Equivariant derived category and representation of real semisimple Lie groups

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1 Introduction

This note is based on five lectures on the geometry of flag manifolds and the representation theory of real semisimple Lie groups, delivered at the CIME summer school "Representation theory and Complex Analysis", June 10–17, 2004, Venezia.

The study of the relation of the geometry of flag manifolds and the representation theory of complex algebraic groups has a long history. However, it is rather recent that we realize the close relation between the representation theory of real semisimple Lie groups and the geometry of the flag manifold and its cotangent bundle. In these relations, there are two facets, complex geometry and real geometry. The Matsuki correspondence is an example: it is a correspondence between the orbits of the real semisimple group on the flag manifold and the orbits of the complexification of its maximal compact subgroup.

Among these relations, we focus on the diagram below.

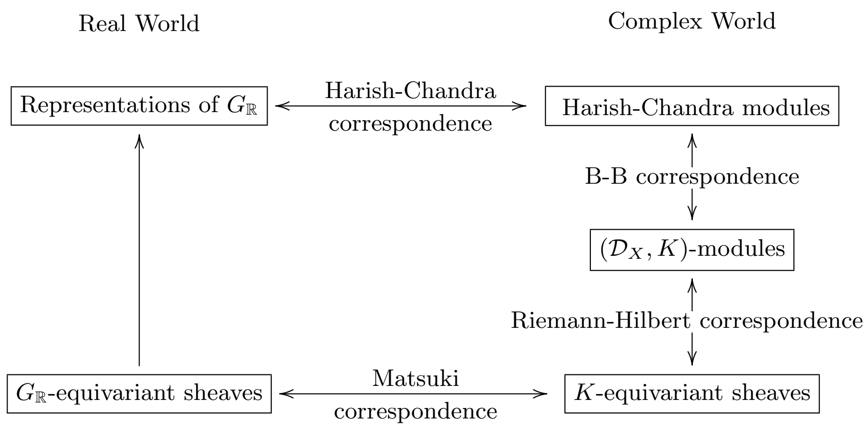

Fig. 1. Correspondences

The purpose of this note is to explain this diagram.

In Introduction, we give the overview of this diagram, and we will explain more details in the subsequent sections. In order to simplify the arguments, we restrict ourselves to the case of the trivial infinitesimal character in Introduction. In order to treat the general case, we need the "twisting" of sheaves and the ring of differential operators. For them, see the subsequent sections.

Considerable parts of this note are a joint work with W. Schmid, and they are announced in [21].

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1.1 Harish-Chandra correspondence

Let $G_{\mathbb{R}}$ be a connected real semisimple Lie group with a finite center, and $K_{\mathbb{R}}$ a maximal compact subgroup of $G_{\mathbb{R}}$. Let $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{k}_{\mathbb{R}}$ be the Lie algebras of $G_{\mathbb{R}}$ and $K_{\mathbb{R}}$, respectively. Let $\mathfrak{g}$ and $\mathfrak{k}$ be their complexifications. Let K be the complexification of $K_{\mathbb{R}}$.

We consider a representation of $G_{\mathbb{R}}$. Here, it means a complete locally convex topological space E with a continuous action of $G_{\mathbb{R}}$. A vector v in E

is called $K_{\mathbb{R}}$-finite if v is contained in a finite-dimensional $K_{\mathbb{R}}$-submodule of E. Harish-Chandra considered

$$\mathrm{HC}(E) := \{v \in E\,;\, v \text{ is } K_{\mathbb{R}}\text{-finite}\}\,.$$

If E has finite $K_{\mathbb{R}}$-multiplicities, i.e., $\dim \mathrm{Hom}_{K_{\mathbb{R}}}(V, E) < \infty$ for any finite-dimensional irreducible representation V of $K_{\mathbb{R}}$, he called E an *admissible* representation. The action of $G_{\mathbb{R}}$ on an admissible representation E can be differentiated on $\mathrm{HC}(E)$, and $\mathfrak{g}$ acts on $\mathrm{HC}(E)$. Since any continuous $K_{\mathbb{R}}$-action on a finite-dimensional vector space extends to a K-action, $\mathrm{HC}(E)$ has a $(\mathfrak{g}, K)$-module structure (see Definition 3.1.1).

Definition 1.1.1. A $(\mathfrak{g}, K)$-module M is called a *Harish-Chandra module* if it satisfies the conditions:

(a) M is $\mathfrak{z}(\mathfrak{g})$-finite,
(b) M has finite K-multiplicities,
(c) M is finitely generated over $U(\mathfrak{g})$.

Here, $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ and $\mathfrak{z}(\mathfrak{g})$ is the center of $U(\mathfrak{g})$. The condition (i) (a) means that the image of $\mathfrak{z}(\mathfrak{g}) \to \mathrm{End}(M)$ is finite-dimensional over $\mathbb{C}$.

In fact, if two of the three conditions (a)–(c) are satisfied, then all of the three are satisfied.

An admissible representation E is of finite length if and only if $\mathrm{HC}(E)$ is a Harish-Chandra module.

The $(\mathfrak{g}, K)$-module $\mathrm{HC}(E)$ is a dense subspace of E, and hence E is the completion of $\mathrm{HC}(E)$ with the induced topology on $\mathrm{HC}(E)$. However, for a Harish-Chandra module M, there exist many representations E such that $\mathrm{HC}(E) \simeq M$. Among them, there exist the smallest one $\mathrm{mg}(M)$ and the largest one $\mathrm{MG}(M)$.

More precisely, we have the following results ([24, 25]). Let $\mathcal{T}^{\mathrm{adm}}_{G_{\mathbb{R}}}$ be the category of admissible representations of $G_{\mathbb{R}}$ of finite length. Let $\mathrm{HC}(\mathfrak{g}, K)$ be the category of Harish-Chandra modules. Then, for any $M \in \mathrm{HC}(\mathfrak{g}, K)$, there exist $\mathrm{mg}(M)$ and $\mathrm{MG}(M)$ in $\mathcal{T}^{\mathrm{adm}}_{G_{\mathbb{R}}}$ satisfying:

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{HC}(\mathfrak{g},K)}(M, \mathrm{HC}(E)) \simeq \mathrm{Hom}_{\mathcal{T}^{\mathrm{adm}}_{G_{\mathbb{R}}}}(\mathrm{mg}(M), E), \\ &\mathrm{Hom}_{\mathrm{HC}(\mathfrak{g},K)}(\mathrm{HC}(E), M) \simeq \mathrm{Hom}_{\mathcal{T}^{\mathrm{adm}}_{G_{\mathbb{R}}}}(E, \mathrm{MG}(M)) \end{aligned} \tag{1.1.1}$$

for any $E \in \mathcal{T}^{\mathrm{adm}}_{G_{\mathbb{R}}}$. In other words, $M \mapsto \mathrm{mg}(M)$ (resp. $M \mapsto \mathrm{MG}(M)$) is a left adjoint functor (resp. right adjoint functor) of the functor $\mathrm{HC}\colon \mathcal{T}^{\mathrm{adm}}_{G_R} \to \mathrm{HC}(\mathfrak{g}, K)$. Moreover we have

$$M \xrightarrow{\sim} \mathrm{HC}(\mathrm{mg}(M)) \xrightarrow{\sim} \mathrm{HC}(\mathrm{MG}(M)) \quad \text{for any } M \in \mathrm{HC}(\mathfrak{g}, K).$$

For a Harish-Chandra module M and a representation E such that $\mathrm{HC}(E) \simeq M$, we have

$$M \subset \mathrm{mg}(M) \subset E \subset \mathrm{MG}(M)\,.$$

We call $\mathrm{mg}(M)$ the *minimal globalization* of M and $\mathrm{MG}(M)$ the *maximal globalization* of M. The space $\mathrm{mg}(M)$ is a dual Fréchet nuclear space and $\mathrm{MG}(M)$ is a Fréchet nuclear space (see Example 2.1.2 (ii)).

Example 1.1.2. Let $P_{\mathbb{R}}$ be a parabolic subgroup of $G_{\mathbb{R}}$ and $Y = G_{\mathbb{R}}/P_{\mathbb{R}}$. Then Y is compact. The space $\mathscr{A}(Y)$ of real analytic functions, the space $\mathrm{C}^\infty(Y)$ of C^∞-functions, the space $\mathrm{L}^2(Y)$ of L^2-functions, the space $\mathscr{D}ist(Y)$ of distributions, and the space $\mathscr{B}(Y)$ of hyperfunctions are admissible representations of $G_{\mathbb{R}}$, and they have the same Harish-Chandra module M. We have

$$\mathrm{mg}(M) = \mathscr{A}(Y) \subset \mathrm{C}^\infty(Y) \subset \mathrm{L}^2(Y) \subset \mathscr{D}ist(Y) \subset \mathscr{B}(Y) = \mathrm{MG}(M)\,.$$

The representation $\mathrm{MG}(M)$ can be explicitly constructed as follows. Let us set

$$M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})^{K\text{-fini}}\,.$$

Here, the superscript "K-fini" means the set of K-finite vectors. Then M^* is again a Harish-Chandra module, and we have

$$\mathrm{MG}(M) \simeq \mathrm{Hom}_{U(\mathfrak{g})}(M^*, \mathrm{C}^\infty(G_{\mathbb{R}}))\,.$$

Here, $\mathrm{C}^\infty(G_{\mathbb{R}})$ is a $U(\mathfrak{g})$-module with respect to the right action of $G_{\mathbb{R}}$ on $G_{\mathbb{R}}$. The module $\mathrm{Hom}_{U(\mathfrak{g})}(M^*, \mathrm{C}^\infty(G_{\mathbb{R}}))$ is calculated with respect to this structure. Since the left $G_{\mathbb{R}}$-action on $G_{\mathbb{R}}$ commutes with the right action, $\mathrm{Hom}_{U(\mathfrak{g})}(M^*, \mathrm{C}^\infty(G_{\mathbb{R}}))$ is a representation of $G_{\mathbb{R}}$ by the left action of $G_{\mathbb{R}}$ on $G_{\mathbb{R}}$. We endow $\mathrm{Hom}_{U(\mathfrak{g})}(M^*, \mathrm{C}^\infty(G_{\mathbb{R}}))$ with the topology induced from the Fréchet nuclear topology of $\mathrm{C}^\infty(G_{\mathbb{R}})$. The minimal globalization $\mathrm{mg}(M)$ is the dual representation of $\mathrm{MG}(M^*)$.

In § 10, we shall give a proof of the fact that $M \mapsto \mathrm{mg}(M)$ and $M \mapsto \mathrm{MG}(M)$ are exact functors, and $\mathrm{mg}(M) \simeq \Gamma_c(G_{\mathbb{R}}; \mathscr{D}ist_{G_{\mathbb{R}}}) \otimes_{U(\mathfrak{g})} M$. Here, $\Gamma_c(G_{\mathbb{R}}; \mathscr{D}ist_{G_{\mathbb{R}}})$ is the space of distributions on $G_{\mathbb{R}}$ with compact support.

1.2 Beilinson-Bernstein correspondence

Beilinson and Bernstein established the correspondence between $U(\mathfrak{g})$-modules and D-modules on the flag manifold.

Let G be a semisimple algebraic group with $\mathfrak{g}$ as its Lie algebra. Let X be the flag manifold of G, i.e., the space of all Borel subgroups of G.

For a $\mathbb{C}$-algebra homomorphism $\chi\colon \mathfrak{z}(\mathfrak{g}) \to \mathbb{C}$ and a $\mathfrak{g}$-module M, we say that M has an *infinitesimal character* χ if $a \cdot u = \chi(a)u$ for any $a \in \mathfrak{z}(\mathfrak{g})$ and $u \in M$. In Introduction, we restrict ourselves to the case of the trivial infinitesimal character, although we treat the general case in the body of this note. Let $\chi_{\mathrm{triv}}\colon \mathfrak{z}(\mathfrak{g}) \to \mathbb{C}$ be the trivial infinitesimal character (the infinitesimal character of the trivial representation). We set $U_{\chi_{\mathrm{triv}}}(\mathfrak{g}) = U(\mathfrak{g})/U(\mathfrak{g})\,\mathrm{Ker}(\chi_{\mathrm{triv}})$.

Then $U_{\chi_{\mathrm{triv}}}(\mathfrak{g})$-modules are nothing but $\mathfrak{g}$-modules with the trivial infinitesimal character.

Let $\mathscr{D}_X$ be the sheaf of differential operators on X. Then we have the following theorem due to Beilinson-Bernstein [1].

Theorem 1.2.1. (i) *The Lie algebra homomorphism* $\mathfrak{g} \to \Gamma(X;\mathscr{D}_X)$ *induces an isomorphism*

$$U_{\chi_{\mathrm{triv}}}(\mathfrak{g}) \xrightarrow{\sim} \Gamma(X;\mathscr{D}_X)\,.$$

(ii) $H^n(X;\mathscr{M}) = 0$ *for any quasi-coherent* $\mathscr{D}_X$*-module* $\mathscr{M}$ *and* $n \neq 0$.

(iii) *The category* $\mathrm{Mod}\,(\mathscr{D}_X)$ *of quasi-coherent* $\mathscr{D}_X$*-modules and the category* $\mathrm{Mod}\,(U_{\chi_{\mathrm{triv}}}(\mathfrak{g}))$ *of* $U_{\chi_{\mathrm{triv}}}(\mathfrak{g})$*-modules are equivalent by*

$$\mathrm{Mod}\,(\mathscr{D}_X) \ni \mathscr{M} \longmapsto \Gamma(X;\mathscr{M}) \in \mathrm{Mod}\,(U_{\chi_{\mathrm{triv}}}(\mathfrak{g}))\,,$$

$$\mathrm{Mod}\,(\mathscr{D}_X) \ni \mathscr{D}_X \otimes_{U(\mathfrak{g})} M \longleftarrow\!\!\dashv M \in \mathrm{Mod}\,(U_{\chi_{\mathrm{triv}}}(\mathfrak{g}))\,.$$

In particular, we have the following corollary.

Corollary 1.2.2. *The category* $\mathrm{HC}_{\chi_{\mathrm{triv}}}(\mathfrak{g},K)$ *of Harish-Chandra modules with the trivial infinitesimal character and the category* $\mathrm{Mod}_{K,\,\mathrm{coh}}(\mathscr{D}_X)$ *of coherent* K*-equivariant* $\mathscr{D}_X$*-modules are equivalent.*

The K-equivariant $\mathscr{D}_X$-modules are, roughly speaking, $\mathscr{D}_X$-modules with an action of K. (For the precise definition, see § 3.) We call this equivalence the B-B correspondence.

The set of isomorphism classes of irreducible K-equivariant $\mathscr{D}_X$-modules is isomorphic to the set of pairs (O,L) of a K-orbit O in X and an isomorphism class L of an irreducible representation of the finite group $K_x/(K_x)^\circ$. Here K_x is the isotropy subgroup of K at a point x of O, and $(K_x)^\circ$ is its connected component containing the identity. Hence the set of isomorphism classes of irreducible Harish-Chandra modules with the trivial infinitesimal character corresponds to the set of such pairs (O,L).

1.3 Riemann-Hilbert correspondence

The flag manifold X has finitely many K-orbits. Therefore any coherent K-equivariant $\mathscr{D}_X$-module is a regular holonomic $\mathscr{D}_X$-module (see [15]). Let $\mathrm{D^b}(\mathscr{D}_X)$ be the bounded derived category of $\mathscr{D}_X$-modules, and let $\mathrm{D^b_{rh}}(\mathscr{D}_X)$ be the full subcategory of $\mathrm{D^b}(\mathscr{D}_X)$ consisting of bounded complexes of $\mathscr{D}_X$-modules with regular holonomic cohomology groups.

Let $Z \longmapsto Z^{\mathrm{an}}$ be the canonical functor from the category of complex algebraic varieties to the one of complex analytic spaces. Then there exists a morphism of ringed space $\pi\colon Z^{\mathrm{an}} \to Z$. For an $\mathscr{O}_Z$-module $\mathscr{F}$, let $\mathscr{F}^{\mathrm{an}} := \mathscr{O}_{Z^{\mathrm{an}}} \otimes_{\pi^{-1}\mathscr{O}_Z} \pi^{-1}\mathscr{F}$ be the corresponding $\mathscr{O}_{Z^{\mathrm{an}}}$-module. Similarly, for a $\mathscr{D}_Z$-module $\mathscr{M}$, let $\mathscr{M}^{\mathrm{an}} := \mathscr{D}_{Z^{\mathrm{an}}} \otimes_{\pi^{-1}\mathscr{D}_Z} \pi^{-1}\mathscr{M} \simeq \mathscr{O}_{Z^{\mathrm{an}}} \otimes_{\pi^{-1}\mathscr{O}_Z} \pi^{-1}\mathscr{M}$ be the corresponding $\mathscr{D}_{Z^{\mathrm{an}}}$-module. For a $\mathscr{D}_Z$-module $\mathscr{M}$ and a $\mathscr{D}_{Z^{\mathrm{an}}}$-module $\mathscr{N}$, we write

$\mathscr{H}om_{\mathscr{D}_Z}(\mathscr{M},\mathscr{N})$ instead of $\mathscr{H}om_{\pi^{-1}\mathscr{D}_Z}(\pi^{-1}\mathscr{M},\mathscr{N}) \simeq \mathscr{H}om_{\mathscr{D}_{Z^{\mathrm{an}}}}(\mathscr{M}^{\mathrm{an}},\mathscr{N})$ for short.

Let us denote by $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{X^{\mathrm{an}}})$ the bounded derived category of sheaves of $\mathbb{C}$-vector spaces on X^{an}. Then the de Rham functor $\mathrm{DR}_X \colon \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X) \to \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{X^{\mathrm{an}}})$, given by $\mathrm{DR}_X(\mathscr{M}) = \mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X,\mathscr{M}^{\mathrm{an}})$, induces an equivalence of triangulated categories, called the *Riemann-Hilbert correspondence* ([12])

$$\mathrm{DR}_X \colon \mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_{X^{\mathrm{an}}})\,.$$

Here $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_{X^{\mathrm{an}}})$ is the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{X^{\mathrm{an}}})$ consisting of bounded complexes of sheaves of $\mathbb{C}$-vector spaces on X^{an} with constructible cohomologies (see [18] and also § 4.4).

Let $\mathrm{RH}(\mathscr{D}_X)$ be the category of regular holonomic $\mathscr{D}_X$-modules. Then it may be regarded as a full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$. Its image by DR_X is a full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_{X^{\mathrm{an}}})$ and denoted by $\mathrm{Perv}(\mathbb{C}_{X^{\mathrm{an}}})$. Since $\mathrm{RH}(\mathscr{D}_X)$ is an abelian category, $\mathrm{Perv}(\mathbb{C}_{X^{\mathrm{an}}})$ is also an abelian category. An object of $\mathrm{Perv}(\mathbb{C}_{X^{\mathrm{an}}})$ is called a *perverse sheaf* on X^{an}.

Then the functor DR_X induces an equivalence between $\mathrm{Mod}_{K,\,\mathrm{coh}}(\mathscr{D}_X)$ and the category $\mathrm{Perv}_{K^{\mathrm{an}}}(\mathbb{C}_{X^{\mathrm{an}}})$ of K^{an}-equivariant perverse sheaves on X^{an}:

$$\mathrm{DR}_X : \mathrm{Mod}_{K,\,\mathrm{coh}}(\mathscr{D}_X) \xrightarrow{\sim} \mathrm{Perv}_{K^{\mathrm{an}}}(\mathbb{C}_{X^{\mathrm{an}}})\,.$$

1.4 Matsuki correspondence

The following theorem is due to Matsuki ([22]).

Proposition 1.4.1. (i) *There are only finitely many K-orbits in X and also finitely many $G_{\mathbb{R}}$-orbits in X^{an}.*

(ii) *There is a one-to-one correspondence between the set of K-orbits and the set of $G_{\mathbb{R}}$-orbits.*

(iii) *A K-orbit U and a $G_{\mathbb{R}}$-orbit V correspond by the correspondence in* (ii) *if and only if $U^{\mathrm{an}} \cap V$ is a $K_{\mathbb{R}}$-orbit.*

Its sheaf-theoretical version is conjectured by Kashiwara [14] and proved by Mirković-Uzawa-Vilonen [23].

In order to state the results, we have to use the equivariant derived category (see [4], and also § 4). Let H be a real Lie group, and let Z be a topological space with an action of H. We assume that Z is locally compact with a finite cohomological dimension. Then we can define the equivariant derived category $\mathrm{D}^{\mathrm{b}}_H(\mathbb{C}_Z)$, which has the following properties:

(a) there exists a forgetful functor $\mathrm{D}^{\mathrm{b}}_H(\mathbb{C}_Z) \to \mathrm{D}^{\mathrm{b}}(\mathbb{C}_Z)$,
(b) for any $F \in \mathrm{D}^{\mathrm{b}}_H(\mathbb{C}_Z)$, its cohomology group $H^n(F)$ is an H-equivariant sheaf on Z for any n,

(c) for any H-equivariant morphism $f\colon Z \to Z'$, there exist canonical functors f^{-1}, $f^!\colon \mathrm{D}^{\mathrm{b}}_H(\mathbb{C}_{Z'}) \to \mathrm{D}^{\mathrm{b}}_H(\mathbb{C}_Z)$ and f_*, $f_!\colon \mathrm{D}^{\mathrm{b}}_H(\mathbb{C}_Z) \to \mathrm{D}^{\mathrm{b}}_H(\mathbb{C}_{Z'})$ which commute with the forgetful functors in (a), and satisfy the usual properties (see § 4),
(d) if H acts freely on Z, then $\mathrm{D}^{\mathrm{b}}_H(\mathbb{C}_Z) \simeq \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{Z/H})$.
(e) if H is a closed subgroup of H', then we have an equivalence

$$\mathrm{Ind}_H^{H'}\colon\ \mathrm{D}^{\mathrm{b}}_H(\mathbb{C}_Z) \xrightarrow{\ \sim\ } \mathrm{D}^{\mathrm{b}}_{H'}(\mathbb{C}_{(Z\times H')/H})\,.$$

Now let us come back to the case of real semisimple groups. We have an equivalence of categories:

$$\mathrm{Ind}_{K^{\mathrm{an}}}^{G^{\mathrm{an}}}\colon\ \mathrm{D}^{\mathrm{b}}_{K^{\mathrm{an}}}(\mathbb{C}_{X^{\mathrm{an}}}) \xrightarrow{\ \sim\ } \mathrm{D}^{\mathrm{b}}_{G^{\mathrm{an}}}(\mathbb{C}_{(X^{\mathrm{an}}\times G^{\mathrm{an}})/K^{\mathrm{an}}}). \tag{1.4.1}$$

Let us set $S = G/K$ and $S_{\mathbb{R}} = G_{\mathbb{R}}/K_{\mathbb{R}}$. Then $S_{\mathbb{R}}$ is a Riemannian symmetric space and $S_{\mathbb{R}} \subset S$. Let $i\colon S_{\mathbb{R}} \hookrightarrow S^{\mathrm{an}}$ be the closed embedding. Since $(X \times G)/K \simeq X \times S$, we obtain an equivalence of categories

$$\mathrm{Ind}_{K^{\mathrm{an}}}^{G^{\mathrm{an}}}\colon\ \mathrm{D}^{\mathrm{b}}_{K^{\mathrm{an}}}(\mathbb{C}_{X^{\mathrm{an}}}) \xrightarrow{\ \sim\ } \mathrm{D}^{\mathrm{b}}_{G^{\mathrm{an}}}(\mathbb{C}_{X^{\mathrm{an}}\times S^{\mathrm{an}}})\,.$$

Let $p_1\colon X^{\mathrm{an}} \times S^{\mathrm{an}} \to X^{\mathrm{an}}$ be the first projection and $p_2\colon X^{\mathrm{an}} \times S^{\mathrm{an}} \to S^{\mathrm{an}}$ the second projection. We define the functor

$$\Phi\colon\ \mathrm{D}^{\mathrm{b}}_{K^{\mathrm{an}}}(\mathbb{C}_{X^{\mathrm{an}}}) \to \mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}}}(\mathbb{C}_{X^{\mathrm{an}}})$$

by

$$\Phi(F) = \mathbf{R}p_{1!}(\mathrm{Ind}_{K^{\mathrm{an}}}^{G^{\mathrm{an}}}(F) \otimes p_2^{-1} i_* \mathbb{C}_{S_{\mathbb{R}}})[\mathrm{d}_S].$$

Here, we use the notation

$$\mathrm{d}_S = \dim S. \tag{1.4.2}$$

Theorem 1.4.2 ([23]). $\Phi\colon\ \mathrm{D}^{\mathrm{b}}_{K^{\mathrm{an}}}(\mathbb{C}_{X^{\mathrm{an}}}) \to \mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}}}(\mathbb{C}_{X^{\mathrm{an}}})$ *is an equivalence of triangulated categories.*

Roughly speaking, there is a correspondence between K^{an}-equivariant sheaves on X^{an} and $G_{\mathbb{R}}$-equivariant sheaves on X^{an}. We call it the (sheaf-theoretical) *Matsuki correspondence.*

1.5 Construction of representations of $G_{\mathbb{R}}$

Let H be an affine algebraic group, and let Z be an algebraic manifold with an action of H. We can in fact define two kinds of H-equivariance on $\mathscr{D}_Z$-modules: a *quasi-equivariance* and an *equivariance*. (For their definitions, see Definition 3.1.3.) Note that $\mathscr{D}_Z \otimes_{\mathscr{O}_Z} \mathscr{F}$ is quasi-H-equivariant for any H-equivariant $\mathscr{O}_Z$-module $\mathscr{F}$, but it is not H-equivariant in general. The $\mathscr{D}_Z$-module $\mathscr{O}_Z$ is

H-equivariant. Let us denote by $\mathrm{Mod}(\mathscr{D}_Z, H)$ (resp. $\mathrm{Mod}_H(\mathscr{D}_Z)$) the category of quasi-$H$-equivariant (resp. H-equivariant) $\mathscr{D}_Z$-modules. Then $\mathrm{Mod}_H(\mathscr{D}_Z)$ is a full abelian subcategory of $\mathrm{Mod}(\mathscr{D}_Z, H)$.

Let $G_\mathbb{R}$ be a real semisimple Lie group contained in a semisimple algebraic group G as a real form. Let $\mathbf{FN}$ be the category of Fréchet nuclear spaces (see Example 2.1.2 (ii)), and let $\mathbf{FN}_{G_\mathbb{R}}$ be the category of Fréchet nuclear spaces with a continuous $G_\mathbb{R}$-action. It is an additive category but not an abelian category. However it is a quasi-abelian category and we can define its bounded derived category $\mathrm{D}^\mathrm{b}(\mathbf{FN}_{G_\mathbb{R}})$ (see § 2).

Let Z be an algebraic manifold with a G-action. Let $\mathrm{D}^\mathrm{b}_\mathrm{coh}(\mathrm{Mod}(\mathscr{D}_Z, G))$ be the full subcategory of $\mathrm{D}^\mathrm{b}(\mathrm{Mod}(\mathscr{D}_Z, G))$ consisting of objects with coherent cohomologies. Let $\mathrm{D}^\mathrm{b}_{G_\mathbb{R},\mathbb{R}\text{-c}}(\mathbb{C}_{Z^\mathrm{an}})$ be the full subcategory of the $G_\mathbb{R}$-equivariant derived category $\mathrm{D}^\mathrm{b}_{G_\mathbb{R}}(\mathbb{C}_{Z^\mathrm{an}})$ consisting of objects with $\mathbb{R}$-constructible cohomologies (see § 4.4). Then for $\mathscr{M} \in \mathrm{D}^\mathrm{b}_\mathrm{coh}(\mathrm{Mod}(\mathscr{D}_Z, G))$ and $F \in \mathrm{D}^\mathrm{b}_{G_\mathbb{R},\mathbb{R}\text{-c}}(\mathbb{C}_{Z^\mathrm{an}})$, we can define

$$\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_Z}(\mathscr{M} \otimes F, \mathscr{O}_{Z^\mathrm{an}})$$

as an object of $\mathrm{D}^\mathrm{b}(\mathbf{FN}_{G_\mathbb{R}})$.

Roughly speaking, it is constructed as follows. (For a precise construction, see § 5.) We can take a bounded complex $\mathscr{D}_Z \otimes \mathcal{V}^\bullet$ of quasi-G-equivariant $\mathscr{D}_Z$-modules which is isomorphic to $\mathscr{M}$ in the derived category, where each $\mathcal{V}^n$ is a G-equivariant vector bundle on Z. On the other hand, we can represent F by a complex $K^\bullet$ of $G_\mathbb{R}$-equivariant sheaves such that each K^n has a form $\oplus_{a \in I_n} L_a$ for an index set I_n, where L_a is a $G_\mathbb{R}$-equivariant locally constant sheaf of finite rank on a $G_\mathbb{R}$-invariant open subset U_a of Z^an.[1] Let $\mathscr{E}^{(0,\bullet)}_{Z^\mathrm{an}}$ be the Dolbeault resolution of $\mathscr{O}_{Z^\mathrm{an}}$ by differential forms with C^∞ coefficients. Then, $\mathrm{Hom}_{\mathscr{D}_Z}((\mathscr{D}_Z \otimes \mathcal{V}^\bullet) \otimes K^\bullet, \mathscr{E}^{(0,\bullet)}_{Z^\mathrm{an}})$ represents $\mathbf{R}\mathrm{Hom}_{\mathscr{D}_Z}(\mathscr{M} \otimes F, \mathscr{O}_{Z^\mathrm{an}}) \in \mathrm{D}^\mathrm{b}(\mathrm{Mod}(\mathbb{C}))$. On the other hand, $\mathrm{Hom}_{\mathscr{D}_Z}((\mathscr{D}_Z \otimes \mathcal{V}^n) \otimes L_a, \mathscr{E}^{(0,q)}_{Z^\mathrm{an}}) = \mathrm{Hom}_{\mathscr{O}_Z}(\mathcal{V}^n \otimes L_a, \mathscr{E}^{(0,q)}_{Z^\mathrm{an}})$ carries a natural topology of Fréchet nuclear spaces and is endowed with a continuous $G_\mathbb{R}$-action. Hence $\mathrm{Hom}_{\mathscr{D}_Z}((\mathscr{D}_Z \otimes \mathcal{V}^\bullet) \otimes K^\bullet, \mathscr{E}^{(0,\bullet)}_{Z^\mathrm{an}})$ is a complex of objects in $\mathbf{FN}_{G_\mathbb{R}}$. It is $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_Z}(\mathscr{M} \otimes F, \mathscr{O}_{Z^\mathrm{an}}) \in \mathrm{D}^\mathrm{b}(\mathbf{FN}_{G_\mathbb{R}})$.

Dually, we can consider the category $\mathbf{DFN}_{G_\mathbb{R}}$ of dual Fréchet nuclear spaces with a continuous $G_\mathbb{R}$-action and its bounded derived category $\mathrm{D}^\mathrm{b}(\mathbf{DFN}_{G_\mathbb{R}})$. Then, we can construct $\mathbf{R}\Gamma^{\mathrm{top}}_\mathrm{c}(Z^\mathrm{an}; F \otimes \Omega_{Z^\mathrm{an}} \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_Z} \mathscr{M})$, which is an object of $\mathrm{D}^\mathrm{b}(\mathbf{DFN}_{G_\mathbb{R}})$. Here, Ω_{Z^an} is the sheaf of holomorphic differential forms with the maximal degree. Let $\mathscr{D}ist^{(\mathrm{d}_Z,\bullet)}$ be the Dolbeault resolution of Ω_{Z^an} by differential forms with distribution coefficients. Then, the complex $\Gamma_c\big(Z^\mathrm{an}; K^\bullet \otimes \mathscr{D}ist^{(\mathrm{d}_Z,\bullet)} \otimes_{\mathscr{D}_Z} (\mathscr{D}_Z \otimes \mathcal{V}^\bullet)\big)$ represents $\mathbf{R}\Gamma_c(Z^\mathrm{an}; F \otimes \Omega_{Z^\mathrm{an}} \otimes_{\mathscr{D}_Z} \mathscr{M}) \in \mathrm{D}^\mathrm{b}(\mathrm{Mod}(\mathbb{C}))$. On the other hand, since $\Gamma_c\big(Z^\mathrm{an}; K^\bullet \otimes \mathscr{D}ist^{(\mathrm{d}_Z,\bullet)} \otimes_{\mathscr{D}_Z}$

[1] In fact, it is not possible to represent F by such a $K^\bullet$ in general. We overcome this difficulty by a resolution of the base space Z (see § 5).

$(\mathscr{D}_Z \otimes \mathcal{V}^\bullet)\big)$ is a complex in $\mathbf{DFN}_{G_\mathbb{R}}$, it may be regarded as an object of $\mathrm{D^b}(\mathbf{DFN}_{G_\mathbb{R}})$. It is $\mathbf{R}\Gamma_\mathrm{c}^\mathrm{top}(Z^\mathrm{an}; F \otimes \Omega_{Z^\mathrm{an}} \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_Z} \mathscr{M})$. We have

$$\mathbf{R}\Gamma_\mathrm{c}^\mathrm{top}(Z^\mathrm{an}; F \otimes \Omega_{Z^\mathrm{an}} \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_Z} \mathscr{M}) \simeq \big(\mathbf{R}\mathrm{Hom}^\mathrm{top}_{\mathscr{D}_Z}(\mathscr{M} \otimes F, \mathscr{O}_{Z^\mathrm{an}})\big)^* .$$

Let us apply it to the flag manifold X with the action of G. Let F be an object of $\mathrm{D}^\mathrm{b}_{G_\mathbb{R},\,\mathbb{R}\text{-c}}(\mathbb{C}_{X^\mathrm{an}})$. Then $\mathbf{R}\mathrm{Hom}^\mathrm{top}_\mathbb{C}(F, \mathscr{O}_{X^\mathrm{an}}) := \mathbf{R}\mathrm{Hom}^\mathrm{top}_{\mathscr{D}_X}(\mathscr{D}_X \otimes F, \mathscr{O}_{X^\mathrm{an}})$ is an object of $\mathrm{D^b}(\mathbf{FN}_{G_\mathbb{R}})$. This is strict, i.e., if we represent $\mathbf{R}\mathrm{Hom}^\mathrm{top}_\mathbb{C}(F, \mathscr{O}_{X^\mathrm{an}})$ as a complex in $\mathbf{FN}_{G_\mathbb{R}}$, the differentials of such a complex have closed ranges. Moreover, its cohomology group $H^n(\mathbf{R}\mathrm{Hom}^\mathrm{top}_\mathbb{C}(F, \mathscr{O}_{X^\mathrm{an}}))$ is the maximal globalization of some Harish-Chandra module (see § 10). Similarly, $\mathbf{R}\mathrm{Hom}^\mathrm{top}_\mathbb{C}(F, \Omega_{X^\mathrm{an}}) := \mathbf{R}\mathrm{Hom}^\mathrm{top}_{\mathscr{D}_X}\big((\mathscr{D}_X \otimes \Omega_X^{\otimes -1}) \otimes F, \mathscr{O}_{X^\mathrm{an}}\big)$ is a strict object of $\mathrm{D^b}(\mathbf{FN}_{G_\mathbb{R}})$ and its cohomology groups are the maximal globalization of a Harish-Chandra module. Here Ω_X is the sheaf of differential forms with degree d_X on X.

Dually, we can consider $\mathbf{R}\Gamma_\mathrm{c}^\mathrm{top}(X^\mathrm{an}; F \otimes \mathscr{O}_{X^\mathrm{an}})$ as an object of $\mathrm{D^b}(\mathbf{DFN}_{G_\mathbb{R}})$, whose cohomology groups are the minimal globalization of a Harish-Chandra module.

This is the left vertical arrow in Fig. 1.

Remark 1.5.1. Note the works by Hecht-Taylor [11] and Smithies-Taylor [27] which are relevant to this note. They considered the $\mathscr{D}_{X^\mathrm{an}}$-module $\mathscr{O}_{X^\mathrm{an}} \otimes F$ instead of F, and construct the left vertical arrow in Fig. 1 in a similar way to the Beilinson-Bernstein correspondence.

Let us denote by $\mathrm{Mod}_f(\mathfrak{g}, K)$ the category of $(\mathfrak{g}, K)$-modules finitely generated over $U(\mathfrak{g})$. Then, $\mathrm{Mod}_f(\mathfrak{g}, K)$ has enough projectives. Indeed, $U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} N$ is a projective object of $\mathrm{Mod}_f(\mathfrak{g}, K)$ for any finite-dimensional K-module N. Hence there exists a right derived functor

$$\mathbf{R}\mathrm{Hom}^\mathrm{top}_{U(\mathfrak{g})}(\,\bullet\,, \mathrm{C}^\infty(G_\mathbb{R})) : \mathrm{D^b}(\mathrm{Mod}_f(\mathfrak{g}, K))^\mathrm{op} \to \mathrm{D^b}(\mathbf{FN}_{G_\mathbb{R}})$$

of the functor $\mathrm{Hom}_{U(\mathfrak{g})}(\,\bullet\,, \mathrm{C}^\infty(G_\mathbb{R}))\colon \mathrm{Mod}_f(\mathfrak{g}, K)^\mathrm{op} \to \mathbf{FN}_{G_\mathbb{R}}$. Similarly, there exists a left derived functor

$$\Gamma_c(G_\mathbb{R}; \mathscr{D}ist_{G_\mathbb{R}}) \overset{\mathbf{L}}{\otimes}_{U(\mathfrak{g})} \bullet\ : \mathrm{D^b}(\mathrm{Mod}_f(\mathfrak{g}, K)) \to \mathrm{D^b}(\mathbf{DFN}_{G_\mathbb{R}})$$

of the functor $\Gamma_c(G_\mathbb{R}; \mathscr{D}ist_{G_\mathbb{R}}) \otimes_{U(\mathfrak{g})} \bullet : \mathrm{Mod}_f(\mathfrak{g}, K) \to \mathbf{DFN}_{G_\mathbb{R}}$.[2] In § 10, we prove $H^n(\mathbf{R}\mathrm{Hom}^\mathrm{top}_{U(\mathfrak{g})}(M, \mathrm{C}^\infty(G_\mathbb{R}))) = 0$, $H^n(\Gamma_c(G_\mathbb{R}; \mathscr{D}ist_{G_\mathbb{R}}) \overset{\mathbf{L}}{\otimes}_{(\mathfrak{g}, K_\mathbb{R})} M) = 0$ for $n \neq 0$, and

[2] They are denoted by $\mathbf{R}\mathrm{Hom}^\mathrm{top}_{(\mathfrak{g}, K_\mathbb{R})}(\,\bullet\,, \mathrm{C}^\infty(G_\mathbb{R}))$ and $\Gamma_c(G_\mathbb{R}; \mathscr{D}ist_{G_\mathbb{R}}) \overset{\mathbf{L}}{\otimes}_{(\mathfrak{g}, K_\mathbb{R})} \bullet$ in Subsection 9.5.

$$\mathrm{MG}(M^*) \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{U(\mathfrak{g})}(M, \mathrm{C}^\infty(G_\mathbb{R})),$$

$$\mathrm{mg}(M) \simeq \Gamma_c(G_\mathbb{R}; \mathscr{D}ist_{G_\mathbb{R}}) \overset{\mathbf{L}}{\otimes}_{(\mathfrak{g},K_\mathbb{R})} M$$

for any Harish-Chandra module M.

1.6 Integral transforms

Let Y and Z be algebraic manifolds, and consider the diagram:

$$\begin{array}{ccc} & Y \times Z & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ Y & & Z\,. \end{array}$$

We assume that Y is projective. For $\mathscr{N} \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_Y)$ and $\mathscr{K} \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_{Y\times Z})$ we define their convolution

$$\mathscr{N} \overset{\mathbf{D}}{\circ} \mathscr{K} := \mathbf{D}p_{2*}(\mathbf{D}p_1^*\mathscr{N} \overset{\mathbf{D}}{\otimes} \mathscr{K}) \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_Z),$$

where $\mathbf{D}p_{2*}$, $\mathbf{D}p_1^*$, $\overset{\mathbf{D}}{\otimes}$ are the direct image, inverse image, tensor product functors for D-modules (see § 3). Similarly, for $K \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{Y^{\mathrm{an}}\times Z^{\mathrm{an}}})$ and $F \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{Z^{\mathrm{an}}})$, we define their convolution

$$K \circ F := \mathbf{R}(p_1^{\mathrm{an}})_!(K \otimes (p_2^{\mathrm{an}})^{-1}F) \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{Y^{\mathrm{an}}})\,.$$

Let $\mathrm{DR}_{Y\times Z}\colon \mathrm{D}^{\mathrm{b}}(\mathscr{D}_{Y\times Z}) \to \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{Y^{\mathrm{an}}\times Z^{\mathrm{an}}})$ be the de Rham functor. Then we have the following integral transform formula.

Theorem 1.6.1. *For $\mathscr{K} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{Y\times Z})$, $\mathscr{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y)$ and $F \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{Z^{\mathrm{an}}})$, set $K = \mathrm{DR}_{Y\times Z}(\mathscr{K}) \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_{Y^{\mathrm{an}}\times Z^{\mathrm{an}}})$. If $\mathscr{N}$ and $\mathscr{K}$ are non-characteristic, then we have an isomorphism*

$$\mathbf{R}\mathrm{Hom}_{\mathscr{D}_Z}((\mathscr{N} \overset{\mathbf{D}}{\circ} \mathscr{K}) \otimes F, \mathscr{O}_{Z^{\mathrm{an}}}) \simeq \mathbf{R}\mathrm{Hom}_{\mathscr{D}_Y}(\mathscr{N} \otimes (K \circ F), \mathscr{O}_{Y^{\mathrm{an}}})[\mathrm{d}_Y - 2\mathrm{d}_Z]\,.$$

Note that $\mathscr{N}$ and $\mathscr{K}$ are non-characteristic if $\big(\mathrm{Ch}(\mathscr{N}) \times T^*_Z Z\big) \cap \mathrm{Ch}(\mathscr{K}) \subset T^*_{Y\times Z}(Y \times Z)$, where Ch denotes the characteristic variety (see § 8).

Its equivariant version also holds.

Let us apply this to the following situation. Let G, $G_\mathbb{R}$, K, $K_\mathbb{R}$, X, S be as before, and consider the diagram:

$$\begin{array}{ccc} & X \times S & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ X & & S\,. \end{array}$$

Theorem 1.6.2. *For* $\mathscr{K} \in \mathrm{D}^{\mathrm{b}}_{G,\mathrm{coh}}(\mathscr{D}_{X\times S})$, $\mathscr{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{Mod}(\mathscr{D}_X, G))$ *and* $F \in \mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}},\mathbb{R}\text{-}c}(\mathbb{C}_{S^{\mathrm{an}}})$, *set* $K = \mathrm{DR}_{X\times S}(\mathscr{K}) \in \mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}},\mathbb{C}\text{-}c}(\mathbb{C}_{X^{\mathrm{an}}\times S^{\mathrm{an}}})$. *Then we have an isomorphism*

$$\begin{aligned}(1.6.1)\qquad &\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_S}((\mathscr{N} \overset{\mathbf{D}}{\circ} \mathscr{K}) \otimes F, \mathscr{O}_{S^{\mathrm{an}}}) \\ &\qquad \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{N} \otimes (K \circ F), \mathscr{O}_{X^{\mathrm{an}}})[\mathrm{d}_X - 2\mathrm{d}_S]\end{aligned}$$

in $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$.

