mathfrak{M})\big) = \Gamma\big(\mathfrak{g}^*;\mu_*\big(\mathrm{Gr}^F(\mathfrak{M})^{\sim}\big)\big).$$

Since μ is a projective morphism, this is a $S(\mathfrak{g})$-module of finite type. Hence $F(M)$ is a coherent filtration of M and $\mathrm{Ch}(M)$ is the support of $\mathrm{Gr}^F(M)$. Hence we obtain

$$\begin{aligned}
\mathrm{Ch}(M) &= \mathrm{Supp}(\mathrm{Gr}^F(M)) \\
&\subset \mathrm{Supp}\Big(\Gamma\big(X;\mathrm{Gr}^F(\mathfrak{M})\big)\Big) \\
&\subset \mu\big(\mathrm{Supp}(\mathrm{Gr}^F(\mathfrak{M})^{\sim})\big).
\end{aligned}$$

Next we shall show (ii). For a point p outside $\mathrm{Ch}(M)$, let us show that p is not contained in $\mathrm{Ch}(\mathfrak{M})$. By the condition, $\mathcal{D}_{X,\lambda} \otimes_{U(\mathfrak{g})_\lambda} M \to \mathfrak{M}$ is an epimorphism. Let u_j $(j = 1,\cdots,r)$ be a system of generators of M. Then there exist m_j and $A_j \in F_{m_j}(U(\mathfrak{g}))$ such that $A_j u_j = 0$ and the image $a_j \in S^{m_j}(\mathfrak{g})$ of A_j does not vanish at p. Hence we have an epimorphism

$$\bigoplus_j \big(\mathcal{D}_{X,\lambda}/\mathcal{D}_{X,\lambda}A_j\big) \to \mathfrak{M}.$$

Therefore

$$\mathrm{Ch}(\mathfrak{M}) \subset \bigcup_j \mathrm{Ch}\big(\mathcal{D}_{X,\lambda}/\mathcal{D}_{X,\lambda}A_j\big) \subset \bigcup_j \mu^{-1}a_j^{-1}(0).$$

□

Now we are ready to prove the following theorem stated in the introduction.

Theorem 4.12. *Let* M *be a Harish-Chandra module. Then* $\mathrm{Ch}(M^*) = c_{\mathrm{u}}(\mathrm{Ch}(M))$.

Proof. We may assume that M is irreducible. Then we may assume that M has an infinitesimal character χ_λ with λ satisfying the conditions (a) and (b) in Theorem 4.6. Set $\mathfrak{M} = \mathcal{D}_{X,\lambda} \otimes_{U(\mathfrak{g})} M$. Then

$\mathfrak{M}$ is a coherent $(\mathcal{D}_{X,\lambda}, K)$-module and $\Gamma(X;\mathfrak{M}) \simeq M$. Theorem 4.10 implies $M^* = \Gamma(X; C_\lambda(\mathfrak{M}))$. By the preceding proposition, $\mathrm{Ch}(M) = \mu(\mathrm{Ch}(\mathfrak{M}))$ and $\mathrm{Ch}(M^*) = \mu(\mathrm{Ch}(C_\lambda(\mathfrak{M})))$.

Now let us denote by $\mathrm{SS}(DR_X^\lambda(\mathfrak{M}))$ the micro-support of $DR_X^\lambda(\mathfrak{M})$ (see [8]). Then by [8, Theorem 11.3.3], we have $\mathrm{Ch}(\mathfrak{M}) = \mathrm{SS}(DR_X^\lambda(\mathfrak{M}))$. On the other hand, we have

$$\mathrm{SS}(c_X^{-1} DR_X^\lambda(\mathfrak{M})) = c_{T^*X}^{-1}(\mathrm{SS}(DR_X^\lambda(\mathfrak{M}))).$$

Here c_{T^*X} is the automorphism of T^*X induced by the automorphism of X. Hence the theorem follows from the commutative diagram

$$\begin{array}{ccc} T^*X & \xrightarrow{c_{T^*X}} & T^*X \\ \mu\downarrow & & \mu\downarrow \\ \mathfrak{g}^* & \xrightarrow{c_{\mathrm{u}}} & \mathfrak{g}^*. \end{array}$$

