RIMS-569

Character, Character Cycle, Fixed Point Theorem and Group Representations

with the appendix
"Open Problems in the Theory Representations of Lie Groups"

By

Masaki KASHIWARA

April 1987

京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

Character, character cycle, fixed point theorem and Group representations

by

Masaki Kashiwara
RIMS, Kyoto University

§0. Introduction.

Among many methods to derive Weyl's character formula, there is an application of the fixed point theorem (à la Atiyah-Singer) to a line bundle on the flag variety. Namely, any finite-dimensional irreducible representation of a reductive group G is obtained as the cohomology group of an equivariant line bundle on the flag variety. Hence the trace of the action of an element g of G is obtained as the sum of the contributions at each fixed point. When g is a regular element, there are as many fixed points as the order of the Weyl group and each of them gives one of the terms $\operatorname{sgn} w\, e^{w\lambda}/\Pi(e^{\alpha/2}-e^{-\alpha/2})$ in Weyl's character formula.

On the other hand, Harich-Chandra [HC] defined the character of an (infinite-dimensional) representation of a real semisimple group $G_{\mathbb{R}}$ as an invariant eigendistribution. In this paper we shall give a character formula in terms of the geometry of flag manifold as a conjecture and prove it for discrete series. The correspondance of Harich-Chandra modules and K-equivariant sheaves is completed by adding representations of $G_{\mathbb{R}}$ and $G_{\mathbb{R}}$-equivariant sheaves (See [K_2] and also the articles of W. Schmid and J. Wolf in the same volume). Then the character would be calculated from $G_{\mathbb{R}}$-equivariant sheaves. We can illustrate this schematically as follows.

{invariant eigendistributions} ←┐

↕

{Harich-Chandra modules} ↔ {representations of $G_{\mathbb{R}}$}

↕ ↕

{K-equivariant sheaves} ↔ {$G_{\mathbb{R}}$-equivariant sheaves}

↕ ↕

{character cycles} {character cycles} ←┘

§1. Formalism around fixed point theorem

§1.0. Let X be a compact manifold and $f: X \to X$ a continuous map. Then $\sum(-1)^i \mathrm{tr}(f: H^i(X))$ is calculated as the intersection number of the graph of f and the diagonal set. We shall generalize this fact.

§1.1. Notations. In this note, for a topological space X, we denote by $D(X)$ the derived category of the abelian category of sheaves of $\mathbb{C}$-vector spaces. If X is a locally compact space with finite cohomological dimension, we denote by ω_X the dualizing sheaf ; i.e. $\omega_X = a_X^! \mathbb{C}$ where $a_X: X \to pt$ is the projection from X to the set (pt) consisting of a single element. Let $\mathbb{D}$ denote the Verdier dual, i.e. $\mathbb{D}(F) = \mathbb{R}\underline{Hom}(F, \omega_X)$. For a topological manifold X, let $\underline{or}_X$ denote the orientation sheaf of X, so that we have $\omega_X = \underline{or}_X[\dim X]$. For a subanalytic (resp. complex analytic) space X, $\mathbb{D}_{\mathbb{R}-c}(X)$ (resp. $D_{\mathbb{C}-c}(X)$) denotes the full subcategory of $D(X)$ consisting of bounded complexes with $\mathbb{R}$-constructible (resp. $\mathbb{C}$-constructible) sheaves as cohomology groups. Here an $\mathbb{R}$-constructible (resp. $\mathbb{C}$-constructible) sheaf is a sheaf F admitting a subanalytic (resp. complex analytic) stratification such that the restriction of F to each stratum is a locally constant sheaf of finite rank.

§1.2. Let X be a compat real analytic manifold, F an $\mathbb{R}$-constructible complex of sheaves on X. Let $\varphi: f^*F \to F$

be a morphism in D(X). We set

$$(1.2.1) \qquad \operatorname{tr} \varphi = \sum_i (-1)^i \operatorname{tr}(\varphi : H^i(X;F)).$$

Then $\operatorname{tr}(\varphi)$ is expressed by local contributions as follows. Let $s: X \hookrightarrow X \times X$ denotes the graph map $x \mapsto (x, f(x))$, $j: X \hookrightarrow X \times X$ the diagonal embedding and $p_i: X \times X \to X$ the i-th projection (i=1,2). Then we have a chain of homomorphisms

$$\begin{aligned}(1.2.2) \quad \mathbb{R}\mathrm{Hom}(f^*F,F) &\overset{\sim}{\to} \mathbb{R}\Gamma(X; \mathbb{R}\underline{\mathrm{Hom}}(s^*p_2^*F,\ s^! p_1^! F)) \\ &\overset{\sim}{\leftarrow} \mathbb{R}\Gamma(X\times X;\ s_! \mathbb{R}\underline{\mathrm{Hom}}(p_2^*F, p_1^!F)) \overset{\sim}{\leftarrow} \mathbb{R}\Gamma_{s(X)}(X\times X; F \boxtimes \mathbb{D}F) \\ &\to \mathbb{R}\Gamma_{j^{-1}s(X)}(X;\ j^*(F\boxtimes \mathbb{D}F)) \\ &\to \mathbb{R}\Gamma_{j^{-1}s(X)}(X;\ F \otimes \mathbb{D}F) \\ &\to \mathbb{R}\Gamma_{j^{-1}s(X)}(X;\ \omega_X),\end{aligned}$$

and

$$(1.2.3) \qquad \mathbb{R}\Gamma_{j^{-1}s(X)}(X;\omega_X) \to \mathbb{R}\Gamma(X;\omega_X) \to \mathbb{C}.$$

Then the image of $\varphi \in \mathrm{Hom}(f^*F,F)$ by their composition $\mathrm{Hom}(f^*F,F) \to \mathbb{C}$ coincides with $\operatorname{tr}(\varphi)$.