Note that the non-characteristic condition in Theorem 1.6.1 is automatically satisfied in this case.

1.7 Commutativity of Fig. 1

Let us apply Theorem 1.6.2 in order to show the commutativity of Fig. 1. Let us start by taking $\mathscr{M} \in \mathrm{Mod}_{K,\mathrm{coh}}(\mathscr{D}_X)$. Then, by the Beilinson-Bernstein correspondence, $\mathscr{M}$ corresponds to the Harish-Chandra module $M := \Gamma(X;\mathscr{M})$. Let us set $\mathscr{K} = \mathrm{Ind}^G_K(\mathscr{M}) \in \mathrm{Mod}_{G,\mathrm{coh}}(\mathscr{D}_{X\times S})$. If we set $\mathscr{N} = \mathscr{D}_X \otimes \Omega_X^{\otimes -1} \in \mathrm{Mod}(\mathscr{D}_X, G)$, then $\mathscr{N} \overset{\mathbf{D}}{\circ} \mathscr{K} \in \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}(\mathscr{D}_S, G))$. By the equivalence of categories $\mathrm{Mod}(\mathscr{D}_S, G) \simeq \mathrm{Mod}(\mathfrak{g}, K)$, $\mathscr{N} \overset{\mathbf{D}}{\circ} \mathscr{K}$ corresponds to $M \in \mathrm{Mod}(\mathfrak{g}, K)$. Now we take $F = \mathbb{C}_{S_{\mathbb{R}}}[-\mathrm{d}_S]$. Then the left-hand side of (1.6.1) coincides with

$$\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_S}\big(\mathscr{N} \overset{\mathbf{D}}{\circ} \mathscr{K}, \mathbf{R}\mathscr{H}om(\mathbb{C}_{S_{\mathbb{R}}}[-\mathrm{d}_S], \mathscr{O}_{S^{\mathrm{an}}})\big) \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_S}(\mathscr{N} \overset{\mathbf{D}}{\circ} \mathscr{K}, \mathscr{B}_{S_{\mathbb{R}}}),$$

where $\mathscr{B}_{S_{\mathbb{R}}}$ is the sheaf of hyperfunctions on $S_{\mathbb{R}}$. Since $\mathscr{N} \overset{\mathbf{D}}{\circ} \mathscr{K}$ is an elliptic $\mathscr{D}_S$-module, we have

$$\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_S}(\mathscr{N} \overset{\mathbf{D}}{\circ} \mathscr{K}, \mathscr{B}_{S_{\mathbb{R}}}) \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_S}(\mathscr{N} \overset{\mathbf{D}}{\circ} \mathscr{K}, \mathscr{C}^{\infty}_{S_{\mathbb{R}}}),$$

where $\mathscr{C}^{\infty}_{S_{\mathbb{R}}}$ is the sheaf of C^{∞}-functions on $S_{\mathbb{R}}$. The equivalence of categories $\mathrm{Mod}(\mathscr{D}_S, G) \simeq \mathrm{Mod}(\mathfrak{g}, K)$ implies

$$\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_S}(\mathscr{N} \overset{\mathbf{D}}{\circ} \mathscr{K}, \mathscr{C}^{\infty}_{S_{\mathbb{R}}}) \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{U(\mathfrak{g})}(M, \mathrm{C}^{\infty}(G_{\mathbb{R}})).$$

Hence we have calculated the left-hand side of (1.6.1):

$$\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_S}((\mathscr{N} \overset{\mathbf{D}}{\circ} \mathscr{K}) \otimes F, \mathscr{O}_{S^{\mathrm{an}}}) \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{U(\mathfrak{g})}(M, \mathrm{C}^{\infty}(G_{\mathbb{R}})).$$

Now let us calculate the right-hand side of (1.6.1). Since we have

$$\begin{aligned}K := \mathrm{DR}_{X\times S}\mathscr{K} &= \mathrm{DR}_{X\times S}\big(\mathrm{Ind}^G_K(\mathscr{M})\big) \\ &\simeq \mathrm{Ind}^{G^{\mathrm{an}}}_{K^{\mathrm{an}}}(\mathrm{DR}_X(\mathscr{M})),\end{aligned}$$

$K \circ F$ is nothing but $\Phi(\mathrm{DR}_X(\mathscr{M}))[-2d_S]$. Therefore the right-hand side of (1.6.1) is isomorphic to $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathbb{C}}(\Phi(\mathrm{DR}_X(\mathscr{M})), \Omega_{X^{\mathrm{an}}}[d_X])$. Finally we obtain

$$\begin{aligned} &\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{U(\mathfrak{g})}(\Gamma(X;\mathscr{M})\,,\ \mathrm{C}^\infty(G_{\mathbb{R}})) \\ &\qquad \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathbb{C}}(\Phi(\mathrm{DR}_X(\mathscr{M})), \Omega_{X^{\mathrm{an}}}[d_X]), \end{aligned} \tag{1.7.1}$$

or

$$\mathrm{MG}(\Gamma(X;\mathscr{M})^*) \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathbb{C}}(\Phi(\mathrm{DR}_X(\mathscr{M})), \Omega_{X^{\mathrm{an}}}[d_X]). \tag{1.7.2}$$

By duality, we have

$$\mathrm{mg}(\Gamma(X;\mathscr{M})) \simeq \mathbf{R}\,\Gamma^{\mathrm{top}}_{\mathrm{c}}(X^{\mathrm{an}}\,;\,\Phi(\mathrm{DR}_X(\mathscr{M})) \otimes \mathscr{O}_{X^{\mathrm{an}}}). \tag{1.7.3}$$

This is the commutativity of Fig. 1.

1.8 Example

Let us illustrate the results explained so far by taking $SL(2,\mathbb{R}) \simeq SU(1,1)$ as an example. We set

$$\begin{aligned} G_{\mathbb{R}} &= SU(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} ; \alpha, \beta \in \mathbb{C},\ |\alpha|^2 - |\beta|^2 = 1 \right\} \subset G = SL(2,\mathbb{C}), \\ K_{\mathbb{R}} &= \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \bar\alpha \end{pmatrix} ; \alpha \in \mathbb{C},\ |\alpha| = 1 \right\} \subset K = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} ; \alpha \in \mathbb{C} \setminus \{0\} \right\}, \\ X &= \mathbb{P}^1. \end{aligned}$$

Here G acts on the flag manifold $X = \mathbb{P}^1 = \mathbb{C} \sqcup \{\infty\}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \longmapsto \frac{az+b}{cz+d}.$$

Its infinitesimal action $L_X\colon \mathfrak{g} \to \Gamma(X;\Theta_X)$ (with the sheaf Θ_X of vector fields on X) is given by

$$\begin{aligned} h &:= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longmapsto -2z\frac{d}{dz}, \\ e &:= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longmapsto -\frac{d}{dz}, \\ f &:= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \longmapsto z^2\frac{d}{dz}. \end{aligned}$$

We have

$$\Gamma(X;\mathscr{D}_X) = U(\mathfrak{g})/U(\mathfrak{g})\Delta,$$

where $\Delta = h(h-2) + 4ef = h(h+2) + 4fe \in \mathfrak{z}(\mathfrak{g})$.

The flag manifold X has three K-orbits:

$$\{0\},\ \{\infty\} \text{ and } X \setminus \{0, \infty\}\,.$$

The corresponding three $G_{\mathbb{R}}$-orbits are

$$X_-,\ X_+ \text{ and } X_{\mathbb{R}},$$

where $X_\pm = \{z \in \mathbb{P}^1\,;\, |z| \gtrless 1\}$ and $X_{\mathbb{R}} = \{z \in \mathbb{C}\,;\, |z| = 1\}$.

Let $j_0 \colon X \setminus \{0\} \hookrightarrow X$, $j_\infty \colon X \setminus \{\infty\} \hookrightarrow X$ and $j_{0,\infty} \colon X \setminus \{0, \infty\} \hookrightarrow X$ be the open embeddings. Then we have K-equivariant $\mathscr{D}_X$-modules $\mathscr{O}_X$, $j_{0*}j_0^{-1}\mathscr{O}_X$, $j_{\infty*}j_\infty^{-1}\mathscr{O}_X$ and $j_{0,\infty*}j_{0,\infty}^{-1}\mathscr{O}_X$. We have the inclusion relation:

$$\begin{array}{ccc} & j_{0,\infty*}j_{0,\infty}^{-1}\mathscr{O}_X & \\ \nearrow & & \nwarrow \\ j_{0*}j_0^{-1}\mathscr{O}_X & & j_{\infty*}j_\infty^{-1}\mathscr{O}_X\,. \\ \nwarrow & & \nearrow \\ & \mathscr{O}_X & \end{array}$$

There exist four irreducible K-equivariant $\mathscr{D}_X$-modules:

$$\begin{aligned}
\mathscr{M}_0 &= \mathcal{H}^1_{\{0\}}(\mathscr{O}_X) \simeq j_{0*}j_0^{-1}\mathscr{O}_X/\mathscr{O}_X \simeq j_{0,\infty*}j_{0,\infty}^{-1}\mathscr{O}_X/j_{\infty*}j_\infty^{-1}\mathscr{O}_X,\\
\mathscr{M}_\infty &= \mathcal{H}^1_{\{\infty\}}(\mathscr{O}_X) \simeq j_{\infty*}j_\infty^{-1}\mathscr{O}_X/\mathscr{O}_X \simeq j_{0,\infty*}j_{0,\infty}^{-1}\mathscr{O}_X/j_{0*}j_0^{-1}\mathscr{O}_X,\\
\mathscr{M}_{0,\infty} &= \mathscr{O}_X,\\
\mathscr{M}_{1/2} &= \mathscr{O}_X\sqrt{z} = \mathscr{D}_X/\mathscr{D}_X(L_X(h)+1).
\end{aligned}$$

Here, $\mathscr{M}_0$ and $\mathscr{M}_\infty$ correspond to the K-orbits $\{0\}$ and $\{\infty\}$, respectively, while both $\mathscr{M}_{0,\infty}$ and $\mathscr{M}_{1/2}$ correspond to the open K-orbit $X \setminus \{0, \infty\}$. Note that the isotropy subgroup K_z of K at $z \in X \setminus \{0, \infty\}$ is isomorphic to $\{1, -1\}$, and $\mathscr{M}_{0,\infty}$ corresponds to the trivial representation of K_z and $\mathscr{M}_{1/2}$ corresponds to the non-trivial one-dimensional representation of K_z. By the Beilinson-Bernstein correspondence, we obtain four irreducible Harish-Chandra modules with the trivial infinitesimal character:

$$\begin{aligned}
M_0 &= \mathscr{O}_X(X \setminus \{0\})/\mathbb{C} = \mathbb{C}[z^{-1}]/\mathbb{C} \simeq U(\mathfrak{g})/(U(\mathfrak{g})(h-2) + U(\mathfrak{g})\,f),\\
M_\infty &= \mathscr{O}_X(X \setminus \{\infty\})/\mathbb{C} \simeq \mathbb{C}[z]/\mathbb{C} \simeq U(\mathfrak{g})/(U(\mathfrak{g})(h+2) + U(\mathfrak{g})\,e),\\
M_{0,\infty} &= \mathscr{O}_X(X) = \mathbb{C} \simeq U(\mathfrak{g})/(U(\mathfrak{g})\,h + U(\mathfrak{g})\,e + U(\mathfrak{g})\,f),\\
M_{1/2} &= \mathbb{C}[z, z^{-1}]\sqrt{z} \simeq U(\mathfrak{g})/(U(\mathfrak{g})(h+1) + U(\mathfrak{g})\Delta).
\end{aligned}$$

Among them, $M_{0,\infty}$ and $M_{1/2}$ are self-dual, namely they satisfy $M^* \simeq M$. We have $(M_0)^* \simeq M_\infty$.

By the de Rham functor, the irreducible K-equivariant $\mathscr{D}_X$-modules are transformed to irreducible K^{an}-equivariant perverse sheaves as follows:

$$\begin{aligned}
\mathrm{DR}_X(\mathscr{M}_0) &= \mathbb{C}_{\{0\}}[-1],\\
\mathrm{DR}_X(\mathscr{M}_\infty) &= \mathbb{C}_{\{\infty\}}[-1],\\
\mathrm{DR}_X(\mathscr{M}_{0,\infty}) &= \mathbb{C}_{X^{\mathrm{an}}},\\
\mathrm{DR}_X(\mathscr{M}_{1/2}) &= \mathbb{C}_{X^{\mathrm{an}}}\sqrt{z}.
\end{aligned}$$

Here $\mathbb{C}_{X^{\mathrm{an}}}\sqrt{z}$ is the locally constant sheaf on $X^{\mathrm{an}} \setminus \{0,\infty\}$ of rank one (extended by zero over X^{an}) with the monodromy -1 around 0 and ∞.

Their images by the Matsuki correspondence (see Proposition 9.4.3) are

$$\begin{aligned}
\Phi(\mathrm{DR}_X(\mathscr{M}_0)) &\simeq \mathbb{C}_{X_-}[1],\\
\Phi(\mathrm{DR}_X(\mathscr{M}_\infty)) &\simeq \mathbb{C}_{X_+}[1],\\
\Phi(\mathrm{DR}_X(\mathscr{M}_{0,\infty})) &\simeq \mathbb{C}_{X^{\mathrm{an}}},\\
\Phi(\mathrm{DR}_X(\mathscr{M}_{1/2})) &\simeq \mathbb{C}_{X_\mathbb{R}}\sqrt{z}.
\end{aligned}$$

Note that $\mathbb{C}_{X_\mathbb{R}}\sqrt{z}$ is a local system on $X_\mathbb{R}$ of rank one with the monodromy -1.

Hence (1.7.2) reads as

$$\begin{aligned}
&\mathrm{MG}(M_0^*) \simeq \mathrm{MG}(M_\infty) \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathbb{C}}(\mathbb{C}_{X_-}[1], \Omega_{X^{\mathrm{an}}}[1]) \simeq \Omega_{X^{\mathrm{an}}}(X_-),\\
&\mathrm{MG}(M_\infty^*) \simeq \mathrm{MG}(M_0) \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathbb{C}}(\mathbb{C}_{X_+}[1], \Omega_{X^{\mathrm{an}}}[1]) \simeq \Omega_{X^{\mathrm{an}}}(X_+),\\
&\mathrm{MG}(M_{0,\infty}^*) \simeq \mathrm{MG}(M_{0,\infty}) \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathbb{C}}(\mathbb{C}_{X^{\mathrm{an}}}, \Omega_{X^{\mathrm{an}}}[1])\\
&\qquad\qquad \simeq H^1(X^{\mathrm{an}}; \Omega_{X^{\mathrm{an}}}) \simeq \mathbb{C},\\
&\mathrm{MG}(M_{1/2}^*) \simeq \mathrm{MG}(M_{1/2}) \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathbb{C}}(\mathbb{C}_{X_\mathbb{R}}\sqrt{z}, \Omega_{X^{\mathrm{an}}}[1])\\
&\qquad\qquad \simeq \Gamma\big(X_\mathbb{R}; \mathscr{B}_{X_\mathbb{R}} \otimes \Omega_{X_\mathbb{R}} \otimes \mathbb{C}_{X_\mathbb{R}}\sqrt{z}\big).
\end{aligned}$$

Here $\mathscr{B}_{X_\mathbb{R}}$ is the sheaf of hyperfunctions on $X_\mathbb{R}$. Note that the exterior differentiation gives isomorphisms

$$\begin{aligned}
\mathscr{O}_{X^{\mathrm{an}}}(X_\pm)/\mathbb{C} &\xrightarrow[d]{\sim} \Omega_{X^{\mathrm{an}}}(X_\pm),\\
\Gamma(X_\mathbb{R}; \mathscr{B}_{X_\mathbb{R}} \otimes \mathbb{C}_{X_\mathbb{R}}\sqrt{z}) &\xrightarrow[d]{\sim} \Gamma(X_\mathbb{R}; \mathscr{B}_{X_\mathbb{R}} \otimes \Omega_{X_\mathbb{R}} \otimes \mathbb{C}_{X_\mathbb{R}}\sqrt{z}).
\end{aligned}$$

In fact, we have

$$\begin{aligned}
\mathrm{mg}(M_0) \simeq \Omega_{X^{\mathrm{an}}}(\overline{X_+}) &\subset \Omega_{X^{\mathrm{an}}}(X_+) \simeq \mathrm{MG}(M_0),\\
\mathrm{mg}(M_\infty) \simeq \Omega_{X^{\mathrm{an}}}(\overline{X_-}) &\subset \Omega_{X^{\mathrm{an}}}(X_-) \simeq \mathrm{MG}(M_\infty),\\
\mathrm{mg}(M_{0,\infty}) &\xrightarrow{\sim} \mathrm{MG}(M_{0,\infty}) \simeq \mathbb{C},\\
\mathrm{mg}(M_{1/2}) \simeq \Gamma\big(X_\mathbb{R}; \mathscr{A}_{X_\mathbb{R}} \otimes \mathbb{C}_{X_\mathbb{R}}\sqrt{z}\big) &\subset \Gamma\big(X_\mathbb{R}; \mathscr{B}_{X_\mathbb{R}} \otimes \mathbb{C}_{X_\mathbb{R}}\sqrt{z}\big) \simeq \mathrm{MG}(M_{1/2}).
\end{aligned}$$

Here $\mathscr{A}_{X_\mathbb{R}}$ is the sheaf of real analytic functions on $X_\mathbb{R}$.

For example, by (1.7.3), $\mathrm{mg}(M_0) \simeq \mathbf{R}\Gamma^{\mathrm{top}}_{\mathrm{c}}(X^{\mathrm{an}}; \mathbb{C}_{X_-}[1] \otimes \mathscr{O}_{X^{\mathrm{an}}})$. The exact sequence

$$0 \to \mathbb{C}_{X_-} \to \mathbb{C}_{X^{\mathrm{an}}} \to \mathbb{C}_{\overline{X_+}} \to 0$$

yields the exact sequence:

$$\begin{aligned} &H^0(X^{\mathrm{an}}; \mathbb{C}_{X_-} \otimes \mathscr{O}_{X^{\mathrm{an}}}) \to H^0(X^{\mathrm{an}}; \mathbb{C}_{X^{\mathrm{an}}} \otimes \mathscr{O}_{X^{\mathrm{an}}}) \to H^0(X^{\mathrm{an}}; \mathbb{C}_{\overline{X_+}} \otimes \mathscr{O}_{X^{\mathrm{an}}}) \\ &\quad \to H^0(X^{\mathrm{an}}; \mathbb{C}_{X_-}[1] \otimes \mathscr{O}_{X^{\mathrm{an}}}) \to H^0(X^{\mathrm{an}}; \mathbb{C}_{X^{\mathrm{an}}}[1] \otimes \mathscr{O}_{X^{\mathrm{an}}}), \end{aligned}$$

in which $H^0(X^{\mathrm{an}}; \mathbb{C}_{X_-} \otimes \mathscr{O}_{X^{\mathrm{an}}}) = \{u \in \mathscr{O}_{X^{\mathrm{an}}}(X^{\mathrm{an}})\,;\, \mathrm{supp}(u) \subset X_-\} = 0$, $H^0(X^{\mathrm{an}}; \mathbb{C}_{X^{\mathrm{an}}} \otimes \mathscr{O}_{X^{\mathrm{an}}}) = \mathscr{O}_{X^{\mathrm{an}}}(X^{\mathrm{an}}) = \mathbb{C}$ and $H^0(X^{\mathrm{an}}; \mathbb{C}_{X^{\mathrm{an}}}[1] \otimes \mathscr{O}_{X^{\mathrm{an}}}) = H^1(X^{\mathrm{an}}; \mathscr{O}_{X^{\mathrm{an}}}) = 0$.

Hence we have

$$\mathbf{R}\,\Gamma_{\mathrm{c}}^{\mathrm{top}}(X^{\mathrm{an}}; \mathbb{C}_{X_-}[1] \otimes \mathscr{O}_{X^{\mathrm{an}}}) \simeq \mathscr{O}_{X^{\mathrm{an}}}(\overline{X_+})/\mathbb{C}.$$

The exterior differentiation gives an isomorphism

$$\mathscr{O}_{X^{\mathrm{an}}}(\overline{X_+})/\mathbb{C} \xrightarrow[d]{\sim} \Omega_{X^{\mathrm{an}}}(\overline{X_+}).$$

Note that we have

$$\begin{aligned} \mathrm{HC}(\Omega_{X^{\mathrm{an}}}(\overline{X_+})) &\simeq \mathrm{HC}(\Omega_{X^{\mathrm{an}}}(X_+)) \\ &\simeq \Omega_X(X \setminus \{0\}) \xleftarrow[d]{\sim} \mathscr{O}_X(X \setminus \{0\})/\mathbb{C} \simeq M_0. \end{aligned}$$

1.9 Organization of the note

So far, we have explained Fig. 1 briefly. We shall explain more details in the subsequent sections.

The category of representations of $G_{\mathbb{R}}$ is not an abelian category, but it is a so-called quasi-abelian category and we can consider its derived category. In § 2, we explain the derived category of a quasi-abelian category following J.-P. Schneiders [26].

In § 3, we introduce the notion of quasi-G-equivariant D-modules, and studies their derived category. We construct the pull-back and push-forward functors for $\mathrm{D^b}(\mathrm{Mod}\,(\mathscr{D}_X, G))$, and prove that they commute with the forgetful functor $\mathrm{D^b}(\mathrm{Mod}\,(\mathscr{D}_X, G)) \to \mathrm{D^b}(\mathrm{Mod}\,(\mathscr{D}_X))$.

In § 4, we explain the equivariant derived category following Bernstein-Lunts [4].

In § 5, we define $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_Z}(\mathscr{M} \otimes F, \mathscr{O}_{Z^{\mathrm{an}}})$ and studies its functorial properties.

In § 6, we prove the ellipticity theorem, which says that, for a real form $i\colon X_{\mathbb{R}} \hookrightarrow X$, $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}^{\infty}_{X_{\mathbb{R}}}) \longrightarrow \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M} \otimes i_* i^! \mathbb{C}_{X_{\mathbb{R}}}, \mathscr{O}_{X^{\mathrm{an}}})$ is an isomorphism when $\mathscr{M}$ is an elliptic D-module. In order to construct this morphism, we use the Whitney functor introduced by Kashiwara-Schapira [20].

If we want to deal with non-trivial infinitesimal characters, we need to twist sheaves and D-modules. In § 7, we explain these twistings.

In § 8, we prove the integral transform formula explained in the subsection 1.6.

In § 9, we apply these results to the representation theory of real semisimple Lie groups. We construct the arrows in Fig. 1

As an application of § 9, we give a proof of the cohomology vanishing theorem $H^j(\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{U(\mathfrak{g})}(M, \mathrm{C}^\infty(G_\mathbb{R})) = 0$ $(j \neq 0)$ and its dual statement $H^j(\Gamma_c(G_\mathbb{R}; \mathscr{D}ist_{G_\mathbb{R}}) \overset{\mathbf{L}}{\otimes}_{U(\mathfrak{g})} M) = 0$ in § 10.

2 Derived categories of quasi-abelian categories

2.1 Quasi-abelian categories

The representations of real semisimple groups are realized on topological vector spaces, and they do not form an abelian category. However, they form a so-called quasi-abelian category. In this section, we shall review the results of J.-P. Schneiders on the theory of quasi-abelian categories and their derived categories. For more details, we refer the reader to [26].

Let $\mathcal{C}$ be an additive category admitting the kernels and the cokernels. Let us recall that, for a morphism $f\colon X \to Y$ in $\mathcal{C}$, $\mathrm{Im}(f)$ is the kernel of $Y \to \mathrm{Coker}(f)$, and $\mathrm{Coim}(f)$ is the cokernel of $\mathrm{Ker}(f) \to X$. Then f decomposes as $X \to \mathrm{Coim}(f) \to \mathrm{Im}(f) \to Y$. We say that f is *strict* if $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism. Note that a monomorphism (resp. epimorphism) $f\colon X \to Y$ is strict if and only if $X \to \mathrm{Im}(f)$ (resp. $\mathrm{Coim}(f) \to Y$) is an isomorphism. Note that, for any morphism $f\colon X \to Y$, the morphisms $\mathrm{Ker}(f) \to X$ and $\mathrm{Im}(f) \to Y$ are strict monomorphisms, and $X \to \mathrm{Coim}(f)$ and $Y \to \mathrm{Coker}(f)$ are strict epimorphisms. Note also that a morphism f is strict if and only if it factors as $i \circ s$ with a strict epimorphism s and a strict monomorphism i.

Definition 2.1.1. A *quasi-abelian* category is an additive category admitting the kernels and the cokernels which satisfies the following conditions:

(i) the strict epimorphisms are stable by base changes,
(ii) the strict monomorphisms are stable by co-base changes.

The condition (i) means that, for any strict epimorphism $u\colon X \to Y$ and a morphism $Y' \to Y$, setting $X' = X \times_Y Y' = \mathrm{Ker}(X \oplus Y' \to Y)$, the composition $X' \to X \oplus Y' \to Y'$ is a strict epimorphism. The condition (ii) is the similar condition obtained by reversing arrows.

Note that, for any morphism $f\colon X \to Y$ in a quasi-abelian category, $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is a monomorphism and an epimorphism.

Remark that if $\mathcal{C}$ is a quasi-abelian category, then its opposite category $\mathcal{C}^{\mathrm{op}}$ is also quasi-abelian.

We recall that an abelian category is an additive category such that it admits the kernels and the cokernels and all the morphisms are strict.

Example 2.1.2. (i) Let **Top** be the category of Hausdorff locally convex topological vector spaces. Then **Top** is a quasi-abelian category. For a morphism $f\colon X \to Y$, $\mathrm{Ker}(f)$ is $f^{-1}(0)$ with the induced topology from X, $\mathrm{Coker}(f)$ is $Y/\overline{f(X)}$ with the quotient topology of Y, $\mathrm{Coim}(f)$ is $f(X)$ with the quotient topology of X and $\mathrm{Im}(f)$ is $\overline{f(X)}$ with the induced topology from Y. Hence f is strict if and only if $f(X)$ is a closed subspace of Y and the topology on $f(X)$ induced from X coincides with the one induced from Y.

(ii) Let E be a Hausdorff locally convex topological vector space. Let us recall that a subset B of E is *bounded* if for any neighborhood U of 0 there exists $c > 0$ such that $B \subset cU$. A family $\{f_i\}$ of linear functionals on E is called *equicontinuous* if there exists a neighborhood U of $0 \in E$ such that $f_i(U) \subset \{c \in \mathbb{C}\,;\, |c| < 1\}$ for any i. For two complete locally convex topological vector spaces E and F, a continuous linear map $f\colon E \to F$ is called *nuclear* if there exist an equicontinuous sequence $\{h_n\}_{n\geq 1}$ of linear functionals on E, a bounded sequence $\{v_n\}_{n\geq 1}$ of elements of F and a sequence $\{c_n\}$ in $\mathbb{C}$ such that $\sum |c_n| < \infty$ and $f(x) = \sum_n c_n h_n(x) v_n$ for all $x \in E$.

A Fréchet nuclear space (FN space, for short) is a Fréchet space E such that any homomorphism from E to a Banach space is nuclear. It is equivalent to saying that E is isomorphic to the projective limit of a sequence of Banach spaces $F_1 \leftarrow F_2 \leftarrow \cdots$ such that $F_n \to F_{n-1}$ are nuclear for all n. We denote by **FN** the full subcategory of **Top** consisting of Fréchet nuclear spaces.

A dual Fréchet nuclear space (DFN space, for short) is the inductive limit of a sequence of Banach spaces $F_1 \to F_2 \to \cdots$ such that $F_n \to F_{n+1}$ are injective and nuclear for all n. We denote by **DFN** the full subcategory of **Top** consisting of dual Fréchet nuclear spaces.

A closed linear subspace of an FN space (resp. a DFN space), as well as the quotient of an FN space (resp. a DFN space) by a closed subspace, is also an FN space (resp. a DFN space). Hence, both **FN** and **DFN** are quasi-abelian.

A morphism $f\colon E \to F$ in **FN** or **DFN** is strict if and only if $f(E)$ is a closed subspace of F.

The category **DFN** is equivalent to the opposite category $\mathbf{FN}^{\mathrm{op}}$ of **FN** by $E \mapsto E^*$, where E^* is the strong dual of E.

Note that if M is a C^∞-manifold (countable at infinity), then the space $\mathrm{C}^\infty(M)$ of C^∞-functions on M is an FN space. The space $\Gamma_c(M; \mathscr{D}ist_M)$ of distributions with compact support is a DFN space. If X is a complex manifold (countable at infinity), the space $\mathscr{O}_X(X)$ of holomorphic functions is an FN space. For a compact subset K of X, the space $\mathscr{O}_X(K)$ of holomorphic functions defined on a neighborhood of K is a DFN space.

(iii) Let G be a Lie group. A Fréchet nuclear G-module is an FN space E with a continuous G-action, namely G acts on E and the action map $G \times E \to E$ is continuous. Let us denote by $\mathbf{FN}_G$ the category of Fréchet nuclear G-modules. It is also a quasi-abelian category. Similarly we define the notion of dual Fréchet nuclear G-modules and the category $\mathbf{DFN}_G$. The category $(\mathbf{FN}_G)^{\mathrm{op}}$ and $\mathbf{DFN}_G$ are equivalent.

2.2 Derived categories

Let $\mathcal{C}$ be a quasi-abelian category. A complex X in $\mathcal{C}$ consists of objects X^n ($n \in \mathbb{Z}$) and morphisms $d_X^n \colon X^n \to X^{n+1}$ such that $d_X^{n+1} \circ d_X^n = 0$. The morphisms d_X^n are called the *differentials* of X. Morphisms between complexes are naturally defined. Then the complexes in $\mathcal{C}$ form an additive category, which will be denoted by $\mathrm{C}(\mathcal{C})$. For a complex X and $k \in \mathbb{Z}$, let $X[k]$ be the complex defined by

$$X[k]^n = X^{n+k} \quad d^n_{X[k]} = (-1)^k d_X^{n+k}.$$

Then $X \mapsto X[k]$ is an equivalence of categories, called the *translation functor*.

We say that a complex X is a *strict* complex if all the differentials d_X^n are strict. We say that a complex X is *strictly exact* if $\mathrm{Coker}(d_X^{n-1}) \to \mathrm{Ker}(d_X^{n+1})$ is an isomorphism for all n. Note that $d_X^n \colon X^n \to X^{n+1}$ decomposes into

$$X^n \twoheadrightarrow \mathrm{Coker}(d_X^{n-1}) \twoheadrightarrow \mathrm{Coim}(d_X^n) \to \mathrm{Im}(d_X^n) \rightarrowtail \mathrm{Ker}(d_X^{n+1}) \rightarrowtail X^{n+1}.$$

If X is strictly exact, then X is a strict complex and $0 \to \mathrm{Ker}(d_X^n) \to X^n \to \mathrm{Ker}(d_X^{n+1}) \to 0$ is strictly exact.

For a morphism $f \colon X \to Y$ in $\mathrm{C}(\mathcal{C})$, its mapping cone $\mathrm{Mc}(f)$ is defined by

$$\mathrm{Mc}(f)^n = X^{n+1} \oplus Y^n \text{ and } d^n_{\mathrm{Mc}(f)} = \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix}.$$

Then we have a sequence of canonical morphisms in $\mathrm{C}(\mathcal{C})$:

$$X \xrightarrow{f} Y \xrightarrow{\alpha(f)} \mathrm{Mc}(f) \xrightarrow{\beta(f)} X[1]. \tag{2.2.1}$$

Let $\mathrm{K}(\mathcal{C})$ be the *homotopy category*, which is defined as follows: $\mathrm{Ob}(\mathrm{K}(\mathcal{C})) = \mathrm{Ob}(\mathrm{C}(\mathcal{C}))$ and, for $X, Y \in \mathrm{K}(\mathcal{C})$, we define

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(X, Y) = \mathrm{Hom}_{\mathrm{C}(\mathcal{C})}(X, Y)/\mathrm{Ht}(X, Y),$$

where

$$\mathrm{Ht}(X, Y) = \{f \in \mathrm{Hom}_{\mathrm{C}(\mathcal{C})}(X, Y)\,;\text{ there exist } h^n \colon X^n \to Y^{n-1} \text{ such that } f^n = d_Y^{n-1} \circ h^n + h^{n+1} \circ d_X^n \text{ for all } n\}.$$

A morphism in $\mathrm{Ht}(X, Y)$ is sometimes called a morphism *homotopic to zero*.

A *triangle* in $\mathrm{K}(\mathcal{C})$ is a sequence of morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

such that $g \circ f = 0$, $h \circ g = 0$, $f[1] \circ h = 0$. For example, the image of (2.2.1) in $\mathrm{K}(\mathcal{C})$ is a triangle for any morphism $f \in \mathrm{C}(\mathcal{C})$. A triangle in $\mathrm{K}(\mathcal{C})$ is called a *distinguished triangle* if it is isomorphic to the image of the triangle (2.2.1) by the functor $\mathrm{C}(\mathcal{C}) \to \mathrm{K}(\mathcal{C})$ for some morphism $f \in \mathrm{C}(\mathcal{C})$. The additive category $\mathrm{K}(\mathcal{C})$ with the translation functor $\bullet[1]$ and the family of distinguished triangles is a *triangulated category* (see e.g. [19]).

Note that if two complexes X and Y are isomorphic in $\mathrm{K}(\mathcal{C})$, and if X is a strictly exact complex, then so is Y. Let $\mathscr{E}$ be the subcategory of $\mathrm{K}(\mathcal{C})$ consisting of strictly exact complexes. Then $\mathscr{E}$ is a triangulated subcategory, namely it is closed by the translation functors $[k]$ $(k \in \mathbb{Z})$, and if $X \to Y \to Z \to X[1]$ is a distinguished triangle and $X, Y \in \mathscr{E}$, then $Z \in \mathscr{E}$.

We define the derived category $\mathrm{D}(\mathcal{C})$ as the quotient category $\mathrm{K}(\mathcal{C})/\mathscr{E}$. It is defined as follows. A morphism $f\colon X \to Y$ is called a *quasi-isomorphism* (qis for short) if, embedding it in a distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$, Z belongs to $\mathscr{E}$. For a chain of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathrm{K}(\mathcal{C})$, if two of f, g and $g \circ f$ are qis, then all the three are qis.

With this terminology, $\mathrm{Ob}(\mathrm{D}(\mathcal{C})) = \mathrm{Ob}(\mathrm{K}(\mathcal{C}))$ and for $X, Y \in \mathrm{D}(\mathcal{C})$,

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(X, Y) &\simeq \varinjlim_{X' \xrightarrow{\text{qis}} X} \mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(X', Y) \\
&\xrightarrow{\sim} \varinjlim_{X' \xrightarrow{\text{qis}} X,\, Y \xrightarrow{\text{qis}} Y'} \mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(X', Y') \\
&\xleftarrow{\sim} \varinjlim_{Y \xrightarrow{\text{qis}} Y'} \mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(X, Y').
\end{aligned}$$

The composition of morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ is visualized by the following diagram:

$$\begin{array}{ccccc}
X & \overset{f}{\dashrightarrow} & Y & \overset{g}{\dashrightarrow} & Z \\
\uparrow{\scriptstyle \text{qis}} & \nearrow & & \searrow & \downarrow{\scriptstyle \text{qis}} \\
X' & \longrightarrow & & \longrightarrow & Z'.
\end{array}$$

A morphism in $\mathrm{K}(\mathcal{C})$ induces an isomorphism in $\mathrm{D}(\mathcal{C})$ if and only if it is a quasi-isomorphism.

A triangle $X \to Y \to Z \to X[1]$ in $\mathrm{D}(\mathcal{C})$ is called a distinguished triangle if it is isomorphic to the image of a distinguished triangle in $\mathrm{K}(\mathcal{C})$. Then $\mathrm{D}(\mathcal{C})$ is also a triangulated category.

Note that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a sequence of morphisms in $\mathrm{C}(\mathcal{C})$ such that $0 \to X^n \to Y^n \to Z^n \to 0$ is strictly exact for all n, then the natural morphism $\mathrm{Mc}(f) \to Z$ is a qis, and we have a distinguished triangle

$$X \to Y \to Z \to X[1]$$

in $\mathrm{D}(\mathcal{C})$.

We denote by $\mathrm{C}^+(\mathcal{C})$ (resp. $\mathrm{C}^-(\mathcal{C})$, $\mathrm{C}^{\mathrm{b}}(\mathcal{C})$) the full subcategory of $\mathrm{C}(\mathcal{C})$ consisting of objects X such that $X^n = 0$ for $n \ll 0$ (resp. $n \gg 0$, $|n| \gg 0$). Let $\mathrm{D}^*(\mathcal{C})$ ($* = +, -, \mathrm{b}$) be the full subcategory of $\mathrm{D}(\mathcal{C})$ whose objects are isomorphic to the image of objects of $\mathrm{C}^*(\mathcal{C})$. Similarly, we define the full subcategory $\mathrm{K}^*(\mathcal{C})$ of $\mathrm{K}(\mathcal{C})$.

We call $\mathrm{D}^{\mathrm{b}}(\mathcal{C})$ the *bounded derived category* of $\mathcal{C}$.

2.3 *t*-structure

Let us define various truncation functors for $X \in \mathrm{C}(\mathcal{C})$:

$$\begin{array}{llllllllll}
\tau^{\le n}X & : \cdots \longrightarrow & X^{n-1} & \longrightarrow & \operatorname{Ker} d_X^n & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \\
\tau^{\le n+1/2}X & : \cdots \longrightarrow & X^{n-1} & \longrightarrow & X^n & \longrightarrow & \operatorname{Im} d_X^n & \longrightarrow & 0 & \longrightarrow \cdots \\
\tau^{\ge n}X & : \cdots \longrightarrow & 0 & \longrightarrow & \operatorname{Coker} d_X^{n-1} & \longrightarrow & X^{n+1} & \longrightarrow & X^{n+2} & \longrightarrow \cdots \\
\tau^{\ge n+1/2}X & : \cdots \longrightarrow & 0 & \longrightarrow & \operatorname{Coim} d_X^n & \longrightarrow & X^{n+1} & \longrightarrow & X^{n+2} & \longrightarrow \cdots
\end{array}$$

for $n \in \mathbb{Z}$. Then we have morphisms

$$\tau^{\le s}X \longrightarrow \tau^{\le t}X \longrightarrow X \longrightarrow \tau^{\ge s}X \longrightarrow \tau^{\ge t}X$$

for $s, t \in \frac{1}{2}\mathbb{Z}$ such that $s \le t$. We can easily check that the functors $\tau^{\le s}, \tau^{\ge s} \colon \mathrm{C}(\mathcal{C}) \to \mathrm{C}(\mathcal{C})$ send the morphisms homotopic to zero to morphisms homotopic to zero and the quasi-isomorphisms to quasi-isomorphisms. Hence, they induce the functors

$$\tau^{\le s}, \tau^{\ge s} \colon \mathrm{D}(\mathcal{C}) \to \mathrm{D}(\mathcal{C})$$

and morphisms $\tau^{\le s} \to \mathrm{id} \to \tau^{\ge s}$. We have isomorphisms of functors:

$$\begin{array}{l}
\tau^{\le s} \circ \tau^{\le t} \simeq \tau^{\le \min(s,t)}, \ \tau^{\ge s} \circ \tau^{\ge t} \simeq \tau^{\ge \max(s,t)}, \text{ and} \\
\tau^{\le s} \circ \tau^{\ge t} \simeq \tau^{\ge t} \circ \tau^{\le s} \qquad \text{for } s, t \in \frac{1}{2}\mathbb{Z}.
\end{array}$$

We set $\tau^{>s} = \tau^{\ge s+1/2}$ and $\tau^{<s} = \tau^{\le s-1/2}$.

We have a distinguished triangle in $\mathrm{D}(\mathcal{C})$:

$$\tau^{\le s}X \longrightarrow X \longrightarrow \tau^{>s}X \longrightarrow (\tau^{\le s}X)[1].$$

For $s \in \frac{1}{2}\mathbb{Z}$, set

$$\begin{aligned}
\mathrm{D}^{\le s}(\mathcal{C}) &= \left\{X \in \mathrm{D}(\mathcal{C})\,;\, \tau^{\le s}X \to X \text{ is an isomorphism}\right\} \\
&= \left\{X \in \mathrm{D}(\mathcal{C})\,;\, \tau^{>s}X \simeq 0\right\}, \\
\mathrm{D}^{\ge s}(\mathcal{C}) &= \left\{X \in \mathrm{D}(\mathcal{C})\,;\, X \to \tau^{\ge s}X \text{ is an isomorphism}\right\} \\
&= \left\{X \in \mathrm{D}(\mathcal{C})\,;\, \tau^{<s}X \simeq 0\right\}.
\end{aligned}$$

Then $\{\mathrm{D}^{\leq s}(\mathcal{C})\}_{s\in\frac{1}{2}\mathbb{Z}}$ is an increasing sequence of full subcategories of $\mathrm{D}(\mathcal{C})$, and $\{\mathrm{D}^{\geq s}(\mathcal{C})\}_{s\in\frac{1}{2}\mathbb{Z}}$ is a decreasing sequence of full subcategories of $\mathrm{D}(\mathcal{C})$.

Note that $\mathrm{D}^{+}(\mathcal{C})$ (resp. $\mathrm{D}^{-}(\mathcal{C})$) is the union of all the $\mathrm{D}^{\geq n}(\mathcal{C})$'s (resp. all the $\mathrm{D}^{\leq n}(\mathcal{C})$'s), and $\mathrm{D}^{\mathrm{b}}(\mathcal{C})$ is the intersection of $\mathrm{D}^{+}(\mathcal{C})$ and $\mathrm{D}^{-}(\mathcal{C})$.

The functor $\tau^{\leq s}\colon \mathrm{D}(\mathcal{C}) \to \mathrm{D}^{\leq s}(\mathcal{C})$ is a right adjoint functor of the inclusion functor $\mathrm{D}^{\leq s}(\mathcal{C}) \hookrightarrow \mathrm{D}(\mathcal{C})$, and $\tau^{\geq s}\colon \mathrm{D}(\mathcal{C}) \to \mathrm{D}^{\geq s}(\mathcal{C})$ is a left adjoint functor of $\mathrm{D}^{\geq s}(\mathcal{C}) \hookrightarrow \mathrm{D}(\mathcal{C})$.

Set $\mathrm{D}^{>s}(\mathcal{C}) = \mathrm{D}^{\geq s+1/2}(\mathcal{C})$ and $\mathrm{D}^{<s}(\mathcal{C}) = \mathrm{D}^{\leq s-1/2}(\mathcal{C})$.