□

5. Proof of Proposition 4.5

Let us denote by $\mathcal{Q}(\lambda)$ the space of $\mathfrak{g}_u$-finite function in $\Gamma(Z_{\mathbb{R}}; \mathcal{C}^\infty_{Z_{\mathbb{R}}}(\lambda, -w_0\lambda))$. Since Z_R is a homogeneous space of G_{u}, the functions in $\mathcal{Q}(\lambda)$ is real analytic, and they are the restrictions of functions in $\Gamma(Z_{\mathrm{an}}; \mathcal{O}^{(\lambda,-w_0\lambda)}_{X_{\mathrm{an}}\times X_{\mathrm{an}}})$. Since $\Gamma(Z; \mathcal{O}^{(\lambda,-w_0\lambda)}_{X\times X})$ is the space of G-finite vectors in $\Gamma(Z_{\mathrm{an}}; \mathcal{O}^{(\lambda,-w_0\lambda)}_{X_{\mathrm{an}}\times X_{\mathrm{an}}})$, we obtain

$$\Gamma(Z; \mathcal{O}^{(\lambda,-w_0\lambda)}_{X\times X}) \xrightarrow{\sim} \mathcal{Q}(\lambda).$$

The space $\mathcal{Q}(\lambda)$ has a $U(\mathfrak{g}\oplus\mathfrak{g})$-module structure, and f_λ is a unique (up to a constant multiple) vector of $\mathcal{Q}(\lambda)$ invariant by the diagonal action of $\mathfrak{g}$.

Let us denote by $M(\lambda)$ the Verma module (with respect to $\mathfrak{b}_0$) with highest weight λ, and by $M^*(\lambda)$ the dual Verma module with highest weight λ. Let us denote by $M_-(\lambda)$ the Verma module with lowest weight λ and by $M^*_-(\lambda)$ the dual Verma module with lowest weight λ. Namely, they are defined by

$$\begin{aligned} M(\lambda) &= U(\mathfrak{g})/(U(\mathfrak{g})\mathfrak{n}_0 + \sum_{H\in\mathfrak{t}_0} U(\mathfrak{g})(H-\lambda(H))), \\ M_-(\lambda) &= U(\mathfrak{g})/(U(\mathfrak{g})\bar{\mathfrak{n}}_0 + \sum_{H\in\mathfrak{t}_0} U(\mathfrak{g})(H-\lambda(H))), \\ M^*(\lambda) &= \mathrm{Hom}_{\mathbb{C}}(M_-(-\lambda), \mathbb{C})_{\mathfrak{t}_0-\mathrm{fin}}, \\ M^*_-(\lambda) &= \mathrm{Hom}_{\mathbb{C}}(M(-\lambda), \mathbb{C})_{\mathfrak{t}_0-\mathrm{fin}}. \end{aligned}$$

Here the subscript "$\mathfrak{t}_0$-fin" means the space of $\mathfrak{t}_0$-finite vectors.

Proposition 5.1. $\mathcal{Q}(\lambda) \simeq \mathrm{Hom}_{\mathbb{C}}(M(-\lambda), M^*(-\lambda))_{\mathfrak{g}-\mathrm{fin}}$.

Proof. Let us denote by p the point $(x_0, \bar{x}_0)$. Then the formal completion of $\mathcal{O}_{X\times X}(\lambda, -w_0\lambda)$ at p is isomorphic to

$$L := \mathrm{Hom}_{\mathbb{C}}(M(-\lambda) \otimes M_-(\lambda), \mathbb{C})$$

as a $\mathfrak{g} \oplus \mathfrak{g}$-module. Hence, $\mathcal{Q}(\lambda)$ is isomorphic to the space $L_{\mathfrak{g}-\mathrm{fin}}$ of $\mathfrak{g}$-finite vectors in L. Here, $\mathfrak{g}$ acts on L by the diagonal action. On the other hand, we have

$$L = \mathrm{Hom}_{\mathbb{C}}\Big(M(-\lambda), \mathrm{Hom}_{\mathbb{C}}(M_-(-\lambda), \mathbb{C})\Big).$$

Hence we obtain

$$L_{\mathfrak{g}-\mathrm{fin}} \simeq \mathrm{Hom}_{\mathbb{C}}(M(-\lambda), M^*(-\lambda))_{\mathfrak{g}-\mathrm{fin}}.$$

□

The canonical morphism $M(\lambda) \to M^*(\lambda)$ gives an element in $\mathcal{Q}(\lambda)$ This corresponds to f_λ.

The following result is due to Nicole Berline.