§1.3. Assume moreover that, in the situation of §1.2, the fixed point set $j^{-1}s(X)$ is discrete. Then to each fixed point $x \in X$, we can associate the image of φ by the composition of $\mathrm{Hom}(f^*F,F) \to H^0_{j^{-1}s(X)}(X;\omega_X) \overset{\sim}{\to} \bigoplus_{y \in j^{-1}s(X)} H^0_{\{y\}}(X;\omega_X)$

$\to H^0_{\{x\}}(X;\omega_X) \to \mathbb{C}.$

We shall denote this by $tr_x(\varphi)$. Then $tr(\varphi)$ is expressed by the local contributions:

$$(1.3.1) \qquad tr(\varphi) = \sum_x tr_x(\varphi),$$

where x ranges over the fixed point set.

§1.4. We shall calculate explicitly $tr_x(\varphi)$. Assume that

(1.4.1) X is a real analytic manifold and the diagonal set and the graph of f intersect transversally.

Let x be a fixed point of f. Then f induces the homomorphism $f_*:T_xX\to T_xX$ of the tangent space and φ induces $\varphi:\nu_x(F)\to\nu_x(F)$. Here ν denotes the normalization functor (See [KS]).

Let V_s be a vector subspace of T_xX invariant by f_* satisfying (1.4.2) and (1.4.3):

(1.4.2) No eigenvalue λ of $f_*|V_s$ satisfies $\lambda > 1$

(1.4.3) No eigenvalue λ of $f_*|T_xX/V_s$ satisfies $0 \leqq \lambda < 1$.

Then we can prove the following proposition.

<u>Proposition 1.4.1</u>. Under the condition (1.4.1), we have

$$tr_x(\varphi) = \sum_i (-1)^i tr(\varphi : H^i_{V_s}(T_xX;\ \nu_x(F))).$$

Corollary 1.4.2. If moreover X is a complex manifold and if F is $\mathbb{C}$-constructible then we have

$$\mathrm{tr}_x(\varphi) = \sum_i (-1)^i \mathrm{tr}(\varphi;\ H^i_{\{x\}}(X;F))$$
$$= \sum_i (-1)^i \mathrm{tr}(\varphi;\ H^i(F)_x).$$

Example 1.4.3. Set $X = \mathbb{R} \cup \infty$, $f: x \mapsto ax$, $F = \mathbb{C}_{\{x \geq 0\}}$ and let $\varphi: f^*F \to F$ be the homomorphism such that $\varphi_0: F_0 \to F_0$ is the identity. Then, at $x = 0$, $\mathrm{tr}_x(\varphi) = 0$ when $a > 1$ and $\mathrm{tr}_x(\varphi) = 1$ when $0 < a < 1$.

§1.5. We shall generalize the situation in 1.4. Let X and Y be subanalytic spaces, $f,g: X \rightrightarrows Y$ two continuous subanalytic maps, and $F \in D_{\mathbb{R}-c}(Y)$. Let $\varphi: f^*F \to g^*F$ be a morphism. Let us denote by $s: X \to Y \times Y$ the map $x \mapsto (f(x), g(x))$ and let Δ_Y denote the diagonal set and $Z = s^{-1}\Delta_Y = \{x \in X;\ f(x) = g(x)\}$. Then we have a chain of homomorphisms:

$$\mathbb{R}\,\mathrm{Hom}(F,F) \xrightarrow{\sim} \mathbb{R}\Gamma_{\Delta_Y}(Y\times Y;\ F \boxtimes \mathbb{D}F) \to \mathbb{R}\Gamma_Z(X;\ s^*(F \boxtimes \mathbb{D}F))$$
$$\xrightarrow{\sim} \mathbb{R}\Gamma_Z(X; f^*F \otimes g^*\mathbb{D}F)$$
$$\xrightarrow{\varphi} \mathbb{R}\Gamma_Z(X;\ g^*F \otimes g^*\mathbb{D}F))$$
$$\xrightarrow{\sim} \mathbb{R}\Gamma_Z(X;\ g^*(F \otimes \mathbb{D}F))$$
$$\to \mathbb{R}\Gamma_Z(X;\ g^*\omega_Y).$$

Hence $id_F \in Hom(F,F) = H^0(\mathbb{R}\ \underline{Hom}(F,F))$ gives an element $c(F,\varphi) \in H^0_Z(X;\ g^*\omega_Y)$. If $X = Y$ and $g = id$, then this construction coincides with the former one.