The pair $(\mathrm{D}^{\leq s}(\mathcal{C}), \mathrm{D}^{>s-1}(\mathcal{C}))$ is a t-structure of $\mathrm{D}(\mathcal{C})$ (see [3] and also [18]) for any $s \in \frac{1}{2}\mathbb{Z}$. Hence, $\mathrm{D}^{\leq s}(\mathcal{C}) \cap \mathrm{D}^{>s-1}(\mathcal{C})$ is an abelian category. The triangulated category $\mathrm{D}(\mathcal{C})$ is equivalent to the derived category of $\mathrm{D}^{\leq s}(\mathcal{C}) \cap \mathrm{D}^{>s-1}(\mathcal{C})$. The full subcategory $\mathrm{D}^{\leq 0}(\mathcal{C}) \cap \mathrm{D}^{\geq 0}(\mathcal{C})$ is equivalent to $\mathcal{C}$.

For $X \in \mathrm{C}(\mathcal{C})$ and an integer n, the following conditions are equivalent:

(i) d_X^n is strict,

(ii) $\tau^{\leq n}X \to \tau^{\leq n+1/2}X$ is a quasi-isomorphism,

(iii) $\tau^{\geq n+1/2}X \to \tau^{\geq n+1}X$ is a quasi-isomorphism.

Hence, for an object X of $\mathrm{D}(\mathcal{C})$, X is represented by some strict complex if and only if all complexes in $\mathrm{C}(\mathcal{C})$ representing X are strict complexes. In such a case, we say that X is *strict*. Then, its cohomology group $H^n(X) := \mathrm{Coker}(X^{n-1} \to \mathrm{Ker}(d_X^n)) \simeq \mathrm{Ker}(\mathrm{Coker}(d_X^{n-1}) \to X^{n+1})$ has a sense as an object of $\mathcal{C}$. The following lemma is immediate.

Lemma 2.3.1. *Let $X \to Y \to Z \xrightarrow{+1} X[1]$ be a distinguished triangle, and assume that X and Y are strict. If $H^n(X) \to H^n(Y)$ is a strict morphism for all n, then Z is strict. Moreover we have a strictly exact sequence:*

$$\cdots \longrightarrow H^n(X) \longrightarrow H^n(Y) \longrightarrow H^n(Z) \longrightarrow H^{n+1}(X) \longrightarrow H^{n+1}(Y) \longrightarrow \cdots.$$

Remark 2.3.2. When $\mathcal{C}$ is either **FN** or **DFN**, a complex X in $\mathcal{C}$ is strictly exact if and only if it is exact as a complex of vector spaces forgetting the topology. A complex X is strict if and only if the image of the differential d_X^n is closed in X^{n+1} for all n. Hence, denoting by $\mathscr{F}$ the functor from $\mathrm{D}(\mathbf{FN})$ (resp. $\mathrm{D}(\mathbf{DFN})$) to $\mathrm{D}(\mathrm{Mod}(\mathbb{C}))$, a morphism f in $\mathrm{D}(\mathbf{FN})$ (resp. $\mathrm{D}(\mathbf{DFN})$) is an isomorphism if and only if so is $\mathscr{F}(u)$.

3 Quasi-equivariant D-modules

3.1 Definition

For the theory of D-modules, we refer the reader to [16].

Let us recall the definition of quasi-equivariant D-modules (cf. [15]).

Let G be an affine algebraic group over $\mathbb{C}$ and $\mathfrak{g}$ its Lie algebra. A *G-module* is by definition a vector space V endowed with an action of G such that $g \mapsto gv$ is a regular function on G for any $v \in V$, i.e., there exist finitely many $\{v_i\}_i$ of vectors in V and regular functions $\{a_i(g)\}_i$ on G such that $gv = \sum_i a_i(g)v_i$ for any $g \in G$. It is equivalent to saying that there is a homomorphism $V \to \mathscr{O}_G(G) \otimes V$ (i.e., $v \mapsto \sum_i a_i(g) \otimes v_i$) such that for any $g \in G$ the action $\mu_g \in \mathrm{End}_{\mathbb{C}}(V)$ is given by $V \to \mathscr{O}_G(G) \otimes V \xrightarrow{i_g^*} V$, where the last arrow i_g^* is induced by the evaluation map $\mathscr{O}_G(G) \to \mathbb{C}$ at g. Hence the G-module structure is equivalent to the co-module structure over the cogebra $\mathscr{O}_G(G)$.

We denote by $\mathrm{Mod}(G)$ the category of G-modules, and by $\mathrm{Mod}_f(G)$ the category of finite-dimensional G-modules. It is well-known that any G-module is a union of finite-dimensional sub-G-modules.

Let us recall the definition of $(\mathfrak{g}, H)$-modules for a subgroup H of G.

Definition 3.1.1. Let H be a closed subgroup of G with a Lie algebra $\mathfrak{h}$. A $(\mathfrak{g}, H)$-module is a vector space M endowed with an H-module structure and a $\mathfrak{g}$-module structure such that

(i) the $\mathfrak{h}$-module structure on M induced by the H-module structure coincides with the one induced by the $\mathfrak{g}$-module structure,
(ii) the multiplication homomorphism $\mathfrak{g} \otimes M \to M$ is H-linear, where H acts on $\mathfrak{g}$ by the adjoint action.

Let us denote by $\mathrm{Mod}(\mathfrak{g}, H)$ the category of $(\mathfrak{g}, H)$-modules.

Let X be a smooth algebraic variety with a G-action (we call it *algebraic G-manifold*). Let $\mu\colon G \times X \to X$ denote the action morphism and $\mathrm{pr}\colon G \times X \to X$ the projection. We shall define $p_k\colon G \times G \times X \to G \times X$ $(k = 0, 1, 2)$ by

$$\begin{aligned} p_0(g_1, g_2, x) &= (g_1, g_2 x), \\ p_1(g_1, g_2, x) &= (g_1 g_2, x), \\ p_2(g_1, g_2, x) &= (g_2, x). \end{aligned} \qquad \begin{aligned} \mu(g, x) &= gx, \\ \mathrm{pr}(g, x) &= x, \end{aligned}$$

Then we have a simplicial diagram

$$G \times G \times X \overset{p_0}{\underset{p_2}{\overset{\xrightarrow{\quad p_1 \quad}}{\rightrightarrows}}} G \times X \overset{\mu}{\underset{\mathrm{pr}}{\rightrightarrows}} X.$$

It means that these morphisms satisfy the commutation relations:

$$\begin{aligned} \mu \circ p_0 &= \mu \circ p_1, \\ \mathrm{pr} \circ p_1 &= \mathrm{pr} \circ p_2, \\ \mu \circ p_2 &= \mathrm{pr} \circ p_0. \end{aligned}$$

Definition 3.1.2. A *G-equivariant $\mathscr{O}_X$*-module is an $\mathscr{O}_X$-module $\mathscr{F}$ endowed with an isomorphism of $\mathscr{O}_{G \times X}$-modules:

$$\beta\colon \mu^* \mathscr{F} \xrightarrow{\sim} \mathrm{pr}^* \mathscr{F} \tag{3.1.1}$$

such that the following diagram commutes (*associative law*):

$$\begin{array}{ccccccc} p_1^*\mu^*\mathscr{F} & & \xrightarrow{p_1^*\beta} & & & & p_1^*\mathrm{pr}^*\mathscr{F} \\ \| & & & & & & \| \\ p_0^*\mu^*\mathscr{F} & \xrightarrow{p_0^*\beta} & p_0^*\mathrm{pr}^*\mathscr{F} & = & p_2^*\mu^*\mathscr{F} & \xrightarrow{p_2^*\beta} & p_2^*\mathrm{pr}^*\mathscr{F}\,. \end{array} \tag{3.1.2}$$

We denote by $\mathrm{Mod}(\mathscr{O}_X, G)$ the category of G-equivariant $\mathscr{O}_X$-modules which are *quasi-coherent* as $\mathscr{O}_X$-modules.

For a G-equivariant $\mathscr{O}_X$-module $\mathscr{F}$, we can define an action of the Lie algebra $\mathfrak{g}$ on $\mathscr{F}$, i.e., a Lie algebra homomorphism:

$$L_{\mathrm{v}}\colon \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(\mathscr{F}) \tag{3.1.3}$$

as follows. Let us denote by

$$L_X\colon \mathfrak{g} \to \Theta_X(X) \to \mathscr{D}_X(X) \tag{3.1.4}$$

the infinitesimal action of G on X. Here, Θ_X denotes the sheaf of vector fields on X, and $\mathscr{D}_X$ denotes the sheaf of differential operators. It is a Lie algebra homomorphism. Let us denote by

$$L_G\colon \mathfrak{g} \to \Gamma(G; \mathscr{D}_G) \tag{3.1.5}$$

the Lie algebra homomorphism derived by the left action of G on itself. Then its image is the space of right invariant vector fields on G. Denoting by $i\colon X \to G \times X$ the map $x \mapsto (e, x)$, we define

$$L_{\mathrm{v}}(A)s = i^*\Big(\big(L_G(A) \boxtimes \mathrm{id}\big)(\beta\mu^*(s))\Big) \quad \text{for } A \in \mathfrak{g} \text{ and } s \in \mathscr{F}. \tag{3.1.6}$$

It is a derivation, namely

$$L_{\mathrm{v}}(A)(as) = \big(L_X(A)a\big)\, s + a\big(L_{\mathrm{v}}(A)s\big) \quad \text{for } A \in \mathfrak{g},\ a \in \mathscr{O}_X \text{ and } s \in \mathscr{F}.$$

The notion of equivariance of D-modules is defined similarly to the one of equivariant $\mathscr{O}$-modules. However, there are two options in the D-module case. Let $\mathscr{O}_G \boxtimes \mathscr{D}_X$ denote the subring $\mathscr{O}_{G\times X} \otimes_{\mathrm{pr}^{-1}\mathscr{O}_X} \mathrm{pr}^{-1}\mathscr{D}_X$ of $\mathscr{D}_{G\times X}$. There are two ring morphisms

$$\mathrm{pr}^{-1}\mathscr{D}_X \to \mathscr{O}_G \boxtimes \mathscr{D}_X \quad \text{and} \quad \mathscr{O}_{G\times X} \to \mathscr{O}_G \boxtimes \mathscr{D}_X.$$

Definition 3.1.3. A *quasi-G-equivariant* $\mathscr{D}_X$-module is a $\mathscr{D}_X$-module $\mathscr{M}$ endowed with an $\mathscr{O}_G \boxtimes \mathscr{D}_X$-linear isomorphism

$$\beta\colon \mathbf{D}\mu^*\mathscr{M} \xrightarrow{\sim} \mathbf{D}\mathrm{pr}^*\mathscr{M} \tag{3.1.7}$$

such that the following diagram commutes (*associative law*):

$$\begin{array}{ccccccc}
\mathbf{D}p_1^*\mathbf{D}\mu^*\mathscr{M} & & \xrightarrow{\qquad\qquad\mathbf{D}p_1^*\beta\qquad\qquad} & & & & \mathbf{D}p_1^*\mathbf{D}\mathrm{pr}^*\mathscr{M} \\
\| & & & & & & \| \\
\mathbf{D}p_0^*\mathbf{D}\mu^*\mathscr{M} & \xrightarrow{\mathbf{D}p_0^*\beta} & \mathbf{D}p_0^*\mathbf{D}\mathrm{pr}^*\mathscr{M} & = & \mathbf{D}p_2^*\mathbf{D}\mu^*\mathscr{M} & \xrightarrow{\mathbf{D}p_2^*\beta} & \mathbf{D}p_2^*\mathbf{D}\mathrm{pr}^*\mathscr{M}.
\end{array}$$

Here $\mathbf{D}\mu^*$, $\mathbf{D}p_0^*$, etc. are the pull-back functors for D-modules (see § 3.4). If moreover β is $\mathscr{D}_{G\times X}$-linear, $\mathscr{M}$ is called G-*equivariant*.

For quasi-G-equivariant $\mathscr{D}_X$-modules $\mathscr{M}$ and $\mathscr{N}$, a G-equivariant morphism $u\colon \mathscr{M} \to \mathscr{N}$ is a $\mathscr{D}_X$-linear homomorphism $u\colon \mathscr{M} \to \mathscr{N}$ such that

$$\begin{array}{ccc}
\mathbf{D}\mu^*\mathscr{M} & \xrightarrow{\beta} & \mathbf{D}\mathrm{pr}^*\mathscr{M} \\
{\scriptstyle \mathbf{D}\mu^*u}\downarrow & & \downarrow{\scriptstyle \mathbf{D}\mathrm{pr}^*u} \\
\mathbf{D}\mu^*\mathscr{N} & \xrightarrow{\beta} & \mathbf{D}\mathrm{pr}^*\mathscr{N}
\end{array}$$

commutes. Let us denote by $\mathrm{Mod}(\mathscr{D}_X, G)$ the category of quasi-coherent quasi-G-equivariant $\mathscr{D}_X$-modules, and by $\mathrm{Mod}_G(\mathscr{D}_X)$ the full subcategory $\mathrm{Mod}(\mathscr{D}_X, G)$ consisting of quasi-coherent G-equivariant $\mathscr{D}_X$-modules. Then they are abelian categories, and the functor $\mathrm{Mod}_G(\mathscr{D}_X) \to \mathrm{Mod}(\mathscr{D}_X, G)$ is fully faithful and exact, and the functors $\mathrm{Mod}(\mathscr{D}_X, G) \to \mathrm{Mod}(\mathscr{D}_X) \to \mathrm{Mod}(\mathscr{O}_X)$ and $\mathrm{Mod}(\mathscr{D}_X, G) \to \mathrm{Mod}(\mathscr{O}_X, G)$ are exact.

Roughly speaking, quasi-equivariance means the following. For $g \in G$ let $\mu_g\colon X \to X$ denotes the multiplication map. Then a $\mathscr{D}_X$-linear isomorphism $\beta_g\colon \mu_g^*\mathscr{M} \xrightarrow{\sim} \mathscr{M}$ is given in such a way that it depends algebraically on g and satisfies the chain condition $\beta_{g_1g_2} = \beta_{g_2} \circ \beta_{g_1}$ for $g_1, g_2 \in G$: the diagram

$$\begin{array}{ccc}
\mu_{g_2}^*\mu_{g_1}^*\mathscr{M} & \xrightarrow{\beta_{g_1}} & \mu_{g_2}^*\mathscr{M} \\
\| & & \downarrow{\scriptstyle \beta_{g_2}} \\
\mu_{g_1g_2}^*\mathscr{M} & \xrightarrow{\beta_{g_1g_2}} & \mathscr{M}
\end{array}$$

is commutative.

Example 3.1. (i) If $\mathscr{F}$ is a G-equivariant $\mathscr{O}_X$-module, then $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{F}$ is a quasi-G-equivariant $\mathscr{D}_X$-module.
(ii) Let $P_1, \dots, P_\ell$ be a family of G-invariant differential operators on X. Then $\mathscr{D}_X / \big(\sum_i \mathscr{D}_X P_i\big)$ is a quasi-G-equivariant $\mathscr{D}_X$-module.

Let $\mathscr{M}$ be a quasi-G-equivariant $\mathscr{D}_X$-module. Then the G-equivariant $\mathscr{O}_X$-module structure on $\mathscr{M}$ induces the Lie algebra homomorphism

$$L_{\mathrm{v}}\colon \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(\mathscr{M}).$$

On the other hand, the $\mathscr{D}_X$-module structure on $\mathscr{M}$ induces the Lie algebra homomorphism

$$\alpha_D\colon \mathfrak{g} \to \Gamma(X;\mathscr{D}_X) \to \mathrm{End}_{\mathbb{C}}(\mathscr{M}).$$

Hence we have:

$$\begin{aligned} \alpha_D(A)s &= i^*\big((L_G(A)\boxtimes 1)(\mu^*(s))\big) \\ L_{\mathrm{v}}(A)s &= i^*\big((L_G(A)\boxtimes 1)(\beta\circ\mu^*(s))\big) \end{aligned} \qquad \text{for } s\in\mathscr{M} \text{ and } A\in\mathfrak{g}.$$

Set

$$\gamma_{\mathscr{M}} = L_{\mathrm{v}} - \alpha_D : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(\mathscr{M}).$$

Since we have

$$[L_{\mathrm{v}}(A), P] = [\alpha_D(A), P] \text{ for any } A\in\mathfrak{g} \text{ and } P\in\mathscr{D}_X,$$

the homomorphism $\gamma_{\mathscr{M}}$ sends $\mathfrak{g}$ to $\mathrm{End}_{\mathscr{D}_X}(\mathscr{M})$. The homomorphism $\gamma_{\mathscr{M}}\colon \mathfrak{g} \to \mathrm{End}_{\mathscr{D}_X}(\mathscr{M})$ vanishes if and only if $L_G(A)\boxtimes 1 \in \Theta_{G\times X}$ commutes with β for all $A\in\mathfrak{g}$. Thus we have obtained the following lemma.

Lemma 3.1.4. *Let $\mathscr{M}$ be a quasi-G-equivariant $\mathscr{D}_X$-module. Let $\gamma_{\mathscr{M}}$ be as above. Then we have*

(i) *$\gamma_{\mathscr{M}}$ is a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{End}_{\mathscr{D}_X}(\mathscr{M})$,*
(ii) *$\mathscr{M}$ is G-equivariant if and only if $\gamma_{\mathscr{M}} = 0$.*

Thus $\mathscr{M}$ has a $(\mathscr{D}_X, U(\mathfrak{g}))$-bimodule structure.

When G acts transitively on X, we have the following description of quasi-equivariant D-modules.

Proposition 3.1.5 ([15]). *Let $X = G/H$ for a closed subgroup H of G, and let $i\colon \mathrm{pt} \to X$ be the map associated with $e \bmod H$. Then $\mathscr{M} \mapsto i^*\mathscr{M}$ gives equivalences of categories*

$$\begin{array}{ccc} \mathrm{Mod}(\mathscr{D}_X, G) & \xrightarrow{\sim} & \mathrm{Mod}(\mathfrak{g}, H) \\ \bigcup & & \bigcup \\ \mathrm{Mod}_G(\mathscr{D}_X) & \xrightarrow{\sim} & \mathrm{Mod}(H/H^\circ), \end{array}$$

where H° is the connected component of H containing the identity.

The $\mathfrak{g}$-module structure on $i^*\mathscr{M}$ is given by $\gamma_{\mathscr{M}}$. We remark that $\mathrm{Mod}(H/H^\circ)$ is embedded in $\mathrm{Mod}(\mathfrak{g}, H)$ in such a way that $\mathfrak{g}$ acts trivially on the vector spaces in $\mathrm{Mod}(H/H^\circ)$.

Remark 3.1.6. The inclusion functor $\mathrm{Mod}_G(\mathscr{D}_X) \to \mathrm{Mod}(\mathscr{D}_X, G)$ has a left adjoint functor and a right adjoint functor

$$\mathscr{M} \longmapsto \mathbb{C}\otimes_{U(\mathfrak{g})}\mathscr{M} \quad \text{and} \quad \mathscr{M} \longmapsto \mathscr{H}om_{U(\mathfrak{g})}(\mathbb{C}, \mathscr{M}).$$

Here $U(\mathfrak{g})$ acts on $\mathscr{M}$ via $\gamma_{\mathscr{M}}$.

3.2 Derived Categories

Recall that $\mathrm{Mod}(\mathscr{D}_X, G)$ denotes the abelian category of quasi-coherent quasi-G-equivariant $\mathscr{D}_X$-modules. There are the forgetful functor

$$\mathrm{Mod}(\mathscr{D}_X, G) \to \mathrm{Mod}(\mathscr{O}_X, G)$$

and

$$\mathscr{D}_X \otimes_{\mathscr{O}_X} \bullet : \mathrm{Mod}(\mathscr{O}_X, G) \to \mathrm{Mod}(\mathscr{D}_X, G).$$

They are adjoint functors to each other. Namely there is a functorial isomorphism in $\mathscr{F} \in \mathrm{Mod}(\mathscr{O}_X, G)$ and $\mathscr{M} \in \mathrm{Mod}(\mathscr{D}_X, G)$

$$\text{(3.2.1)} \qquad \mathrm{Hom}_{\mathrm{Mod}(\mathscr{O}_X,G)}(\mathscr{F}, \mathscr{M}) \cong \mathrm{Hom}_{\mathrm{Mod}(\mathscr{D}_X,G)}(\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{F}, \mathscr{M}).$$

Note that, for $\mathscr{F} \in \mathrm{Mod}(\mathscr{O}_X, G)$, the morphism $\gamma_{\mathscr{M}} : \mathfrak{g} \to \mathrm{End}_{\mathscr{D}_X}(\mathscr{M})$ for $\mathscr{M} = \mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{F}$ is given by $\gamma_{\mathscr{M}}(A)(P \otimes s) = -PL_X(A) \otimes s + P \otimes L_{\mathrm{v}}(A)s$ for $A \in \mathfrak{g}$, $P \in \mathscr{D}_X$, $s \in \mathscr{F}$. Hence $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{F}$ is not a G-equivariant $\mathscr{D}_X$-module in general.

Let $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X, G)$ denote the full subcategory of $\mathrm{Mod}(\mathscr{D}_X, G)$ consisting of coherent quasi-G-equivariant $\mathscr{D}_X$-modules. Similarly let us denote by $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_X, G)$ the category of coherent G-equivariant $\mathscr{O}_X$-modules.

We shall introduce the following intermediate category.

Definition 3.2.1. A quasi-coherent $\mathscr{O}_X$-module (resp. $\mathscr{D}_X$-module) is called *countably coherent* if it is locally generated by countably many sections.

Note that if $\mathscr{F}$ is a countably coherent $\mathscr{O}_X$-module, then there exists locally an exact sequence $\mathscr{O}_X^{\oplus I} \to \mathscr{O}_X^{\oplus J} \to \mathscr{F} \to 0$ where I and J are countable sets.

Note also that any coherent $\mathscr{D}_X$-module is countably coherent over $\mathscr{O}_X$. Hence a quasi-coherent $\mathscr{D}_X$-module is countably coherent, if and only if so is it as an $\mathscr{O}_X$-module.

Note that countably coherent $\mathscr{O}$-modules are stable by inverse images, direct images and tensor products.

Let $\mathrm{Mod}_{\mathrm{cc}}(\mathscr{D}_X, G)$ denote the full subcategory of $\mathrm{Mod}(\mathscr{D}_X, G)$ consisting of countably coherent quasi-G-equivariant $\mathscr{D}_X$-modules.

Let us denote by $\mathrm{D}(\mathscr{D}_X, G)$ the derived category of $\mathrm{Mod}(\mathscr{D}_X, G)$. Let $\mathrm{D}_{\mathrm{cc}}(\mathscr{D}_X, G)$ (resp. $\mathrm{D}_{\mathrm{coh}}(\mathscr{D}_X, G)$) denotes the full subcategory of $\mathrm{D}(\mathscr{D}_X, G)$ consisting of objects whose cohomologies belong to $\mathrm{Mod}_{\mathrm{cc}}(\mathscr{D}_X, G)$ (resp. $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X, G)$).

Let us denote by $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X, G)$ the full subcategory of $\mathrm{D}(\mathscr{D}_X, G)$ consisting of objects with bounded cohomologies. We define similarly $\mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_X, G)$ and $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X, G)$.

Proposition 3.2.2. *The functors*

$$\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}_{\mathrm{cc}}(\mathscr{D}_X, G)) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_X, G) \qquad \textit{and}$$
$$\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X, G)) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X, G)$$

are equivalences of categories.

This follows easily from the following lemma and a standard argument (e.g. cf. [19])

Lemma 3.2.3. *Any quasi-coherent G-equivariant $\mathscr{O}_X$-module is a union of coherent G-equivariant $\mathscr{O}_X$-submodules. Similarly, any quasi-coherent quasi-G-equivariant $\mathscr{D}_X$-module is a union of coherent quasi-G-equivariant $\mathscr{D}_X$-submodules.*

3.3 Sumihiro's result

Hereafter we shall assume that X is *quasi-projective*, i.e., X is isomorphic to a subscheme of the projective space $\mathbb{P}^n$ for some n. In such a case, $\mathrm{Mod}(\mathscr{D}_X, G)$ has enough objects so that $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X, G)$ is a desired derived category, namely, the forgetful functor $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X, G) \to \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)$ commutes with various functors such as pull-back functors, push-forward functors, etc. This follows from the following result due to Sumihiro [28].

Proposition 3.3.1. *Let X be a quasi-projective G-manifold.*

(i) *There exists a G-equivariant ample invertible $\mathscr{O}_X$-module.*
(ii) *There exists a G-equivariant open embedding from X into a projective G-manifold.*

In the sequel, we assume

$$X \text{ is a quasi-projective } G\text{-manifold.} \tag{3.3.1}$$

Let $\mathscr{L}$ be a G-equivariant ample invertible $\mathscr{O}_X$-module.

Lemma 3.3.2. *Let $\mathscr{F}$ be a coherent G-equivariant $\mathscr{O}_X$-module. Then, for $n \gg 0$, there exist a finite-dimensional G-module V and a G-equivariant surjective homomorphism*

$$\mathscr{L}^{\otimes -n} \otimes V \twoheadrightarrow \mathscr{F}. \tag{3.3.2}$$

Proof. For $n \gg 0$, $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ is generated by global sections. Take a finite-dimensional G-submodule V of the G-module $\Gamma(X; \mathscr{F} \otimes \mathscr{L}^{\otimes n})$ such that $V \otimes \mathscr{O}_X \to \mathscr{F} \otimes \mathscr{L}^{\otimes n}$ is surjective. Then this gives a desired homomorphism. Q.E.D.

Lemma 3.3.2 implies the following exactitude criterion.

Lemma 3.3.3. *Let $\mathscr{M}' \to \mathscr{M} \to \mathscr{M}''$ be a sequence in $\mathrm{Mod}(\mathscr{O}_X, G)$. If $\mathrm{Hom}_{\mathrm{Mod}(\mathscr{O}_X,G)}(\mathscr{E}, \mathscr{M}') \to \mathrm{Hom}_{\mathrm{Mod}(\mathscr{O}_X,G)}(\mathscr{E}, \mathscr{M}) \to \mathrm{Hom}_{\mathrm{Mod}(\mathscr{O}_X,G)}(\mathscr{E}, \mathscr{M}'')$ is exact for any locally free G-equivariant $\mathscr{O}_X$-module $\mathscr{E}$ of finite rank, then $\mathscr{M}' \to \mathscr{M} \to \mathscr{M}''$ is exact.*

Let us denote by $\mathrm{Mod}_{lf}(\mathscr{D}_X, G)$ the full subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X, G)$ consisting of objects of the form $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{E}$ for a locally free coherent G-equivariant $\mathscr{O}_X$-module $\mathscr{E}$. By Lemma 3.3.2, for any $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X, G)$, there exists a surjective G-equivariant homomorphism $\mathscr{N} \to \mathscr{M}$ with $\mathscr{N} \in \mathrm{Mod}_{lf}(\mathscr{D}_X, G)$.

Lemma 3.3.2 together with standard arguments (see e.g. [19]), we obtain

Proposition 3.3.4. *For any* $\mathscr{M} \in \mathrm{K}^-\big(\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X, G)\big)$ *there exist* $\mathscr{N} \in \mathrm{K}^-\big(\mathrm{Mod}_{lf}(\mathscr{D}_X, G)\big)$ *and a quasi-isomorphism* $\mathscr{N} \to \mathscr{M}$*.*

The abelian category $\mathrm{Mod}(\mathscr{D}_X, G)$ is a Grothendieck category. By a general theory of homological algebra, we have the following proposition (see e.g. [19]).

Proposition 3.3.5. *Any object of* $\mathrm{Mod}(\mathscr{D}_X, G)$ *is embedded in an injective object of* $\mathrm{Mod}(\mathscr{D}_X, G)$*.*

Injective objects of $\mathrm{Mod}(\mathscr{D}_X, G)$ have the following properties.

Lemma 3.3.6. *The forgetful functor* $\mathrm{Mod}(\mathscr{D}_X, G) \to \mathrm{Mod}(\mathscr{O}_X, G)$ *sends the injective objects to injective objects.*

This follows from (3.2.1) and the exactitude of $\mathscr{F} \mapsto \mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{F}$.

Lemma 3.3.7. *Let* $\mathscr{I}$ *be an injective object of* $\mathrm{Mod}(\mathscr{O}_X, G)$*. Then the functor* $\mathscr{F} \mapsto \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{I})$ *is an exact functor from* $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_X, G)^{\mathrm{op}}$ *to* $\mathrm{Mod}(\mathscr{O}_X, G)$*.*

Proof. By Lemma 3.3.3, it is enough to remark that, for any locally free $\mathscr{E} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_X, G)$,

$$\mathrm{Hom}_{\mathrm{Mod}(\mathscr{O}_X,G)}\big(\mathscr{E}, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{I})\big) \cong \mathrm{Hom}_{\mathrm{Mod}(\mathscr{O}_X,G)}(\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{F}, \mathscr{I})$$

is an exact functor in $\mathscr{F}$. Q.E.D.

Proposition 3.3.8. *Let* $\mathscr{I}$ *be an injective object of* $\mathrm{Mod}(\mathscr{O}_X, G)$*. Then for any* $\mathscr{F} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_X, G)$*,*

$$\mathscr{E}xt^k_{\mathscr{O}_X}(\mathscr{F}, \mathscr{I}) = 0 \quad \textit{and} \quad \mathrm{Ext}^k_{\mathscr{O}_X}(\mathscr{F}, \mathscr{I}) = 0 \quad \textit{for } k > 0. \tag{3.3.3}$$

Proof. Let us prove first the global case.
(1) Projective case. Assume first that X is projective. We have

$$\mathrm{Ext}^k_{\mathscr{O}_X}(\mathscr{F}, \mathscr{I}) = \varinjlim_{\mathscr{E}} \mathrm{Ext}^k_{\mathscr{O}_X}(\mathscr{F}, \mathscr{E})$$

where $\mathscr{E}$ ranges over the set of coherent G-equivariant $\mathscr{O}_X$-submodules of $\mathscr{I}$. Hence it is enough to show that for such an $\mathscr{E}$

$$\beta\colon \mathrm{Ext}^k_{\mathscr{O}_X}(\mathscr{F},\mathscr{E}) \to \mathrm{Ext}^k_{\mathscr{O}_X}(\mathscr{F},\mathscr{I})$$

vanishes. We shall prove this by the induction on $k > 0$.

For $n \gg 0$, there exists a G-equivariant surjective morphism $V \otimes \mathscr{L}^{\otimes -n} \to \mathscr{F} \to 0$ by Lemma 3.3.2, which induces an exact sequence $0 \to \mathscr{F}' \to V \otimes \mathscr{L}^{\otimes -n} \to \mathscr{F} \to 0$. We may assume that n is so large that $H^m(X; \mathscr{E} \otimes \mathscr{L}^{\otimes n}) = 0$ for any $m > 0$. Then $\mathrm{Ext}^m_{\mathscr{O}_X}(V \otimes \mathscr{L}^{\otimes -n}, \mathscr{E}) = V^* \otimes H^m(X; \mathscr{E} \otimes \mathscr{L}^{\otimes n}) = 0$ for $m > 0$, and hence we obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Ext}^{k-1}_{\mathscr{O}_X}(V \otimes \mathscr{L}^{\otimes -n}, \mathscr{E}) & \longrightarrow & \mathrm{Ext}^{k-1}_{\mathscr{O}_X}(\mathscr{F}', \mathscr{E}) & \xrightarrow[\alpha]{} & \mathrm{Ext}^{k}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{E}) & \longrightarrow & 0. \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\beta} & & \\
\mathrm{Ext}^{k-1}_{\mathscr{O}_X}(V \otimes \mathscr{L}^{\otimes -n}, \mathscr{I}) & \xrightarrow[\gamma]{} & \mathrm{Ext}^{k-1}_{\mathscr{O}_X}(\mathscr{F}', \mathscr{I}) & \xrightarrow[\delta]{} & \mathrm{Ext}^{k}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{I}). & &
\end{array}$$

The homomorphism γ is surjective, because $\mathscr{I}$ is injective for $k = 1$ and the induction hypothesis implies $\mathrm{Ext}^{k-1}_{\mathscr{O}_X}(\mathscr{F}', \mathscr{I}) = 0$ for $k > 1$. Hence we have $\delta = 0$, and the surjectivity of α implies $\beta = 0$.

(2) General case. Let us embed X in a projective G-manifold $\bar{X}$ and let $j\colon X \hookrightarrow \bar{X}$ be the open embedding. Since

$$\mathrm{Hom}_{\mathscr{O}_{\bar{X}}}(\mathscr{N}, j_*\mathscr{I}) = \mathrm{Hom}_{\mathscr{O}_X}(j^{-1}\mathscr{N}, \mathscr{I})$$

for $\mathscr{N} \in \mathrm{Mod}(\mathscr{O}_{\bar{X}}, G)$, $j_*\mathscr{I}$ is an injective object of $\mathrm{Mod}(\mathscr{O}_{\bar{X}}, G)$. Let J be the defining ideal of $\bar{X} \setminus X$. Then J is a coherent G-equivariant ideal of $\mathscr{O}_{\bar{X}}$. Let us take a coherent G-equivariant $\mathscr{O}_{\bar{X}}$-module $\overline{\mathscr{F}}$ such that $\overline{\mathscr{F}}|_X \simeq \mathscr{F}$. Then, the isomorphism (see [6])

$$\mathrm{Ext}^k_{\mathscr{O}_X}(\mathscr{F}, \mathscr{I}) = \varinjlim_n \mathrm{Ext}^k_{\mathscr{O}_{\bar{X}}}(\overline{\mathscr{F}} \otimes_{\mathscr{O}_X} J^n, j_*\mathscr{I})$$

implies the desired result.

The local case can be proved similarly to the proof in (1) by using Lemma 3.3.7. Q.E.D.

Proposition 3.3.9. *Let $f\colon X \to Y$ be a G-equivariant morphism of quasi-projective G-manifolds. Then for any injective object $\mathscr{I}$ of* $\mathrm{Mod}(\mathscr{O}_X, G)$ *and* $\mathscr{F} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_X, G)$*, we have*

$$R^k f_*\big(\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{I})\big) = 0 \quad \text{for } k > 0.$$

Proof. The proof is similar to the proof of the preceding proposition. The morphism $f\colon X \to Y$ can be embedded in $\bar{f}\colon \bar{X} \to \bar{Y}$ for projective G-manifolds $\bar{X}$ and $\bar{Y}$. Let $j\colon X \to \bar{X}$ be the open embedding. Let J be the defining ideal of $\bar{X} \setminus X$. Then, extending $\mathscr{F}$ to a coherent G-equivariant $\mathscr{O}_{\bar{X}}$-module $\overline{\mathscr{F}}$, one has

$$R^k f_*\big(\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{I})\big) \simeq \varinjlim_n R^k \bar{f}_*\big(\mathscr{H}om_{\mathscr{O}_X}(\bar{\mathscr{F}} \otimes J^n, j_*\mathscr{I})\big)|_Y.$$

Hence, we may assume from the beginning that X and Y are projective. Then we can argue similarly to (1) in the proof of Proposition 3.3.8, once we prove

(3.3.4) $\mathscr{F} \mapsto f_* \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{I})$ is an exact functor in $\mathscr{F} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_X, G)$.

This follows from Lemma 3.3.3 and the exactitude of the functor

$$\mathrm{Hom}_{\mathrm{Mod}(\mathscr{O}_Y,G)}(\mathscr{E}, f_* \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{I})) \simeq \mathrm{Hom}_{\mathrm{Mod}(\mathscr{O}_X,G)}(f^*\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{F}, \mathscr{I})$$

in $\mathscr{F}$ for any locally free G-equivariant coherent $\mathscr{O}_Y$-module $\mathscr{E}$. Q.E.D.

By this proposition, we obtain the following corollary.

Corollary 3.3.10. *Let $\mathscr{I}$ be an injective object of $\mathrm{Mod}(\mathscr{D}_X, G)$. Then for any morphism $f\colon X \to Y$ of quasi-projective G-manifolds and a coherent locally free G-equivariant $\mathscr{O}_X$-module $\mathscr{E}$*

(3.3.5) $$R^k f_*(\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{I}) = 0 \quad \text{for } k > 0.$$

Lemma 3.3.11. *For any morphism $f\colon X \to Y$ and $\mathscr{M} \in \mathrm{Mod}_{\mathrm{cc}}(\mathscr{D}_X, G)$ and a coherent locally free G-equivariant $\mathscr{O}_X$-module $\mathscr{E}$, there exists a monomorphism $\mathscr{M} \to \mathscr{M}'$ in $\mathrm{Mod}_{\mathrm{cc}}(\mathscr{D}_X, G)$ such that $R^k f_*(\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{M}') = 0$ for any $k \neq 0$.*

Proof. Let us take a monomorphism $\mathscr{M} \to \mathscr{I}$ where $\mathscr{I}$ is an injective object of $\mathrm{Mod}(\mathscr{D}_X, G)$. Let us construct, by the induction on n, an increasing sequence $\{\mathscr{M}_n\}_{n \geq 0}$ of countably coherent subobjects of $\mathscr{I}$ such that $\mathscr{M}_0 = \mathscr{M}$ and

(3.3.6) $$R^k f_*(\mathscr{E} \otimes \mathscr{M}_n) \to R^k f_*(\mathscr{E} \otimes \mathscr{M}_{n+1}) \text{ vanishes for } k \neq 0.$$

Assuming that $\mathscr{M}_n$ has been constructed, we shall construct $\mathscr{M}_{n+1}$. We have

$$\varinjlim_{\mathscr{N} \subset \mathscr{I}} R^k f_*(\mathscr{E} \otimes \mathscr{N}) \cong R^k f_*(\mathscr{E} \otimes \mathscr{I}) = 0 \quad \text{for } k \neq 0.$$

Here $\mathscr{N}$ ranges over the set of countably coherent subobjects of $\mathscr{I}$. Since $R^k f_*(\mathscr{E} \otimes \mathscr{M}_n)$ is countably coherent, there exists a countably coherent subobject $\mathscr{M}_{n+1}$ of $\mathscr{I}$ such that $\mathscr{M}_n \subset \mathscr{M}_{n+1}$ and the morphism $R^k f_*(\mathscr{E} \otimes \mathscr{M}_n) \to R^k f_*(\mathscr{E} \otimes \mathscr{M}_{n+1})$ vanishes for $k \neq 0$.

Then $\mathscr{M}' := \varinjlim_n \mathscr{M}_n$ satisfies the desired condition, because (3.3.6) implies

$$R^k f_*(\mathscr{E} \otimes \mathscr{M}') \simeq \varinjlim_n R^k f_*(\mathscr{E} \otimes \mathscr{M}_n) \simeq 0$$

for $k \neq 0$. Q.E.D.

3.4 Pull-back functors

Let $f\colon X \to Y$ be a morphism of quasi-projective algebraic manifolds. Set $\mathscr{D}_{X\to Y} = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y$. Then $\mathscr{D}_{X\to Y}$ has a structure of a $(\mathscr{D}_X, f^{-1}\mathscr{D}_Y)$-bimodule. It is countably coherent as a $\mathscr{D}_X$-module. Then

$$f^*\colon \mathscr{N} \mapsto \mathscr{D}_{X\to Y} \otimes_{\mathscr{D}_Y} \mathscr{N} = \mathscr{O}_X \otimes_{\mathscr{O}_Y} \mathscr{N}$$

gives a right exact functor from $\mathrm{Mod}(\mathscr{D}_Y)$ to $\mathrm{Mod}(\mathscr{D}_X)$. It is left derivable, and we denote by $\mathbf{D}f^*$ its left derived functor:

$$\mathbf{D}f^*\colon\ \mathrm{D^b}(\mathscr{D}_Y) \to \mathrm{D^b}(\mathscr{D}_X).$$

Now let $f\colon X \to Y$ be a G-equivariant morphism of quasi-projective algebraic G-manifolds. Then $f^*\colon \mathscr{N} \mapsto \mathscr{D}_{X\to Y} \otimes_{\mathscr{D}_Y} \mathscr{N} = \mathscr{O}_X \otimes_{\mathscr{O}_Y} \mathscr{N}$ gives also a right exact functor:

$$f^*\colon \mathrm{Mod}(\mathscr{D}_Y, G) \to \mathrm{Mod}(\mathscr{D}_X, G).$$

Lemma 3.3.2 implies that any quasi-coherent quasi-G-equivariant $\mathscr{D}_Y$-module has a finite resolution by quasi-coherent quasi-G-equivariant $\mathscr{D}_Y$-modules flat over $\mathscr{O}_Y$. Hence the functor $f^*\colon \mathrm{Mod}(\mathscr{D}_Y, G) \to \mathrm{Mod}(\mathscr{D}_X, G)$ is left derivable. We denote its left derived functor by $\mathbf{D}f^*$:

(3.4.1) $$\mathbf{D}f^*\colon\ \mathrm{D^b}(\mathscr{D}_Y, G) \to \mathrm{D^b}(\mathscr{D}_X, G).$$

By the construction, the diagram

$$\begin{array}{ccccc}
\mathrm{D^b}(\mathscr{D}_Y, G) & \longrightarrow & \mathrm{D^b}(\mathscr{D}_Y) & \longrightarrow & \mathrm{D^b}(\mathscr{O}_Y) \\
\downarrow{\scriptstyle \mathbf{D}f^*} & & \downarrow{\scriptstyle \mathbf{D}f^*} & & \downarrow{\scriptstyle \mathbf{L}f^*} \\
\mathrm{D^b}(\mathscr{D}_X, G) & \longrightarrow & \mathrm{D^b}(\mathscr{D}_X) & \longrightarrow & \mathrm{D^b}(\mathscr{O}_X)
\end{array}$$

commutes. The functor $\mathbf{D}f^*$ sends $\mathrm{D^b_{cc}}(\mathscr{D}_Y, G)$ to $\mathrm{D^b_{cc}}(\mathscr{D}_X, G)$. If f is a smooth morphism, then $\mathbf{D}f^*$ sends $\mathrm{D^b_{coh}}(\mathscr{D}_Y, G)$ to $\mathrm{D^b_{coh}}(\mathscr{D}_X, G)$.

3.5 Push-forward functors

Let $f\colon X \to Y$ be a morphism of quasi-projective algebraic manifolds. Recall that the push-forward functor

(3.5.1) $$\mathbf{D}f_*\colon\ \mathrm{D^b}(\mathscr{D}_X) \to \mathrm{D^b}(\mathscr{D}_Y)$$

is defined by $Rf_*(\mathscr{D}_{Y\leftarrow X} \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M})$. Here $\mathscr{D}_{Y\leftarrow X}$ is an $(f^{-1}\mathscr{D}_Y, \mathscr{D}_X)$-bimodule $f^{-1}\mathscr{D}_Y \otimes_{f^{-1}\mathscr{O}_Y} \Omega_{X/Y}$, where we use the notations:

$$\Omega_X := \Omega_X^{d_X} \text{ and } \Omega_{X/Y} := \Omega_X \otimes \Omega_Y^{\otimes -1}.$$

Let $f\colon X \to Y$ be a G-equivariant morphism of quasi-projective algebraic G-manifolds. Let us define the push-forward functor

$$\mathbf{D}f_*\colon \mathrm{D^b}(\mathscr{D}_X, G) \to \mathrm{D^b}(\mathscr{D}_Y, G) \tag{3.5.2}$$

in the equivariant setting.