Proposition 5.2 ([3])**.** *If $\lambda+\rho$ is anti-dominant (i.e. $\langle \lambda+\rho, \alpha^\vee\rangle \notin \mathbb{Z}_{>0}$ for any $\alpha \in \Delta^+$), then $\mathrm{Hom}_{\mathbb{C}}(M(\lambda), M^*(\lambda))_{\mathfrak{g}-\mathrm{fin}}$ is generated by the canonical element as a $U(\mathfrak{g} \oplus \mathfrak{g})$-module.*

In fact, because $M(\lambda) \xrightarrow{\sim} M^*(\lambda)$ in this case and they are irreducible, we can apply [3, Corollary 6.9, page 398].

Combining these two propositions, we obtain the following result stated in § 4.

Proposition 4.5. *Assume that $\lambda - \rho$ is integrally dominant. Then the $U(\mathfrak{g} \oplus \mathfrak{g})$-submodule generated by f_λ is dense in the Frechet space $\Gamma(Z_{\mathbb{R}}; \mathcal{C}^\infty_{Z_R}(\lambda, -w_0\lambda))$.*

Alternative proof of Proposition 4.5 We can prove Proposition 4.5, by a different method. Let $q_1 \colon Z \to X$ be the first projection, and set $\mathfrak{N} = q_{1*}(\mathcal{O}_{X\times X}(\lambda, -w_0\lambda)|_Z)$. Then $\mathfrak{N}$ is a quasi-G-equivariant $\mathcal{D}_{X,\lambda}$-module (see [4]). By Theorem 4.10.2 in [4], the category of quasi-G-equivariant $\mathcal{D}_{X,\lambda}$-modules is equivalent to the category of locally $\mathfrak{b}_0$-finite $\mathfrak{g}$-modules on which $\mathfrak{t}_0$ acts semisimply with characters in $-\lambda+P$. Now, the $\mathfrak{g}$-module corresponding to $\mathfrak{N}$ is

$$\mathfrak{N}(x_0) = \Gamma(B\bar{x}_0; \mathcal{O}_X(-w_0\lambda)).$$

It is isomorphic to $M^*(-\lambda)$ (e.g. by [7]). Since $-\lambda+\rho$ is integrally anti-dominant, $M^*(-\lambda)$ is isomorphic to $M(-\lambda)$. Hence $\mathfrak{N}$ is isomorphic to $\mathcal{D}_{X,\lambda}$. Therefore $\Gamma(Z; \mathcal{O}_{X\times X}(\lambda, -w_0\lambda))$ is isomorphic to $\Gamma(X; \mathcal{D}_{X,\lambda})$.

By this correspondence, f_λ corresponds to the identity in $\Gamma(X; \mathcal{D}_{X,\lambda}) \simeq U(\mathfrak{g})_\lambda$. Hence $\Gamma(Z; \mathcal{O}_{X\times X}(\lambda, -w_0\lambda))$ is generated by f_λ.

References

[1] A. Beilinson, J. Bernstein, *Localisation de $\mathfrak{g}$-modules*, C. R. Acad. Sci. Paris Sér. I Math., **292** (1981) 15–18.

[2] W. Borho and J.-L. Brylinski, *Differential operators on homogeneous spaces. III. Characteristic varieties of Harish-Chandra modules and of primitive ideals,* Invent. Math., **80** (1985) 1–68.

[3] Nicole Conze, *Algèbres d'opérateurs différentiels et quotients des algèbres enveloppantes*, Bull. Soc. Math. France, **102** (1974), 379–415.

[4] M. Kashiwara, *Representation theory and D-modules on flag varieties*, Astérisque **173–174**, *Orbites Unipotentes et Représentations*, Soc. Math. France (1989) 55–109.

[5] ———, *Regular holonomic D-modules and distributions on complex manifolds*, Adv. Studies in Pure Math. **8**, *Complex analytic singularities* (1986) 199–206.

[6] ———, *D-modules and representation theory of Lie groups*, Ann. Inst. Fourier, Grenoble **43**, no.5 (1994) 1597–1618.

[7] M. Kashiwara and T. Tanisaki, *Kazhdan-Lusztig conjecture for affine Lie algebra with negative level. II. Non-integral case*, Duke Math. J., **84** (1996) 771–813.

[8] M. Kashiwara and P. Schapira, *Sheaves on Manifolds*, Grundlehren der mathematischen Wissenschaften **292**, Springer-Verlag (1990).

[9] T. Matsuki, *Orbits on affine symmetric spaces under the action of parabolic subgroup*, Hiroshima Math. J. **12** (1982) 307–320.

(D. Barlet) Institut E. Cartan, BP 239, 54506 Vandœvres-les Nancy, Cedex, France

(M. Kashiwara) Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606–8502, Japan