§1.6. We shall specialize the preceding construction to a group action case. Let X be a subanalytic space, G a Lie group operating on X. Let F be a G-equivariant $\mathbb{R}$-constructible complex of sheaves. In this note, we shall not investigate systematically the notion of G-equivariant $\mathbb{R}$-constructible complex of sheaves. However, this notion implies an isomorphism φ: $\mu^*F \to pr^*F$, where μ: $G\times X \to X$ is the composition map $(g,x) \mapsto gx$ and pr: $G\times X \to X$ is the second projection. Thus we can apply the result of §1.5, and we obtain $c(F,\varphi) \in H^0_{\tilde{G}}(G\times X;\ pr^*\omega_X)$. Here $\tilde{G}$ denotes the fixed point set $\{(g,x) \in G\times X;\ gx = x\}$. Note that $H^0_{\tilde{G}}(G\times X;\ pr^*\omega_X) = H^0_{\tilde{G}}(G\times X;\ \omega_{G\times X} \otimes \underline{or}_G[-\dim G]) = H^{-\dim G}(\tilde{G};\ \omega_{\tilde{G}} \otimes \underline{or}_G)$ $= H^{inf}_{\dim G}(\tilde{G};\ \underline{or}_G)$. Here $H^{inf}_n(\tilde{G};\ \underline{or}_G)$ is the n-th homology group of $\underline{or}_G$-valued locally finite chains. We shall denote $c(F,\varphi)$ by $\underline{ch}(F) \in H^{inf}_{\dim G}(\tilde{G};\ \underline{or}_G)$ and call it the <u>character cycle</u> of F.

§1.7. When X has only finitely many G-orbits, the situation is simple. In fact, in such a case, we have $\tilde{G} = \cup \pi^{-1}(S)$, where π is the projection $\tilde{G} \to X$. Then $\pi^{-1}(S)$ being the fiber bundle over S with the isotropy subgroup as a fiber, $\pi^{-1}(S)$ is a $(\dim G)$-dimensional manifold. Hence we have $H^{inf}_{\dim G}(\tilde{G};\ \underline{or}_H) \subset A^N$, where N is the number of connected components of the regular locus of $\tilde{G}$.

§1.8. In ([K_1]), we defined the characteristic cycle for constructible sheaves. Let us reformulate this. Let X be a real analytic manifold and $F \in \mathbb{D}_{\mathbb{R}-c}(X)$. Then, denoting by Δ_X the diagonal set and by μ the microlocalization functor (see [KS]), we have

$$\begin{aligned}
&\mathbb{R}Hom(F,F) \to \mathbb{R}\Gamma_{\Delta_X}(X\times X; F \boxtimes \mathbb{D}F) \\
&\to \mathbb{R}\Gamma(T^*X;\ \mu_{\Delta_X}(F \boxtimes \mathbb{D}F)) \\
&= \mathbb{R}\Gamma_{SSF}(T^*X\ ;\ \mu_{\Delta_X}(F \boxtimes \mathbb{D}F)) \\
&\to \mathbb{R}\Gamma_{SSF}(T^*X;\ \mu_{\Delta_X}(\mathbb{R}j_* j^*(F \boxtimes \mathbb{D}F))) \\
&\to \mathbb{R}\Gamma_{SSF}(T^*X;\ \mu_{\Delta_X}(j_*\omega_X)) \\
&\simeq \mathbb{R}\Gamma_{SSF}(T^*X;\ \pi^*\omega_X)
\end{aligned}$$

Here $\pi\colon T^*X \to X$ is the cotangent bundle and $j\colon X \hookrightarrow X\times X$ is the diagonal embedding. We have furthermore

$$\mathbb{R}\Gamma_{SSF}(T^*X;\ \pi^*\omega_X) = \mathbb{R}\Gamma_{SSF}(T^*X;\ \omega_{T^*X} \otimes \underline{or}_X\ [-\dim X]).$$

The image of $id_F \in Hom(F,F)$ by the homomophism

$$Hom(F,F) \to H^{-\dim X}_{SSF}(T^*X;\ \omega_{T^*X} \otimes \underline{or}_X) = H^{\inf}_{\dim X}(SSF;\ \underline{or}_X)$$

is called the <u>characteristic cycle</u> of F, and denoted by $\underline{SS}(F)$. This definition coincides with the one given in [K].

§1.9. Let X be a homogeneous space of a Lie group G and let H be a subgroup of G. Let $\mathfrak{g}$ and $\underline{h}$ denote the Lie algebra of G and H, respectively. Let F be an H-equivariant $\mathbb{R}$-constructible complex on X. Let us investigate the relation between the character cycle $\underline{ch}(F)$ and the characteristie cycle $SS(F)$ of F. Let $\rho\colon \tilde{G}\to G$ and $\pi\colon \tilde{G}\to X$ denote the projections. Let us consider the following chain of homomorphisms

$$(1.9.1)\quad \mathbb{R}\Gamma_{\rho^{-1}(H)}(H\times X;\ \mathbb{C}_H\boxtimes\omega_X) = \mathbb{R}\Gamma(G\times X;\ \mathbb{R}\underline{Hom}(\mathbb{C}_{\tilde{G}},\ \mathbb{C}_H\boxtimes\omega_X))$$
$$\to \mathbb{R}\Gamma(T^*_{e\times X}(G\times X);\ \mathbb{R}\underline{Hom}(\mu_{e\times X}(\mathbb{C}_{\tilde{G}}),\ \mu_{e\times X}(\mathbb{C}_H\boxtimes\omega_X))).$$

Here $\mu_{e\times X}$ is the microlocalization functor along $\{e\}\times X$. Note that $T^*_{e\times X}(G\times X)=\mathfrak{g}^*\times X$ and

$$\mu_{e\times X}(\mathbb{C}_{\tilde{G}}) = \mathbb{C}_{T^*X}\otimes\underline{or}_{\mathfrak{g}}\otimes\underline{or}_X[\dim X-\dim G]$$

and

$$\mu_{e\times X}(\mathbb{C}_H\boxtimes\omega_X) = (\mathbb{C}_{\underline{h}^{\perp}}\boxtimes\omega_X)\otimes\underline{or}_h[\dim H] = \omega_{h^{\perp}\times X}\otimes\underline{or}_{\mathfrak{g}}[-\dim G].$$