In order to calculate $\mathscr{D}_{Y\leftarrow X} \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M}$, let us take a resolution of $\mathscr{D}_{Y\leftarrow X}$ by flat $\mathscr{D}_X$-modules:

$$\begin{aligned} 0 \leftarrow \mathscr{D}_{Y\leftarrow X} &\leftarrow f^{-1}(\mathscr{D}_Y \otimes \Omega_Y^{\otimes -1}) \otimes \Omega_X^{d_X} \otimes \mathscr{D}_X \\ &\leftarrow f^{-1}(\mathscr{D}_Y \otimes \Omega_Y^{\otimes -1}) \otimes \Omega_X^{d_X - 1} \otimes \mathscr{D}_X \\ &\leftarrow \cdots\cdots\cdots\cdots\cdots \leftarrow \\ &\leftarrow f^{-1}(\mathscr{D}_Y \otimes \Omega_Y^{\otimes -1}) \otimes \Omega_X^{0} \otimes \mathscr{D}_X \leftarrow 0. \end{aligned} \tag{3.5.3}$$

It is an exact sequence of $(f^{-1}\mathscr{D}_Y, \mathscr{D}_X)$-bimodules. Thus, for a complex $\mathscr{M}$ of $\mathscr{D}_X$-modules, $\mathscr{D}_{Y\leftarrow X} \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M}$ is represented by the complex of $f^{-1}\mathscr{D}_Y$-modules

$$f^{-1}(\mathscr{D}_Y \otimes_{\mathscr{O}_Y} \Omega_Y^{\otimes -1}) \otimes_{f^{-1}\mathscr{O}_Y} \Omega_X^{\bullet} \otimes_{\mathscr{O}_X} \mathscr{M}\,[d_X]\,. \tag{3.5.4}$$

The differential of the complex (3.5.4) is given as follows. First note that there is a left $\mathscr{D}_Y$-linear homomorphism

$$d\colon \mathscr{D}_Y \otimes_{\mathscr{O}_Y} \Omega_Y^{\otimes -1} \to \mathscr{D}_Y \otimes_{\mathscr{O}_Y} \Omega_Y^{\otimes -1} \otimes_{\mathscr{O}_Y} \Omega_Y^1$$

given by

$$d(P \otimes dy^{\otimes -1}) = -\sum_j P \frac{\partial}{\partial y_i} \otimes dy^{\otimes -1} \otimes dy_j.$$

Here $(y_1, \dots, y_m)$ is a local coordinate system of Y, $dy^{\otimes -1} = (dy_1 \wedge \cdots \wedge dy_m)^{\otimes -1}$ and $P \in \mathscr{D}_Y$. We define the morphism

$$\varphi\colon f^{-1}(\mathscr{D}_Y \otimes \Omega_Y^{\otimes -1} \otimes \Omega_Y^{\bullet}) \otimes \Omega_X^{\bullet} \to f^{-1}(\mathscr{D}_Y \otimes \Omega_Y^{\otimes -1}) \otimes \Omega_X^{\bullet}$$

by $a \otimes \theta \otimes \omega \mapsto a \otimes (f^*\theta \wedge \omega)$ for $a \in \mathscr{D}_Y \otimes \Omega_Y^{\otimes -1}$, $\theta \in \Omega_Y^{\bullet}$ and $\omega \in \Omega_X^{\bullet}$. Then, taking a local coordinate system $(x_1, \dots, x_n)$ of X, the differential d of (3.5.4) is given by

$$\begin{aligned} &d(a \otimes \omega \otimes u) \\ &\quad = \varphi(da \otimes \omega) \otimes u + a \otimes d\omega \otimes u \\ &\qquad + \sum_i a \otimes (dx_i \wedge \omega) \otimes \frac{\partial}{\partial x_i} u + (-1)^p a \otimes \omega \otimes du \end{aligned}$$

for $a \in \mathscr{D}_Y \otimes_{\mathscr{O}_Y} \Omega_Y^{\otimes -1}$, $\omega \in \Omega_X^p$ and $u \in \mathscr{M}$.

We now define the functor

$$\mathrm{K}\, f_*\colon\ \mathrm{K}^+(\mathrm{Mod}\,(\mathscr{D}_X, G)) \to \mathrm{K}^+(\mathrm{Mod}\,(\mathscr{D}_Y, G))$$

by

$$\begin{aligned}\mathrm{K}\, f_*(\mathscr{M}) &:= f_*\bigl(f^{-1}(\mathscr{D}_Y \otimes_{\mathscr{O}_Y} \Omega_Y^{\otimes -1}) \otimes_{f^{-1}\mathscr{O}_Y} \Omega_X^\bullet \otimes_{\mathscr{O}_X} \mathscr{M}\bigr)[\mathrm{d}_X] \\ &\cong \mathscr{D}_Y \otimes_{\mathscr{O}_Y} \Omega_Y^{\otimes -1} \otimes_{\mathscr{O}_Y} f_*(\Omega_X^\bullet \otimes_{\mathscr{O}_X} \mathscr{M})[\mathrm{d}_X].\end{aligned}$$

For an injective object $\mathscr{M}$ of $\mathrm{Mod}\,(\mathscr{D}_X, G)$, Corollary 3.3.10 implies

$$R^k f_*(\Omega_X^p \otimes_{\mathscr{O}_X} \mathscr{M}) = 0 \quad \text{for any } p \text{ and any } k > 0. \tag{3.5.5}$$

Hence if $\mathscr{I}^\bullet$ is an exact complex in $\mathrm{Mod}\,(\mathscr{D}_X, G)$ such that all $\mathscr{I}^n$ are injective, then $\mathrm{K}\, f_*(\mathscr{I}^\bullet)$ is exact. Hence $\mathrm{K}\, f_*$ is right derivable. Let $\mathbf{D}f_*$ be its right derived functor:

$$\mathbf{D}f_*\colon\ \mathrm{D}^+(\mathrm{Mod}\,(\mathscr{D}_X, G)) \to \mathrm{D}^+(\mathrm{Mod}\,(\mathscr{D}_Y, G)).$$

For a complex $\mathscr{M}$ in $\mathrm{Mod}\,(\mathscr{D}_X, G)$ bounded from below, we have

$$\begin{aligned}&\mathrm{K}\, f_*\mathscr{M} \xrightarrow{\sim} \mathbf{D}f_*(\mathscr{M}) \\ &\qquad \text{as soon as } R^k f_*(\Omega_X^p \otimes_{\mathscr{O}_X} \mathscr{M}^n) = 0 \text{ for all } k \neq 0 \text{ and } p,\, n.\end{aligned} \tag{3.5.6}$$

By the construction, the following diagram commutes.

$$\begin{array}{ccc} \mathrm{D}^+(\mathrm{Mod}\,(\mathscr{D}_X, G)) & \xrightarrow{\mathbf{D}f_*} & \mathrm{D}^+(\mathrm{Mod}\,(\mathscr{D}_Y, G)) \\ \downarrow & & \downarrow \\ \mathrm{D}^+(\mathrm{Mod}\,(\mathscr{D}_X)) & \xrightarrow{\mathbf{D}f_*} & \mathrm{D}^+(\mathrm{Mod}\,(\mathscr{D}_Y)). \end{array}$$

Since $\mathbf{D}f_*$ sends $\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\,(\mathscr{D}_X))$ to $\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\,(\mathscr{D}_Y))$, we conclude that $\mathbf{D}f_*$ sends $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X, G)$ to $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_Y, G)$, and $\mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_X, G)$ to $\mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_Y, G)$.

Proposition 3.5.1. *The restriction*

$$\mathrm{K}_{\mathrm{cc}}\, f_*\colon\ \mathrm{K}^{\mathrm{b}}(\mathrm{Mod}_{\mathrm{cc}}(\mathscr{D}_X, G)) \to \mathrm{K}^{\mathrm{b}}(\mathrm{Mod}_{\mathrm{cc}}(\mathscr{D}_Y, G))$$

of $\mathrm{K}\, f_*$ *is right derivable and the diagram*

$$\begin{array}{ccc} \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}_{\mathrm{cc}}(\mathscr{D}_X, G)) & \xrightarrow{\sim} & \mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_X, G) \\ {\scriptstyle \mathbf{R}(\mathrm{K}_{\mathrm{cc}}\, f_*)}\downarrow & & \downarrow{\scriptstyle \mathbf{D}f_*} \\ \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}_{\mathrm{cc}}(\mathscr{D}_Y, G)) & \xrightarrow{\sim} & \mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_Y, G) \end{array}$$

quasi-commutes.

Proof. It is enough to show that, for any $\mathscr{M} \in \mathrm{K}^{\mathrm{b}}(\mathrm{Mod}_{\mathrm{cc}}(\mathscr{D}_X, G))$, we can find a quasi-isomorphism $\mathscr{M} \to \mathscr{M}'$ such that the morphism $\mathrm{K}f_*(\mathscr{M}') \to \mathbf{D}f_*(\mathscr{M})$ is an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_Y, G)$. In order to have such an isomorphism, it is enough to show that $\mathscr{M}'$ satisfies the condition in (3.5.6). By Lemma 3.3.11, we have a quasi-isomorphism $\mathscr{M} \to \mathscr{M}'$ such that $\mathscr{M}'$ is a complex in $\mathrm{Mod}_{\mathrm{cc}}(\mathscr{D}_X, G)$ bounded below and satisfies the condition in (3.5.6). Since the cohomological dimension of $\mathbf{R}f_*$ is finite, by taking n sufficiently large, the truncated complex $\tau^{\le n}\mathscr{M}'$ satisfies the condition in (3.5.6), and $\mathscr{M} \to \tau^{\le n}\mathscr{M}'$ is a quasi-isomorphism. Q.E.D.

Note that, if f is projective, $\mathbf{D}f_*$ sends $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X, G)$ to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y, G)$ (see [16]).

3.6 External and internal tensor products

Let X and Y be two algebraic G-manifolds. Let $q_1\colon X \times Y \to X$ and $q_2\colon X \times Y \to Y$ be the projections. Then for $\mathscr{M}_1 \in \mathrm{Mod}(\mathscr{D}_X, G)$ and $\mathscr{M}_2 \in \mathrm{Mod}(\mathscr{D}_Y, G)$, $\mathscr{M}_1 \boxtimes \mathscr{M}_2 = (\mathscr{O}_{X\times Y} \otimes_{q_1^{-1}\mathscr{O}_X} q_1^{-1}\mathscr{M}_1) \otimes_{q_2^{-1}\mathscr{O}_Y} q_2^{-1}\mathscr{M}_2$ has a structure of quasi-G-equivariant $\mathscr{D}_{X\times Y}$-module. Since this is an exact bi-functor, we obtain

$$\bullet \boxtimes \bullet \quad : \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X, G) \times \mathrm{D}^{\mathrm{b}}(\mathscr{D}_Y, G) \to \mathrm{D}^{\mathrm{b}}(\mathscr{D}_{X\times Y}, G).$$

Taking pt as Y, we obtain

$$\bullet \otimes \bullet \quad : \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X, G) \times \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}(G)) \to \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X, G).$$

Here $\mathrm{Mod}(G)$ denotes the category of G-modules.

For two quasi-G-equivariant $\mathscr{D}_X$-modules $\mathscr{M}_1$ and $\mathscr{M}_2$, the $\mathscr{O}_X$-module $\mathscr{M}_1 \otimes_{\mathscr{O}_X} \mathscr{M}_2$ has a structure of $\mathscr{D}_X$-module by

$$v(s_1 \otimes s_2) = (vs_1) \otimes s_2 + s_1 \otimes (vs_2) \quad \text{for } v \in \Theta_X \text{ and } s_\nu \in \mathscr{M}_\nu.$$

Since this is G-equivariant, we obtain the right exact bi-functor

$$\bullet \otimes \bullet \quad : \mathrm{Mod}(\mathscr{D}_X, G) \times \mathrm{Mod}(\mathscr{D}_X, G). \to \mathrm{Mod}(\mathscr{D}_X, G).$$

Taking its left derived functor, we obtain

$$\bullet \overset{\mathbf{D}}{\otimes} \bullet \quad : \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X, G) \times \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X, G) \longrightarrow \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X, G).$$

We have

$$\mathscr{M}_1 \overset{\mathbf{D}}{\otimes} \mathscr{M}_2 \simeq \mathscr{M}_1 \otimes_{\mathscr{O}_X} \mathscr{M}_2$$

if either $\mathscr{M}_1$ or $\mathscr{M}_2$ are complexes in $\mathrm{Mod}(\mathscr{D}_X, G)$ flat over $\mathscr{O}_X$.

The functor $\bullet \overset{\mathbf{D}}{\otimes} \bullet$ sends $\mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_X, G) \times \mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_X, G)$ to $\mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_X, G)$.

Note that, denoting by $\delta\colon X \to X \times X$ the diagonal embedding, we have

$$\mathscr{M}_1 \overset{\mathbf{D}}{\otimes} \mathscr{M}_2 \simeq \mathbf{D}\delta^*(\mathscr{M}_1 \boxtimes \mathscr{M}_2).$$

Lemma 3.6.1. *For* $\mathscr{F} \in \mathrm{Mod}(\mathscr{O}_X, G)$ *and* $\mathscr{M} \in \mathrm{Mod}(\mathscr{D}_X, G)$*, there exists a canonical isomorphism in* $\mathrm{Mod}(\mathscr{D}_X, G)$ *:*

$$(\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{F}) \otimes \mathscr{M} \simeq \mathscr{D}_X \otimes_{\mathscr{O}_X} (\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{M}). \tag{3.6.1}$$

Here $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{M}$ *in the right-hand side is regarded as a* G*-equivariant* $\mathscr{O}_X$*-module.*

The proof is similar to the one in [16] in the non-equivariant case.

3.7 Semi-outer hom

Let $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X, G)$ and $\mathscr{M}' \in \mathrm{Mod}(\mathscr{D}_X, G)$. Then the vector space $\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{M}')$ has a structure of G-modules as follows:

$$\begin{aligned}
&\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{M}') \to \mathrm{Hom}_{\mathscr{D}_{G\times X}}(\mu^*\mathscr{M}, \mu^*\mathscr{M}') \\
&\quad \to \mathrm{Hom}_{\mathscr{O}_G \boxtimes \mathscr{D}_X}(\mu^*\mathscr{M}, \mu^*\mathscr{M}') \simeq \mathrm{Hom}_{\mathscr{O}_G \boxtimes \mathscr{D}_X}(\mathrm{pr}^*\mathscr{M}, \mathrm{pr}^*\mathscr{M}') \\
&\quad \simeq \mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, \mathrm{pr}_*\mathrm{pr}^*\mathscr{M}') \simeq \mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_G(G) \otimes_{\mathbb{C}} \mathscr{M}') \\
&\quad \simeq \mathscr{O}_G(G) \otimes \mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{M}').
\end{aligned}$$

Here the last isomorphism follows from the fact that $\mathscr{M}$ is coherent.

We can easily see that for any $V \in \mathrm{Mod}(G)$

$$\mathrm{Hom}_{\mathrm{Mod}(G)}(V, \mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{M}')) \cong \mathrm{Hom}_{\mathrm{Mod}(\mathscr{D}_X, G)}(V \otimes \mathscr{M}, \mathscr{M}'). \tag{3.7.1}$$

Since $V \mapsto V \otimes \mathscr{M}$ is an exact functor, (3.7.1) implies the following lemma.

Lemma 3.7.1. *Let* $\mathscr{I}$ *be an injective object of* $\mathrm{Mod}(\mathscr{D}_X, G)$ *and* $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X, G)$*. Then* $\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{I})$ *is an injective object of* $\mathrm{Mod}(G)$*.*

Let $\mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, \bullet)$ be the right derived functor of $\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, \bullet)$:

$$\mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\bullet, \bullet) \colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X, G)^{\mathrm{op}} \times D^+(\mathscr{D}_X, G) \to D^+(\mathrm{Mod}(G)).$$

By (3.7.1) and Lemma 3.7.1, we have

$$\mathrm{Hom}_{\mathrm{D^b}(\mathscr{D}_X, G)}(V \otimes \mathscr{M}, \mathscr{M}') \cong \mathrm{Hom}_{\mathrm{D^b}(\mathrm{Mod}(G))}(V, \mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{M}')) \tag{3.7.2}$$

for $V \in \mathrm{D^b}(\mathrm{Mod}(G))$, $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X, G)$ and $\mathscr{M}' \in D^+(\mathscr{D}_X, G)$. In particular we have

$$\mathrm{Hom}_{\mathrm{D^b}(\mathscr{D}_X, G)}(\mathscr{M}, \mathscr{M}') \cong \mathrm{Hom}_{\mathrm{D^b}(\mathrm{Mod}(G))}(\mathbb{C}, \mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{M}')). \tag{3.7.3}$$

Lemma 3.7.2. (i) $\mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\bullet, \bullet)$ *sends* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X, G)^{\mathrm{op}} \times \mathrm{D^b}(\mathscr{D}_X, G)$ *to* $\mathrm{D^b}(\mathrm{Mod}(G))$*.*

(ii) *Let $\mathscr{F}_G$ denote the functors forgetting G-structures:*

$$\mathscr{F}_G : \begin{array}{l} \mathrm{D^b}(\mathscr{D}_X, G) \to \mathrm{D^b}(\mathscr{D}_X), \\ \mathrm{D^b}(\mathrm{Mod}(G)) \to \mathrm{D^b}(\mathbb{C}). \end{array}$$

Then $\mathscr{F}_G \mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N}) \cong \mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{F}_G \mathscr{M}, \mathscr{F}_G \mathscr{N})$ for any $\mathscr{M} \in \mathrm{D^b_{coh}}(\mathscr{D}_X, G)$ and $\mathscr{N} \in \mathrm{D^b}(\mathscr{D}_X, G)$.

Proof. We may assume that $\mathscr{M} \in \mathrm{D^b}(\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X, G))$ by Proposition 3.2.1. Then, for an injective complex $\mathscr{N}$ in $\mathrm{Mod}(\mathscr{D}_X, G)$, we have

$$\mathscr{F}_G \mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N}) \simeq \mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N}) \simeq \mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{F}_G \mathscr{M}, \mathscr{F}_G \mathscr{N})$$

by Proposition 3.3.8. This shows (ii), and (i) follows from the fact that the global homological dimension of $\mathrm{Mod}(\mathscr{D}_X)$ is at most $2\dim X$ (see [16]). Q.E.D.

Remark that this shows that the global homological dimension of $\mathrm{Mod}(\mathscr{D}_X, G)$ is finite. Indeed, the arguments of the preceding lemma shows that for $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X, G)$ and $\mathscr{N} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X, G), H^n(\mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N})) = 0$ for $n > 2\dim X$. On the other hand, the global homological dimension of $\mathrm{Mod}(G)$ is at most $\dim G$ (or more precisely the dimension of the unipotent radical of G). Thus (3.7.3) shows $\mathrm{Hom}_{\mathrm{D}(\mathscr{D}_X, G)}(\mathscr{M}, \mathscr{N}[n]) \simeq \mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}(G))}(\mathbb{C}, \mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N})[n]) = 0$ for $n > \dim G + 2\dim X$. Therefore, the global homological dimension of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X, G)$ is at most $\dim G + 2\dim X$. Hence so is $\mathrm{Mod}(\mathscr{D}_X, G)$,

3.8 Relations of Push-forward and Pull-back functors

Statements

Let $f\colon X \to Y$ be a G-equivariant morphism of quasi-projective G-manifolds. Then $\mathbf{D}f^*$ and $\mathbf{D}f_*$ are adjoint functors in two ways. We use the notations: $\mathrm{d}_{X/Y} = \dim X - \dim Y$.

Theorem 3.8.1. *Let $f\colon X \to Y$ be a G-equivariant morphism of quasi-projective G-manifolds.*

(i) *Assume that f is* smooth. *Then there exists a functorial isomorphism in $\mathscr{M} \in \mathrm{D^b_{coh}}(\mathscr{D}_X, G)$ and $\mathscr{N} \in \mathrm{D^b_{coh}}(\mathscr{D}_Y, G)$:*

$$\text{(3.8.1)}\quad \mathrm{Hom}_{\mathrm{D^b}(\mathscr{D}_Y, G)}(\mathscr{N}, \mathbf{D}f_*\mathscr{M}) \cong \mathrm{Hom}_{\mathrm{D^b}(\mathscr{D}_X, G)}(\mathbf{D}f^*\mathscr{N}[-\mathrm{d}_{X/Y}], \mathscr{M}).$$

(ii) *Assume that f is* smooth and projective. *Then there exists a functorial isomorphism in $\mathscr{M} \in \mathrm{D^b_{coh}}(\mathscr{D}_X, G)$ and $\mathscr{N} \in \mathrm{D^b_{coh}}(\mathscr{D}_Y, G)$:*

$$\text{(3.8.2)}\quad \mathrm{Hom}_{\mathrm{D^b}(\mathscr{D}_Y, G)}(\mathbf{D}f_*\mathscr{M}, \mathscr{N}) \cong \mathrm{Hom}_{\mathrm{D^b}(\mathscr{D}_X, G)}(\mathscr{M}, \mathbf{D}f^*\mathscr{N}[\mathrm{d}_{X/Y}]).$$

This theorem will be proved at the end of this subsection.

By Theorem 3.8.1, we obtain the following morphisms for a smooth and projective morphism $f\colon X \to Y$, $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X, G)$ and $\mathscr{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y, G)$.

$$\mathbf{D}f^*\mathbf{D}f_*\mathscr{M}[-\mathrm{d}_{X/Y}] \to \mathscr{M}, \tag{3.8.3}$$

$$\mathscr{M} \to \mathbf{D}f^*\mathbf{D}f_*\mathscr{M}[\mathrm{d}_{X/Y}], \tag{3.8.4}$$

$$\mathbf{D}f_*\mathbf{D}f^*\mathscr{N}[\mathrm{d}_{X/Y}] \to \mathscr{N}, \tag{3.8.5}$$

$$\mathscr{N} \to \mathbf{D}f_*\mathbf{D}f^*\mathscr{N}[-\mathrm{d}_{X/Y}]. \tag{3.8.6}$$

Residue morphism

In order to prove Theorem 3.8.1, we shall first define the morphism

$$\mathbf{D}f_*\mathscr{O}_X[\mathrm{d}_{X/Y}] \to \mathscr{O}_Y. \tag{3.8.7}$$

Let $f\colon X \to Y$ be a smooth and projective morphism. Let $\mathscr{F}_G$ be the functor from $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_Y, G)$ to $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_Y)$. Then we have, by the theory of D-modules

$$\mathscr{F}_G(\mathbf{D}f_*\mathscr{O}_X[\mathrm{d}_{X/Y}]) \to \mathscr{O}_Y \tag{3.8.8}$$

in $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_Y)$. This morphism (3.8.8) gives a $\mathscr{D}_Y$-linear homomorphism

$$H^{\mathrm{d}_{X/Y}}(\mathbf{D}f_*\mathscr{O}_X) \to \mathscr{O}_Y.$$

Since this is canonical, this commutes with the action of any element of $G(\mathbb{C})$. Hence this is a morphism in $\mathrm{Mod}(\mathscr{D}_Y, G)$. On the other hand, we have

$$H^j(\mathbf{D}f_*\mathscr{O}_X) = 0 \quad \text{for } j > \mathrm{d}_{X/Y}.$$

We have therefore a morphism in $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_Y, G)$.

$$\mathbf{D}f_*\mathscr{O}_X[\mathrm{d}_{X/Y}] \to \tau^{\geq 0}(\mathbf{D}f_*\mathscr{O}_X[\mathrm{d}_{X/Y}]) = H^{\mathrm{d}_{X/Y}}(\mathbf{D}f_*\mathscr{O}_X).$$

Therefore, we obtain a morphism $\mathbf{D}f_*\mathscr{O}_X[\mathrm{d}_{X/Y}] \to \mathscr{O}_Y$ in $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_Y, G)$.

Lemma 3.8.2 (Projection formula). *There is a functorial isomorphism in $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X, G)$ and $\mathscr{N} \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_Y, G)$*

$$\mathbf{D}f_*(\mathscr{M} \overset{\mathbf{D}}{\otimes} \mathbf{D}f^*\mathscr{N}) \simeq (\mathbf{D}f_*\mathscr{M}) \overset{\mathbf{D}}{\otimes} \mathscr{N}.$$

Since this is proved similarly to the non-equivariant case, we omit the proof (see e.g. [16]).

By this lemma, we obtain the residue morphism:

$$\mathrm{Res}_{X/Y}\colon \mathbf{D}f_*\mathbf{D}f^*\mathscr{N}[\mathrm{d}_{X/Y}] \to \mathscr{N}, \tag{3.8.9}$$

as the compositions of a chain of morphisms

$$\begin{aligned}\mathbf{D}f_*\mathbf{D}f^*\mathscr{N}[\mathrm{d}_{X/Y}] &\simeq \mathbf{D}f_*(\mathscr{O}_X[\mathrm{d}_{X/Y}] \overset{\mathbf{D}}{\otimes} \mathbf{D}f^*\mathscr{N}) \\ &\simeq (\mathbf{D}f_*\mathscr{O}_X[\mathrm{d}_{X/Y}]) \overset{\mathbf{D}}{\otimes} \mathscr{N} \\ &\to \mathscr{O}_Y \overset{\mathbf{D}}{\otimes} \mathscr{N} \simeq \mathscr{N}.\end{aligned}$$

Proof of Theorem 3.8.1

We shall prove first the isomorphism (3.8.1) in Theorem 3.8.1. For $\mathscr{N} \in \mathrm{K}^+(\mathrm{Mod}(\mathscr{D}_Y, G))$, we have a quasi-isomorphism

$$\mathscr{N} \leftarrow \mathscr{D}_Y \otimes \Omega_Y^{\otimes -1} \otimes \Omega_Y^{\bullet} \otimes \mathscr{N}[\mathrm{d}_Y]$$

and a morphism

$$\mathscr{D}_Y \otimes \Omega_Y^{\otimes -1} \otimes \Omega_Y^{\bullet} \otimes \mathscr{N} \to \mathscr{D}_Y \otimes \Omega_Y^{\otimes -1} \otimes f_*(\Omega_X^{\bullet} \otimes f^*\mathscr{N}) \simeq \mathrm{K}\, f_*(f^*\mathscr{N})[-\mathrm{d}_X].$$

Thus we obtain a morphism in $\mathrm{D^b}(\mathscr{D}_Y, G)$:

$$\mathscr{N} \to \mathbf{D}f_*\mathbf{D}f^*\mathscr{N}[-\mathrm{d}_{X/Y}], \tag{3.8.10}$$

even if f is not assumed to be smooth projective. This gives a chain of homomorphisms

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{D^b}(\mathscr{D}_X,G)}(\mathbf{D}f^*\mathscr{N}[-\mathrm{d}_{X/Y}], \mathscr{M})\\ &\quad\to \mathrm{Hom}_{\mathrm{D^b}(\mathscr{D}_Y,G)}(\mathbf{D}f_*\mathbf{D}f^*\mathscr{N}[-\mathrm{d}_{X/Y}], \mathbf{D}f_*\mathscr{M})\\ &\quad\to \mathrm{Hom}_{\mathrm{D^b}(\mathscr{D}_Y,G)}(\mathscr{N}, \mathbf{D}f_*\mathscr{M}).\end{aligned}$$

Let us prove that the composition is an isomorphism when f is smooth. Similarly as above, we have a morphism in $\mathrm{D}(\mathrm{Mod}(G))$

$$\mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathbf{D}f^*\mathscr{N}[-\mathrm{d}_{X/Y}], \mathscr{M}) \to \mathbf{R}\mathrm{Hom}_{\mathscr{D}_Y}(\mathscr{N}, \mathbf{D}f_*\mathscr{M}).$$

By the theory of D-modules, forgetting the equivariance, this is an isomorphism in $\mathrm{D^b}(\mathbb{C})$, assuming that f is smooth (see [16]). Hence this is an isomorphism in $\mathrm{D^b}(\mathrm{Mod}(G))$. Finally we obtain by (3.7.3)

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{D^b}(\mathscr{D}_X,G)}(\mathbf{D}f^*\mathscr{N}[-\mathrm{d}_{X/Y}], \mathscr{M})\\ &\quad\simeq \mathrm{Hom}_{\mathrm{D^b}(\mathrm{Mod}(G))}\big(\mathbb{C}, \mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathbf{D}f^*\mathscr{N}[-\mathrm{d}_{X/Y}], \mathscr{M})\big)\\ &\quad\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D^b}(\mathrm{Mod}(G))}(\mathbb{C}, \mathbf{R}\mathrm{Hom}_{\mathscr{D}_Y}(\mathscr{N}, \mathbf{D}f_*\mathscr{M}))\\ &\quad\simeq \mathrm{Hom}_{\mathrm{D^b}(\mathscr{D}_Y,G)}(\mathscr{N}, \mathbf{D}f_*\mathscr{M}).\end{aligned}$$

The proof of (3.8.2) is similar using $\mathrm{Res}_{X/Y} \colon \mathbf{D}f_*\mathbf{D}f^*\mathscr{N}[\mathrm{d}_{X/Y}] \to \mathscr{N}$ given in (3.8.9) instead of (3.8.10).

3.9 Flag manifold case

We shall apply Theorem 3.8.1 when $X = G/P$ and $Y = \{\mathrm{pt}\}$, where P is a parabolic subgroup of a reductive group G. Note that X is a projective G-manifold. Then, we obtain the following duality isomorphism.

Lemma 3.9.1. *For any finite-dimensional G-module E and a $(\mathfrak{g}, P)$-module M finitely generated over $U(\mathfrak{g})$, we have an isomorphism*

$$\operatorname{Ext}^{2\dim(G/P)-j}_{(\mathfrak{g},P)}(M,E) \cong \operatorname{Hom}_{\mathbb{C}}\big(\operatorname{Ext}^{j}_{(\mathfrak{g},P)}(E,M),\mathbb{C}\big). \tag{3.9.1}$$

Proof. The category $\operatorname{Mod}_{\mathrm{coh}}(\mathscr{D}_X, G)$ is equivalent to the category $\operatorname{Mod}_f(\mathfrak{g}, P)$ of $(\mathfrak{g}, P)$-modules finitely generated over $U(\mathfrak{g})$, and $\operatorname{Mod}_{\mathrm{coh}}(\mathscr{D}_Y, G)$ is equivalent to the category $\operatorname{Mod}_f(G)$ of finite-dimensional G-modules (see Proposition 3.1.5). The functor $\mathbf{D}f^*$ is induced by the functor $V \mapsto V$ from $\operatorname{Mod}(G)$ to $\operatorname{Mod}(\mathfrak{g}, P)$. The right adjoint functor to the last functor is given by

$$M \longmapsto \bigoplus_V V \otimes \operatorname{Hom}_{(\mathfrak{g},P)}(V, M). \tag{3.9.2}$$

Here V ranges over the isomorphism classes of irreducible G-modules. Hence the functor $\mathbf{D}f_*[-\mathrm{d}_{X/Y}]$, the right adjoint functor of $\mathbf{D}f^*$, is the right derived functor of the functor (3.9.2). Hence (3.8.2) implies that

$$\prod_V \operatorname{Hom}_{\mathrm{D}(\operatorname{Mod}(G))}(V \otimes \mathbf{R}\operatorname{Hom}_{(\mathfrak{g},P)}(V, M)[\mathrm{d}_X], E[j])$$
$$\simeq \operatorname{Hom}_{\mathrm{D}(\operatorname{Mod}(\mathfrak{g},P))}(M, E[j][\mathrm{d}_X]).$$

The last term is isomorphic to $\operatorname{Ext}^{\mathrm{d}_X+j}_{(\mathfrak{g},P)}(M,E)$, and the first term is isomorphic to $\operatorname{Hom}_{\mathbb{C}}\big(\operatorname{Ext}^{\mathrm{d}_X-j}_{(\mathfrak{g},P)}(E,M),\mathbb{C}\big)$ because, when E and V are irreducible, we have

$$\operatorname{Hom}_{\mathrm{D}(\operatorname{Mod}(G))}(V, E[j]) = \begin{cases} \mathbb{C} & \text{if } V \simeq E \text{ and } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Q.E.D.

4 Equivariant derived category

4.1 Introduction

In the case of quasi-equivariant D-modules, the category has enough objects, and it is enough to consider the derived category of the abelian category of quasi-equivariant D-modules. However the categories of equivariant sheaves have not enough objects, and the derived category of the abelian category of equivariant sheaves is not an appropriate category. In order to avoid this difficulty, we have to enrich spaces itself. In this paper, we follow a definition of the equivariant derived categories due to Bernstein-Lunts [4].

4.2 Sheaf case

Let G be a real Lie group and X a (separated) locally compact space with G-action. We assume that X has a finite soft dimension (e.g. a finite-dimensional topological manifold). We call such an X a G-space. If X is a manifold, we call it a G-manifold.

In this paper, we say that G acts *freely* if the morphism $\tilde{\mu}\colon G \times X \to X \times X$ $((g,x) \mapsto (gx,x))$ is a closed embedding. Therefore, if X is a G-manifold with a free action of G, then X/G exists as a (separated) topological manifold.

Let $\mathrm{Mod}(\mathbb{C}_X)$ be the category of sheaves of $\mathbb{C}$-vector spaces on X. We denote by $\mathrm{D^b}(\mathbb{C}_X)$ the bounded derived category of $\mathrm{Mod}(\mathbb{C}_X)$.

Let $\mu\colon G \times X \to X$ be the action map and $\mathrm{pr}\colon G \times X \to X$ the projection.

Definition 4.2.1. A sheaf F of $\mathbb{C}$-vector spaces is called G-*equivariant* if it is endowed with an isomorphism $\mu^{-1}F \xrightarrow{\sim} \mathrm{pr}^{-1}F$ satisfying the associative law as in (3.1.2).

Let us denote by $\mathrm{Mod}_G(\mathbb{C}_X)$ the abelian categories of G-equivariant sheaves.

If G acts freely, then we have the equivalence of categories:

$$\mathrm{Mod}(\mathbb{C}_{X/G}) \xrightarrow{\sim} \mathrm{Mod}_G(\mathbb{C}_X).$$

We will construct the equivariant derived category $\mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_X)$ which has suitable functorial properties and satisfies the condition:

$$\text{if } G \text{ acts freely on } X, \text{ then } \mathrm{D^b}(\mathbb{C}_{X/G}) \simeq \mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_X).$$

Assume that there is a sequence of G-equivariant morphisms

$$V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow \cdots$$

where V_k is a connected G-manifold with a free action and

$$\text{(4.2.1)} \quad \begin{array}{ll} \text{(i)} & H^n(V_k;\mathbb{C}) \text{ is finite-dimensional for any } n,k, \\ \text{(ii)} & \text{for each } n>0,\ H^n(V_k;\mathbb{C}) = 0 \text{ for } k \gg 0. \end{array}$$

Any real semisimple Lie group with finite center has such a sequence $\{V_k\}$. If G is embedded in some $GL_N(\mathbb{C})$ as a closed subgroup, we can take $V_k = \big\{f \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^N, \mathbb{C}^{N+k})\,;\, f \text{ is injective}\big\}$. If G is a connected real semi-simple group with finite center, then we can take $(G \times V_k)/K$ as V_k, where K is a maximal compact subgroup of G and V_k is the one for K. Note that G/K is contractible.

The condition (4.2.1) implies

$$\mathbb{C} \xrightarrow{\sim} \text{“}\varprojlim_k\text{”}\, \mathbf{R}\Gamma(V_k;\mathbb{C}).$$

This follows from the following lemma (see e.g. [19, Exercise 15.1]).

Lemma 4.2.2. *Let $\mathcal{C}$ be an abelian category. Let $\{X_n\}_{n\in\mathbb{Z}_{\geq 0}}$ be a projective system in $\mathrm{D}^{\mathrm{b}}(\mathcal{C})$. Assume that it satisfies the conditions:*

(i) *for any $k \in \mathbb{Z}$, $\text{“}\varprojlim_n\text{”}\, H^k(X_n)$ is representable by an object of $\mathcal{C}$,*

(ii) *one of the following conditions holds:*
  (a) *there exist $a \leq b$ such that $H^k(X_n) \simeq 0$ for $k > b$ and $\text{“}\varprojlim_n\text{”}\, H^k(X_n) \simeq 0$ for $k < a$,*
  (b) *$\mathcal{C}$ has finite homological dimension, and there exist $a \leq b$ such that $\text{“}\varprojlim_n\text{”}\, H^k(X_n) \simeq 0$ unless $a \leq k \leq b$,*

Then $\text{“}\varprojlim_n\text{”}\, X_n$ is representable by an object of $\mathrm{D}^{\mathrm{b}}(\mathcal{C})$.

For example, we say that $\text{“}\varprojlim_n\text{”}\, X_n$ is representable by $X \in \mathrm{D}^{\mathrm{b}}(\mathcal{C})$ if there exists an isomorphism $\varinjlim_n \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathcal{C})}(X_n, Y) \simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathcal{C})}(X, Y)$ functorially in $Y \in \mathrm{D}^{\mathrm{b}}(\mathcal{C})$. In such a case, X is unique up to an isomorphism, and we write $X = \text{“}\varprojlim_n\text{”}\, X_n$.

Let us denote by $p_k\colon V_k \times X \to X$ the second projection and by $\pi_k\colon V_k \times X \to (V_k \times X)/G$ the quotient map. Here the action of G on $V_k \times X$ is the diagonal action. We denote by the same letter i_k the maps $V_k \times X \to V_{k+1} \times X$ and $(V_k \times X)/G \to (V_{k+1} \times X)/G$.

Definition 4.2.3. Let $\mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_X)$ be the category whose objects are $F = \big(F_\infty, F_k, j_k, \varphi_k\ (k = 1, 2, \ldots)\big)$ where $F_\infty \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_X)$, $F_k \in \mathrm{D}^{\mathrm{b}}\big(\mathbb{C}_{(V_k\times X)/G}\big)$ and $j_k\colon i_k^{-1}F_{k+1} \xrightarrow{\sim} F_k$ and $\varphi_k\colon p_k^{-1}F_\infty \xrightarrow{\sim} \pi_k^{-1}F_k$ such that the diagram

$$\begin{array}{ccc} i_k^{-1}p_{k+1}^{-1}F_\infty & \xrightarrow{\ \sim\ } & p_k^{-1}F_\infty \\ \Big\downarrow{\scriptstyle \varphi_{k+1}} & & \Big\downarrow{\scriptstyle \varphi_k} \\ i_k^{-1}\pi_{k+1}^{-1}F_{k+1} & \xrightarrow[j_k]{\ \sim\ } & \pi_k^{-1}F_k \end{array}$$

commutes. The morphisms in $\mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_X)$ are defined in an evident way.

The category $\mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_X)$ is a triangulated category in an obvious way, and the triangulated category $\mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_X)$ does not depend on the choice of a sequence $\{V_k\}_k$ (see [4]). We call $\mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_X)$ the *equivariant derived category.*

By the condition (4.2.1), we have

$$\text{“}\varprojlim_k\text{”}\, \mathbf{R}p_{k*}\pi_k^{-1}F_k \cong F_\infty. \tag{4.2.2}$$

Indeed, we have

$$\text{“}\varprojlim_k\text{”}\, \mathbf{R}p_{k*}\pi_k^{-1}F_k \cong \text{“}\varprojlim_k\text{”}\, \mathbf{R}p_{k*}p_k^{-1}F_\infty \cong \text{“}\varprojlim_k\text{”}\Big(F_\infty \otimes \mathbf{R}\,\Gamma(V_k;\mathbb{C})\Big)$$

and $\text{“}\varprojlim_k\text{”}\, \mathbf{R}\,\Gamma(V_k;\mathbb{C}) \simeq \mathbb{C}$ by (4.2.1).

There exists a functor of triangulated categories (called the *forgetful functor*):

$$\mathscr{F}_G : \mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_X) \to \mathrm{D}^{\mathrm{b}}(\mathbb{C}_X).$$

Note that a morphism u in $\mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_X)$ is an isomorphism if and only if $\mathscr{F}_G(u)$ is an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_X)$.

By taking the cohomology groups, we obtain cohomological functors:

$$H^n \colon \mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_X) \to \mathrm{Mod}_G(\mathbb{C}_X).$$

Lemma 4.2.4. *Assume that G acts freely on X. Then $\mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_X)$ is equivalent to $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{X/G})$.*

Proof. The functor $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_{X/G}) \to \mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_X)$ is obviously defined, and its quasi-inverse $\mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_X) \to \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{X/G})$ is given by $F \mapsto \text{“}\varprojlim_k\text{”}\, \mathbf{R}q_{k*}(F_k)$, where q_k is the map $(V_k \times X)/G \to X/G$. Note that $\text{“}\varprojlim_k\text{”}\, \mathbf{R}q_{k*}(F_k) \cong \tau^{\le a}\mathbf{R}q_{l*}(F_l)$ for $l \gg a \gg 0$. Q.E.D.

Since $\mathrm{Mod}(\mathbb{C}_{X/G})$ is equivalent to $\mathrm{Mod}_G(\mathbb{C}_X)$ in such a case, we have

(4.2.3) if G acts freely on X, then $\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}_G(\mathbb{C}_X)) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_X)$.

For a G-equivariant map $f \colon X \to Y$, we can define the functors

$$f^{-1},\ f^! \colon \mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_Y) \to \mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_X)$$

and

$$\mathbf{R}f_!,\ \mathbf{R}f_* \colon \mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_X) \to \mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_Y).$$

The functors $\mathbf{R}f_!$ and f^{-1} are left adjoint functors of $f^!$ and $\mathbf{R}f_*$, respectively. Moreover they commute with the forgetful functor $\mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_X) \to \mathrm{D}^{\mathrm{b}}(\mathbb{C}_X)$.

4.3 Induction functor

The following properties are easily checked.

(4.3.1) For a group morphism $H \to G$ and a G-manifold X, there exists a canonical functor (*restriction functor*)
$$\mathrm{Res}^G_H\colon \mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_X) \to \mathrm{D}^{\mathrm{b}}_H(\mathbb{C}_X).$$

(4.3.2) If H is a closed normal subgroup of G and if H acts freely on a G-manifold X, then
$$\mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_X) \simeq \mathrm{D}^{\mathrm{b}}_{G/H}(\mathbb{C}_{X/H}).$$

For $F \in \mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_X)$, we denote by F/H the corresponding object of $\mathrm{D}^{\mathrm{b}}_{G/H}(\mathbb{C}_{X/H})$.

Let H be a closed subgroup of G and X an H-manifold. Then we have a chain of equivalences of triangulated categories
$$\mathrm{D}^{\mathrm{b}}_H(\mathbb{C}_X) \simeq \mathrm{D}^{\mathrm{b}}_{H\times G}(\mathbb{C}_{X\times G}) \simeq \mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_{(X\times G)/H})$$
by (4.3.2). Here $H \times G$ acts on $X \times G$ by $(h,g)(x,g') = (hx, gg'h^{-1})$. Let us denote the composition by

$$\mathrm{Ind}^G_H\colon \mathrm{D}^{\mathrm{b}}_H(\mathbb{C}_X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_{(X\times G)/H}). \tag{4.3.3}$$

When X is a G-manifold, we have $(X\times G)/H \simeq X \times (G/H)$, and we obtain an equivalence of categories

$$\mathrm{Ind}^G_H\colon \mathrm{D}^{\mathrm{b}}_H(\mathbb{C}_X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_{X\times(G/H)}) \quad \text{when } X \text{ is a } G\text{-manifold}. \tag{4.3.4}$$

Note that the action of G on $X \times (G/H)$ is the diagonal action.

4.4 Constructible sheaves

Assume that X is a complex algebraic variety and a real Lie group G acts real analytically on the associated complex manifold X^{an}. We denote by $\mathrm{D}^{\mathrm{b}}_{G,\mathbb{R}\text{-c}}(\mathbb{C}_{X^{\mathrm{an}}})$ the full subcategory of $\mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_{X^{\mathrm{an}}})$ consisting of $\mathbb{R}$-constructible objects. Here $F \in \mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_{X^{\mathrm{an}}})$ is called $\mathbb{R}$-*constructible* if it satisfies the following two conditions:

(i) $\dim H^j(F)_x < \infty$ for any $x \in X^{\mathrm{an}}$.
(ii) there exists a finite family $\{Z_\alpha\}$ of locally closed subsets of X^{an} such that
  (a) $X^{\mathrm{an}} = \bigcup_\alpha Z_\alpha$,
  (b) each Z_α is subanalytic in $(\overline{X})^{\mathrm{an}}$ for any (or equivalently, some) compactification $X \hookrightarrow \overline{X}$ of X,
  (c) $H^j(F)|_{Z_\alpha}$ is locally constant .

For subanalyticity and $\mathbb{R}$-constructibility, see e.g. [18].

We say that F is $\mathbb{C}$-*constructible* (or *constructible*, for short) if we assume further that each Z_α is the associated topological set of a subscheme of X.

We denote by $\mathrm{D}^{\mathrm{b}}_{G,\mathbb{R}\text{-c}}(\mathbb{C}_{X^{\mathrm{an}}})$ (resp. $\mathrm{D}^{\mathrm{b}}_{G,\mathbb{C}\text{-c}}(\mathbb{C}_{X^{\mathrm{an}}})$) the full subcategory of $\mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_{X^{\mathrm{an}}})$ consisting of $\mathbb{R}$-constructible (resp. constructible) objects.

4.5 D-module case

The construction of the equivariant derived category for sheaves can be applied similarly to the equivariant derived categories of D-modules.

Let G be an affine algebraic group. Let us take a sequence of connected algebraic G-manifolds

$$V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow \cdots \tag{4.5.1}$$

such that

$$\begin{aligned} &G \text{ acts freely on } V_k, \text{ and} \\ &\text{for any } n > 0,\ \mathrm{Ext}^n_{\mathscr{D}_{V_k}}(\mathscr{O}_{V_k}, \mathscr{O}_{V_k}) \cong H^n(V_k^{\mathrm{an}}; \mathbb{C}) = 0 \text{ for } k \gg 0. \end{aligned} \tag{4.5.2}$$

Such a sequence $\{V_k\}_k$ exists. With the aid of $\{V_k\}_k$, we can define the equivariant derived category of D-modules similarly to the sheaf case. Let X be a quasi-projective algebraic G-manifold. Let us denote by $p_k \colon V_k \times X \to X$ the second projection and by $\pi_k \colon V_k \times X \to (V_k \times X)/G$ the quotient morphism.[3] We denote by the same letter i_k the maps $V_k \times X \to V_{k+1} \times X$ and $(V_k \times X)/G \to (V_{k+1} \times X)/G$.

Definition 4.5.1. Let $\mathrm{D}^{\mathrm{b}}_G(\mathscr{D}_X)$ be the category whose objects are $\mathscr{M} = \bigl(\mathscr{M}_\infty,\ \mathscr{M}_k,\ j_k,\ \varphi_k\ (k \in \mathbb{Z}_{\geq 1})\bigr)$ where $\mathscr{M}_\infty \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)$, $\mathscr{M}_k \in \mathrm{D}^{\mathrm{b}}\bigl(\mathscr{D}_{(V_k \times X)/G}\bigr)$ and $j_k \colon \mathbf{D}i_k^*\mathscr{M}_{k+1} \xrightarrow{\sim} \mathscr{M}_k$ and $\varphi_k \colon \mathbf{D}p_k^*\mathscr{M}_\infty \xrightarrow{\sim} \mathbf{D}\pi_k^*\mathscr{M}_k$ such that the diagram

$$\begin{array}{ccc} \mathbf{D}i_k^*\mathbf{D}p_{k+1}^*\mathscr{M}_\infty & \overset{\sim}{=\!=\!=} & \mathbf{D}p_k^*\mathscr{M}_\infty \\ \big\downarrow{\scriptstyle \varphi_{k+1}} & & \big\downarrow{\scriptstyle \varphi_k} \\ \mathbf{D}i_k^*\mathbf{D}\pi_{k+1}^*\mathscr{M}_{k+1} & \xrightarrow{j_k} & \mathbf{D}\pi_k^*\mathscr{M}_k \end{array}$$

commutes.

Note that we have a canonical functor

$$\mathrm{D}^{\mathrm{b}}_G(\mathscr{D}_X) \to \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X, G).$$

We denote by $\mathrm{D}^{\mathrm{b}}_{G,\mathrm{coh}}(\mathscr{D}_X)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_G(\mathscr{D}_X)$ consisting of objects $\mathscr{M}$ with coherent cohomologies.

Similarly to the sheaf case, we have the following properties.

[3] The quotient $(V_k \times X)/G$ may not exist as a scheme, but it exists as an algebraic space. Although we do not develop here, we have the theory of D-modules on algebraic spaces. Alternatively, we can use $\mathrm{Mod}_G(\mathscr{D}_{V_k \times X})$ instead of $\mathrm{Mod}(\mathscr{D}_{(V_k \times X)/G})$.

(4.5.3) For a morphism $f\colon X \to Y$ of quasi-projective G-manifolds, we can define the pull-back functor $\mathbf{D}f^*\colon \mathrm{D}^{\mathrm{b}}_G(\mathscr{D}_Y) \to \mathrm{D}^{\mathrm{b}}_G(\mathscr{D}_X)$ and the push-forward functor $\mathbf{D}f_*\colon \mathrm{D}^{\mathrm{b}}_G(\mathscr{D}_X) \to \mathrm{D}^{\mathrm{b}}_G(\mathscr{D}_Y)$.

(4.5.4) The canonical functor $\mathrm{D}^{\mathrm{b}}_G(\mathscr{D}_X) \to \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X, G)$ commutes with the pull-back and push-forward functors.

(4.5.5) For a closed algebraic subgroup H of G and an algebraic G-manifold X, there exists a canonical functor $\mathrm{Res}^G_H\colon \mathrm{D}^{\mathrm{b}}_G(\mathscr{D}_X) \to \mathrm{D}^{\mathrm{b}}_H(\mathscr{D}_X)$.

(4.5.6) If H is a normal subgroup of G and if H acts freely on X and if X/H exists, then $\mathrm{D}^{\mathrm{b}}_G(\mathscr{D}_X) \simeq \mathrm{D}^{\mathrm{b}}_{G/H}(\mathscr{D}_{X/H})$.

(4.5.7) If H is a closed algebraic subgroup of G and X is an algebraic G-manifold, then we have
$$\mathrm{Ind}^G_H\colon \mathrm{D}^{\mathrm{b}}_H(\mathscr{D}_X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_G(\mathscr{D}_{X\times(G/H)}).$$

4.6 Equivariant Riemann-Hilbert correspondence

Let X be a quasi-projective manifold. Let us denote by X^{an} the associated complex manifold. Accordingly, $\mathscr{O}_{X^{\mathrm{an}}}$ is the sheaf of holomorphic functions on X^{an}. Then there exists a morphism of ringed spaces $\pi\colon X^{\mathrm{an}} \to X$. We denote by $\mathscr{D}_{X^{\mathrm{an}}}$ the sheaf of differential operators with holomorphic coefficients on X^{an}. For a $\mathscr{D}_X$-module $\mathscr{M}$, we denote by $\mathscr{M}^{\mathrm{an}}$ the associated $\mathscr{D}_{X^{\mathrm{an}}}$-module $\mathscr{D}_{X^{\mathrm{an}}} \otimes_{\pi^{-1}\mathscr{D}_X} \pi^{-1}\mathscr{M} \simeq \mathscr{O}_{X^{\mathrm{an}}} \otimes_{\pi^{-1}\mathscr{O}_X} \pi^{-1}\mathscr{M}$.

Let us denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$ (resp. $\mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$) the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)$ consisting of objects with holonomic cohomologies (resp. regular holonomic cohomologies) (see [16]). Then the de Rham functor
$$\mathrm{DR}_X := \mathbf{R}\mathscr{H}om_{\mathscr{D}_{X^{\mathrm{an}}}}(\mathscr{O}_{X^{\mathrm{an}}}, \bullet^{\mathrm{an}}) : \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X) \to \mathrm{D}^{\mathrm{b}}(\mathbb{C}_X)$$
sends $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$ to $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_{X^{\mathrm{an}}})$.

Then we have the following Riemann-Hilbert correspondence.

Theorem 4.6.1 ([12]). *The functor* DR_X *gives an equivalence of categories:*

$$\mathrm{DR}_X : \mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_{X^{\mathrm{an}}}). \tag{4.6.1}$$

Now, let G be an affine algebraic group and X a quasi-projective G-manifold. Then we define similarly $\mathrm{D}^{\mathrm{b}}_{G,\mathrm{hol}}(\mathscr{D}_X)$ and $\mathrm{D}^{\mathrm{b}}_{G,\mathrm{rh}}(\mathscr{D}_X)$ as full subcategories of $\mathrm{D}^{\mathrm{b}}_G(\mathscr{D}_X)$. Then we can define the equivariant de Rham functor:
$$\mathrm{DR}_X\colon \mathrm{D}^{\mathrm{b}}_{G,\mathrm{hol}}(\mathscr{D}_X) \to \mathrm{D}^{\mathrm{b}}_{G^{\mathrm{an}},\mathbb{C}\text{-c}}(\mathbb{C}_{X^{\mathrm{an}}}).$$

Theorem 4.6.1 implies the following theorem.

Theorem 4.6.2. *The functor* DR_X *gives an equivalence of categories:*

$$\mathrm{DR}_X : \mathrm{D}^{\mathrm{b}}_{G,\mathrm{rh}}(\mathscr{D}_X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{G^{\mathrm{an}},\mathbb{C}\text{-}c}(\mathbb{C}_{X^{\mathrm{an}}}). \tag{4.6.2}$$

5 Holomorphic solution spaces

5.1 Introduction

Let G be an affine complex algebraic group and let X be a quasi-projective G-manifold. Recall that we denote by X^{an} the associated complex manifold and, for a $\mathscr{D}_X$-module $\mathscr{M}$, we denote by $\mathscr{M}^{\mathrm{an}}$ the associated $\mathscr{D}_{X^{\mathrm{an}}}$-module $\mathscr{D}_{X^{\mathrm{an}}} \otimes_{\pi^{-1}\mathscr{D}_X} \pi^{-1}\mathscr{M} \simeq \mathscr{O}_{X^{\mathrm{an}}} \otimes_{\pi^{-1}\mathscr{O}_X} \pi^{-1}\mathscr{M}$. Here $\pi\colon X^{\mathrm{an}} \to X$ is the canonical morphism of ringed spaces.

Let $G_{\mathbb{R}}$ be a real Lie group and let $G_{\mathbb{R}} \to G^{\mathrm{an}}$ be a morphism of Lie groups. Hence $G_{\mathbb{R}}$ acts on X^{an}.

Recall that $\mathbf{FN}_{G_{\mathbb{R}}}$ is the category of Fréchet nuclear $G_{\mathbb{R}}$-modules (see Example 2.1.2 (iii)). We denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_X, G)$ the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X, G)$ consisting of objects with countably coherent cohomologies, by $\mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}},\mathrm{ctb}}(\mathbb{C}_{X^{\mathrm{an}}})$ the full subcategory of $\mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}}}(\mathbb{C}_{X^{\mathrm{an}}})$ consisting of objects with countable sheaves as cohomology groups (see § 5.2), and by $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$ the bounded derived category of $\mathbf{FN}_{G_{\mathbb{R}}}$.

In this section, we shall define

$$\mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M} \otimes K, \mathscr{O}_{X^{\mathrm{an}}})$$

as an object of $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$ for $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_X, G)$ and $K \in \mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}},\mathrm{ctb}}(\mathbb{C}_{X^{\mathrm{an}}})$. Here, we write $\mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M} \otimes K, \mathscr{O}_{X^{\mathrm{an}}})$ instead of $\mathbf{R}\mathrm{Hom}_{\pi^{-1}\mathscr{D}_X}(\pi^{-1}\mathscr{M} \otimes K, \mathscr{O}_{X^{\mathrm{an}}}) \simeq \mathbf{R}\mathrm{Hom}_{\mathscr{D}_{X^{\mathrm{an}}}}(\mathscr{M}^{\mathrm{an}} \otimes K, \mathscr{O}_{X^{\mathrm{an}}})$ for short.

We also prove the dual statement. Let $\mathbf{DFN}_{G_{\mathbb{R}}}$ be the category of dual Fréchet nuclear $G_{\mathbb{R}}$-modules. We will define

$$\mathbf{R}\Gamma_{\mathrm{c}}(X^{\mathrm{an}}; K \otimes \Omega_{X^{\mathrm{an}}} \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M})$$

as an object of $\mathrm{D}^{\mathrm{b}}(\mathbf{DFN}_{G_{\mathbb{R}}})$ for $\mathscr{M}$ and K as above. We then prove that $\mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M} \otimes K, \mathscr{O}_{X^{\mathrm{an}}})$ and $\mathbf{R}\Gamma_{\mathrm{c}}(X^{\mathrm{an}}; K \otimes \Omega_{X^{\mathrm{an}}} \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M})[\mathrm{d}_X]$ are dual to each other.

5.2 Countable sheaves

Let X be a topological manifold (countable at infinity).

Proposition 5.2.1. *Let F be a sheaf of $\mathbb{C}$-vector spaces on X. Then the following conditions are equivalent.*

(i) *for any compact subset K of X, $\Gamma(K; F)$ is countable-dimensional,*
(ii) *for any compact subset K of X, $H^n(K; F)$ is countable-dimensional for all n,*
(iii) *for any x and an open neighborhood U of x, there exists an open neighborhood V of x such that $V \subset U$ and $\mathrm{Im}(\Gamma(U; F) \to \Gamma(V; F))$ is countable-dimensional,*

(iv) *there exist a countable family of open subsets* $\{U_i\}_i$ *of* X *and an epimorphism* $\oplus_i \mathbb{C}_{U_i} \twoheadrightarrow F$.

If X *is a real analytic manifold, then the above conditions are also equivalent to*

(a) *there exist a countable family of subanalytic open subsets* $\{U_i\}_i$ *of* X *and an epimorphism* $\oplus_i \mathbb{C}_{U_i} \twoheadrightarrow F$.

Proof. For compact subsets K_1 and K_2, we have an exact sequence

$$H^{n-1}(K_1 \cap K_2; F) \longrightarrow H^n(K_1 \cup K_2; F) \longrightarrow H^n(K_1; F) \oplus H^n(K_2; F).$$

Hence, if K_1, K_2 and $K_1 \cap K_2$, satisfy the condition (i) or (ii), then so does $K_1 \cup K_2$. Hence the conditions (i) and (ii) are local properties. Since the other conditions are also local, we may assume from the beginning that X is real analytic.

(ii)⇒(i)⇒ (iii) are obvious.
(iii)⇒(iv) Let us take a countable base of open subsets $\{U_s\}_{s \in S}$ of X. Then, for each $s \in S$, there exists a countable open covering $\{V_i\}_{i \in I(s)}$ of U_s such that $\mathrm{Im}(\Gamma(U_s; F) \to \Gamma(V_i; F))$ is countable-dimensional. Then the natural morphism

$$\bigoplus_{s \in S,\, i \in I(s)} \mathrm{Im}(\Gamma(U_s; F) \to \Gamma(V_i; F)) \otimes \mathbb{C}_{V_i} \to F$$

is an epimorphism.
(iv)⇒(a) follows from the fact that each $\mathbb{C}_{U_i}$ is a quotient of a countable direct sum of sheaves of the form $\mathbb{C}_V$ with a subanalytic open subset V.
(a)⇒(ii) We shall prove it by the descending induction on n. Assume that F satisfies the condition (a). Let us take an exact sequence

$$0 \to F' \to L \to F \to 0,$$

such that $L \simeq \oplus_i \mathbb{C}_{U_i}$ for a countable family $\{U_i\}_i$ of subanalytic open subsets of X. Then, for any relatively compact subanalytic open subset W, $H^k(W; \mathbb{C}_{U_i})$ is finite-dimensional (see e.g. [18]). Hence, the cohomology group $H^k(K; \mathbb{C}_{U_i}) \cong \varinjlim_{K \subset W} H^k(W; \mathbb{C}_{U_i})$ is countable-dimensional, and so is $H^k(K; L) \simeq \oplus_i H^k(K; \mathbb{C}_{U_i})$. Therefore L satisfies (i), which implies that F' also satisfies the condition (i) and hence the condition (a). By the induction hypothesis, $H^{n+1}(K; F')$ is countable-dimensional. By the exact sequence

$$H^n(K; L) \to H^n(K; F) \to H^{n+1}(K; F'),$$

$H^n(K; F)$ is countable-dimensional. Q.E.D.

Definition 5.2.2. A sheaf F of complex vector spaces on X is called a *countable sheaf* if F satisfies the equivalent conditions in Proposition 5.2.1.

Let us denote by $\mathrm{Mod}_{\mathrm{ctb}}(\mathbb{C}_X)$ the full subcategory of $\mathrm{Mod}(\mathbb{C}_X)$ consisting of countable sheaves. Then, $\mathrm{Mod}_{\mathrm{ctb}}(\mathbb{C}_X)$ is closed by subobjects, quotients and extensions. Moreover it is closed by a countable inductive limits. Let us denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{ctb}}(\mathbb{C}_X)$ the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_X)$ consisting of objects whose cohomology groups are countable sheaves. It is a triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_X)$.

Lemma 5.2.3. (i) *If F, $F' \in \mathrm{D}^{\mathrm{b}}_{\mathrm{ctb}}(\mathbb{C}_X)$, then $F \otimes F'' \in \mathrm{D}^{\mathrm{b}}_{\mathrm{ctb}}(\mathbb{C}_X)$.*

(ii) *For $F \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_X)$, the following conditions are equivalent.*

(a) *$F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{ctb}}(\mathbb{C}_X)$,*

(b) *$H^n(K; F)$ is countable-dimensional for any compact subset K and any integer n,*

(c) *$H^n_c(U; F)$ is countable-dimensional for any open subset U and any integer n.*

(iii) *Let $f\colon X \to Y$ is a continuous map of topological manifolds. Then $\mathbf{R}f_! F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{ctb}}(\mathbb{C}_Y)$ for any $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{ctb}}(\mathbb{C}_X)$.*

Proof. (i) follows from (iv) in Proposition 5.2.1.
(ii) (b)⇒(c) If U is relatively compact, it follows from the exact sequence $H^{n-1}(K \setminus U; F) \to H^n_c(U; F) \to H^n(K; F)$ for a compact set $K \supset U$, and if U is arbitrary, it follows from $H^n_c(U; F) = \varinjlim_{V \subset\subset U} H^n_c(V; F)$.
(c)⇒(b) follows from the exact sequence

$$H^n_c(X; F) \to H^n(K; F) \to H^{n+1}_c(X \setminus K; F).$$

(a)⇒(b) Let us show that $H^n(K; \tau^{\leq k} F)$ is countable-dimensional by the induction on k. If $H^n(K; \tau^{\leq k-1} F)$ is countable-dimensional, the exact sequence

$$H^n(K; \tau^{\leq k-1} F) \to H^n(K; \tau^{\leq k} F) \to H^{n-k}(K; H^k(F))$$

shows that $H^n(K; \tau^{\leq k} F)$ is countable-dimensional.
(b)⇒(a) We shall show that $H^k(F)$ is a countable sheaf by the induction on k. Assume that $\tau^{<k} F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{ctb}}(\mathbb{C}_X)$. Then, for any compact subset K, we have the exact sequence

$$H^n(K; F) \to H^n(K; \tau^{\geq k} F)) \to H^{n+1}(K; \tau^{<k} F).$$

Since $H^{n+1}(K; \tau^{<k} F)$ is countable-dimensional by (a)⇒(b), $H^n(K; \tau^{\geq k} F))$ is also countable-dimensional. In particular, $\Gamma(K; H^k(F)) = H^k(K; \tau^{\geq k} F)$ is countable-dimensional.
(iii) For any open subset V of Y, $H^n_c(V; \mathbf{R}f_! F) \simeq H^n_c(f^{-1}(V); F)$ is countable-dimensional. Q.E.D.

The following lemma is immediate.

Lemma 5.2.4. *Let F be a countable sheaf and let $H \twoheadrightarrow F$ be an epimorphism. Then there exist a countable sheaf F' and a morphism $F' \to H$ such that the composition $F' \to H \to F$ is an epimorphism.*

By Lemma 5.2.4, we have the following lemma.

Lemma 5.2.5. *The functor* $\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}_{\mathrm{ctb}}(\mathbb{C}_X)) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{ctb}}(\mathbb{C}_X)$ *is an equivalence of triangulated categories.*

More precisely, we have the following.

Lemma 5.2.6. *Let F be a bounded complex of sheaves such that all the cohomology groups are countable. Then we can find a bounded complex F' of countable sheaves and a quasi-isomorphism $F' \to F$.*

If a Lie group G acts on a real analytic manifold X, we denote by $\mathrm{Mod}_{G,\mathrm{ctb}}(\mathbb{C}_X)$ the category of G-equivariant sheaves of $\mathbb{C}$-vector spaces which are countable.

Remark 5.2.7. A sheaf F of $\mathbb{C}$-vector spaces on X is not necessarily countable even if F_x is finite-dimensional for all $x \in X$. Indeed, the sheaf $\oplus_{x\in X}\mathbb{C}_{\{x\}}$ on X is such an example.

5.3 C$^\infty$-solutions

Let X, G and $G_{\mathbb{R}}$ be as in § 5.1. Let $X_{\mathbb{R}}$ be a real analytic submanifold of X^{an} invariant by the $G_{\mathbb{R}}$-action such that $T_xX \cong \mathbb{C}\otimes_{\mathbb{R}} T_xX_{\mathbb{R}}$ for any $x \in X_{\mathbb{R}}$. Let M be a differentiable $G_{\mathbb{R}}$-manifold. Let us denote by $\mathscr{C}^{\infty}_{X_{\mathbb{R}}\times M}$ the sheaf of C^∞-functions on $X_{\mathbb{R}} \times M$. Then, $\mathscr{C}^{\infty}_{X_{\mathbb{R}}\times M}$ is an $f^{-1}\mathscr{D}_X$-module, where $f\colon X^{\mathrm{an}} \times M \to X$ is a canonical map. For $\mathscr{M} \in \mathrm{Mod}(\mathscr{D}_X)$ and $K \in \mathrm{Mod}(\mathbb{C}_{X_{\mathbb{R}}\times M})$, we write $\mathrm{Hom}_{\mathscr{D}_X}\left(\mathscr{M}\otimes K, \mathscr{C}^{\infty}_{X_{\mathbb{R}}\times M}\right)$ instead of $\mathrm{Hom}_{f^{-1}\mathscr{D}_X}\left(f^{-1}\mathscr{M}\otimes K, \mathscr{C}^{\infty}_{X_{\mathbb{R}}\times M}\right)$ for short.

Lemma 5.3.1. *For any countable sheaf K on $X_{\mathbb{R}} \times M$ and a countably coherent $\mathscr{D}_X$-module $\mathscr{M}$,* $\mathrm{Hom}_{\mathscr{D}_X}\left(\mathscr{M}\otimes K, \mathscr{C}^{\infty}_{X_{\mathbb{R}}\times M}\right)$ *has a structure of Fréchet nuclear space.*

Proof. The topology is the weakest topology such that, for any open subset U of X, any open subset V of $(U^{\mathrm{an}} \cap X_{\mathbb{R}}) \times M$ and $s \in \Gamma(U;\mathscr{M})$, $t \in \Gamma(V;K)$, the homomorphism

$$\mathrm{Hom}_{\mathscr{D}_X}\left(\mathscr{M}\otimes K, \mathscr{C}^{\infty}_{X_{\mathbb{R}}\times M}\right) \ni \varphi \mapsto \varphi(s\otimes t) \in \mathrm{C}^\infty(V) \tag{5.3.1}$$

is a continuous map. Here, $\mathrm{C}^\infty(V)$ is the space of C^∞-functions on V.

There exist a countable index set A and a family of open subsets $\{U_a\}_{a\in A}$ of X, open subsets $\{V_a\}_{a\in A}$ of $X_{\mathbb{R}} \times M$ and $s_a \in \Gamma(U_a;\mathscr{M})$, $t_a \in \Gamma(V_a;K)$ satisfying the following properties:

(i) $V_a \subset U_a^{\mathrm{an}} \times M$,
(ii) $\{s_a\}_{a\in A}$ generates $\mathscr{M}$, namely, $\mathscr{M}_x = \sum_{x\in U_a}(\mathscr{D}_X)_x(s_a)_x$ for any $x \in X$,
(iii) $\{t_a\}_{a\in A}$ generates K, namely, $K_x \simeq \sum_{x\in V_a}\mathbb{C}(t_a)_x$ for any $x \in X_{\mathbb{R}} \times M$.

Then by the morphisms (5.3.1), $\{s_a\}$ and $\{t_a\}$ induce an injection

$$\mathrm{Hom}_{\mathscr{D}_X}\left(\mathscr{M}\otimes K, \mathscr{C}^{\infty}_{X_{\mathbb{R}}\times M}\right) \rightarrowtail \prod_{a\in A} \mathrm{C}^{\infty}(V_a).$$

We can easily see that its image is a closed subspace of $\prod_{a\in A}\mathrm{C}^{\infty}(V_a)$, and the induced topology coincides with the weakest topology introduced in the beginning. Since $\mathrm{C}^{\infty}(V_a)$ is a Fréchet nuclear space and a countable product of Fréchet nuclear spaces is also a Fréchet nuclear space, $\prod_{a\in A}\mathrm{C}^{\infty}(V_a)$ is a Fréchet nuclear space. Hence, its closed subspace $\mathrm{Hom}_{\mathscr{D}_X}\left(\mathscr{M}\otimes K, \mathscr{C}^{\infty}_{X_{\mathbb{R}}\times M}\right)$ is also a Fréchet nuclear space. Q.E.D.

Let $\mathscr{E}^{(p,q,r)}_{X^{\mathrm{an}}\times M}$ denote the sheaf of differential forms on $X^{\mathrm{an}}\times M$ with C^{∞}-coefficients which are (p,q)-forms with respect to X^{an}, and r-forms with respect to M. We set $\mathscr{E}^{(0,n)}_{X^{\mathrm{an}}\times M} = \oplus_{n=q+r}\mathscr{E}^{(0,q,r)}_{X^{\mathrm{an}}\times M}$. Then $\mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}}\times M}$ is a complex of $p^{-1}\mathscr{D}_{X^{\mathrm{an}}}$-modules, and it is quasi-isomorphic to $p^{-1}\mathscr{O}_{X^{\mathrm{an}}}$, where $p\colon X^{\mathrm{an}}\times M\to X^{\mathrm{an}}$ is the projection.

Lemma 5.3.2. *For any $K\in\mathrm{Mod}_{G_{\mathbb{R}},\mathrm{ctb}}(\mathbb{C}_{X^{\mathrm{an}}\times M})$ and $\mathscr{M}\in\mathrm{Mod}_{\mathrm{cc}}(\mathscr{D}_X,G)$, $\mathrm{Hom}_{\mathscr{D}_X}\left(\mathscr{M}\otimes K,\mathscr{E}^{(0,q)}_{X^{\mathrm{an}}\times M}\right)$ has a Fréchet nuclear $G_{\mathbb{R}}$-module structure.*

The proof is similar to the previous lemma.

We denote by $\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}\left(\mathscr{M}\otimes K,\mathscr{C}^{\infty}_{X_{\mathbb{R}}\times M}\right)$ and $\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}\left(\mathscr{M}\otimes K,\mathscr{E}^{(0,n)}_{X^{\mathrm{an}}\times M}\right)$ the corresponding space endowed with the Fréchet nuclear $G_{\mathbb{R}}$-module structure.

5.4 Definition of $\mathbf{RHom^{top}}$

Let us take a differentiable $G_{\mathbb{R}}$-manifold M with a free $G_{\mathbb{R}}$-cation. Then we have an equivalence of categories:

$$\begin{array}{ccc} \mathrm{Mod}_{G_{\mathbb{R}}}(\mathbb{C}_{X^{\mathrm{an}}\times M}) & \simeq & \mathrm{Mod}(\mathbb{C}_{(X^{\mathrm{an}}\times M)/G_{\mathbb{R}}}) \\ \bigcup & & \bigcup \\ \mathrm{Mod}_{G_{\mathbb{R}},\mathrm{ctb}}(\mathbb{C}_{X^{\mathrm{an}}\times M}) & \simeq & \mathrm{Mod}_{\mathrm{ctb}}(\mathbb{C}_{(X^{\mathrm{an}}\times M)/G_{\mathbb{R}}}). \end{array} \tag{5.4.1}$$

Definition 5.4.1. A countable $G_{\mathbb{R}}$-equivariant sheaf K on $X^{\mathrm{an}}\times M$ is called *standard* if K is isomorphic to $\bigoplus_{j\in J}(E_j)_{U_j}$, where $\{U_j\}_{j\in J}$ is a countable family of $G_{\mathbb{R}}$-invariant open subsets of X^{an} and E_j is a $G_{\mathbb{R}}$-equivariant local system on U_j of finite rank. Note the $(E_j)_{U_j}$ is the extension of E_j to the sheaf on $X^{\mathrm{an}}\times M$ such that $(E_j)_{U_j}|_{(X^{\mathrm{an}}\times M)\setminus U_j}=0$.

Let us denote by $\mathrm{Mod}_{G_{\mathbb{R}},\mathrm{stand}}(\mathbb{C}_{X^{\mathrm{an}}\times M})$ the full abelian subcategory of $\mathrm{Mod}_{G_{\mathbb{R}}}(\mathbb{C}_{X^{\mathrm{an}}\times M})$ consisting of standard sheaves. With this terminology, we obtain the following lemma by (5.4.1) and Proposition 5.2.1.

Lemma 5.4.2. *For any $K\in\mathrm{C}^{-}\bigl(\mathrm{Mod}_{G_{\mathbb{R}}}(\mathbb{C}_{X^{\mathrm{an}}\times M})\bigr)$ with countable sheaves as cohomologies, there exist $K'\in\mathrm{C}^{-}\bigl(\mathrm{Mod}_{G_{\mathbb{R}},\mathrm{stand}}(\mathbb{C}_{X^{\mathrm{an}}\times M})\bigr)$ and a quasi-isomorphism $K'\to K$ in $\mathrm{C}^{-}\bigl(\mathrm{Mod}_{G_{\mathbb{R}}}(\mathbb{C}_{X^{\mathrm{an}}\times M})\bigr)$.*

Similarly we introduce the following notion.

Definition 5.4.3. A countably coherent quasi-G-equivariant $\mathscr{D}_X$-module $\mathscr{M}$ is called *standard* if $\mathscr{M}$ is isomorphic to $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{E}$ where $\mathscr{E}$ is a countably coherent locally free G-equivariant $\mathscr{O}_X$-module.

We denote by $\mathrm{Mod}_{\mathrm{stand}}(\mathscr{D}_X, G)$ the full subcategory of $\mathrm{Mod}(\mathscr{D}_X, G)$ consisting of standard modules.

For $K \in \mathrm{K^b}\left(\mathrm{Mod}_{G_{\mathbb{R}},\mathrm{ctb}}(\mathbb{C}_{X^{\mathrm{an}}\times M})\right)$ and $\mathscr{M} \in \mathrm{K^b}\left(\mathrm{Mod}_{\mathrm{cc}}(\mathscr{D}_X, G)\right)$, we define the complex $\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}\left(\mathscr{M}\otimes K, \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}}\times M}\right)$ of Fréchet nuclear $G_{\mathbb{R}}$-modules in Lemma 5.3.2.

Lemma 5.4.4. (i) *Let* $\mathscr{N} \in \mathrm{Mod}_{\mathrm{stand}}(\mathscr{D}_X)$ *and* $L \in \mathrm{Mod}_{\mathrm{stand}}(\mathbb{C}_{X^{\mathrm{an}}\times M})$. *Then, we have*

$$\mathrm{Ext}^j_{\mathscr{D}_X}(\mathscr{N}\otimes L, \mathscr{E}^{(0,q)}_{X^{\mathrm{an}}\times M}) = 0$$

for any $j \neq 0$ *and any* q.

(ii) *We have isomorphisms in* $\mathrm{D^b}(\mathbb{C})$*:*

$$\begin{aligned}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}\otimes K, \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}}\times M}) &\xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}\otimes K, \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}}\times M})\\ &\simeq \mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}\otimes K, p^{-1}\mathscr{O}_{X^{\mathrm{an}}})\end{aligned}$$

for $\mathscr{M} \in \mathrm{K}^-(\mathrm{Mod}_{\mathrm{stand}}(\mathscr{D}_X))$ *and* $K \in \mathrm{K}^-(\mathrm{Mod}_{\mathrm{stand}}(\mathbb{C}_{X^{\mathrm{an}}\times M}))$. *Here* $p\colon X^{\mathrm{an}}\times M \to X^{\mathrm{an}}$ *is the projection.*

Proof. (i) Since $\mathscr{N}$ is a locally free $\mathscr{D}_X$-module and $\mathscr{E}^{(0,q)}_{X^{\mathrm{an}}\times M}$ is a soft sheaf, we have $\mathscr{E}xt^j_{\mathscr{D}_X}(\mathscr{N}, \mathscr{E}^{(0,q)}_{X^{\mathrm{an}}\times M}) = 0$ for $j \neq 0$. Hence, $\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{N}, \mathscr{E}^{(0,q)}_{X^{\mathrm{an}}\times M})$ is represented by $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{N}, \mathscr{E}^{(0,q)}_{X^{\mathrm{an}}\times M})$. Since $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{N}, \mathscr{E}^{(0,q)}_{X^{\mathrm{an}}\times M})$ has locally a $\mathscr{C}^\infty_{X^{\mathrm{an}}\times M}$-module structure, it is a soft sheaf. Hence, we obtain $\mathrm{Ext}^j_{\mathscr{D}_X}(L, \mathscr{H}om_{\mathscr{D}_X}(\mathscr{N}, \mathscr{E}^{(0,q)}_{X^{\mathrm{an}}\times M})) = 0$ for $j \neq 0$. Finally, we conclude that $\mathscr{H}om_{\mathbb{C}}(L, \mathscr{H}om_{\mathscr{D}_X}(\mathscr{N}, \mathscr{E}^{(0,q)}_{X^{\mathrm{an}}\times M}))$ represents

$$\mathbf{R}\mathscr{H}om_{\mathbb{C}}(L, \mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{N}, \mathscr{E}^{(0,q)}_{X^{\mathrm{an}}\times M})) \simeq \mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{N}\otimes L, \mathscr{E}^{(0,q)}_{X^{\mathrm{an}}\times M}).$$

(ii) follows immediately from (i). Q.E.D.

Proposition 5.4.5. *Let us assume that* $K \in \mathrm{K^b}\left(\mathrm{Mod}_{G_{\mathbb{R}},\mathrm{ctb}}(\mathbb{C}_{X^{\mathrm{an}}\times M})\right)$ *and* $\mathscr{M} \in \mathrm{K^b}\left(\mathrm{Mod}_{\mathrm{cc}}(\mathscr{D}_X, G)\right)$. *Then,*

$$\text{“}\varinjlim_{\mathscr{M}',K'}\text{”}\ \mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}\left(\mathscr{M}'\otimes K', \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}}\times M}\right)$$

is representable in $\mathrm{D^b}(\mathbf{FN}_{G_{\mathbb{R}}})$. *Here,* $\mathscr{M}' \to \mathscr{M}$ *ranges over the quasi-isomorphisms in* $\mathrm{K}^-\left(\mathrm{Mod}_{\mathrm{cc}}(\mathscr{D}_X, G)\right)$ *and* $K' \to K$ *ranges over the quasi-isomorphisms in* $\mathrm{K}^-\left(\mathrm{Mod}_{G_{\mathbb{R}},\mathrm{ctb}}(\mathbb{C}_{X^{\mathrm{an}}\times M})\right)$. *Moreover, forgetting the topology and the equivariance, it is isomorphic to* $\mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}\otimes K, p^{-1}\mathscr{O}_{X^{\mathrm{an}}})$. *Here* $p\colon X^{\mathrm{an}}\times M \to X^{\mathrm{an}}$ *is the projection.*

Proof. There exist $\mathscr{M}' \in \mathrm{K}^-(\mathrm{Mod}_{\mathrm{stand}}(\mathscr{D}_X, G))$ and a quasi-isomorphism $\mathscr{M}' \to \mathscr{M}$. Similarly by Lemma 5.4.2, there exist $K' \in \mathrm{K}^-(\mathrm{Mod}_{G_{\mathbb{R}},\mathrm{stand}}(\mathbb{C}_X))$ and a quasi-isomorphism $K' \to K$.

Then

$$\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}' \otimes K', \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}}\times M}) \to \mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M} \otimes K, p^{-1}\mathscr{O}_{X^{\mathrm{an}}}) \tag{5.4.2}$$

is an isomorphism in $\mathrm{D}(\mathbb{C})$ by the preceding lemma.

To complete the proof, it is enough to remark that, if a morphism in $\mathrm{K}(\mathbf{FN}_{G_{\mathbb{R}}})$ is a quasi-isomorphism in $\mathrm{K}(\mathrm{Mod}(\mathbb{C}))$ forgetting the topology and the equivariance, then it is a quasi-isomorphism in $\mathrm{K}(\mathbf{FN}_{G_{\mathbb{R}}})$. Q.E.D.

Definition 5.4.6. Assume that $G_{\mathbb{R}}$ acts freely on M. For $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_X, G)$ and $K \in \mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}},\mathrm{ctb}}(\mathbb{C}_{X^{\mathrm{an}}\times M})$, we define $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M} \otimes K, \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}}\times M})$ as the object

$$\text{“}\varinjlim_{\mathscr{M}',K'}\text{”}\ \mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}' \otimes K', \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}}\times M})$$

of $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$. Here, $\mathscr{M}'$ ranges over the set of objects of $\mathrm{K}^-\big(\mathrm{Mod}_{\mathrm{cc}}(\mathscr{D}_X, G)\big)$ isomorphic to $\mathscr{M}$ in $\mathrm{D}_{\mathrm{cc}}(\mathscr{D}_X, G)$, and K' ranges over the set of objects of $\mathrm{K}^-\big(\mathrm{Mod}_{G_{\mathbb{R}},\mathrm{ctb}}(\mathbb{C}_{X^{\mathrm{an}}\times M})\big)$ isomorphic to K in $\mathrm{D}\big(\mathrm{Mod}_{G_{\mathbb{R}}}(\mathbb{C}_{X^{\mathrm{an}}\times M})\big)$.

Let us take a sequence of $G_{\mathbb{R}}$-manifolds with a free $G_{\mathbb{R}}$-action:

$$V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow \cdots \tag{5.4.3}$$

as in (4.2.1).

Lemma 5.4.7. *For $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_X, G)$ and $K \in \mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}},\mathrm{ctb}}(\mathbb{C}_{X^{\mathrm{an}}})$, Then*

$$\tau^{\le a}\,\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M} \otimes p_k^{-1}K, \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}}\times V_k})$$

does not depend on $k \gg a \gg 0$ as an object of $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$. Here $p_k\colon X^{\mathrm{an}} \times V_k \to X^{\mathrm{an}}$ is the projection.

Proof. Forgetting the topology and the equivariance, we have

$$\begin{aligned}\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M} \otimes p_k^{-1}K, \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}}\times V_k}) &\simeq \mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M} \otimes p_k^{-1}K, p_k^{-1}\mathscr{O}_{X^{\mathrm{an}}})\\ &\simeq \mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M} \otimes K, \mathscr{O}_{X^{\mathrm{an}}}) \otimes \mathbf{R}\Gamma(V_k;\mathbb{C}),\end{aligned}$$

and

$$\tau^{\le a}\Big(\mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M} \otimes K, \mathscr{O}_{X^{\mathrm{an}}}) \otimes \mathbf{R}\Gamma(V_k;\mathbb{C})\Big) \simeq \mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M} \otimes K, \mathscr{O}_{X^{\mathrm{an}}})$$

for $k \gg a \gg 0$. Q.E.D.

Definition 5.4.8. We define

$$\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}\otimes K,\mathscr{O}_{X^{\mathrm{an}}})$$

as $\tau^{\le a}\,\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}\otimes p_k^{-1}K,\mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}}\times V_k})$ for $k\gg a\gg 0$.

Note that

$$\begin{aligned}&\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}\otimes K,\mathscr{O}_{X^{\mathrm{an}}})\\ &\qquad\simeq \text{“}\varprojlim_k\text{”}\,\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}\otimes p_k^{-1}K,\mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}}\times V_k}).\end{aligned}$$

Note that, forgetting the topology and the equivariance, $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}\otimes K,\mathscr{O}_{X^{\mathrm{an}}})$ is isomorphic to $\mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}\otimes K,\mathscr{O}_{X^{\mathrm{an}}})\in\mathrm{D}(\mathbb{C})$.

5.5 DFN version

In this subsection, let us define $\mathbf{R}\,\Gamma^{\mathrm{top}}_{\mathrm{c}}(X^{\mathrm{an}};K\otimes\Omega_{X^{\mathrm{an}}}\overset{\mathbf{L}}{\otimes}_{\mathscr{D}_X}\mathscr{M})$, which is the dual of $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}\otimes K,\mathscr{O}_{X^{\mathrm{an}}})$. Since the construction is similar to the one of $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}\otimes K,\mathscr{O}_{X^{\mathrm{an}}})$, we shall be brief.