Here we identify T^*X with the subset of $\mathfrak{g}^*\times X$ by the moment map $\bar{p}\colon T^*X\to\mathfrak{g}^*$ and $h^{\perp}\subset\mathfrak{g}^*$ denotes the orthogonal complement. Hence the last term of (1.9.1) coincides with

$$\mathbb{R}\Gamma(\mathfrak{g}^*\times X;\ \mathbb{R}\Gamma_{T^*X}(\omega_{h^{\perp}\times X})\otimes\underline{or}_X[-\dim X])$$
$$= \mathbb{R}\Gamma(\bar{p}^{-1}(h^{\perp});\ \omega_{\bar{p}^{-1}(h^{\perp})}\otimes\underline{or}_X[-\dim X]).$$

Thus we obtained

$$(1.9.2) \qquad H^{0}_{\bar{\rho}^{1}(H)}(H\times X;\ \mathbb{C}_H \boxtimes \omega_X) \to H^{-\dim X}_{\bar{\rho}^{-1}(h^{\perp})}(T^*X;\ \omega_{T^*X} \otimes \underline{or}_X)$$

Since F is H-equivariant, SS(F) is contained in $\bar{\rho}^{-1}(h^{\perp})$ and we obtain

$$(1.9.3) \qquad H^{-\dim X}_{SS(F)}(T^*X;\ \omega_{T^*X} \otimes \underline{or}_X)$$
$$\to H^{-\dim X}_{\bar{\rho}^{-1}(h^{\perp})}(T^*X;\ \omega_{T^*X} \otimes \underline{or}_X)$$

One can easily prove the following proposition.

Proposition 1.9.1. The image of the character cycle $\underline{ch}(F)$ by the homomorphism (1.9.2) coincides with the image of the characteristic cycle $\underline{SS}(F)$ by the homomorphism (1.9.3).

If X has finitely many H-orbits, then the homomorphism (1.9.3) is injective because $\dim \bar{\rho}^{-1}(h^{*}) \leq \dim X$. Therefore $\underline{SS}(F)$ is determined by $\underline{ch}(F)$.

§2. Representation of semisimple groups.

§2.1. Let G be a connected complex semisimple group and let $G_{\mathbb{R}} \hookrightarrow G$ be a real form of G. Let $K_{\mathbb{R}}$ be a maximal compact subgroup of $G_{\mathbb{R}}$ and K the complexification of $K_{\mathbb{R}}$. Let X be the flag manifold of G. Set $\tilde{G}=\{(g,x)\in G\times X;\ gx=x\}$ and let $\rho:\tilde{G}\to G$ and $\pi:\tilde{G}\to X$ denote the projections. For $x\in X$, let $B(x)$ denote the Borel subgroup $\pi^{-1}(x)$ corresponding to x. Let $\mathfrak{g}, \mathfrak{k}, \mathfrak{g}_{\mathbb{R}}, \mathfrak{k}_{\mathbb{R}}$ and $b(x)$ denote the Lie algebra of G, K, $G_{\mathbb{R}}$, $K_{\mathbb{R}}$ and $B(x)$, respectively.

Let us denote by G_{reg} the set of regular semisimple elements of G.

§2.2. Let $\mathcal{M}$ denote the $\mathcal{D}_G$-module for invariant eigendistributions. Hence $\mathcal{M}$ is a $\mathcal{D}_G$-module generated by a section u with the relation

$$(2.2.1)\qquad \begin{cases} (\mathrm{Ad}\,\mathfrak{g})\cdot u = 0 \\ Pu = \chi(P)u \quad \text{for} \quad P\in \mathfrak{z}(\mathfrak{g}) \end{cases}$$

Here $\mathrm{Ad}\,\mathfrak{g}$ is the image of $\mathfrak{g}\to\Gamma(G;\Theta_G)$ derived by the adjoint action of G on G, and $\mathfrak{z}(\mathfrak{g})$ is the center of $U(\mathfrak{g})$ considered as the space of bi-invariant differential operators on G and χ is the trivial infinitesimal character $\mathfrak{z}(\mathfrak{g})\to\mathfrak{z}(\mathfrak{g})/(\mathfrak{z}(\mathfrak{g})\cap U(\mathfrak{g})\mathfrak{g})\to\mathbb{C}$. A similar argument to [HK] leads us the following proposition.

Proposition 2.2.1. (i) $\mathcal{M}$ is a regular holonomic $\mathcal{D}_G$-module.

(ii) $\mathbb{R}\underline{Hom}(\mathcal{M}, \mathcal{O}_G)=\mathbb{R}\rho_*\mathbb{C}_{\tilde{G}}$.

(iii) $\mathbb{R}\rho_*\mathbb{C}_{\tilde{G}}$ is the minimal extension of $\rho_*\mathbb{C}_{\tilde{G}}|_{G_{reg}}$.