Let us denote by $\mathscr{D}ist^{(p,q)}_{X^{\mathrm{an}}}$ the sheaf of (p,q)-forms on X^{an} with distributions as coefficients. Then for any open subset U of X^{an}, $\Gamma_c(U;\mathscr{D}ist^{(p,q)}_{X^{\mathrm{an}}})$ is endowed with a DFN-topology and it is the dual topological space of the FN-space $\mathscr{E}^{(\mathrm{d}_X-p,\mathrm{d}_X-q)}_{X^{\mathrm{an}}}(U)$. Hence for $\mathscr{M}\in\mathrm{K}^-(\mathrm{Mod}_{\mathrm{stand}}(\mathscr{D}_X))$ and $F\in\mathrm{K}^-(\mathrm{Mod}_{\mathrm{stand}}(\mathbb{C}_{X^{\mathrm{an}}}))$, $\Gamma_c(X^{\mathrm{an}};K\otimes\mathscr{D}ist^{(\mathrm{d}_X,\bullet)}_{X^{\mathrm{an}}}\otimes_{\mathscr{D}_X}\mathscr{M})[\mathrm{d}_X]$ is a complex of DFN-spaces, and it is the dual of $\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}\otimes K,\mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}}})$. We denote by $\Gamma^{\mathrm{top}}_c(X^{\mathrm{an}};K\otimes\mathscr{D}ist^{(\mathrm{d}_X,\bullet)}_{X^{\mathrm{an}}}\otimes_{\mathscr{D}_X}\mathscr{M})$ the complex of DFN-spaces $\Gamma_c(X^{\mathrm{an}};K\otimes\mathscr{D}ist^{(\mathrm{d}_X,\bullet)}_{X^{\mathrm{an}}}\otimes_{\mathscr{D}_X}\mathscr{M})$. If we forget the topology, it is isomorphic to $\mathbf{R}\,\Gamma_c(X^{\mathrm{an}};K\otimes\Omega_{X^{\mathrm{an}}}\otimes_{\mathscr{D}_X}\mathscr{M})\in\mathrm{D}^{\mathrm{b}}(\mathbb{C})$. Thus we have defined a functor:

$$\mathbf{R}\,\Gamma^{\mathrm{top}}_{\mathrm{c}}(X^{\mathrm{an}};\,\bullet\otimes\Omega_{X^{\mathrm{an}}}\overset{\mathbf{L}}{\otimes}_{\mathscr{D}_X}\bullet)\colon\ \mathrm{D}^{\mathrm{b}}_{\mathrm{ctb}}(\mathbb{C}_{X^{\mathrm{an}}})\times\mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_X)\to\mathrm{D}^{\mathrm{b}}(\mathbf{DFN}).$$

When X is a quasi-projective G-manifold, we can define its equivariant version

$$\mathbf{R}\,\Gamma^{\mathrm{top}}_{\mathrm{c}}(X^{\mathrm{an}};\,\bullet\otimes\Omega_{X^{\mathrm{an}}}\overset{\mathbf{L}}{\otimes}_{\mathscr{D}_X}\bullet)\colon\ \mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}},\mathrm{ctb}}(\mathbb{C}_{X^{\mathrm{an}}})\times\mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_X,G)\to\mathrm{D}^{\mathrm{b}}(\mathbf{DFN}_{G_{\mathbb{R}}}).$$

We have

(5.5.1) $$\mathbf{R}\,\Gamma^{\mathrm{top}}_{\mathrm{c}}(X^{\mathrm{an}};K\otimes\Omega_{X^{\mathrm{an}}}\overset{\mathbf{L}}{\otimes}_{\mathscr{D}_X}\mathscr{M})[\mathrm{d}_X]\cong\big(\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}\otimes K,\mathscr{O}_{X^{\mathrm{an}}})\big)^*.$$

Here $(\,\bullet\,)^*\colon\ \mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})^{\mathrm{op}}\xrightarrow{\sim}\mathrm{D}^{\mathrm{b}}(\mathbf{DFN}_{G_{\mathbb{R}}})$ is the functor induced by the duality.

If we forget the topology and the equivariance, $\mathbf{R}\,\Gamma^{\mathrm{top}}_{\mathrm{c}}(X^{\mathrm{an}};K\otimes\Omega_{X^{\mathrm{an}}}\otimes_{\mathscr{D}_X}\mathscr{M})$ is isomorphic to $\mathbf{R}\,\Gamma_c(X^{\mathrm{an}};K\otimes\Omega_{X^{\mathrm{an}}}\otimes_{\mathscr{D}_X}\mathscr{M})\in\mathrm{D}^{\mathrm{b}}(\mathbb{C})$.

5.6 Functorial Properties of $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}$

Statements

We shall study how $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}$ behaves under G-equivariant morphisms of G-manifolds. We shall keep the notations $G, G_{\mathbb{R}}$ as in § 5.1.

Let $f\colon X \to Y$ be a G-equivariant morphism of quasi-projective algebraic G-manifolds. Let $f^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ be the associated holomorphic map.

Theorem 5.6.1. (i) *Assume that f is smooth and projective. Then, there exists a canonical isomorphism in* $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$:

$$\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}\otimes(f^{\mathrm{an}})^{-1}L, \mathscr{O}_{X^{\mathrm{an}}}) \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_Y}(\mathbf{D}f_*\mathscr{M}\otimes L, \mathscr{O}_{Y^{\mathrm{an}}})\,[-\mathrm{d}_{X/Y}]$$

for $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X, G)$ *and* $L \in \mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}},\mathrm{ctb}}(\mathbb{C}_Y)$.

(ii) *Assume that f is smooth. Then, there exists a canonical isomorphism in* $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$:

$$\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_Y}(\mathscr{N}\otimes\mathbf{R}(f^{\mathrm{an}})_!K, \mathscr{O}_{Y^{\mathrm{an}}}) \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathbf{D}f^*\mathscr{N}\otimes K, \mathscr{O}_{X^{\mathrm{an}}})[2\mathrm{d}_{X/Y}]$$

for $\mathscr{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y, G)$ *and* $K \in \mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}},\mathrm{ctb}}(\mathbb{C}_X)$.

Preparation

Let us take a sequence $\{V_k\}$ as in (4.2.1).

Let $\mathscr{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_Y, G)$ and $L \in \mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}},\mathrm{ctb}}(\mathbb{C}_{Y^{\mathrm{an}}})$. Then, by the definition, we have

$$\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_Y}(\mathscr{N}\otimes(L\boxtimes\mathbb{C}_{V_k}), \mathscr{E}^{(0,\bullet)}_{Y^{\mathrm{an}}\times V_k}) = \text{``}\varinjlim_{\mathscr{N}',L'}\text{''}\,\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_Y}(\mathscr{N}'\otimes L', \mathscr{E}^{(0,\bullet)}_{Y^{\mathrm{an}}\times V_k}).$$

Here, $\mathscr{N}'$ ranges over the objects of $\mathrm{K}^-\big(\mathrm{Mod}_{\mathrm{cc}}(\mathscr{D}_Y, G)\big)$ isomorphic to $\mathscr{N}$ in $\mathrm{D}(\mathrm{Mod}(\mathscr{D}_Y, G))$, and L' ranges over the objects of $\mathrm{K}^-\big(\mathrm{Mod}_{G_{\mathbb{R}},\mathrm{ctb}}(\mathbb{C}_{Y^{\mathrm{an}}\times V_k})\big)$ isomorphic to $L\boxtimes\mathbb{C}_{V_k}$ in $\mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}},\mathrm{ctb}}(\mathbb{C}_{Y^{\mathrm{an}}\times V_k})$. Then the morphism

$$\mathscr{D}_{X\to Y}\underset{\mathscr{D}_Y}{\otimes}\mathscr{E}^{(0,\bullet)}_{Y^{\mathrm{an}}\times V_k} \to \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}}\times V_k}$$

induces morphisms in $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$:

$$\begin{aligned}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_Y}(\mathscr{N}'\otimes L', \mathscr{E}^{(0,\bullet)}_{Y^{\mathrm{an}}\times V_k}) &\to \mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(f^*\mathscr{N}'\otimes(f^{\mathrm{an}}\times\mathrm{id}_{V_k})^{-1}L', \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}}\times V_k})\\ &\to \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}\big(\mathbf{D}f^*\mathscr{N}\otimes((f^{\mathrm{an}})^{-1}L\boxtimes\mathbb{C}_{V_k}), \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}}\times V_k}\big).\end{aligned}$$

Thus we obtain a morphism

$$\begin{aligned}&\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_Y}(\mathscr{N}\otimes(L\boxtimes\mathbb{C}_{V_k}), \mathscr{E}^{(0,\bullet)}_{Y^{\mathrm{an}}\times V_k})\\ &\qquad\to \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathbf{D}f^*\mathscr{N}\otimes((f^{\mathrm{an}})^{-1}L\boxtimes\mathbb{C}_{V_k}), \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}}\times V_k})\end{aligned}$$

for $\mathscr{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_Y, G)$ and $L \in \mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}},\mathrm{ctb}}(\mathbb{C}_{Y_{\mathrm{an}}})$. Taking the projective limit with respect to k, we obtain

$$\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_Y}(\mathscr{N} \otimes L, \mathscr{O}_{Y^{\mathrm{an}}}) \to \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathbf{D}f^*\mathscr{N} \otimes (f^{\mathrm{an}})^{-1}L, \mathscr{O}_{X^{\mathrm{an}}}). \tag{5.6.1}$$

Here, f is arbitrary.

Proof of Theorem 5.6.1

Let us first prove (i). For $\mathscr{M}$ and L as in (i), we have morphisms

$$\begin{aligned}
&\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_Y}\big(\mathbf{D}f_*\mathscr{M} \otimes L[\mathrm{d}_{X/Y}], \mathscr{O}_{Y^{\mathrm{an}}}\big)\\
&\to \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}\big(\mathbf{D}f^*\mathbf{D}f_*\mathscr{M}[\mathrm{d}_{X/Y}] \otimes (f^{\mathrm{an}})^{-1}L, \mathscr{O}_{X^{\mathrm{an}}}\big)\\
&\to \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M} \otimes (f^{\mathrm{an}})^{-1}L, \mathscr{O}_{X^{\mathrm{an}}}).
\end{aligned}$$

Here, the first arrow is given by (5.6.1) and the last arrow is given by $\mathscr{M} \to \mathbf{D}f^*\mathbf{D}f_*\mathscr{M}[\mathrm{d}_{X/Y}]$ (see (3.8.4)).

We shall prove that the composition

$$\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_Y}\big(\mathbf{D}f_*\mathscr{M}[\mathrm{d}_{X/Y}] \otimes L, \mathscr{O}_{Y^{\mathrm{an}}}\big) \to \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M} \otimes (f^{\mathrm{an}})^{-1}L, \mathscr{O}_{X^{\mathrm{an}}})$$

is an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$.

In order to see this, it is enough to show that it is an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbb{C})$. Then the result follows from the result of D-modules:

$$\mathbf{R}\mathscr{H}om_{\mathscr{D}_Y}\big(\mathbf{D}f_*\mathscr{M}[\mathrm{d}_{X/Y}], \mathscr{O}_{Y^{\mathrm{an}}}\big) \cong \mathbf{R}(f^{\mathrm{an}})_*\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_{X^{\mathrm{an}}}).$$

The proof of (ii) is similar. Let $\mathscr{N}$ and K be as in (ii), then we have a sequence of morphisms

$$\begin{aligned}
&\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_Y}(\mathscr{N} \otimes \mathbf{R}f^{\mathrm{an}}{}_!K, \mathscr{O}_{Y^{\mathrm{an}}})\\
&\to \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathbf{D}f^*\mathscr{N} \otimes (f^{\mathrm{an}})^{-1}\mathbf{R}f^{\mathrm{an}}{}_!K, \mathscr{O}_{X^{\mathrm{an}}})\\
&\to \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathbf{D}f^*\mathscr{N} \otimes K, \mathscr{O}_{X^{\mathrm{an}}})[2\mathrm{d}_{X/Y}].
\end{aligned}$$

Here the last arrow is obtained by

$$K \to (f^{\mathrm{an}})^!\mathbf{R}(f^{\mathrm{an}})_!K \cong (f^{\mathrm{an}})^{-1}\mathbf{R}(f^{\mathrm{an}})_!K[2\mathrm{d}_{X/Y}].$$

The rest of arguments is similar to the proof of (i) by reducing it to the corresponding result in the D-module theory:

$$\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathbf{D}f^*\mathscr{N}, \mathscr{O}_{X^{\mathrm{an}}}) \simeq (f^{\mathrm{an}})^{-1}\,\mathbf{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{N}, \mathscr{O}_{Y^{\mathrm{an}}}).$$

5.7 Relation with the De Rham functor

Let X be an algebraic G-manifold. First assume that G acts freely on X. Let $p\colon X \to X/G$ be the projection. Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_X, G)$ and $K \in \mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}},\mathrm{ctb}}(\mathbb{C}_{X^{\mathrm{an}}})$. Let $\mathscr{L}$ be an object of $\mathrm{D}^{\mathrm{b}}_{G,\mathrm{hol}}(\mathscr{D}_X)$. Let $\mathscr{L}/G$ be the object of $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{X/G})$ corresponding to $\mathscr{L}$. Set $L = \mathrm{DR}_X(\mathscr{L}) \in \mathrm{D}^{\mathrm{b}}_{G^{\mathrm{an}},\,\mathbb{C}\text{-c}}(\mathbb{C}_{X^{\mathrm{an}}})$ (see Subsection 4.6). Then the corresponding object $L/G^{\mathrm{an}} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_{(X/G)^{\mathrm{an}}})$ is isomorphic to $\mathrm{DR}_{X/G}(\mathscr{L}/G)$.

Let us represent $\mathscr{M}$ by an object of $\mathrm{K}^{-}_{\mathrm{stand}}(\mathscr{D}_X, G)$ and $\mathscr{L}/G$ by an object $\widetilde{\mathscr{L}} \in \mathrm{K}^{-}(\mathrm{Mod}_{\mathrm{stand}}(\mathscr{D}_{X/G}))$. Then $\mathscr{L}$ is represented by $p^*\widetilde{\mathscr{L}}$. Since $L/G^{\mathrm{an}} \simeq \mathscr{H}om_{\mathscr{D}_{(X/G)^{\mathrm{an}}}}(\mathscr{D}_{(X/G)^{\mathrm{an}}} \otimes \bigwedge^{\bullet} \Theta_{(X/G)^{\mathrm{an}}}, \widetilde{\mathscr{L}}^{\mathrm{an}})$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{ctb}}(\mathbb{C}_{(X/G)^{\mathrm{an}}})$, there exist $F \in \mathrm{K}^{-}(\mathrm{Mod}_{\mathrm{stand}}(\mathbb{C}_{(X/G)^{\mathrm{an}}}))$ and a quasi-isomorphism

$$F \to \mathscr{H}om_{\mathscr{D}_{(X/G)^{\mathrm{an}}}}(\mathscr{D}_{(X/G)^{\mathrm{an}}} \otimes \bigwedge^{\bullet} \Theta_{(X/G)^{\mathrm{an}}}, \widetilde{\mathscr{L}}^{\mathrm{an}})$$

by Lemma 5.2.6. Thus we obtain a morphism of complexes of $\mathscr{D}_{(X/G)^{\mathrm{an}}}$-modules:

$$\mathscr{D}_{(X/G)^{\mathrm{an}}} \otimes \bigwedge^{\bullet} \Theta_{(X/G)^{\mathrm{an}}} \otimes F \to \widetilde{\mathscr{L}}^{\mathrm{an}}. \tag{5.7.1}$$

Let M be a differentiable manifold with a free $G_{\mathbb{R}}$-action. Then for any $E \in K^{-}_{G_{\mathbb{R}},\mathrm{stand}}(\mathbb{C}_{X^{\mathrm{an}} \times M})$, the morphism (5.7.1) induces morphisms

$$\begin{aligned}
&\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{L} \otimes E, \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}} \times M}) \\
&\quad\simeq \mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_{X^{\mathrm{an}}}}(\mathscr{M}^{\mathrm{an}} \otimes_{(p^{\mathrm{an}})^{-1}\mathscr{O}_{(X/G)^{\mathrm{an}}}} (p^{\mathrm{an}})^{-1}\widetilde{\mathscr{L}}^{\mathrm{an}} \otimes_{\mathbb{C}} E, \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}} \times M}) \\
&\quad\to \mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_{X^{\mathrm{an}}}}(\mathscr{M}^{\mathrm{an}} \underset{(p^{\mathrm{an}})^{-1}\mathscr{O}_{(X/G)^{\mathrm{an}}}}{\otimes} (p^{\mathrm{an}})^{-1}(\mathscr{D}_{(X/G)^{\mathrm{an}}} \otimes \bigwedge^{\bullet} \Theta_{(X/G)^{\mathrm{an}}} \otimes F) \otimes_{\mathbb{C}} E, \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}} \times M}) \\
&\quad\simeq \mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}\Big(\big(\mathscr{M} \underset{p^{-1}\mathscr{O}_{X/G}}{\otimes} p^{-1}(\mathscr{D}_{X/G} \otimes \bigwedge^{\bullet} \Theta_{X/G})\big) \otimes \big((p^{\mathrm{an}})^{-1}F \otimes E\big), \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}} \times M}\Big).
\end{aligned}$$

On the other hand, we have an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$:

$$\begin{aligned}
&\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M} \otimes_{\mathbb{C}} (p^{-1}F \otimes_{\mathbb{C}} E), \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}} \times M}) \\
&\quad\simeq \mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}\Big(\big(\mathscr{M} \underset{p^{-1}\mathscr{O}_{X/G}}{\otimes} p^{-1}(\mathscr{D}_{X/G} \otimes \bigwedge^{\bullet} \Theta_{X/G})\big) \otimes \big((p^{\mathrm{an}})^{-1}F \otimes E\big), \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}} \times M}\Big),
\end{aligned}$$

because $\mathscr{M} \otimes_{p^{-1}\mathscr{O}_{X/G}} p^{-1}(\mathscr{D}_{X/G} \otimes \bigwedge^{\bullet} \Theta_{X/G}) \to \mathscr{M}$ is a quasi-isomorphism, and $\mathscr{M} \otimes_{p^{-1}\mathscr{O}_{X/G}} p^{-1}(\mathscr{D}_{X/G} \otimes \bigwedge^{\bullet} \Theta_{X/G})$ and $(p^{\mathrm{an}})^{-1}F \otimes E$ are standard complexes. Thus we obtain a morphism in $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$

$$\begin{aligned}
&\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}((\mathscr{M} \overset{\mathbf{D}}{\otimes} \mathscr{L}) \otimes E, \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}} \times M}) \\
&\qquad\to \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}\big(\mathscr{M} \otimes \big(\mathrm{DR}_X(\mathscr{L}) \otimes E\big), \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}} \times M}\big)
\end{aligned} \tag{5.7.2}$$

for $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_X, G)$, $\mathscr{L} \in \mathrm{D}^{\mathrm{b}}_{G,\mathrm{hol}}(\mathscr{D}_X)$ and $E \in \mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}},\mathrm{ctb}}(\mathbb{C}_{X^{\mathrm{an}} \times M})$.

Let us take a sequence $\{V_k\}$ as in (4.2.1). Let $K \in \mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}},\mathrm{ctb}}(\mathbb{C}_{X^{\mathrm{an}}})$. Setting $M = V_k$, $E = K \boxtimes \mathbb{C}_{V_k}$ in (5.7.2), and then taking the projective limit with respect to k, we obtain

$$\begin{aligned}(5.7.3)\qquad &\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}((\mathscr{M} \overset{\mathbf{D}}{\otimes} \mathscr{L}) \otimes K, \mathscr{O}_{X^{\mathrm{an}}}) \\ &\qquad \to \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M} \otimes (\mathrm{DR}_X(\mathscr{L}) \otimes K), \mathscr{O}_{X^{\mathrm{an}}}).\end{aligned}$$

When the action of G is not free, we can also define the morphism (5.7.3) replacing X with $V_k \times X$, and then taking the projective limit with respect to k. Here $\{V_k\}$ is as in (4.5.1). Thus we obtain the following lemma.

Lemma 5.7.1. *Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_X, G)$ and $K \in \mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}},\mathrm{ctb}}(\mathbb{C}_{X^{\mathrm{an}}})$. Then for any $\mathscr{L} \in \mathrm{D}^{\mathrm{b}}_{G,\mathrm{hol}}(\mathscr{D}_X)$, there exists a canonical morphism in $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$:*

$$\begin{aligned}&\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}\big((\mathscr{M} \overset{\mathbf{D}}{\otimes} \mathscr{L}) \otimes K, \mathscr{O}_{X^{\mathrm{an}}}\big) \\ &\qquad \to \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M} \otimes (\mathrm{DR}_X(\mathscr{L}) \otimes K), \mathscr{O}_{X^{\mathrm{an}}}).\end{aligned}$$

For a coherent $\mathscr{D}_X$-module $\mathscr{N}$, let us denote by $\mathrm{Ch}(\mathscr{N}) \subset T^*X$ the characteristic variety of $\mathscr{N}$ (see [16]). For a submanifold Y of X, we denote by $T^*_Y X$ the conormal bundle to Y. In particular, $T^*_X X$ is nothing but the zero section of the cotangent bundle T^*X.

Theorem 5.7.2. *Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X, G)$, $\mathscr{L} \in \mathrm{D}^{\mathrm{b}}_{G,\mathrm{hol}}(\mathscr{D}_X)$. Assume that $\mathscr{M}$ and $\mathscr{L}$ are non-characteristic, i.e.*

$$(5.7.4)\qquad \mathrm{Ch}(\mathscr{M}) \cap \mathrm{Ch}(\mathscr{L}) \subset T^*_X X.$$

Then, for any $K \in \mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}},\mathrm{ctb}}(\mathbb{C}_{X^{\mathrm{an}}})$, we have an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$:

$$\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}((\mathscr{M} \overset{\mathbf{D}}{\otimes} \mathscr{L}) \otimes K, \mathscr{O}_{X^{\mathrm{an}}}) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M} \otimes (\mathrm{DR}_X(\mathscr{L}) \otimes K), \mathscr{O}_{X^{\mathrm{an}}}).$$

Proof. It is enough to show the result forgetting the topology and the equivariance. Then this follows from the well-known result

$$\begin{aligned}&\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M} \overset{\mathbf{D}}{\otimes} \mathscr{L}, \mathscr{O}_{X^{\mathrm{an}}}) \\ &\quad \xleftarrow{\sim} \mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_{X^{\mathrm{an}}}) \otimes_{\mathbb{C}} \mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{L}, \mathscr{O}_{X^{\mathrm{an}}}) \\ &\quad \xrightarrow{\sim} \mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_{X^{\mathrm{an}}}) \otimes_{\mathbb{C}} \mathbf{R}\mathscr{H}om_{\mathbb{C}_{X^{\mathrm{an}}}}(\mathrm{DR}_X(\mathscr{L}), \mathbb{C}_{X^{\mathrm{an}}}) \\ &\quad \xrightarrow{\sim} \mathbf{R}\mathscr{H}om_{\mathbb{C}}\big(\mathrm{DR}_X(\mathscr{L}), \mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_{X^{\mathrm{an}}})\big) \\ &\quad \xrightarrow{\sim} \mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M} \otimes \mathrm{DR}_X(\mathscr{L}), \mathscr{O}_{X^{\mathrm{an}}}).\end{aligned}$$

Here, the first and the third isomorphisms need the non-characteristic condition (see [18]). Q.E.D.

6 Whitney functor

6.1 Whitney functor

In § 5, we defined $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}\otimes K, \mathscr{O}_{X^{\mathrm{an}}})$ as an object of $\mathrm{D^b}(\mathbf{FN}_{G_\mathbb{R}})$. In this section, we introduce its C^∞-version. We use the Whitney functor developed in Kashiwara-Schapira [20].

Theorem 6.1.1 ([20]). *Let M be a real analytic manifold. Then there exists an exact functor*

$$\bullet \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_M : \mathrm{Mod}_{\mathbb{R}\text{-}c}(\mathbb{C}_M) \to \mathrm{Mod}(\mathscr{D}_M).$$

Moreover, for any $F \in \mathrm{Mod}_{\mathbb{R}\text{-}c}(\mathbb{C}_M)$, $\Gamma(M; F\overset{\mathrm{w}}{\otimes}\mathscr{C}^\infty_M)$ is endowed with a Fréchet nuclear topology, and

$$\Gamma(M; \bullet \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_M) : \mathrm{Mod}_{\mathbb{R}\text{-}c}(\mathbb{C}_M) \to \mathbf{FN}$$

is an exact functor.

Remark 6.1.2. (i) For a subanalytic open subset U, $\Gamma(M; \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_M)$ is the set of C^∞-functions f defined on M such that all the derivatives of f vanish at any point outside U. Its topology is the induced topology of $\mathrm{C}^\infty(M)$.

(ii) For a closed real analytic submanifold N of M, the sheaf $\mathbb{C}_N \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_M$ is isomorphic to the completion $\varprojlim_n \mathscr{C}^\infty_M / I^n$, where I is the ideal of $\mathscr{C}^\infty_M$ consisting of C^∞-functions vanishing on N.

(iii) In this paper, the Whitney functor is used only for the purpose of the construction of the morphism in Proposition 6.3.2. However, with this functor and *Thom* (see [20]), we can construct the C^∞globalization and the distribution globalization of Harish-Chandra modules.

Hence we can define the functor

$$\begin{aligned}\bullet \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_M &: \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_M) \to \mathrm{D^b}(\mathscr{D}_M),\\ \mathbf{R}\,\Gamma^{\mathrm{top}}(M; \bullet \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_M) &: \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_M) \to \mathrm{D^b}(\mathbf{FN}).\end{aligned}$$

For any $F \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$, we have a morphism

$$F \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_M \to \mathscr{H}om_{\mathbb{C}}(\mathscr{H}om_{\mathbb{C}}(F, \mathbb{C}_M), \mathscr{C}^\infty_M),$$

which induces a morphism in $\mathrm{D^b}(\mathbf{FN})$

$$\mathbf{R}\,\Gamma^{\mathrm{top}}(M; F \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_M) \longrightarrow \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathbb{C}}(F^*, \mathscr{C}^\infty_M) \tag{6.1.1}$$

for $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$, where $F^* := \mathbf{R}\mathscr{H}om(F, \mathbb{C}_M)$.

If a real Lie group H acts on M, we can define

$$\Gamma^{\mathrm{top}}(M; \bullet \overset{\mathrm{w}}{\otimes} \mathscr{C}^{\infty}_M) : \mathrm{Mod}_{H,\mathbb{R}\text{-c}}(\mathbb{C}_M) \to \mathbf{FN}_H.$$

Note that, for a complex manifold X and $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$, $F \overset{\mathrm{w}}{\otimes} \mathscr{O}_X \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)$ is defined as $F \overset{\mathrm{w}}{\otimes} \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}}}$.

6.2 The functor $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\bullet, \bullet \overset{\mathrm{w}}{\otimes} \mathscr{O}_{X^{\mathrm{an}}})$

Let X, G, $G_{\mathbb{R}}$ be as in § 5.1.

For $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_X)$, $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$, let us define $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}, F \overset{\mathrm{w}}{\otimes} \mathscr{O}_{X^{\mathrm{an}}})$ as an object of $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$, which is isomorphic to $\mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, F \overset{\mathrm{w}}{\otimes} \mathscr{O}_{X^{\mathrm{an}}})$ forgetting the topology and the equivariance. The construction is similar to the one in § 5.

Let M be a $G_{\mathbb{R}}$-manifold with a free $G_{\mathbb{R}}$-action. For $\mathscr{M} \in \mathrm{Mod}_{\mathrm{cc}}(\mathscr{D}_X, G)$ and $F \in \mathrm{Mod}_{G_{\mathbb{R}},\mathbb{R}\text{-c}}(\mathbb{C}_{X^{\mathrm{an}} \times M})$, we endow $\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, F \overset{\mathrm{w}}{\otimes} \mathscr{E}^{(0,p)}_{X^{\mathrm{an}} \times M})$ with a Fréchet nuclear $G_{\mathbb{R}}$-module structure as in Lemma 5.3.1. Hence, for $\mathscr{M} \in \mathrm{K}^-(\mathrm{Mod}_{\mathrm{cc}}(\mathscr{D}_X, G))$ and $F \in \mathrm{K}^-(\mathrm{Mod}_{G_{\mathbb{R}},\mathbb{R}\text{-c}}(\mathbb{C}_{X^{\mathrm{an}} \times M}))$, we can regard the complex $\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, F \overset{\mathrm{w}}{\otimes} \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}} \times M})$ as an object of $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$. Taking the inductive limit with respect to $\mathscr{M}$, we obtain $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}, F \overset{\mathrm{w}}{\otimes} \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}} \times M}) \in \mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$ for $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_X, G)$ and $F \in \mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}},\mathbb{R}\text{-c}}(\mathbb{C}_{X^{\mathrm{an}} \times M})$.

Let us take a sequence $\{V_k\}$ as in (4.2.1). Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_X, G)$ and $F \in \mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}},\mathbb{R}\text{-c}}(\mathbb{C}_{X^{\mathrm{an}}})$. Forgetting the topology and the equivariance, we have

$$\begin{aligned}&\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}, (F \boxtimes \mathbb{C}_{V_k}) \overset{\mathrm{w}}{\otimes} \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}} \times V_k})\\ &\qquad \simeq \mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, F \overset{\mathrm{w}}{\otimes} \mathscr{O}_{X^{\mathrm{an}}}) \otimes \mathbf{R}\Gamma(V_k; \mathbb{C}) \quad \text{in } \mathrm{D}^{\mathrm{b}}(\mathbb{C}).\end{aligned}$$

As in Definition 5.4.8, we define

$$\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}, F \overset{\mathrm{w}}{\otimes} \mathscr{O}_{X^{\mathrm{an}}}) = \tau^{\leq a}\, \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}, (F \boxtimes \mathbb{C}_{V_k}) \overset{\mathrm{w}}{\otimes} \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}} \times V_k})$$

for $k \gg a \gg 0$.

Thus we have defined the functor

$$\begin{aligned}&\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\bullet, \bullet \overset{\mathrm{w}}{\otimes} \mathscr{O}_{X^{\mathrm{an}}})\\ &\qquad : \mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_X, G)^{\mathrm{op}} \times \mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}},\mathbb{R}\text{-c}}(\mathbb{C}_{X^{\mathrm{an}}}) \to \mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}}).\end{aligned} \tag{6.2.1}$$

By (6.1.1), we have a morphism

$$\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}, F \overset{\mathrm{w}}{\otimes} \mathscr{O}_{X^{\mathrm{an}}}) \to \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M} \otimes F^*, \mathscr{O}_{X^{\mathrm{an}}}) \tag{6.2.2}$$

in $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$.

6.3 Elliptic case

Let $X_{\mathbb{R}}$ be a closed real analytic submanifold of X^{an} invariant by $G_{\mathbb{R}}$. Let $i\colon X_{\mathbb{R}} \hookrightarrow X^{\mathrm{an}}$ be the inclusion.

Assume that $T_xX \cong \mathbb{C} \otimes_{\mathbb{R}} T_xX_{\mathbb{R}}$ for any $x \in X_{\mathbb{R}}$.

In Lemma 5.3.2, we define $\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}^{\infty}_{X_{\mathbb{R}}}) \in \mathbf{FN}_{G_{\mathbb{R}}}$ for a countably coherent quasi-G-equivariant $\mathscr{D}_X$-module $\mathscr{M}$. It is right derivable and we can define the functor

$$\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\,\bullet\,, \mathscr{C}^{\infty}_{X_{\mathbb{R}}}) : \mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_X, G)^{\mathrm{op}} \to \mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}}).$$

Proposition 6.3.1. *For $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_X, G)$, we have*

$$\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}^{\infty}_{X_{\mathbb{R}}}) \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}, i_*\mathbb{C}_{X_{\mathbb{R}}} \overset{\mathrm{w}}{\otimes} \mathscr{O}_{X^{\mathrm{an}}}).$$

Proof. Let $\{V_k\}$ be as in the preceding section. The restriction map

$$(i_*\mathbb{C}_{X_{\mathbb{R}}} \boxtimes \mathbb{C}_{V_k}) \overset{\mathrm{w}}{\otimes} \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}} \times V_k} \to \mathscr{E}^{(0,\bullet)}_{X_{\mathbb{R}} \times V_k}$$

induces $\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}, (i_*\mathbb{C}_{X_{\mathbb{R}}} \boxtimes \mathbb{C}_{V_k}) \overset{\mathrm{w}}{\otimes} \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}}}) \to \mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{E}^{(0,\bullet)}_{X_{\mathbb{R}} \times V_k})$ in $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$. It induces a morphism

$$\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}, (i_*\mathbb{C}_{X_{\mathbb{R}}} \boxtimes \mathbb{C}_{V_k}) \overset{\mathrm{w}}{\otimes} \mathscr{E}^{(0,\bullet)}_{X^{\mathrm{an}} \times V_k}) \to \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{E}^{(0,\bullet)}_{X_{\mathbb{R}} \times V_k}).$$

Taking the projective limit with respect to k, we obtain

$$\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}, i_*\mathbb{C}_{X_{\mathbb{R}}} \overset{\mathrm{w}}{\otimes} \mathscr{O}_{X^{\mathrm{an}}}) \to \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}^{\infty}_{X_{\mathbb{R}}}).$$

Forgetting the topology and the equivariance, it is an isomorphism since $i_*\mathbb{C}_{X_{\mathbb{R}}} \overset{\mathrm{w}}{\otimes} \mathscr{O}_{X^{\mathrm{an}}} \simeq \mathscr{C}^{\infty}_{X_{\mathbb{R}}}$ (see [20]). Q.E.D.

Proposition 6.3.2. *There exists a canonical morphism in* $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$:

$$\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}^{\infty}_{X_{\mathbb{R}}}) \longrightarrow \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M} \otimes i_*i^!\mathbb{C}_{X^{\mathrm{an}}}, \mathscr{O}_{X^{\mathrm{an}}})$$

for $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_X, G)$.

Proof. This follows from the preceding proposition, $(i_*\mathbb{C}_{X_{\mathbb{R}}})^* \simeq i_*i^!\mathbb{C}_{X^{\mathrm{an}}}$ and (6.2.2). Q.E.D.

Proposition 6.3.3. *Let us assume that $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X, G)$ is elliptic i.e. $\mathrm{Ch}(\mathscr{M}) \cap T^*_{X_{\mathbb{R}}}X \subset T^*_XX$ (cf. e.g. [16]). Then we have*

$$\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}^{\infty}_{X_{\mathbb{R}}}) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M} \otimes i_*i^!\mathbb{C}_{X^{\mathrm{an}}}, \mathscr{O}_{X^{\mathrm{an}}})$$

in $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$.

Proof. Let $\mathscr{B}_{X_\mathbb{R}} = \mathbf{R}\mathscr{H}om_\mathbb{C}(i_* i^! \mathbb{C}_{X^{\mathrm{an}}}, \mathscr{O}_{X^{\mathrm{an}}})$ be the sheaf of hyperfunctions on $X_\mathbb{R}$. Forgetting the topology and the equivariance, we have isomorphisms:

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M} \otimes i_* i^! \mathbb{C}_{X^{\mathrm{an}}}, \mathscr{O}_{X^{\mathrm{an}}}) &\simeq \mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_{X_\mathbb{R}}) \\ &\simeq \mathbf{R}\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}^\infty_{X_\mathbb{R}}).\end{aligned}$$

Here, the last isomorphism follows from the ellipticity of $\mathscr{M}$. Hence we obtain the desired result. Q.E.D.

7 Twisted Sheaves

7.1 Twisting data

If we deal with the non-integral infinitesimal character case in the representation theory by a geometric method, we need to *twist* sheaves. In this note, we shall not go into a systematic study of twisted sheaves, but introduce it here in an ad hoc manner by using the notion of twisting data. (See [15] for more details.)

A *twisting data* τ (for twisting sheaves) over a topological space X is a triple $(X_0 \xrightarrow{\pi} X, L, m)$. Here $\pi \colon X_0 \to X$ is a continuous map admitting a section locally on X, L is an invertible $\mathbb{C}_{X_0 \times_X X_0}$-module and m is an isomorphism

$$m \colon p_{12}^{-1} L \otimes p_{23}^{-1} L \xrightarrow{\sim} p_{23}^{-1} L \quad \text{on } X_2.$$

Here and hereafter, we denote by X_n the fiber product of (n+1) copies of X_0 over X, by p_i $(i = 1, 2)$ the i-th projection from X_1 to X_0, by p_{ij} $(i, j = 1, 2, 3)$ the (i, j)-th projection from X_2 to X_1, and so on. We assume that the isomorphism m satisfies the associative law: the following diagram of morphisms of sheaves on X_3 is commutative.

$$\tag{7.1.1} \begin{array}{ccc}
p_{12}^{-1}L \otimes p_{23}^{-1}L \otimes p_{34}^{-1}L & = \!= \!= & p_{12}^{-1}L \otimes p_{234}^{-1}\big(p_{12}^{-1}L \otimes p_{23}^{-1}L\big) \\
\| & & \downarrow m \\
p_{123}^{-1}\big(p_{12}^{-1}L \otimes p_{23}^{-1}L\big) \otimes p_{34}^{-1}L & & p_{12}^{-1}L \otimes p_{234}^{-1}p_{13}^{-1}L \\
m \downarrow & & \| \\
p_{123}^{-1}p_{13}^{-1}L \otimes p_{34}^{-1}L & & p_{12}^{-1}L \otimes p_{24}^{-1}L \\
\| & & \| \\
p_{13}^{-1}L \otimes p_{34}^{-1}L & & p_{124}^{-1}\big(p_{12}^{-1}L \otimes p_{23}^{-1}L\big) \\
\| & & m \downarrow \\
p_{134}^{-1}\big(p_{12}^{-1}L \otimes p_{23}^{-1}L\big) & & p_{124}^{-1}p_{13}^{-1}L \\
m \downarrow & & \| \\
p_{134}^{-1}p_{13}^{-1}L & = \!= \!= & p_{14}^{-1}L\,.
\end{array}$$

In other words, for $(x_1, x_2, x_3) \in X_0 \times_X X_0 \times_X X_0$, an isomorphism

$$m(x_1, x_2, x_3)\colon L_{(x_1,x_2)} \otimes L_{(x_2,x_3)} \xrightarrow{\sim} L_{(x_1,x_3)}$$

is given in a locally constant manner in (x_1, x_2, x_3) such that the diagram

$$\begin{array}{ccccc} & & L_{(x_1,x_2)} \otimes L_{(x_2,x_3)} \otimes L_{(x_3,x_4)} & & \\ & \swarrow^{m(x_1,x_2,x_3)\otimes L_{(x_3,x_4)}} & & \searrow^{L_{(x_1,x_2)}\otimes m(x_2,x_3,x_4)} & \\ L_{(x_1,x_3)} \otimes L_{(x_3,x_4)} & & & & L_{(x_1,x_2)} \otimes L_{(x_2,x_4)} \\ & \searrow_{m(x_1,x_3,x_4)} & & \swarrow_{m(x_1,x_2,x_4)} & \\ & & L_{(x_1,x_4)} & & \end{array}$$

is commutative for $(x_1,\ x_2,\ x_3,\ x_4) \in X_0 \times_X X_0 \times_X X_0 \times_X X_0$.

In particular, we have $i^{-1}L \cong \mathbb{C}_{X_0}$, where $i\colon X_0 \hookrightarrow X_1$ is the diagonal embedding. Indeed, for $x' \in X_0$, $m(x', x', x')$ gives $L_{(x',x')} \otimes L_{(x',x')} \xrightarrow{\sim} L_{(x',x')}$ and hence an isomorphism $L_{(x',x')} \xrightarrow{\sim} \mathbb{C}$.

7.2 Twisted Sheaf

Let $\tau = (X_0 \xrightarrow{\pi} X, L, m)$ be a twisting data on X. A *twisted sheaf* F on X with twist τ (or simply τ-twisted sheaf) is a sheaf F on X_0 equipped with an isomorphism $\beta\colon L \otimes p_2^{-1}F \xrightarrow{\sim} p_1^{-1}F$ such that we have a commutative diagram on X_2

$$\begin{array}{ccc} p_{12}^{-1}L \otimes p_{23}^{-1}L \otimes p_3^{-1}F & = & p_{12}^{-1}L \otimes p_{23}^{-1}(L \otimes p_2^{-1}F) \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle \beta} \\ p_{13}^{-1}L \otimes p_3^{-1}F & & p_{12}^{-1}L \otimes p_{23}^{-1}p_1^{-1}F \\ \| & & \| \\ p_{13}^{-1}(L \otimes p_2^{-1}F) & & p_{12}^{-1}(L \otimes p_2^{-1}F) \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta} \\ p_{13}^{-1}p_1^{-1}F & = & p_{12}^{-1}p_1^{-1}F. \end{array}$$

In particular, F is locally constant on each fiber of π. We can similarly define a twisted sheaf on an open subset U of X. Let $\mathrm{Mod}_\tau(\mathbb{C}_U)$ denote the category of τ-twisted sheaves on U. Then $\mathfrak{Mod}_\tau(\mathbb{C}_X)\colon U \mapsto \mathrm{Mod}_\tau(\mathbb{C}_U)$ is a stack (a sheaf of categories) on X (see e.g. [19]).

If $\pi\colon X_0 \to X$ admits a section $s\colon X \to X_0$, then the category $\mathrm{Mod}_\tau(\mathbb{C}_X)$ is equivalent to the category $\mathrm{Mod}(\mathbb{C}_X)$ of sheaves on X. Indeed, the functor $\mathrm{Mod}_\tau(\mathbb{C}_X) \to \mathrm{Mod}(\mathbb{C}_X)$ is given by $F \mapsto s^{-1}F$ and the quasi-inverse is given by $G \mapsto \tilde{s}^{-1}L \otimes \pi^{-1}G$, where $\tilde{s}$ is the map $x' \mapsto \big(x', s\pi(x')\big)$ from X_0 to X_1. Hence the stack $\mathfrak{Mod}_\tau(\mathbb{C}_X)$ is locally equivalent to the stack $\mathfrak{Mod}(\mathbb{C}_X)$ of sheaves on X. Conversely, a stack locally equivalent to the stack $\mathfrak{Mod}(\mathbb{C}_X)$ is equivalent to $\mathfrak{Mod}_\tau(\mathbb{C}_X)$ for some twisting data τ (see [15]).

Let tr be the twisting data $(X \xrightarrow{\mathrm{id}} X, \mathbb{C}_X)$. Then $\mathrm{Mod}_{\mathrm{tr}}(\mathbb{C}_X)$ is equivalent to $\mathrm{Mod}(\mathbb{C}_X)$.

For a twisting data τ on X, we denote by $\mathrm{D}^{\mathrm{b}}_\tau(\mathbb{C}_X)$ the bounded derived category $\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}_\tau(\mathbb{C}_X))$.

7.3 Morphism of Twisting Data

Let $\tau = (X_0 \xrightarrow{\pi} X, L, m)$ and $\tau' = (X'_0 \xrightarrow{\pi'} X, L', m')$ be two twisting data. A morphism from τ to τ' is a pair $u = (f, \varphi)$ of a map $f\colon X_0 \to X'_0$ over X and an isomorphism $\varphi\colon L \xrightarrow{\sim} {f_1}^{-1}L'$ compatible with m and m'. Here f_1 is the map $f \underset{X}{\times} f\colon X_0 \underset{X}{\times} X_0 \to X'_0 \underset{X}{\times} X'_0$. One can easily see that a morphism $u\colon \tau \to \tau'$ gives an equivalence of categories $u^*\colon \mathrm{Mod}_{\tau'}(\mathbb{C}_X) \xrightarrow{\sim} \mathrm{Mod}_\tau(\mathbb{C}_X)$ by $F \mapsto f^{-1}F$. Hence we say that twisting data τ and τ' are equivalent in this case.

Let us discuss briefly what happens if there are two morphisms $u = (f, \varphi)$ and $u' = (f', \varphi')$ from τ to τ'. Let $g\colon X_0 \to X'_0 \underset{X}{\times} X'_0$ be the map $x' \mapsto \big(f(x'), f'(x')\big)$. Then an invertible sheaf $K' = g^{-1}L'$ on X_0 satisfies $p_1^{-1}K' \cong p_2^{-1}K'$, and there exists an invertible sheaf K on X such that

$$\pi^{-1}K \cong g^{-1}L'.$$

Then, $\bullet \otimes K$ gives an equivalence from $\mathrm{Mod}_\tau(\mathbb{C}_X)$ to itself, and the diagram

$$\begin{array}{ccc} \mathrm{Mod}_{\tau'}(\mathbb{C}_X) & \xrightarrow{u'^*} & \mathrm{Mod}_\tau(\mathbb{C}_X) \\ & \searrow^{u^*} & \downarrow{\scriptstyle \bullet\otimes K} \\ & & \mathrm{Mod}_\tau(\mathbb{C}_X) \end{array}$$

quasi-commutes (i.e. $(\bullet \otimes K) \circ u'^*$ and u^* are isomorphic).

7.4 Tensor Product

Let $\tau' = (X'_0 \to X, L', m')$ and $\tau'' = (X''_0 \to X, L'', m'')$ be two twisting data on X. Then their tensor product $\tau' \otimes \tau''$ is defined as follows: $\tau' \otimes \tau'' = (X_0 \to X, L, m)$, where $X_0 = X'_0 \underset{X}{\times} X''_0$, $L = q_1^{-1}L' \otimes q_2^{-1}L''$ with the projections $q_1\colon X_1 \simeq X'_1 \times X''_1 \to X'_1$ and $q_2\colon X_1 \to X''_1$, and $m = m' \otimes m''$. Then we can define the bi-functor

(7.4.1) $$\bullet\otimes\bullet\colon \mathfrak{Mod}_{\tau'}(\mathbb{C}_X)\times\mathfrak{Mod}_{\tau''}(\mathbb{C}_X)\to\mathfrak{Mod}_{\tau'\otimes\tau''}(\mathbb{C}_X)$$

by $(F',F'')\mapsto r_1^{-1}F'\otimes r_2^{-1}F''$, where $r_1\colon X_0\to X_0'$ and $r_2\colon X_0\to X_0''$ are the projections.

For a twisting data $\tau=(X_0\to X,L,m)$, let $\tau^{\otimes-1}$ be the twisting data $\tau^{\otimes-1}:=(X_0\to X,L^{\otimes-1},m^{\otimes-1})$. Note that $L^{\otimes-1}\simeq r^{-1}L$, where $r\colon X_1\to X_1$ is the map $(x',x'')\mapsto(x'',x')$. Then we can easily see that $\tau\otimes\tau^{\otimes-1}$ is canonically equivalent to the trivial twisting data. Hence we obtain

$$\bullet\otimes\bullet:\mathfrak{Mod}_{\tau}(\mathbb{C}_X)\times\mathfrak{Mod}_{\tau^{\otimes-1}}(\mathbb{C}_X)\to\mathfrak{Mod}(\mathbb{C}_X).$$

For twisting data τ and τ', we have a functor

(7.4.2) $$\mathscr{H}om(\bullet,\bullet):\mathfrak{Mod}_{\tau}(\mathbb{C}_X)^{\mathrm{op}}\times\mathfrak{Mod}_{\tau'}(\mathbb{C}_X)\to\mathfrak{Mod}_{\tau^{\otimes-1}\otimes\tau'}(\mathbb{C}_X).$$

They induce functors:

(7.4.3) $$\begin{aligned}\bullet\otimes\bullet\ &:\ \mathrm{D}^{\mathrm{b}}_{\tau}(\mathbb{C}_Y)\otimes\mathrm{D}^{\mathrm{b}}_{\tau'}(\mathbb{C}_Y)\to\mathrm{D}^{\mathrm{b}}_{\tau\otimes\tau'}(\mathbb{C}_Y)\quad\text{and}\\ \mathbf{R}\mathscr{H}om(\bullet,\bullet)\ &:\ \mathrm{D}^{\mathrm{b}}_{\tau}(\mathbb{C}_Y)^{\mathrm{op}}\times\mathrm{D}^{\mathrm{b}}_{\tau'}(\mathbb{C}_Y)\to\mathrm{D}^{\mathrm{b}}_{\tau^{\otimes-1}\otimes\tau'}(\mathbb{C}_Y).\end{aligned}$$

7.5 Inverse and Direct Images

Let $f\colon X\to Y$ be a continuous map and let $\tau=(Y_0\xrightarrow{\pi}Y,L_Y,m_Y)$ be a twisting data on Y. Then one can define naturally the pull-back $f^*\tau$. This is the twisting data $(X_0\to X,L_X,m_X)$ on X, where X_0 is the fiber product $X\times_Y Y_0$, L_X is the inverse image of L_Y by the map $X_1\to Y_1$ and m_X is the isomorphism induced by m_Y.

Then, similarly to the non-twisted case, we can define

(7.5.1) $$\begin{aligned}f^{-1}\ &:\quad \mathrm{Mod}_{\tau}(\mathbb{C}_Y)\to\mathrm{Mod}_{f^*\tau}(\mathbb{C}_X),\\ f_*,\,f_!\ &:\ \mathrm{Mod}_{f^*\tau}(\mathbb{C}_X)\to\mathrm{Mod}_{\tau}(\mathbb{C}_Y).\end{aligned}$$

They have right derived functors:

(7.5.2) $$\begin{aligned}f^{-1}\ &:\ \mathrm{D}^{\mathrm{b}}_{\tau}(\mathbb{C}_Y)\to\mathrm{D}^{\mathrm{b}}_{f^*\tau}(\mathbb{C}_X),\\ \mathbf{R}f_*,\,\mathbf{R}f_!\ &:\ \mathrm{D}^{\mathrm{b}}_{f^*\tau}(\mathbb{C}_X)\to\mathrm{D}^{\mathrm{b}}_{\tau}(\mathbb{C}_Y).\end{aligned}$$

The functor $Rf_!$ has a right adjoint functor

(7.5.3) $$f^!\colon\ \mathrm{D}^{\mathrm{b}}_{\tau}(\mathbb{C}_Y)\to\mathrm{D}^{\mathrm{b}}_{f^*\tau}(\mathbb{C}_X).$$

7.6 Twisted Modules

Let $\tau=(X_0\xrightarrow{\pi}X,L,m)$ be a twisting data on X. Let $\mathscr{A}$ be a sheaf of $\mathbb{C}$-algebras on X. Then we can define the category $\mathrm{Mod}_{\tau}(\mathscr{A})$ of τ-twisted $\mathscr{A}$-modules. A τ-twisted $\mathscr{A}$-module is a pair (F,β) of a $\pi^{-1}\mathscr{A}$-module F on X_0 and a $p^{-1}\mathscr{A}$-linear isomorphism $\beta\colon L\otimes p_2^{-1}F\xrightarrow{\sim}p_1^{-1}F$ satisfying the chain condition (7.2.1). Here $p\colon X_1\to X$ is the projection. The stack $\mathfrak{Mod}_{\tau}(\mathscr{A})$ of τ-twisted $\mathscr{A}$-modules is locally equivalent to the stack $\mathfrak{Mod}(\mathscr{A})$ of $\mathscr{A}$-modules.

7.7 Equivariant twisting data

Let G be a Lie group, and let X be a topological G-manifold. A G-equivariant twisting data on X is a twisting data $\tau = (X_0 \xrightarrow{\pi} X, L, m)$ such that X_0 is a G-manifold, π is G-equivariant and L is G-equivariant, as well as m. Let $\mu\colon G \times X \to X$ be the multiplication map and $\mathrm{pr}\colon G \times X \to X$ the projection. Then the two twisting data $\mu^*\tau$ and $\mathrm{pr}^*\tau$ on $G \times X$ are canonically isomorphic. We can then define the G-equivariant derived category $\mathrm{D}^{\mathrm{b}}_{G,\tau}(\mathbb{C}_X)$, similarly to the non-twisted case.

If G acts freely on X, then denoting by $p\colon X \to X/G$ the projection, we can construct the quotient twisting data τ/G on X/G such that $\tau \cong p^*(\tau/G)$, and we have an equivalence

$$\mathrm{D}^{\mathrm{b}}_{G,\tau}(\mathbb{C}_X) \simeq \mathrm{D}^{\mathrm{b}}_{\tau/G}(\mathbb{C}_{X/G}).$$

7.8 Character local system

In order to construct twisting data, the following notion is sometimes useful.

Let H be a real Lie group. Let $\mu\colon H \times H \to H$ be the multiplication map and $q_j\colon H \times H \to H$ be the j-th projection ($j = 1, 2$). A *character local system* on H is by definition an invertible $\mathbb{C}_H$-module L equipped with an isomorphism $m\colon q_1^{-1}L \otimes q_2^{-1}L \xrightarrow{\sim} \mu^{-1}L$ satisfying the associativity law: denoting by $m(h_1, h_2)\colon L_{h_1} \otimes L_{h_2} \to L_{h_1h_2}$ the morphism given by m, the following diagram commutes for $h_1, h_2, h_3 \in H$

$$\begin{array}{ccc} L_{h_1} \otimes L_{h_2} \otimes L_{h_3} & \xrightarrow{m(h_1,h_2)} & L_{h_1h_2} \otimes L_{h_3} \\ {\scriptstyle m(h_2,h_3)}\downarrow & & \downarrow{\scriptstyle m(h_1h_2,h_3)} \\ L_{h_1} \otimes L_{h_2h_3} & \xrightarrow[m(h_1,h_2h_3)]{} & L_{h_1h_2h_3}. \end{array} \tag{7.8.1}$$

Let $\mathfrak{h}$ be the Lie algebra of H. For $A \in \mathfrak{h}$, let $L_H(A)$ and $R_H(A)$ denote the vector fields on H defined by

$$\big(L_H(A)f\big)(h) = \frac{d}{dt} f(e^{-tA}h)\Big|_{t=0} \text{ and } \big(R_H(A)f\big)(h) = \frac{d}{dt} f(he^{tA})\Big|_{t=0}. \tag{7.8.2}$$

Let us take an H-invariant element λ of $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{h}, \mathbb{C}) \simeq \mathrm{Hom}_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{h}, \mathbb{C})$. Hence λ satisfies $\lambda([\mathfrak{h}, \mathfrak{h}]) = 0$. Let L_λ be the sheaf of functions f on H satisfying $R_H(A)f = \lambda(A)f$ for all $A \in \mathfrak{h}$, or equivalently $L_H(A)f = -\lambda(A)f$ for all $A \in \mathfrak{h}$. Then L_λ is a local system on H of rank one. Regarding $q_1^{-1}L_\lambda$, $q_2^{-1}L_\lambda$ and $\mu^{-1}L_\lambda$ as subsheaves of the sheaf $\mathscr{O}_{H\times H}$ of functions on $H \times H$, the multiplication morphism $\mathscr{O}_{H\times H} \otimes \mathscr{O}_{H\times H} \to \mathscr{O}_{H\times H}$ induces an isomorphism

$$m\colon q_1^{-1}L_\lambda \otimes q_2^{-1}L_\lambda \xrightarrow{\sim} \mu^{-1}L_\lambda. \tag{7.8.3}$$

With this data, L_λ has a structure of a character local system.

If λ lifts to a character $\chi\colon H \to \mathbb{C}^*$, then L_λ is isomorphic to the trivial character local system $\mathbb{C}_H = L_0$ by $\mathbb{C}_H \xrightarrow{\sim} L_\lambda \subset \mathscr{O}_H$ given by χ.

For $\lambda, \lambda' \in \mathrm{Hom}_H(\mathfrak{h}, \mathbb{C})$, we have

$$L_\lambda \otimes L_{\lambda'} \cong L_{\lambda+\lambda'} \tag{7.8.4}$$

compatible with m.

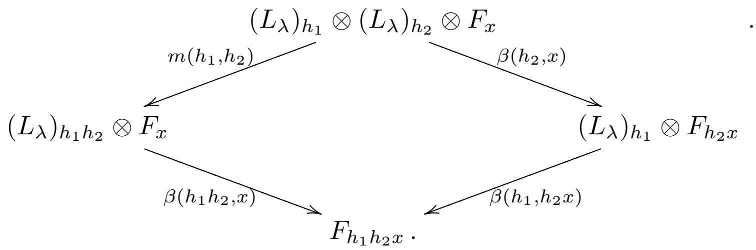
7.9 Twisted equivariance

Let H, λ, L_λ be as in the preceding subsection. Let X be an H-manifold. Let $\mathrm{pr}\colon H \times X \to X$ and $q\colon H \times X \to H$ be the projections and $\mu\colon H \times X \to X$ the multiplication map.

Definition 7.9.1. An (H, λ)-*equivariant sheaf on* X *is a pair* (F, β) *where* F *is a* $\mathbb{C}_X$*-module and* β *is an isomorphism*

$$\beta\colon q^{-1}L_\lambda \otimes \mathrm{pr}^{-1}F \simeq \mu^{-1}F \tag{7.9.1}$$

satisfying the following associativity law: letting $\beta(h, x)\colon (L_\lambda)_h \otimes F_x \xrightarrow{\sim} F_{hx}$ *be the induced morphism for* $(h, x) \in H \times X$*, the following diagram commutes for* $(h_1, h_2, x) \in H \times H \times X$*:*



Let us denote by $\mathrm{Mod}_{(H,\lambda)}(\mathbb{C}_X)$ the category of (H, λ)-equivariant sheaves on X. It is an abelian category.

If $\lambda = 0$, then $\mathrm{Mod}_{(H,\lambda)}(\mathbb{C}_X) \simeq \mathrm{Mod}_H(\mathbb{C}_X)$.

For $x \in X$ and $h \in H$, we have a chain of isomorphisms

$$F_x \xrightarrow[\beta]{\sim} (L_\lambda)_{h^{-1}} \otimes F_{hx} \xrightarrow{\sim} \mathbb{C} \otimes F_{hx} \simeq F_{hx}. \tag{7.9.2}$$

Here $(L_\lambda)_{h^{-1}} \xrightarrow{\sim} \mathbb{C}$ is induced by the evaluation map $(\mathscr{O}_H)_{h^{-1}} \to \mathbb{C}$. Let H_x be the isotropy subgroup at $x \in X$ and $\mathfrak{h}_x$ its Lie algebra. Then, (7.9.2) gives a group homomorphism

$$H_x \to \mathrm{Aut}(F_x).$$

Its infinitesimal representation coincides with $\mathfrak{h}_x \xrightarrow{-\lambda} \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(F_x)$.

Lemma 7.9.2 ([15]). *Let* X *be a homogeneous space of* H *and* $x \in X$*. Then* $\mathrm{Mod}_{(H,\lambda)}(\mathbb{C}_X)$ *is equivalent to the category of* H_x*-modules* M *such that its infinitesimal representation coincides with* $\mathfrak{h}_x \xrightarrow{-\lambda} \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(M)$*.*

7.10 Twisting data associated with principal bundles

Let $\pi\colon X_0 \to X$ be a principal bundle with a real Lie group H as a structure group. We use the convention that H acts from the left on X_0. Let $\mathfrak{h}$ be the Lie algebra of H and λ an H-invariant element of $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{h}, \mathbb{C})$. Let L_λ be the corresponding character local character system. Let us identify $X_0 \times_X X_0$ with $H \times X_0$ by the isomorphism $H \times X_0 \xrightarrow{\sim} X_0 \times_X X_0$ given by $(h, x') \mapsto (hx', x')$. Then the projection map $H \times X_0 \to H$ gives $q\colon X_0 \times_X X_0 \to H((hx', x') \mapsto h)$. Then the multiplication isomorphism (7.8.3) induces

$$p_{12}^{-1}(q^{-1}L_\lambda) \otimes p_{23}^{-1}(q^{-1}L_\lambda) \xrightarrow{\sim} p_{13}^{-1}(q^{-1}L_\lambda).$$

Thus $(X_0 \to X, q^{-1}L_\lambda)$ is a twisting data on X. We denote it by τ_λ. By the definition, we have an equivalence of categories:

$$\mathrm{Mod}_{\tau_\lambda}(\mathbb{C}_X) \cong \mathrm{Mod}_{(H,\lambda)}(\mathbb{C}_{X_0}). \tag{7.10.1}$$

For λ, $\lambda' \in \mathrm{Hom}_H(\mathfrak{h}, \mathbb{C})$, we have

$$\tau_\lambda \otimes \tau_{\lambda'} \cong \tau_{\lambda+\lambda'}.$$

Assume that X, X_0 are complex manifolds, $X_0 \to X$ and H are complex analytic and λ is an H-invariant element of $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$. Let $\mathscr{O}_X(\lambda)$ be the sheaf on X_0 given by

$$\mathscr{O}_X(\lambda) = \{\varphi \in \mathscr{O}_{X_0}\,;\, L_X(A)\varphi = -\langle A, \lambda\rangle\varphi \quad \text{for any } A \in \mathfrak{h}\}\,. \tag{7.10.2}$$

Then $\mathscr{O}_X(\lambda)$ is (H, λ)-equivariant and we regard it as an object of $\mathrm{Mod}_{\tau_\lambda}(\mathscr{O}_X)$.

7.11 Twisting (*D*-module case)

So far, we discussed the twisting in the topological framework. Now let us investigate the twisting in the D-module framework. This is similar to the topological case. Referring the reader to [15] for treatments in a more general situation, we restrict ourselves to the twisting arising from a principal bundle as in § 7.10.

Let H be a complex affine algebraic group, $\mathfrak{h}$ its Lie algebra and let R_H, $L_H\colon \mathfrak{h} \to \Theta_H$ be the Lie algebra homomorphisms defined by (7.8.2). For $\lambda \in \mathrm{Hom}_H(\mathfrak{h}, \mathbb{C})$, let us define the $\mathscr{D}_H$-module $\mathscr{L}_\lambda = \mathscr{D}_H u_\lambda$ by the defining relation $R_H(A)u_\lambda = \lambda(A)u_\lambda$ for any $A \in \mathfrak{h}$ (which is equivalent to the relation: $L_H(A)u_\lambda = -\lambda(A)u_\lambda$ for any $A \in \mathfrak{h}$). Hence we have $L_\lambda \cong \mathscr{H}om_{\mathscr{D}_H}(\mathscr{L}_\lambda, \mathscr{O}_{H^{\mathrm{an}}})$. Let $\mu\colon H \times H \to H$ be the multiplication morphism. Then we have $\mathscr{D}_{H\times H}$-linear isomorphism

$$m\colon \mathscr{L}_\lambda \overset{\mathbf{D}}{\boxtimes} \mathscr{L}_\lambda \xrightarrow{\sim} \mathbf{D}\mu^*\mathscr{L}_\lambda \tag{7.11.1}$$

by $m(u_\lambda \boxtimes u_\lambda) = \mu^*(u_\lambda)$. It satisfies the associative law similar to (7.8.1) (i.e., (7.11.2) with $\mathscr{M} = \mathscr{L}_\lambda$ and $\beta = m$). For $\lambda, \lambda' \in \operatorname{Hom}_H(\mathfrak{h}, \mathbb{C})$, there is an isomorphism

$$\mathscr{L}_\lambda \overset{\mathbf{D}}{\otimes} \mathscr{L}_{\lambda'} \cong \mathscr{L}_{\lambda+\lambda'}$$

that is compatible with m.

Let X be a complex algebraic H-manifold. Then we can define the notion of (H, λ)-equivariant $\mathscr{D}_X$-module as in §7.9. Let us denote by $\mu\colon H \times X \to X$ the multiplication morphism.

Definition 7.11.1. An (H, λ)-equivariant $\mathscr{D}_X$-module is a pair $(\mathscr{M}, \beta)$ where $\mathscr{M}$ is a $\mathscr{D}_X$-module and β is a $\mathscr{D}_{H\times X}$-linear isomorphism

$$\beta\colon \mathscr{L}_\lambda \overset{\mathbf{D}}{\boxtimes} \mathscr{M} \xrightarrow{\sim} \mathbf{D}\mu^*\mathscr{M}$$

satisfying the associativity law: the following diagram on $H \times H \times X$ commutes.

$$\begin{array}{ccc}
 & \mathscr{L}_\lambda \overset{\mathbf{D}}{\boxtimes} \mathscr{L}_\lambda \overset{\mathbf{D}}{\boxtimes} \mathscr{M} & \\
 \swarrow^{m} & & \searrow^{\beta} \\
\mathbf{D}\mu^*\mathscr{L}_\lambda \overset{\mathbf{D}}{\boxtimes} \mathscr{M} & & \mathscr{L}_\lambda \overset{\mathbf{D}}{\boxtimes} \mathbf{D}\mu^*\mathscr{M} \\
\| & & \| \\
\mathbf{D}(\mu\times \mathrm{id})^*(\mathscr{L}_\lambda \overset{\mathbf{D}}{\boxtimes} \mathscr{M}) & & \mathbf{D}(\mathrm{id}\times\mu)^*(\mathscr{L}_\lambda \overset{\mathbf{D}}{\boxtimes} \mathscr{M}) \\
\downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} \\
\mathbf{D}(\mu\times \mathrm{id})^*\mathbf{D}\mu^*\mathscr{M} & \overset{\sim}{=\!=\!=\!=} & \mathbf{D}(\mathrm{id}\times\mu)^*\mathbf{D}\mu^*\mathscr{M}\,.
\end{array} \tag{7.11.2}$$

Then the quasi-coherent (H, λ)-equivariant $\mathscr{D}_X$-modules form an abelian category. We denote it by $\operatorname{Mod}_{(H,\lambda)}(\mathscr{D}_X)$.

Note that any (H, λ)-equivariant $\mathscr{D}_X$-module may be regarded as a quasi-H-equivariant $\mathscr{D}_X$-module since $\mathscr{L}_\lambda = \mathscr{O}_H u_\lambda \cong \mathscr{O}_H$ as an $\mathscr{O}_H$-module, and $m(u_\lambda \boxtimes u_\lambda) = \mu^* u_\lambda$. Thus we have a fully faithful exact functor

$$\operatorname{Mod}_{(H,\lambda)}(\mathscr{D}_X) \to \operatorname{Mod}(\mathscr{D}_X, H).$$

Similarly to Lemma 3.1.4, we can prove the following lemma (see [15]).

Lemma 7.11.2. *An object $\mathscr{M}$ of $\operatorname{Mod}(\mathscr{D}_X, H)$ is isomorphic to the image of an object of $\operatorname{Mod}_{(H,\lambda)}(\mathscr{D}_X)$ if and only if $\gamma_{\mathscr{M}}\colon \mathfrak{h} \to \operatorname{End}_{\mathscr{D}_X}(\mathscr{M})$ coincides with the composition $\mathfrak{h} \xrightarrow{\lambda} \mathbb{C} \to \operatorname{End}_{\mathscr{D}_X}(\mathscr{M})$.*

Note that for λ, $\lambda' \in \operatorname{Hom}_H(\mathfrak{h}, \mathbb{C})$, $\mathscr{L}_\lambda \otimes \mathscr{L}_{\lambda'} \cong \mathscr{L}_{\lambda+\lambda'}$ gives the right exact functor

$$\bullet \overset{\mathbf{D}}{\otimes} \bullet = \bullet \underset{\mathscr{O}_X}{\otimes} \bullet : \operatorname{Mod}_{(H,\lambda)}(\mathscr{D}_X) \times \operatorname{Mod}_{(H,\lambda')}(\mathscr{D}_X) \to \operatorname{Mod}_{(H,\,\lambda+\lambda')}(\mathscr{D}_X).$$

Note that for $\mathscr{M} \in \operatorname{Mod}_{(H,\lambda)}(\mathscr{D}_X)$, the sheaf $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_{X^{\mathrm{an}}})$ is an $(H^{\mathrm{an}}, \lambda)$-equivariant sheaf on X^{an}.

7.12 Ring of twisted differential operators

Let $\pi\colon X_0 \to X$ be a principal H-bundle over X, and $\lambda \in \mathrm{Hom}_H(\mathfrak{h}, \mathbb{C})$. Let $\mathscr{N}_\lambda = \mathscr{D}_{X_0} v_\lambda$ be the $\mathscr{D}_{X_0}$-module defined by the defining relation $L_{X_0}(A)v_\lambda = -\lambda(A)v_\lambda$. Then $\mathscr{N}_\lambda$ is an (H,λ)-equivariant $\mathscr{D}_{X_0}$-module in an evident way. We set

$$\mathscr{D}_{X,\lambda} = \left\{ f \in \pi_*\mathscr{E}nd_{\mathscr{D}_{X_0}}(\mathscr{N}_\lambda)\,;\, f \text{ is } H\text{-equivariant} \right\}^{\mathrm{op}}.$$

Here op means the opposite ring. Then $\mathscr{D}_{X,\lambda}$ is a ring on X, and $\mathscr{N}_\lambda$ is a right $\pi^{-1}\mathscr{D}_{X,\lambda}$-module.

If there is a section s of $\pi\colon X_0 \to X$, then the composition $\mathscr{D}_{X,\lambda} \to \mathscr{E}nd_{\mathscr{D}_X}(s^*\mathscr{N}_\lambda)^{\mathrm{op}} = \mathscr{E}nd_{\mathscr{D}_X}(\mathscr{D}_X)^{\mathrm{op}} = \mathscr{D}_X$ is an isomorphism. Hence $\mathscr{D}_{X,\lambda}$ is locally isomorphic to $\mathscr{D}_X$ (with respect to the étale topology), and hence it is a ring of twisted differential operators on X (cf. e.g. [15]). We have

Lemma 7.12.1. *We have an equivalence* $\mathrm{Mod}_{(H,\lambda)}(\mathscr{D}_{X_0}) \cong \mathrm{Mod}(\mathscr{D}_{X,\lambda})$. *The equivalence is given by:*

$$\begin{aligned} \mathrm{Mod}_{(H,\lambda)}(\mathscr{D}_{X_0}) \ni \widetilde{\mathscr{M}} \mapsto \pi_*\,\mathscr{H}om_{(\mathscr{D}_{X_0},H)}(\mathscr{N}_\lambda, \widetilde{\mathscr{M}}) \in \mathrm{Mod}(\mathscr{D}_{X,\lambda}) \quad \textit{and} \\ \mathrm{Mod}(\mathscr{D}_{X_0,\lambda}) \ni \mathscr{M} \mapsto \mathscr{N}_\lambda \otimes_{\mathscr{D}_{X,\lambda}} \mathscr{M} \in \mathrm{Mod}_{(H,\lambda)}(\mathscr{D}_{X_0}). \end{aligned}$$

Here $\pi_*\,\mathscr{H}om_{(\mathscr{D}_{X_0},H)}(\mathscr{N}_\lambda, \widetilde{\mathscr{M}})$ *is the sheaf which associates*

$$\mathrm{Hom}_{\mathrm{Mod}_{(H,\lambda)}(\mathscr{D}_{\pi^{-1}U})}(\mathscr{N}_\lambda|_{\pi^{-1}U}, \widetilde{\mathscr{M}}|_{\pi^{-1}U})$$

to an open set U *of* X.

Note that $\mathscr{O}_{X^{\mathrm{an}}}(\lambda) \cong \mathscr{H}om_{\mathscr{D}_{X_0}}(\mathscr{N}_\lambda, \mathscr{O}_{X_0^{\mathrm{an}}})$ is an $(H^{\mathrm{an}},\lambda)$-equivariant sheaf and it may be regarded as a τ_λ-twisted $\mathscr{D}_{X^{\mathrm{an}},\lambda}$-module:

$$\mathscr{O}_{X^{\mathrm{an}}}(\lambda) \in \mathrm{Mod}_{\tau_\lambda}(\mathscr{D}_{X^{\mathrm{an}},\lambda}).$$

The twisted module $\mathscr{O}_{X^{\mathrm{an}}}(\lambda)$ plays the role of $\mathscr{O}_{X^{\mathrm{an}}}$ for $\mathscr{D}_X$-modules. For example, defining by

$$\begin{aligned} &\mathrm{DR}_X(\mathscr{M}) := \mathbf{R}\mathscr{H}om_{\mathscr{D}_{X^{\mathrm{an}},\lambda}}(\mathscr{O}_{X^{\mathrm{an}}}(\lambda), \mathscr{M}^{\mathrm{an}}) \quad \text{and} \\ &\mathrm{Sol}_X(\mathscr{M}) := \mathbf{R}\mathscr{H}om_{\mathscr{D}_{X,\lambda}}(\mathscr{M}, \mathscr{O}_{X^{\mathrm{an}}}(\lambda)), \end{aligned}$$

we obtain the functors

$$\begin{aligned} \mathrm{DR}_X &\colon \mathrm{D^b}(\mathscr{D}_{X,\lambda}) \to \mathrm{D}^{\mathrm{b}}_{\tau_{-\lambda}}(\mathbb{C}_{X^{\mathrm{an}}}), \\ \mathrm{Sol}_X &\colon \mathrm{D^b}(\mathscr{D}_{X,\lambda})^{\mathrm{op}} \to \mathrm{D}^{\mathrm{b}}_{\tau_\lambda}(\mathbb{C}_{X^{\mathrm{an}}}). \end{aligned} \tag{7.12.1}$$

Note that we have

$$\mathrm{Mod}(\mathscr{D}_{X^{\mathrm{an}},\lambda}) \simeq \mathrm{Mod}_{\tau_{-\lambda}}(\mathscr{D}_{X^{\mathrm{an}}})$$

by $\mathscr{M} \mapsto \mathscr{O}_{X^{\mathrm{an}}}(-\lambda) \otimes \mathscr{M}$.

7.13 Equivariance of twisted sheaves and twisted D-modules

Let $\pi\colon X_0 \to X$ be a principal bundle with an affine group H as a structure group, and let $\lambda \in \mathrm{Hom}_H(\mathfrak{h}, \mathbb{C})$. Assume that an affine group G acts on X_0 and X such that π is G-equivariant and the action of G commutes with the action of H. Then, as we saw in §7.6, we can define the notion of G^{an}-equivariant τ_λ-twisted $\mathbb{C}_{X^{\mathrm{an}}}$-modules, and the equivariant derived category $\mathrm{D}^{\mathrm{b}}_{G^{\mathrm{an}},\tau_\lambda}(\mathbb{C}_{X^{\mathrm{an}}})$.

Let $\mathfrak{g}$ be the Lie algebra of G. Then, for any $A \in \mathfrak{g}$, $\mathscr{N}_\lambda \ni v_\lambda \mapsto L_{X_0}(A)v_\lambda \in \mathscr{N}_\lambda$ extends to a $\mathscr{D}_{X_0}$-linear endomorphism of $\mathscr{N}_\lambda$ and it gives an element of $\mathscr{D}_{X,\lambda}$. Hence we obtain a Lie algebra homomorphism

$$\mathfrak{g} \to \Gamma(X; \mathscr{D}_{X,\lambda}).$$

We can define the notion of quasi-G-equivariant $\mathscr{D}_{X,\lambda}$-modules and G-equivariant $\mathscr{D}_{X,\lambda}$-modules. Moreover the results in the preceding sections for non-twisted case hold with a suitable modification, and we shall not repeat them. For example, for λ, $\mu \in \mathrm{Hom}_H(\mathfrak{h}, \mathbb{C})$, we have a functor

$$\bullet \overset{\mathbf{D}}{\otimes} \bullet \,:\, \mathrm{D}^{\mathrm{b}}(\mathscr{D}_{X,\lambda}, G) \times \mathrm{D}^{\mathrm{b}}(\mathscr{D}_{X,\mu}, G) \to \mathrm{D}^{\mathrm{b}}(\mathscr{D}_{X,\lambda+\mu}, G).$$

If $G_\mathbb{R}$ is a real Lie group with a Lie group morphism $G_\mathbb{R} \to G^{\mathrm{an}}$,

$$\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_{X,\lambda}}(\mathscr{M} \otimes F, \mathscr{O}_{X^{\mathrm{an}}}(\lambda)) \in \mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_\mathbb{R}})$$

is well-defined for $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathscr{D}_{X,\lambda}, G)$ and $F \in \mathrm{D}^{\mathrm{b}}_{G_\mathbb{R},\tau_\lambda,\mathrm{ctb}}(\mathbb{C}_{X^{\mathrm{an}}})$. Note that $\mathscr{H}om_\mathbb{C}(F, \mathscr{O}_{X^{\mathrm{an}}}(\lambda)) \in \mathrm{Mod}(\mathscr{D}_{X^{\mathrm{an}},\lambda})$ because $\mathscr{O}_{X^{\mathrm{an}}}(\lambda) \in \mathrm{Mod}_{\tau_\lambda}(\mathscr{D}_{X^{\mathrm{an}},\lambda})$.

7.14 Riemann-Hilbert correspondence

Let $\pi\colon X_0 \to X$, H, G and $\lambda \in \mathrm{Hom}_H(\mathfrak{h}, \mathbb{C})$ be as in the preceding subsection. Assume that λ vanishes on the Lie algebra of the unipotent radical of H. Then $\mathscr{L}_\lambda$ is a regular holonomic $\mathscr{D}_H$-module. Hence we can define the notion of regular holonomic $\mathscr{D}_{X,\lambda}$-module (i.e. a $\mathscr{D}_{X,\lambda}$-module $\mathscr{M}$ is regular holonomic if $\mathscr{N}_\lambda \otimes_{\mathscr{D}_{X,\lambda}} \mathscr{M}$ is a regular holonomic $\mathscr{D}_{X_0}$-module).

Assume that there are finitely many G-orbits in X. Then any coherent holonomic G-equivariant $\mathscr{D}_{X,\lambda}$-module is regular holonomic (see [15]). Hence the Riemann-Hilbert correspondence (see Subsection 4.6) implies the following result.

Theorem 7.14.1. *Assume that λ vanishes on the Lie algebra of the unipotent radical of H. If there are only finitely many G-orbits in X, then the functor*

$$\mathrm{DR}_X := \mathbf{R}\mathscr{H}om_{\mathscr{D}_{X^{\mathrm{an}},\lambda}}(\mathscr{O}_{X^{\mathrm{an}}}(\lambda), \bullet^{\mathrm{an}})\colon \mathrm{D}^{\mathrm{b}}_{G,\mathrm{coh}}(\mathscr{D}_{X,\lambda}) \to \mathrm{D}^{\mathrm{b}}_{G^{\mathrm{an}},\tau_{-\lambda},\mathbb{C}\text{-}\mathrm{c}}(\mathbb{C}_{X^{\mathrm{an}}})$$

is an equivalence of triangulated categories.

8 Integral transforms

8.1 Convolutions

Let X, Y and Z be topological manifolds.

Let us consider a diagram

$$\begin{array}{ccccc} & & X\times Y\times Z & & \\ & \swarrow^{p_{12}} & \downarrow{\scriptstyle p_{13}} & {}^{p_{23}}\searrow & \\ X\times Y & & X\times Z & & Y\times Z. \end{array}$$

For $F \in \mathrm{D^b}(\mathbb{C}_{X\times Y})$ and $G \in \mathrm{D^b}(\mathbb{C}_{Y\times Z})$, we define the object $F\circ G$ of $\mathrm{D^b}(\mathbb{C}_{X\times Z})$ by

$$F\circ G := \mathbf{R}p_{13!}({p_{12}}^{-1}F\otimes {p_{23}}^{-1}G). \tag{8.1.1}$$

We call it the *convolution* of F and G.

Hence we obtain the functor

$$\bullet\circ\bullet:\ \mathrm{D^b}(\mathbb{C}_{X\times Y})\times\mathrm{D^b}(\mathbb{C}_{Y\times Z})\longrightarrow \mathrm{D^b}(\mathbb{C}_{X\times Z}).$$

In particular, letting X or Z be $\{\mathrm{pt}\}$, we obtain

$$\bullet\circ\bullet:\ \begin{array}{l}\mathrm{D^b}(\mathbb{C}_{Y})\times\mathrm{D^b}(\mathbb{C}_{Y\times Z})\longrightarrow \mathrm{D^b}(\mathbb{C}_{Z})\\ \mathrm{D^b}(\mathbb{C}_{X\times Y})\times\mathrm{D^b}(\mathbb{C}_{Y})\longrightarrow \mathrm{D^b}(\mathbb{C}_{X}).\end{array}$$

This functor satisfies the associative law

$$(F\circ G)\circ H\simeq F\circ(G\circ H)$$

for $F\in\mathrm{D^b}(\mathbb{C}_{X\times Y})$, $G\in\mathrm{D^b}(\mathbb{C}_{Y\times Z})$ and $H\in\mathrm{D^b}(\mathbb{C}_{Z\times W})$.

This can be generalized to the twisted case. Let τ_X (resp. τ_Y, τ_Z) be a twisting data on X (resp. Y, Z). Then we have a functor

$$\bullet\circ\bullet:\ \mathrm{D}^{\mathrm{b}}_{\tau_X\boxtimes(\tau_Y)^{\otimes-1}}(\mathbb{C}_{X\times Y})\times\mathrm{D}^{\mathrm{b}}_{\tau_Y\boxtimes(\tau_Z)^{\otimes-1}}(\mathbb{C}_{Y\times Z})\longrightarrow \mathrm{D}^{\mathrm{b}}_{\tau_X\boxtimes(\tau_Z)^{\otimes-1}}(\mathbb{C}_{X\times Z}).$$

Similarly, we can define the convolutions of D-modules. Let X, Y and Z be algebraic manifolds. Then we can define, for $\mathscr{M}\in\mathrm{D^b}(\mathscr{D}_{X\times Y})$ and $\mathscr{N}\in\mathrm{D^b}(\mathscr{D}_{Y\times Z})$, the object $\mathscr{M}\overset{\mathbf{D}}{\circ}\mathscr{N}$ of $\mathrm{D^b}(\mathscr{D}_{X\times Z})$ by

$$\mathscr{M}\overset{\mathbf{D}}{\circ}\mathscr{N} := \mathbf{D}p_{13*}(\mathbf{D}{p_{12}}^{*}\mathscr{M}\overset{\mathbf{D}}{\otimes}\mathbf{D}{p_{23}}^{*}\mathscr{N}). \tag{8.1.2}$$

We call it the *convolution* of $\mathscr{M}$ and $\mathscr{N}$.

Hence we obtain the functor

$$\bullet\overset{\mathbf{D}}{\circ}\bullet:\ \mathrm{D^b}(\mathscr{D}_{X\times Y})\times\mathrm{D^b}(\mathscr{D}_{Y\times Z})\longrightarrow \mathrm{D^b}(\mathscr{D}_{X\times Z}).$$

If X, Y and Z are quasi-projective G-manifolds, we can define

$$\bullet\overset{\mathbf{D}}{\circ}\bullet:\ \mathrm{D^b}(\mathscr{D}_{X\times Y},G)\times\mathrm{D^b}(\mathscr{D}_{Y\times Z},G)\longrightarrow \mathrm{D^b}(\mathscr{D}_{X\times Z},G).$$

These definitions also extend to the twisted case.

8.2 Integral transform formula

Let G be an affine algebraic group, and let $G_{\mathbb{R}}$ be a real Lie group with a Lie group morphism $G_{\mathbb{R}} \to G^{\mathrm{an}}$.

Let X be a projective algebraic G-manifold and Y a quasi-projective G-manifold. Let us consider the diagram

$$\begin{array}{ccc} & X \times Y & \\ {\scriptstyle p_1} \swarrow & & \searrow {\scriptstyle p_2} \\ X & & Y. \end{array}$$

For $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X, G)$, $\mathscr{K} \in \mathrm{D}^{\mathrm{b}}_{G,\mathrm{hol}}(\mathscr{D}_{X\times Y})$ and $F \in \mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}},\mathrm{ctb}}(\mathbb{C}_{Y^{\mathrm{an}}})$, let us calculate $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_Y}\big((\mathscr{M} \overset{\mathbf{D}}{\circ} \mathscr{K}) \otimes F, \mathscr{O}_{Y^{\mathrm{an}}}\big)$. Note that $\mathscr{M} \overset{\mathbf{D}}{\circ} \mathscr{K} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y, G)$. We have by Theorem 5.6.1

$$\begin{aligned} &\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_Y}((\mathscr{M} \overset{\mathbf{D}}{\circ} \mathscr{K}) \otimes F, \mathscr{O}_{Y^{\mathrm{an}}}) \\ &\quad = \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_Y}(\mathbf{D}p_{2*}(\mathbf{D}p_1{}^*\mathscr{M} \overset{\mathbf{D}}{\otimes} \mathscr{K}) \otimes F, \mathscr{O}_{Y^{\mathrm{an}}}) \\ &\quad \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_{X\times Y}}((\mathbf{D}p_1{}^*\mathscr{M} \overset{\mathbf{D}}{\otimes} \mathscr{K}) \otimes (p_2^{\mathrm{an}})^{-1}F, \mathscr{O}_{(X\times Y)^{\mathrm{an}}})[\mathrm{d}_X]. \end{aligned} \tag{8.2.1}$$

If we assume the non-characteristic condition:

$$\big(\mathrm{Ch}(\mathscr{M}) \times T^*_Y Y\big) \cap \mathrm{Ch}(\mathscr{K}) \subset T^*_{X\times Y}(X \times Y),$$

Theorem 5.7.2 implies that

$$\begin{aligned} &\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_{X\times Y}}((\mathbf{D}p_1{}^*\mathscr{M} \overset{\mathbf{D}}{\otimes} \mathscr{K}) \otimes (p_2^{\mathrm{an}})^{-1}F, \mathscr{O}_{(X\times Y)^{\mathrm{an}}}) \\ &\quad \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_{X\times Y}}(\mathbf{D}p_1{}^*\mathscr{M} \otimes (K \otimes (p_2^{\mathrm{an}})^{-1}F), \mathscr{O}_{(X\times Y)^{\mathrm{an}}}). \end{aligned} \tag{8.2.2}$$

Here, $K{:=}\mathrm{DR}_{X\times Y}(\mathscr{K}) \in \mathrm{D}^{\mathrm{b}}_{G^{\mathrm{an}},\,\mathbb{C}\text{-c}}(\mathbb{C}_{(X\times Y)^{\mathrm{an}}})$. Then, again by Theorem 5.6.1, we have

$$\begin{aligned} &\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_{X\times Y}}(\mathbf{D}p_1{}^*\mathscr{M} \otimes (K \otimes (p_2^{\mathrm{an}})^{-1}F), \mathscr{O}_{(X\times Y)^{\mathrm{an}}}) \\ &\quad \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}\big(\mathscr{M} \otimes \mathbf{R}(p_1^{\mathrm{an}})_!(K \otimes (p_2^{\mathrm{an}})^{-1}F), \mathscr{O}_{X^{\mathrm{an}}}\big)[-2\mathrm{d}_Y] \\ &\quad = \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}\big(\mathscr{M} \otimes (K \circ F), \mathscr{O}_{X^{\mathrm{an}}}\big)[-2\mathrm{d}_Y]. \end{aligned}$$

Combining this with (8.2.1) and (8.2.2), we obtain

$$\begin{aligned} &\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_Y}((\mathscr{M} \overset{\mathbf{D}}{\circ} \mathscr{K}) \otimes F, \mathscr{O}_{Y^{\mathrm{an}}}) \\ &\qquad\qquad \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M} \otimes (K \circ F), \mathscr{O}_{X^{\mathrm{an}}})[\mathrm{d}_X - 2\mathrm{d}_Y]. \end{aligned}$$

Thus we obtain the following theorem.