§2.3. Let us investigate the space of invariant eigendistributions. This is equal to $H^0(G_{\mathbb{R}};\mathbb{R}\underline{Hom}_{\mathcal{D}_G}(\mathcal{M}, \mathcal{B}_{G_{\mathbb{R}}}))$. Here, $\mathcal{B}_{G_{\mathbb{R}}}=\mathcal{H}^{\dim G_{\mathbb{R}}}_{G_{\mathbb{R}}}(\mathcal{O}_X)\otimes\underline{or}_G\otimes\underline{or}_{G_{\mathbb{R}}}$ is the sheaf of hyperfunctios. We have

$$\mathbb{R}\Gamma(G_{\mathbb{R}};\mathbb{R}\underline{Hom}_{\mathcal{D}_G}(\mathcal{M}, \mathcal{B}_{G_{\mathbb{R}}}))$$

$$= \mathbb{R}\Gamma(G;\mathbb{R}\Gamma_{G_{\mathbb{R}}}(\mathbb{R}\underline{Hom}_{\mathcal{D}_G}(\mathcal{M};\mathcal{O}_X)) \otimes \underline{or}_G \otimes \underline{or}_{G_{\mathbb{R}}})[\dim G_{\mathbb{R}}]$$

$$= \mathbb{R}\Gamma(G;\mathbb{R}\Gamma_{G_{\mathbb{R}}}(\mathbb{R}\rho_*\mathbb{C}_{\tilde{G}}) \otimes \underline{or}_G \otimes \underline{or}_{G_{\mathbb{R}}})[\dim G_{\mathbb{R}}]$$

$$= \mathbb{R}\Gamma(\tilde{G};\mathbb{R}\Gamma_{\rho^{-1}G_{\mathbb{R}}}(\mathbb{C}_{\tilde{G}}) \otimes \underline{or}_G \otimes \underline{or}_{G_{\mathbb{R}}})[\dim G_{\mathbb{R}}]$$

$$= \mathbb{R}\Gamma(\rho^{-1}G_{\mathbb{R}};\omega_{\rho^{-1}G_{\mathbb{R}}} \otimes \underline{or}_{G_{\mathbb{R}}})[-\dim G_{\mathbb{R}}].$$

Hence we have

Proposition 2.3.1. The space of invariant eigendistribution on $G_{\mathbb{R}}$ coincides with $H^{inf}_{\dim G_{\mathbb{R}}}(\rho^{-1}G_{\mathbb{R}};\underline{or}_{G_{\mathbb{R}}})$.

We shall call this correspondence the Hirai correspondence by the reason explained in the next paragraph.

§2.4. Let us write explicitly the correspondence in

Proposition 2.3.1. Let $p=(g,x)$ be a point of $\tilde{G}$. The isotropy group $G_x=\pi^{-1}(x)$ acts on $T_x^*X=(\mathfrak{g}/b(x))^*$. Let us denote $\psi(p)=1/\det(1-g:T_x^*X)$. Then ψ is the meromorphic function with the pole in $\rho^{-1}G_{reg}$. If T is a Cartan subgroup containing g and contained in $B(x)$, we have

$$\psi = \frac{1}{\prod_{\alpha>0}(1-e^{-\alpha(a)})}$$

for $g=e^a$, $a\in \mathrm{Lie}(T)$. For $\sigma\in H^{\inf}_{\dim G_{\mathbb{R}}}(\rho^{-1}(G_{\mathbb{R}});or_{G_{\mathbb{R}}})$, let us denote by f_σ the corresponding invariant eigendistribution. Then for a regular semisimple element g of $G_{\mathbb{R}}$, we have

$$(2.4.1)\quad f_\sigma(g) = \sum_{p\in\rho^{-1}(g)} \sigma(p)\psi(p).$$

Here, $\sigma(p)$ is the intersection number of σ and $\rho^{-1}(g)$ at p.

If an invariant eigendistribution f on $G_{\mathbb{R},reg}$ is given, then it determines the $(\dim G)$-chain α in $\rho^{-1}G_{\mathbb{R}}$. Then f is extended to an invariant eigendistribution on $G_{\mathbb{R}}$ if and only if the boundary of α vanishes. If we write it down, we obtain Hirai's connection formula for invariant eigendistributions ([H]).

§2.5. By Matsuki [M], there exists a correspondence between K-orbits and $G_{\mathbb{R}}$-orbits. This correspondence $S\leftrightarrow S^a$ is characterized by the following property:

(2.5.1) $S\cap S^a$ is compact and non empty.

In such a case, $S \cap S^a$ is a homogeneous space over $K_{\mathbb{R}}$. Moreover, we have

$$(2.5.2) \qquad K_{\mathbb{R}x}/K_{\mathbb{R}x}^{\circ} \cong K/K_x^{\circ} \quad \text{and} \quad K_{\mathbb{R}x}/K_{\mathbb{R}x}^{\circ} \cong G_{\mathbb{R}x}/G_{\mathbb{R}x}^{\circ} \quad \text{for} \quad x \in S \cap S^a.$$

Here the subscript x signifies the isotropy subgroup at x and $\circ$ means the connected component containing the identity.

This shows immediately the following lemma.

Lemma 2.5.1. The set of pairs (S,F) of a K-orbit S and a K-equivariant local system F on S is isomorphic to the set of pairs (S^a,F^a) of a $G_{\mathbb{R}}$-orbit S^a and a $G_{\mathbb{R}}$-equivariant local system F^a on S^a.

Note that (S,F) and (S^a,F^a) correspond if we have (2.5.1) and

$$(2.5.3) \qquad F|_{S \cap S^a} \cong F^a|_{S \cap S^a} \quad \text{as } K_{\mathbb{R}}\text{-equivariant local systems.}$$

We call this correspondence the Matsuki correspondence.

§2.6. Let $\sigma=(S,F)$ be a pair of a K-orbit S and a K-equivariant local system F on S. Let $j:S \hookrightarrow X$ be the embedding. Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_X$-module such that $j_!F[-\operatorname{codim} S]=\mathbb{R}\underline{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$. Then $\mathcal{M}$ is a K-equivariant $\mathcal{D}_X$-module. Then $E=H^0(X;\mathcal{M})$ is a Harich-Chandramodule. To E we can associate a representation of $G_{\mathbb{R}}$ such that the space of its $K_{\mathbb{R}}$-finite vectors is E. Let $\chi(E)$ be its character, which is an invariant eigendistribution on $G_{\mathbb{R}}$ (Harich-Chandra [HC]).

Let (S^a, F^a) be the associated pair to σ by the Matsuki correspondence. Denoting by $j_a : S^a \hookrightarrow X$ the imbedding, we set $F' = \mathbb{R}j_{a_*}(F^a \otimes j_a^! \mathbb{C}_X)$ $[-\mathrm{codim}_{\mathbb{C}} S]$ (See [K_2]). Since F' is a $G_{\mathbb{R}}$-equivariant sheaf, we can define its character cycle $\underline{ch}(F')$ as in §1.6. We have

$$\underline{ch}(F') \in H^{\inf}_{\dim G_{\mathbb{R}}}(\rho^{-1}G_{\mathbb{R}}, \underline{or}_{G_{\mathbb{R}}}).$$

<u>Conjecture</u>. The character $\chi(E)$ of E is equal to the invariant eigendistribution corresponding to $\underline{ch}(F')$ by the Hirai correspondence (§2.3).

§2.7. We shall prove the conjecture when the representation E is a discrete series. In such a case, S is a closed orbit, S^a is an open orbit, and F is a trivial local system. Let f be the invariant eigendistribution corresponding to $\underline{ch}(F')$. By the characterization of $\chi(E)$, it is enough to show

(2.7.1) $f = \chi(E)$ on the regular part of a compact Cartan subgroup,

(2.7.2) $|Df|$ is bounded on $G_{\mathbb{R};\mathrm{reg}}$.

Here D is the discriminant $\left|\prod_{\alpha>0}(e^{\alpha/2}-e^{-\alpha/2})\right|$.

§2.8. In order to prove (2.7.1) and (2.7.2), we shall calculate $\underline{ch}(F)$ for a $G_{\mathbb{R}}$-equivariant sheaf F. Let g be a regular semi-simple element of $G_{\mathbb{R}}$. Let $p=(g,x)$ be a point in $\tilde{G}$ above g.

Let H be a Cartan subgroup containing g and let $\mathfrak{g}=(\bigoplus_{\alpha\in\Delta}\mathfrak{g}_\alpha)\oplus \mathrm{Lie}(H)$ be the root space decomposition with respect to $\mathrm{Lie}(H)$. Let $b=\bigoplus_{\alpha\in\Delta_-}\mathfrak{g}_\alpha\oplus\mathrm{Lie}(H)$ be the isotropy subalgebra at x and

(2.8.1) $$n(p) = \bigoplus_{\alpha\in\Delta_+,\,|(e^\alpha)(g)|<1}\mathfrak{g}_\alpha .$$

Here $\Delta=\Delta_+\cup\Delta_-$ is the corresponding positive and negative roots system. Then $n(p)$ is a nilpotent Lie algebra. Set $U(p)=\exp n(p)$. Then by Proposition 1.4.1, we can see easily

<u>Proposition 2.8.1</u>. $$(ch(F)\cdot\rho^{-1}(g))_p = tr(g:\mathbb{R}\Gamma_{U(p)\cdot x}(X;F)).$$

§2.9. Coming back to the situation of §2.7, we shall prove (2.7.1).

Let us take a compact Cartan subgroup H.

Then any fixed point of H in X is contained in an open $G_{\mathbb{R}}$-orbit. Let us take a fixed point $x_0\in S^a$ and choose a positive ordering $\Delta_+(x_0)$ of the root system of $(\mathfrak{g},\mathrm{Lie}(H))$ such that $\Delta_+(x_0)=\Delta(T_{x_0}X)$.

Set $W_{\mathbb{R}}=N(H)/H$. Then we have

(2.9.1) $$L = \{x\in S^a \,;\, Hx = x\} = W_{\mathbb{R}}\cdot x_0 .$$

By Harich-Chandra [HC], we have

$$(2.9.2)\qquad (-1)^q\chi(E) = \frac{\sum_{w\in W_{\mathbb{R}}} (\operatorname{sgn} w)e^{w\rho(x_0)}}{\prod_{\alpha\in\Delta^+(x_0)} (e^{\alpha/2}-e^{-\alpha/2})} = \sum_{x\in L} \frac{1}{\prod_{\alpha\in\Delta^+(x)} (1-e^{-\alpha})}$$

where $q = \frac{1}{2}\dim(G_{\mathbb{R}}/K_{\mathbb{R}})$.

On the other hand, for $h\in H_{reg}$ and $x\in L$

$$\operatorname{tr}(h : \mathbb{R}\Gamma_{\{x\}}(X ; F)) = (-1)^{\operatorname{codim}S}$$

Hence by Proposition the value of f at h equals

$$\sum_{x\in L} \frac{1}{\prod_{\alpha\in\Delta_+(x)} (1-e^{-\alpha})} (-1)^{\operatorname{codim}S}.$$

Hence (2.7.1) follows from $q=\operatorname{codim}S$.

§2.10. Finally, we shall prove (2.7.2). Let us take $g\in G_{\mathbb{R}}$ and $p=(g,x)\in\tilde{G}$. As seen in §2.8, the contribution from p to $f(g)$ is

$$(2.10.1)\qquad c\cdot\frac{1}{\prod_{\alpha\in\Delta_+(x)} (1-e^{-\alpha})} = \frac{ce^{\rho}}{\prod(e^{\alpha/2}-e^{-\alpha/2})}$$

Here $\rho=\frac{1}{2}\sum_{\alpha>0}\alpha$, $c=\operatorname{tr}(g:\mathbb{R}\Gamma_{U(p)x}(F'))$ and $e^{\rho}=\sqrt{\det(g:T_xX)}$.

Hence in order to prove (2.7.2) it is enough to show

(2.10.2) If $c\neq 0$, then $|\det(g : T_xX)| \leqq 1$.

We have $\mathbb{R}\Gamma_{U(p)x}(X;F')=\mathbb{R}\mathrm{Hom}((F^a\otimes j_a{}^!\mathbb{C}_X)_{U(p)x};\mathbb{C}_X)$. Hence it is zero if $U(p)x\cap S^a=\phi$. Therefore (2.10.2) is a consequence of

(2.10.3) If $U(p)x \cap S^a \neq \phi$, then $|\det(g:T_x X)| \leqq 1$.

This is an easy consequence of Lemma 7 of [OM](p.378). Thus we obtain

Proposition 2.10.1. Conjecture is true if E is a discrete series.

References

[H] T. Hirai, Invariant eigendistributions of Laplace operators on real simple Lie groups II, Japan. J. Math., 2 (1976), 28-89.

[HC] Harich-Chandra, 1. Representations of a semisimple Lie group on a Banach space, I, Trans. Amer. Math. Soc., 74 (1953), 185-243,

2. The characters of semisimple Lie groups, ibid., 83 (1956), 98-163,

3. Invariant eigendistributions on semisimple Lie groups, ibid., 119 (1965), 457-508,

4. Discrete series for semisimple Lie groups I, Acta Math., 113 (1965), 241-318.

[HK] R. Hotta and M. Kashiwara, The invariant holonomic system on a semisimple Lie algebra, Invent. Math., 75 (1984), 327-358.

[K] M. Kashiwara, 1. Index theorem for constructible sheaves, Astérisque, 130 (1985), 193-209,

2. Open problems in group representation theory, Proceeding of Taniguchi Symposium held in 1986.

[KS] ——— and P. Schapira, Microlocal study of Sheaves, Astérisque 128, 1985.

[M] T. Matsuki, Orbits on affine symmetric spaces under the action of parabolic subgroups, Hiroshima Math. J., 12 (1982), 307-320.

[OM] T. Oshima and T. Matsuki, A description of discrete series for semisimple symmetric spaces, Advanced Studies in Pure Math., 4 (1984), 331-390.

Open Problems

in the Theory of Representations of Lie Groups

—— Proceeding of Taniguchi Symposium at Katata ——

By

Masaki KASHIWARA

RIMS, Kyoto University

A. Suggested by the talk of W. Schmid on his work with J. Wolf, I will give the following conjectures which restate their works.

The general scheme can be illustrated as follows:

$$\begin{array}{ccc}
\{\text{Harich-Chandra modules}\} & \longleftrightarrow & \{\text{the representations of } G_{\mathbb{R}}\} \\
\updownarrow & & \\
\{(D,K)\text{-modules}\} & & \updownarrow \\
\updownarrow & & \\
\{K\text{-equivariant sheaves}\} & \longleftrightarrow & \{G_{\mathbb{R}}\text{-equivariant sheaves}\}
\end{array}$$

B. Let G be a complex semi-simple group, $G_{\mathbb{R}}$ a real form of G, T the Cartan subgroup of G, t the Lie algebra of T. Let X be the flag manifold of X. For $\lambda \in t^*$, let $\mathcal{O}_X(\lambda)$ denote the twisted invertible $\mathcal{O}_X$-module corresponding to λ. We normalize $\mathcal{O}_X(\lambda)$ such that $\mathcal{O}_X(\rho) = \mathcal{O}_X, \mathcal{O}_X(-\rho) = \Omega_X^n$, where ρ is the half sum of positive roots and $n = \dim X$. Let $\mathcal{D}_\lambda$ denote the sheaf of differential endomorphisms of $\mathcal{O}_X(\lambda)$. Let $K_{\mathbb{R}}$ be a maximal compact subgroup of $G_{\mathbb{R}}$ and let K be the complexification of K. Let us fix a <u>regular</u> $\lambda \in t^*$. Let us denote by $D_K(X)_\lambda$ (resp. $D_{G_{\mathbb{R}}}(X)_\lambda$) the derived category of K-equivariant (resp. $G_{\mathbb{R}}$ -equivariant) constructible complexes of twisted sheaves with the same twist as $\underline{O}_X(\lambda)$.

Conjecture 1. Two triangulated categories $D_K(X)_\lambda$ and $D_{G_{\mathbb{R}}}(X)_\lambda$ are equivalent.

Let Φ be the functor from $D_K(X)_\lambda$ to $D_{G_{\mathbb{R}}}(X)_\lambda$.

Conjecture 2. For any $F \in D_{G_{\mathbb{R}}}(X)_\lambda$, we have

a) $\mathrm{Ext}^i(F, \underline{O}_X(\lambda))$ is an F.S- (Frechet-Schwartz)-space,

b) $H^{n-i}(X; F^* \otimes \underline{O}_X(-\lambda))$ is a DFS-(dual of F-S)-space, and

c) They are dual to each other.

Conjectrue 3. Let $\underline{M}$ be a holonomic $(\underline{D}_\lambda, K)$-module, and $F = \mathrm{Sol}(\underline{M}) = \mathbb{R}\,\underline{\mathrm{Hom}}_{\underline{D}_\lambda}(\underline{M}, \underline{O}_X(\lambda))$. Then the space of $K_{\mathbb{R}}$ -finite vectors of $\mathrm{Ext}^i(\Phi(F), \underline{O}_X(\lambda))$ coincides with $H^i(X; \underline{M})$.

Conjecture 4. The natural map $H^i(X; F^* \otimes \underline{O}_X(\lambda)) \longrightarrow \mathrm{Ext}^i(F; \underline{O}_X(\lambda))$ is injective with dense image (and hence they have the same space of $K_{\mathbb{R}}$ -finite vectors).

C. Taking ρ for λ for the sake of simplicity, we shall explain about these conjectures.

C1. Conjecture 1 generalizes the Matsuki correspondance of K-orbits and $G_{\mathbb{R}}$ -orbits. The functors $\Phi: D_K(X) \to D_{G_{\mathbb{R}}}(X)$ and $\Psi: D_{G_{\mathbb{R}}}(X) \to D_K(X)$, which give the equivalence, are given as follows. Let us consider $X \xleftarrow{p} K\times X \xrightarrow{r} K \overset{K_{\mathbb{R}}}{\times} X \xrightarrow{q} X$. Here p is the projection, r is the quotient map by $K_{\mathbb{R}}$ and q is the action map $(k,x) \longmapsto kx$. For a K-equivariant F on X, there exists $\tilde{F}$ on $K \overset{K_{\mathbb{R}}}{\times} X$ such that $p^{\#}F = r^{\#}\tilde{F}$. Then $\phi(F)$ is given as $\mathbb{R}\, q_{*}\tilde{F}$. Similarly, we shall consider $X \xleftarrow{p} G_{\mathbb{R}} \times X \xrightarrow{r} G_{\mathbb{R}} \overset{K_{\mathbb{R}}}{\times} X \xrightarrow{q} X$. Then, for a $G_{\mathbb{R}}$-equivariant F on X, there exists $\tilde{F}$ on $G_{\mathbb{R}} \overset{K_{\mathbb{R}}}{\times} X$ such that $p^{\#}F = r^{\#}\tilde{F}$, and $\Psi(F)$ is given as $\mathbb{R}\, q_{!}\tilde{F}$ $[\dim K_{\mathbb{R}}]$. Then Conjecture 1 claims that ϕ and Ψ are a quasi-inverse to each other.

We have also the following conjecture. Let S be a K-orbit and S^a the associated $G_{\mathbb{R}}$-orbit by the Matsuki correspondance. Let F be a K-equivariant local system on S and let F^a be the $G_{\mathbb{R}}$-equivariant local system on S^a such that $F|_{S\cap S^a} = F^a|_{S\cap S^a}$. Let $j: S \hookrightarrow X$ and $j^a \hookrightarrow X$ denote the imbeddings.

<u>Conjecture 5</u>. $\phi(j_{!}F) = \mathbb{R}\, j^a_{*}(F^a \otimes j^{a!}\mathbb{C}_X)$.

<u>C2</u>. For $F \in D_{G_{\mathbb{R}}}(X)$, there exists an isomorphism $F \simeq F'$ in $D(X)$ where F' has the form $F'^n = \bigoplus_j \mathbb{C}_{Z_{n,j}}$ and $X_{n,j}$ are closed subanalytic sets. Let $\underline{O}_X \to \underline{B}^{\cdot}$ be the Dolbeault resolution of $\underline{O}_X$ by hyperfunctions. Then $E^{\cdot} = \mathrm{Hom}(F';\underline{B}^{\cdot})$ is a complex of FS-spaces because the space of hyperfunctions with support in a given compact set carries a natural topology of an FS-space. Then Conjecture 2(a) asks that the differential of $E^{\cdot}$ has closed range so that $H^i(E^{\cdot})$ is an FS-space.

Note that, if so, the topology on $\mathrm{Ext}^i(F;\underline{O}_X)$ does not depend on the choice of quasi-isomorphism $F \simeq F'$:

Similarly, letting $\Omega^n_X \to \underline{A}^{\cdot}_X$ be the Dolbeault resolution of Ω^n_X by the sheaf of real analytic functions, $H^i(X;F\otimes\Omega^n_X)$ is the cohomology group of $\tilde{E}^{\cdot} = \Gamma(X;F'\otimes\underline{A}^{\cdot}_X)$. Hence Conjecture 2(b) means that the differential of $\tilde{E}^{\cdot}$ has closed range. Note that a) and b) are equivalent and they imply c) (Serre duality).

D. Problem 6. Describe the t-structure of $D_{G_{\mathbb{R}}}(X)_\lambda$ corresponding to that of $D_K(X)_\lambda$.