Theorem 8.2.1 (Integral transform formula). *Let G be an affine algebraic group, and let $G_{\mathbb{R}}$ be a Lie group with a Lie group morphism $G_{\mathbb{R}} \to G^{\mathrm{an}}$. Let X be a projective G-manifold and Y a quasi-projective G-manifold. Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X, G)$, $\mathscr{K} \in \mathrm{D}^{\mathrm{b}}_{G,\mathrm{hol}}(\mathscr{D}_{X\times Y})$ and $F \in \mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}},\mathrm{ctb}}(\mathbb{C}_Y)$, If the non-characteristic condition*

$$\big(\mathrm{Ch}(\mathscr{M}) \times T^*_Y Y\big) \cap \mathrm{Ch}(\mathscr{K}) \subset T^*_{X\times Y}(X \times Y) \tag{8.2.3}$$

is satisfied, then we have an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$

$$\begin{aligned} &\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_Y}((\mathscr{M} \overset{\mathbf{D}}{\circ} \mathscr{K}) \otimes F, \mathscr{O}_{Y^{\mathrm{an}}}) \\ &\quad \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_X}(\mathscr{M} \otimes (\mathrm{DR}_{X\times Y}(\mathscr{K}) \circ F), \mathscr{O}_{X^{\mathrm{an}}})[\mathrm{d}_X - 2\mathrm{d}_Y]. \end{aligned} \tag{8.2.4}$$

Remark 8.2.2. If G acts transitively on X, then the non-characteristic condition (8.2.3) is always satisfied. Indeed, let $\mu_X : T^*X \to \mathfrak{g}^*$, $\mu_Y : T^*Y \to \mathfrak{g}^*$ and $\mu_{X\times Y} : T^*(X \times Y) \to \mathfrak{g}^*$ be the moment maps. Then we have $\mu_{X\times Y}(\xi, \eta) = \mu_X(\xi) + \mu_Y(\eta)$ for $\xi \in T^*X$ and $\eta \in T^*Y$. Since $\mathscr{K} \in \mathrm{D}^{\mathrm{b}}_{G,\mathrm{hol}}(\mathscr{D}_X)$, we have $\mathrm{Ch}(\mathscr{K}) \subset \mu^{-1}_{X\times Y}(0)$ (see [15]). Hence we have $(T^*X \times T^*_Y Y) \cap \mathrm{Ch}(\mathscr{K}) \subset \mu_X^{-1}(0) \times T^*_Y Y$. Since G acts transitively on X, we have $\mu_X^{-1}(0) = T^*_X X$.

Remark 8.2.3. Although we don't repeat here, there is a twisted version of Theorem 8.2.1.

9 Application to the representation theory

9.1 Notations

In this section, we shall apply the machinery developed in the earlier sections to the representation theory of real semisimple Lie groups.

Let $G_{\mathbb{R}}$ be a connected real semisimple Lie group with a finite center, and let $K_{\mathbb{R}}$ be a maximal compact subgroup of $G_{\mathbb{R}}$. Let $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{k}_{\mathbb{R}}$ be the Lie algebra of $G_{\mathbb{R}}$ and $K_{\mathbb{R}}$, respectively. Let $\mathfrak{g}$ and $\mathfrak{k}$ be their complexifications. Let K be the complexification of $K_{\mathbb{R}}$. Let G be a connected semisimple algebraic group with the Lie algebra $\mathfrak{g}$, and assume that there is an *injective* morphism $G_{\mathbb{R}} \to G^{\mathrm{an}}$ of real Lie groups which induces the embedding $\mathfrak{g}_{\mathbb{R}} \hookrightarrow \mathfrak{g}$.[4]

Thus we obtain the diagrams:

$$\begin{array}{ccc} K_{\mathbb{R}} & \hookrightarrow & K \\ \downarrow & & \downarrow \\ G_{\mathbb{R}} & \hookrightarrow & G \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathfrak{k}_{\mathbb{R}} & \hookrightarrow & \mathfrak{k} \\ \downarrow & & \downarrow \\ \mathfrak{g}_{\mathbb{R}} & \hookrightarrow & \mathfrak{g}. \end{array}$$

[4] In this note, we assume that $G_{\mathbb{R}} \to G$ is injective. However, we can remove this condition, by regarding G/K as an orbifold.

Let us take an Iwasawa decomposition

$$\begin{aligned} G_{\mathbb{R}} &= K_{\mathbb{R}}\, A_{\mathbb{R}}\, N_{\mathbb{R}}, \\ \mathfrak{g}_{\mathbb{R}} &= \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{a}_{\mathbb{R}} \oplus \mathfrak{n}_{\mathbb{R}}. \end{aligned} \tag{9.1.1}$$

Let $\mathfrak{a}$, $\mathfrak{n}$ be the complexification of $\mathfrak{a}_{\mathbb{R}}$ and $\mathfrak{n}_{\mathbb{R}}$. Let A and N be the connected closed subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$, respectively.

Let $M_{\mathbb{R}} = Z_{K_{\mathbb{R}}}(\mathfrak{a}_{\mathbb{R}})$ and $\mathfrak{m}_{\mathbb{R}} = Z_{\mathfrak{k}_{\mathbb{R}}}(\mathfrak{a}_{\mathbb{R}})$. Let M and $\mathfrak{m}$ be the complexification of $M_{\mathbb{R}}$ and $\mathfrak{m}_{\mathbb{R}}$. Then we have $M = Z_K(A)$. Let P be the parabolic subgroup of G with $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ as its Lie algebra, and $P_{\mathbb{R}} = M_{\mathbb{R}} A_{\mathbb{R}} N_{\mathbb{R}} \subset G_{\mathbb{R}}$.

Let us fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ such that

$$\mathfrak{t} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{t}_{\mathbb{R}} \quad \text{where} \quad \mathfrak{t}_{\mathbb{R}} = (\mathfrak{t} \cap \mathfrak{m}_{\mathbb{R}}) \oplus \mathfrak{a}_{\mathbb{R}}. \tag{9.1.2}$$

Let T be the maximal torus of G with $\mathfrak{t}$ as its Lie algebra.

We take a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{t}$ and $\mathfrak{n}$, and let B be the Borel subgroup with $\mathfrak{b}$ as its Lie algebra.

We have

$$K \cap P = M \quad \text{and} \quad K \cap B = M \cap B, \quad K \cap T = M \cap T,$$

and $M/(M \cap B) \simeq P/B$ is the flag manifold for M.

Let Δ be the root system of $(\mathfrak{g}, \mathfrak{t})$, and take the positive root system $\Delta^+ = \{\alpha \in \Delta\,;\, \mathfrak{g}_\alpha \subset \mathfrak{b}\}$. Let $\Delta_k = \{\alpha \in \Delta\,;\, \mathfrak{g}_\alpha \subset \mathfrak{k}\} = \{\alpha \in \Delta\,;\, \mathfrak{g}_\alpha \subset \mathfrak{m}\} = \{\alpha \in \Delta\,;\, \alpha|_{\mathfrak{a}} = 0\}$ be the set of compact roots, and set $\Delta_k^+ = \Delta_k \cap \Delta^+$. Let ρ be the half sum of positive roots.

An element λ of $\mathfrak{t}^*$ is called *integral* if it can be lifted to a character of T. We say that $\lambda|_{\mathfrak{k}\cap\mathfrak{t}}$ is integral if it can be lifted to a character of $K \cap T = M \cap T$.

Let $\mathfrak{z}(\mathfrak{g})$ denote the center of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Let $\chi\colon \mathfrak{z}(\mathfrak{g}) \to \mathbb{C}[\mathfrak{t}^*] = S(\mathfrak{t})$ be the ring morphism given by:

$$a - (\chi(a))(\lambda) \in \operatorname{Ker}(\mathfrak{b} \xrightarrow{\lambda} \mathbb{C})U(\mathfrak{g}) \quad \text{for any } \lambda \in \mathfrak{t}^* \text{ and } a \in \mathfrak{z}(\mathfrak{g}).$$

It means that $a \in \mathfrak{z}(\mathfrak{g})$ acts on the lowest weight module with lowest weight λ through the multiplication by the scalar $(\chi(a))(\lambda)$. For $\lambda \in \mathfrak{t}^*$, let

$$\chi_\lambda\colon \mathfrak{z}(\mathfrak{g}) \to \mathbb{C}$$

be the ring homomorphism given by $\chi_\lambda(a) := (\chi(a))(\lambda)$. Note that

(9.1.3) for $\lambda, \mu \in \mathfrak{t}^*$, $\chi_\lambda = \chi_\mu$ if and only if $w \circ \lambda = \mu$ for some $w \in W$.

Here $w \circ \lambda = w(\lambda - \rho) + \rho$ is the shifted action of the Weyl group W. We set

$$U_\lambda(\mathfrak{g}) = U(\mathfrak{g})/\big(U(\mathfrak{g}) \operatorname{Ker}(\chi_\lambda)\big).$$

Then $U_\lambda(\mathfrak{g})$-modules are nothing but $\mathfrak{g}$-modules with infinitesimal character χ_λ.

Let X be the flag manifold of G (the set of Borel subgroups of G). Then X is a projective G-manifold and $X \simeq G/B$. For $x \in X$, we set $B(x) = \{g \in G\,;\, gx = x\}$, $\mathfrak{b}(x) = \mathrm{Lie}(B(x))$ the Lie algebra of $B(x)$, and $\mathfrak{n}(x) = [\mathfrak{b}(x), \mathfrak{b}(x)]$ the nilpotent radical of $\mathfrak{b}(x)$. Let $x_0 \in X$ be the point of X such that $\mathfrak{b}(x_0) = \mathfrak{b}$. Then, for any $x \in X$, there exists a unique Lie algebra homomorphism $\mathfrak{b}(x) \to \mathfrak{t}$ which is equal to the composition $\mathfrak{b}(x) \xrightarrow{\mathrm{Ad}(g)} \mathfrak{b} \to \mathfrak{t}$ for any $g \in G$ such that $gx = x_0$.

Let $X_{\min} = G/P$. Let

$$\pi\colon X \to X_{\min}$$

be the canonical projection. We set $x_0^{\min} = \pi(x_0) = e \bmod P$.

Let $\tilde{p}\colon G \to X$ be the G-equivariant projection such that $\tilde{p}(e) = x_0$. Then this is a principal B-bundle. For $\lambda \in \mathfrak{t}^* = (\mathfrak{b}/\mathfrak{n})^* = \mathrm{Hom}_B(\mathfrak{b}, \mathbb{C})$, let $\mathscr{D}_{X,\lambda}$ be the ring of twisted differential operators on X with twist λ. Let τ_λ denote the G^{an}-equivariant twisting data on X^{an} corresponding to λ (see §7.10, 7.12).

Note the following lemma (see [15] and Lemma 7.9.2).

Lemma 9.1.1. (i) *Let H be a closed algebraic subgroup of G with a Lie algebra $\mathfrak{h} \subset \mathfrak{g}$, Z an H-orbit in X and $x \in Z$. Then the category* $\mathrm{Mod}_H(\mathscr{D}_{Z,\lambda})$ *of H-equivariant $\mathscr{D}_{Z,\lambda}$-modules is equivalent to the category of $H \cap B(x)$-modules V whose infinitesimal representation coincides with* $\mathfrak{h} \cap \mathfrak{b}(x) \to \mathfrak{b}(x) \to \mathfrak{t} \xrightarrow{\lambda} \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(V)$.

(ii) *Let H be a closed real Lie subgroup of G^{an} with a Lie algebra $\mathfrak{h} \subset \mathfrak{g}$, Z an H-orbit in X^{an} and $x \in Z$. Then the category* $\mathrm{Mod}_{H,\tau_\lambda}(\mathbb{C}_Z)$ *of H-equivariant τ_λ-twisted sheaves on Z is equivalent to the category of $H \cap B(x)$-modules V whose infinitesimal representation coincides with* $\mathfrak{h} \cap \mathfrak{b}(x) \to \mathfrak{b}(x) \to \mathfrak{t} \xrightarrow{-\lambda} \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(V)$.

Note that, in the situation of (i), the de Rham functor gives an equivalence

$$\mathrm{Mod}_H(\mathscr{D}_{Z,\lambda}) \xrightarrow{\sim} \mathrm{Mod}_{H^{\mathrm{an}},\tau_{-\lambda}}(\mathbb{C}_{Z^{\mathrm{an}}}).$$

9.2 Beilinson-Bernstein Correspondence

Let us recall a result of Beilinson-Bernstein [1] on the correspondence of $U(\mathfrak{g})$-modules and D-modules on the flag manifold.

For $\alpha \in \Delta$, let $\alpha^\vee \in \mathfrak{t}$ be the corresponding co-root.

Definition 9.2.1. Let $\lambda \in \mathfrak{t}^*$.

(i) We say that λ is *regular* if $\langle \alpha^\vee, \lambda \rangle$ does not vanish for any $\alpha \in \Delta^+$.

(ii) We say that a weight $\lambda \in \mathfrak{t}^*$ is *integrally anti-dominant* if $\langle \alpha^\vee, \lambda \rangle \neq 1, 2, 3, \ldots$ for any $\alpha \in \Delta^+$.

Recall that $\tilde{p}\colon G \to X = G/B$ is the projection. For $\lambda \in \mathfrak{t}^* = \mathrm{Hom}_B(\mathfrak{b}, \mathbb{C})$, we have defined the twisting data τ_λ on X^{an} and the ring of twisted differential operators $\mathscr{D}_{X,\lambda}$. We defined also $\mathscr{O}_{X^{\mathrm{an}}}(\lambda)$. Recall that $\mathscr{O}_{X^{\mathrm{an}}}(\lambda)$ is a twisted $\mathscr{D}_{X^{\mathrm{an}},\lambda}$-module, and it is an object of $\mathrm{Mod}_{\tau_\lambda}(\mathscr{D}_{X^{\mathrm{an}},\lambda})$.

If λ is an integral weight, then the twisting data τ_λ is trivial and $\mathscr{O}_{X^{\mathrm{an}}}(\lambda)$ is the invertible $\mathscr{O}_{X^{\mathrm{an}}}$-module associated with the invertible $\mathscr{O}_X$-module $\mathscr{O}_X(\lambda)$:

$$\mathscr{O}_X(\lambda) = \left\{u \in \tilde{p}_*\mathscr{O}_G\,;\, u(gb^{-1}) = b^\lambda u(g) \text{ for any } b \in B\right\}. \tag{9.2.1}$$

Here $B \ni b \mapsto b^\lambda \in \mathbb{C}^*$ is the character of B corresponding to $\lambda \in \mathrm{Hom}_B(\mathfrak{b}, \mathbb{C})$. Note that we have

$$\begin{aligned}\mathscr{D}_{X,\lambda+\mu} &= \mathscr{O}_X(\mu) \otimes \mathscr{D}_{X,\lambda} \otimes \mathscr{O}_X(-\mu)\\ &\qquad \text{for any } \lambda \in \mathfrak{t} \text{ and any integral } \mu \in \mathfrak{t}.\end{aligned} \tag{9.2.2}$$

If λ is an anti-dominant integral weight (i.e., $\langle \alpha^\vee, \lambda\rangle \in \mathbb{Z}_{\le 0}$ for any $\alpha \in \Delta^+$), let $V(\lambda)$ be the irreducible G-module with lowest weight λ. Then we have

$$\Gamma(X; \mathscr{O}_X(\lambda)) \simeq V(\lambda). \tag{9.2.3}$$

Here the isomorphism $V(\lambda) \xrightarrow{\sim} \Gamma(X; \mathscr{O}_X(\lambda))$ is given as follows. Let us fix a highest weight vector $u_{-\lambda}$ of $V(-\lambda) = V(\lambda)^*$. Then, for any $v \in V(\lambda)$, the function $\langle v, gu_{-\lambda}\rangle$ in $g \in G$ is the corresponding global section of $\mathscr{O}_X(\lambda)$.

The following theorem is due to Beilinson-Bernstein ([1]).

Theorem 9.2.2. *Let λ be an element of $\mathfrak{t}^* \cong \mathrm{Hom}_B(\mathfrak{b}, \mathbb{C})$.*

(i) *We have*

$$H^k(X; \mathscr{D}_{X,\lambda}) \simeq \begin{cases} U_\lambda(\mathfrak{g}) & \text{for } k = 0,\\ 0 & \text{otherwise.}\end{cases}$$

(ii) *Assume that $\lambda - \rho$ is integrally anti-dominant. Then we have*

(a) *for any quasi-coherent $\mathscr{D}_{X,\lambda}$-module $\mathscr{M}$, we have*

$$H^n(X; \mathscr{M}) = 0 \quad \text{for any } n \neq 0,$$

(b) *for any $U_\lambda(\mathfrak{g})$-module M, we have an isomorphism*

$$M \xrightarrow{\sim} \Gamma(X; \mathscr{D}_{X,\lambda} \otimes_{U(\mathfrak{g})} M),$$

namely, the diagram

$$\begin{array}{ccc} \mathrm{Mod}\,(U_\lambda(\mathfrak{g})) & \xrightarrow{\mathscr{D}_{X,\lambda}\otimes_{U(\mathfrak{g})}\bullet} & \mathrm{Mod}\,(\mathscr{D}_{X,\lambda}) \\ & {}_{\mathrm{id}}\searrow & \downarrow{\scriptstyle \Gamma(X;\bullet)} \\ & & \mathrm{Mod}\,(U_\lambda(\mathfrak{g})) \end{array} \tag{9.2.4}$$

quasi-commutes.

(iii) *Assume that $\lambda - \rho$ is regular and integrally anti-dominant. Then*

$$\mathscr{M} \simeq \mathscr{D}_{X,\lambda} \otimes_{U(\mathfrak{g})} \Gamma(X; \mathscr{M}),$$

and we have an equivalence of categories

$$\mathrm{Mod}\,(U_\lambda(\mathfrak{g})) \simeq \mathrm{Mod}\,(\mathscr{D}_{X,\lambda}).$$

9.3 Quasi-equivariant D-modules on the symmetric space

We set $S = G/K$. Let $j\colon \mathrm{pt} \hookrightarrow S$ be the morphism given by the origin $s_0 \in S$. By Proposition 3.1.5, $j^*\colon \mathrm{Mod}\,(\mathscr{D}_S, G) \xrightarrow{\sim} \mathrm{Mod}\,(\mathfrak{g}, K)$ is an equivalence of categories. Since $\mathrm{D^b}(\mathscr{D}_S, G) = \mathrm{D^b}(\mathrm{Mod}\,(\mathscr{D}_S, G))$ by the definition, j^* induces an equivalence

(9.3.1) $$\mathbf{L}j^*\colon \mathrm{D^b}(\mathscr{D}_S, G) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{Mod}\,(\mathfrak{g}, K)).$$

Let

$$\Psi : \mathrm{D^b}(\mathrm{Mod}\,(\mathfrak{g}, K)) \xrightarrow{\sim} \mathrm{D^b}(\mathscr{D}_S, G)$$

be its quasi-inverse.

Consider the diagram:

$$\begin{array}{ccc} & X \times S & \\ {\swarrow}^{p_1} & & {\searrow}^{p_2} \\ X & & S. \end{array}$$

Set

$$\mathscr{M}_0 := \mathscr{D}_{X,-\lambda} \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1}.$$

It is an object of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_{X,-\lambda}, G)$.

For $\mathscr{L} \in \mathrm{D}^{\mathrm{b}}_{K,\mathrm{coh}}(\mathscr{D}_{X,\lambda})$, set $\mathscr{L}_0 = \mathrm{Ind}_K^G(\mathscr{L}) \in \mathrm{D}^{\mathrm{b}}_{G,\mathrm{coh}}(\mathscr{D}_{X\times S,\lambda})$. Let us calculate $\mathscr{M}_0 \overset{\mathbf{D}}{\circ} \mathscr{L}_0 \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_S, G)$.

We have

$$\begin{aligned} \mathscr{M}_0 \overset{\mathbf{D}}{\circ} \mathscr{L}_0 &= \mathbf{D}p_{2*}(\mathbf{D}p_1^*\mathscr{M}_0 \overset{\mathbf{D}}{\otimes} \mathscr{L}_0) \\ &\simeq \mathbf{R}p_{2*}(\mathscr{D}_{S\leftarrow X\times S} \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_{X\times S}} (p_1^*(\mathscr{D}_{X,-\lambda} \otimes \Omega_X^{\otimes -1}) \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{X\times S}} \mathscr{L}_0)). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\mathscr{D}_{S\leftarrow X\times S} \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_{X\times S}} (p_1^*(\mathscr{D}_{X,-\lambda} \otimes \Omega_X^{\otimes -1}) \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{X\times S}} \mathscr{L}_0) \\ &\qquad \simeq \Omega_X \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_X} (p_1^*(\mathscr{D}_{X,-\lambda} \otimes \Omega_X^{\otimes -1}) \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{X\times S}} \mathscr{L}_0) \\ &\qquad \simeq \mathscr{L}_0. \end{aligned}$$

Hence we obtain

$$\mathscr{M}_0 \overset{\mathbf{D}}{\circ} \mathscr{L}_0 \cong \mathbf{R}p_{2*}\mathscr{L}_0. \tag{9.3.2}$$

It is an object of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_S, G)$.

Let $\tilde{j}\colon X \to X \times S$ be the induced morphism $(x \mapsto (x, s_0))$. Then, we have

$$\mathbf{L}j^*\mathbf{R}p_{2*}\mathscr{L}_0 \simeq \mathbf{R}\Gamma(X; \mathbf{D}\tilde{j}^*\mathscr{L}_0) \simeq \mathbf{R}\Gamma(X; \mathscr{L}).$$

Thus we obtain the following proposition.

Proposition 9.3.1. *The diagram*

$$\begin{array}{ccc}
\mathrm{D}^{\mathrm{b}}_{K,\mathrm{coh}}(\mathscr{D}_{X,\lambda}) & \xrightarrow{\mathbf{R}\Gamma(X;\,\bullet\,)} & \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}_f(\mathfrak{g}, K)) \\
 & \searrow^{(\mathscr{D}_{X,-\lambda}\otimes\Omega_X^{\otimes -1})\overset{\mathbf{D}}{\circ}\mathrm{Ind}_K^G(\,\bullet\,)} & \downarrow{\scriptstyle \Psi}\,\wr \\
 & & \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_S, G)
\end{array}$$

quasi-commutes.

Proposition 9.3.2. *For any* $\lambda \in \mathfrak{t}^*$*, any* $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_{X,-\lambda}, G)$*, any* $\mathscr{L} \in \mathrm{D}^{\mathrm{b}}_{K,\mathrm{coh}}(\mathscr{D}_{X,\lambda})$ *and any integer* n*,*

$$H^n(\mathscr{M} \overset{\mathbf{D}}{\circ} \mathrm{Ind}_K^G \mathscr{L}) \simeq \Psi(M)$$

for some Harish-Chandra module M*.*

Proof. We know that $H^n(\mathscr{M} \overset{\mathbf{D}}{\circ} \mathrm{Ind}_K^G \mathscr{L}) \simeq \Psi(M)$ for some $M \in \mathrm{Mod}_f(\mathfrak{g}, K)$. Hence we need to show that M is $\mathfrak{z}(\mathfrak{g})$-finite. Since $\mathscr{M}$ has a resolution whose components are of the form $\mathscr{D}_{X,-\lambda}\otimes(\Omega_X^{\otimes -1}\otimes\mathscr{O}_X(\mu)\otimes V)$ for an integrable $\mu \in \mathfrak{t}^*$ and a finite-dimensional G-module V, we may assume from the beginning that $\mathscr{M} = \mathscr{D}_{X,-\lambda} \otimes \big(\Omega_X^{\otimes -1} \otimes \mathscr{O}_X(\mu)\big)$. In this case, we have by (9.2.2)

$$\begin{aligned}
\mathscr{M} &\simeq \big(\mathscr{O}_X(\mu) \otimes \mathscr{D}_{X,-\lambda-\mu} \otimes \mathscr{O}_X(-\mu)\big) \otimes \big(\Omega_X^{\otimes -1} \otimes \mathscr{O}_X(\mu)\big) \\
&\simeq \mathscr{O}_X(\mu) \overset{\mathbf{D}}{\otimes} (\mathscr{D}_{X,-\lambda-\mu} \otimes \Omega_X^{\otimes -1}),
\end{aligned}$$

which implies that

$$\begin{aligned}
\mathbf{D}p_1^*\mathscr{M} \overset{\mathbf{D}}{\otimes} \mathrm{Ind}_K^G(\mathscr{L}) &\simeq \mathbf{D}p_1^*(\mathscr{D}_{X,-\lambda-\mu} \otimes \Omega_X^{\otimes -1}) \overset{\mathbf{D}}{\otimes} \mathbf{D}p_1^*\mathscr{O}_X(\mu) \overset{\mathbf{D}}{\otimes} \mathrm{Ind}_K^G(\mathscr{L}) \\
&\simeq \mathbf{D}p_1^*(\mathscr{D}_{X,-\lambda-\mu} \otimes \Omega_X^{\otimes -1}) \overset{\mathbf{D}}{\otimes} \mathrm{Ind}_K^G(\mathscr{O}_X(\mu) \overset{\mathbf{D}}{\otimes} \mathscr{L}).
\end{aligned}$$

Hence, Proposition 9.3.1 implies

$$\begin{aligned}
\mathscr{M} \overset{\mathbf{D}}{\circ} \mathrm{Ind}_K^G(\mathscr{L}) &\simeq (\mathscr{D}_{X,-\lambda-\mu} \otimes \Omega_X^{\otimes -1}) \overset{\mathbf{D}}{\circ} \mathrm{Ind}_K^G(\mathscr{O}_X(\mu) \overset{\mathbf{D}}{\otimes} \mathscr{L}) \\
&\simeq \Psi\big(\mathbf{R}\Gamma(X; \mathscr{O}_X(\mu) \overset{\mathbf{D}}{\otimes} \mathscr{L})\big).
\end{aligned}$$

Since $\mathscr{O}_X(\mu) \overset{\mathbf{D}}{\otimes} \mathscr{L} \in \mathrm{D}^{\mathrm{b}}_{K,\mathrm{coh}}(\mathscr{D}_{X,\lambda+\mu})$, its cohomology $H^n(X; \mathscr{O}_X(\mu) \overset{\mathbf{D}}{\otimes} \mathscr{L})$ is a Harish-Chandra module. Q.E.D.

9.4 Matsuki correspondence

The following theorem is due to Matsuki ([22]).

Theorem 9.4.1. (i) *There are only finitely many K-orbits in X and also finitely many $G_{\mathbb{R}}$-orbits in X^{an}.*
(ii) *There is a one-to-one correspondence between the set of K-orbits in X and the one of $G_{\mathbb{R}}$-orbits.*

More precisely, a K-orbit E and a $G_{\mathbb{R}}$-orbit F correspond by the correspondence above if and only if one of the following equivalent conditions are satisfied:

(1) $E^{\mathrm{an}} \cap F$ is a $K_{\mathbb{R}}$-orbit,
(2) $E^{\mathrm{an}} \cap F$ is non-empty and compact.

Its sheaf-theoretic version is conjectured by the author [14] and proved by Mirković-Uzawa-Vilonen [23]. Let $S_{\mathbb{R}} = G_{\mathbb{R}}/K_{\mathbb{R}}$ be the Riemannian symmetric space and set $S = G/K$. Then S is an affine algebraic manifold. The canonical map $i\colon S_{\mathbb{R}} \hookrightarrow S^{\mathrm{an}}$ is a closed embedding.

We have the functor

$$\mathrm{Ind}_{K^{\mathrm{an}}}^{G^{\mathrm{an}}}\colon \mathrm{D}^{\mathrm{b}}_{K^{\mathrm{an}},\tau_\lambda}(\mathbb{C}_{X^{\mathrm{an}}}) \to \mathrm{D}^{\mathrm{b}}_{G^{\mathrm{an}},\tau_\lambda}(\mathbb{C}_{(X\times S)^{\mathrm{an}}}). \tag{9.4.1}$$

We define the functor

$$\Phi\colon\ \mathrm{D}^{\mathrm{b}}_{K^{\mathrm{an}},\tau_\lambda}(\mathbb{C}_{X^{\mathrm{an}}}) \to \mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}},\tau_\lambda}(\mathbb{C}_{X^{\mathrm{an}}})$$

by

$$\begin{aligned}\Phi(F) &= \mathrm{Ind}_{K^{\mathrm{an}}}^{G^{\mathrm{an}}}(F) \circ i_* i^! \mathbb{C}_{S^{\mathrm{an}}}[2\mathrm{d}_S] \\ &= \mathbf{R}(p_1^{\mathrm{an}})_!(\mathrm{Ind}_{K^{\mathrm{an}}}^{G^{\mathrm{an}}}(F) \otimes (p_2^{\mathrm{an}})^{-1} i_* i^! \mathbb{C}_{S^{\mathrm{an}}})[2\mathrm{d}_S].\end{aligned}$$

Here, p_1^{an} and p_2^{an} are the projections from $X^{\mathrm{an}} \times S^{\mathrm{an}}$ to X^{an} and S^{an}, respectively. Note that $i_* i^! \mathbb{C}_{S^{\mathrm{an}}}$ is isomorphic to $i_* \mathbb{C}_{S_{\mathbb{R}}}[-\mathrm{d}_S]$ (once we give an orientation of $S_{\mathbb{R}}$). Hence we have

$$\Phi(F) \simeq \mathbf{R}p_{1\mathbb{R}!}(\mathrm{Ind}_{K^{\mathrm{an}}}^{G^{\mathrm{an}}}(F)|_{X\times S_{\mathbb{R}}})[\mathrm{d}_S] \simeq \mathbf{R}p_{1\mathbb{R}!}(\mathrm{Ind}_{K_{\mathbb{R}}}^{G_{\mathbb{R}}}(F))[\mathrm{d}_S], \tag{9.4.2}$$

where $p_{1\mathbb{R}}\colon X^{\mathrm{an}} \times S_{\mathbb{R}} \to X^{\mathrm{an}}$ is the projection.

Theorem 9.4.2 ([23]). *The functor Φ induces equivalences of triangulated categories:*

$$\Phi\ :\ \begin{array}{ccc} \mathrm{D}^{\mathrm{b}}_{K^{\mathrm{an}},\tau_\lambda}(\mathbb{C}_{X^{\mathrm{an}}}) & \xrightarrow{\ \sim\ } & \mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}},\tau_\lambda}(\mathbb{C}_{X^{\mathrm{an}}}) \\ \bigcup & & \bigcup \\ \mathrm{D}^{\mathrm{b}}_{K^{\mathrm{an}},\tau_\lambda,\mathbb{C}\text{-}c}(\mathbb{C}_{X^{\mathrm{an}}}) & \xrightarrow{\ \sim\ } & \mathrm{D}^{\mathrm{b}}_{G_{\mathbb{R}},\tau_\lambda,\mathbb{R}\text{-}c}(\mathbb{C}_{X^{\mathrm{an}}})\,. \end{array}$$

We call Φ the *Matsuki correspondence.*

For some equivariant sheaves, the Matsuki correspondence is given as follows.

Proposition 9.4.3 ([23]). *Let* $i_Z\colon Z \hookrightarrow X$ *be a* K*-orbit in* X *and let* $i_{Z^a}: Z^a \hookrightarrow X^{\mathrm{an}}$ *be the* $G_{\mathbb{R}}$*-orbit corresponding to* Z*.*

(i) *The restriction functors induce equivalences of categories:*

$$\mathrm{Mod}_{K^{\mathrm{an}},\tau_\lambda}(\mathbb{C}_{Z^{\mathrm{an}}}) \xrightarrow{\sim} \mathrm{Mod}_{K_{\mathbb{R}},\tau_\lambda}(\mathbb{C}_{Z^{\mathrm{an}}\cap Z^a}) \xleftarrow{\sim} \mathrm{Mod}_{G_{\mathbb{R}},\tau_\lambda}(\mathbb{C}_{Z^a}).$$

(ii) *Assume that* $F \in \mathrm{Mod}_{K^{\mathrm{an}},\tau_\lambda}(\mathbb{C}_{Z^{\mathrm{an}}})$ *and* $F^a \in \mathrm{Mod}_{G_{\mathbb{R}},\tau_\lambda}(\mathbb{C}_{Z^a})$ *correspond by the equivalence above. Then we have*

$$\Phi(\mathbf{R}(i_Z^{\mathrm{an}})_*F) \simeq \mathbf{R}(i_{Z^a})_!F^a[2\,\mathrm{codim}_X Z].$$

The K-orbit $Kx_0 \subset X$ is a unique open K-orbit in X and $G_{\mathbb{R}}x_0 \subset X^{\mathrm{an}}$ is a unique closed $G_{\mathbb{R}}$-orbit in X^{an}. Set $X^{\mathbb{R}}_{\mathrm{min}} = G_{\mathbb{R}}/P_{\mathbb{R}}$. Then $X^{\mathbb{R}}_{\mathrm{min}} = G_{\mathbb{R}}x_0^{\mathrm{min}} = K_{\mathbb{R}}x_0^{\mathrm{min}}$ and it is a unique closed $G_{\mathbb{R}}$-orbit in $X^{\mathrm{an}}_{\mathrm{min}}$. We have

$$(Kx_0)^{\mathrm{an}} = (\pi^{-1}(Kx_0^{\mathrm{min}}))^{\mathrm{an}} \supset G_{\mathbb{R}}x_0 = K_{\mathbb{R}}x_0 = (\pi^{\mathrm{an}})^{-1}(X^{\mathbb{R}}_{\mathrm{min}}). \tag{9.4.3}$$

Let $j\colon Kx_0 \hookrightarrow X$ be the open embedding and $j^a\colon G_{\mathbb{R}}x_0 \hookrightarrow X^{\mathrm{an}}$ the closed embedding. Then as a particular case of Proposition 9.4.3, we have an isomorphism:

$$\Phi(\mathbf{R}j^{\mathrm{an}}_*F) \simeq j^a_*(F|_{G_{\mathbb{R}}x_0}) \tag{9.4.4}$$

for any K^{an}-equivariant local system F on $K^{\mathrm{an}}x_0$.

9.5 Construction of representations

For $M \in \mathrm{Mod}_f(\mathfrak{g}, K)$, let $\mathrm{Hom}_{(\mathfrak{g},K_{\mathbb{R}})}(M, \mathrm{C}^\infty(G_{\mathbb{R}}))$ be the set of homomorphisms from M to $\mathrm{C}^\infty(G_{\mathbb{R}})$ which commute with the actions of $\mathfrak{g}$ and $K_{\mathbb{R}}$. Here, $\mathfrak{g}$ and $K_{\mathbb{R}}$ act on $\mathrm{C}^\infty(G_{\mathbb{R}})$ through the right $G_{\mathbb{R}}$-action on $G_{\mathbb{R}}$. Then $G_{\mathbb{R}}$ acts on $\mathrm{Hom}_{(\mathfrak{g},K_{\mathbb{R}})}(M, \mathrm{C}^\infty(G_{\mathbb{R}}))$ through the left $G_{\mathbb{R}}$-action on $G_{\mathbb{R}}$.

Let us write by $\mathrm{C}^\infty(G_{\mathbb{R}})^{K_{\mathbb{R}}\text{-fini}}$ the set of $K_{\mathbb{R}}$-finite vectors of $\mathrm{C}^\infty(G_{\mathbb{R}})$. Then $\mathrm{C}^\infty(G_{\mathbb{R}})^{K_{\mathbb{R}}\text{-fini}}$ is a $(\mathfrak{g}, K)$-module and

$$\mathrm{Hom}_{(\mathfrak{g},K_{\mathbb{R}})}(M, \mathrm{C}^\infty(G_{\mathbb{R}})) \simeq \mathrm{Hom}_{(\mathfrak{g},K)}(M, \mathrm{C}^\infty(G_{\mathbb{R}})^{K_{\mathbb{R}}\text{-fini}}).$$

Note that, in our context, $K_{\mathbb{R}}$ is connected and hence the $\mathfrak{g}$-invariance implies the $K_{\mathbb{R}}$-invariance. Therefore, we have

$$\mathrm{Hom}_{(\mathfrak{g},K_{\mathbb{R}})}(M, \mathrm{C}^\infty(G_{\mathbb{R}})) \simeq \mathrm{Hom}_{\mathfrak{g}}(M, \mathrm{C}^\infty(G_{\mathbb{R}}))\,.$$

We endow $\mathrm{Hom}_{(\mathfrak{g},K_{\mathbb{R}})}(M, \mathrm{C}^\infty(G_{\mathbb{R}}))$ with the Fréchet nuclear topology as in Lemma 5.3.1.

In any way, $\mathrm{Hom}_{(\mathfrak{g},K_\mathbb{R})}(M, \mathrm{C}^\infty(G_\mathbb{R}))$ has a Fréchet nuclear $G_\mathbb{R}$-module structure. We denote it by $\mathrm{Hom}^{\mathrm{top}}_{(\mathfrak{g},K_\mathbb{R})}(M, \mathrm{C}^\infty(G_\mathbb{R}))$. Let us denote by

$$\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{(\mathfrak{g},K_\mathbb{R})}(\,\bullet\,, \mathrm{C}^\infty(G_\mathbb{R})) : \mathrm{D}^\mathrm{b}(\mathrm{Mod}_f(\mathfrak{g},K)) \to \mathrm{D}^\mathrm{b}(\mathbf{FN}_{G_\mathbb{R}})$$

its right derived functor.[5] Note that $\mathrm{Mod}_f(\mathfrak{g},K)$ has enough projectives, and any M can be represented by a complex P of projective objects in $\mathrm{Mod}_f(\mathfrak{g},K)$, and then $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{(\mathfrak{g},K_\mathbb{R})}(M, \mathrm{C}^\infty(G_\mathbb{R}))$ is represented by a complex $\mathrm{Hom}^{\mathrm{top}}_{(\mathfrak{g},K_\mathbb{R})}(P, \mathrm{C}^\infty(G_\mathbb{R}))$ of Fréchet nuclear $G_\mathbb{R}$-modules. Note that for a finite-dimensional K-module V, $U(\mathfrak{g}) \otimes_\mathfrak{k} V$ is a projective object of $\mathrm{Mod}_f(\mathfrak{g},K)$, and $\mathrm{Hom}^{\mathrm{top}}_{(\mathfrak{g},K_\mathbb{R})}(U(\mathfrak{g}) \otimes_\mathfrak{k} V, \mathrm{C}^\infty(G_\mathbb{R})) \simeq \mathrm{Hom}_{K_\mathbb{R}}(V, \mathrm{C}^\infty(G_\mathbb{R}))$.

Since we have $\mathrm{Hom}^{\mathrm{top}}_{(\mathfrak{g},K_\mathbb{R})}(M, \mathrm{C}^\infty(G_\mathbb{R})) \simeq \mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_S}(\Psi(M), \mathscr{C}^\infty_{S_\mathbb{R}})$ for any $M \in \mathrm{Mod}_f(\mathfrak{g},K)$, their right derived functors are isomorphic:

$$\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{(\mathfrak{g},K_\mathbb{R})}(M, \mathrm{C}^\infty(G_\mathbb{R})) \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_S}(\Psi(M), \mathscr{C}^\infty_{S_\mathbb{R}})$$

for any $M \in \mathrm{D}^\mathrm{b}(\mathrm{Mod}_f(\mathfrak{g},K))$.

Lemma 9.5.1. *Let M be a Harish-Chandra module. Then $\Psi(M)$ is an elliptic $\mathscr{D}_S$-module.*

Proof. Let Δ be a Casimir element of $U(\mathfrak{g})$. Then there exists a non-zero polynomial $a(t)$ such that $a(\Delta)M = 0$. Hence the characteristic variety $\mathrm{Ch}(\Psi(M)) \subset T^*S$ of $\Psi(M)$ is contained in the zero locus of the principal symbol of $L_S(\Delta)$. Then the result follows from the well-known fact that the Laplacian $L_S(\Delta)|_{S^\mathbb{R}}$ is an elliptic differential operator on $S_\mathbb{R}$. Q.E.D.

If the cohomologies of $M \in \mathrm{D}^\mathrm{b}(\mathrm{Mod}_f(\mathfrak{g},K))$ are Harish-Chandra modules, then $\Psi(M)$ is elliptic, and Proposition 6.3.3 implies

$$\text{(9.5.1)}\quad \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{(\mathfrak{g},K_\mathbb{R})}(M, \mathrm{C}^\infty(G_\mathbb{R})) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_S}(\Psi(M) \otimes i_*i^!\mathbb{C}_{S^{\mathrm{an}}}, \mathscr{O}_{S^{\mathrm{an}}}).$$

There is a dual notion. For $M \in \mathrm{Mod}_f(\mathfrak{g},K)$, let $\Gamma_c(G_\mathbb{R}; \mathscr{D}ist_{G_\mathbb{R}}) \otimes_{(\mathfrak{g},K_\mathbb{R})} M$ be the quotient of $\Gamma_c(G_\mathbb{R}; \mathscr{D}ist_{G_\mathbb{R}}) \otimes_\mathbb{C} M$ by the linear subspace spanned by vectors $(R_{G_\mathbb{R}}(A)u) \otimes v + u \otimes (Av)$ and $(ku) \otimes (kv) - u \otimes v$ ($u \in \Gamma_c(G_\mathbb{R}; \mathscr{D}ist_{G_\mathbb{R}})$, $v \in M$, $A \in \mathfrak{g}$, $k \in K_\mathbb{R}$). Here, we consider it as a vector space (not considering the topology). In our case, $K_\mathbb{R}$ is connected, and $K_\mathbb{R}$ acts trivially on $\Gamma_c(G_\mathbb{R}; \mathscr{D}ist_{G_\mathbb{R}}) \otimes_{U(\mathfrak{k})} M$. Therefore, we have

$$\Gamma_c(G_\mathbb{R}; \mathscr{D}ist_{G_\mathbb{R}}) \otimes_{(\mathfrak{g},K_\mathbb{R})} M \simeq \Gamma_c(G_\mathbb{R}; \mathscr{D}ist_{G_\mathbb{R}}) \otimes_{U(\mathfrak{g})} M.$$

It is a right exact functor from $\mathrm{Mod}_f(\mathfrak{g},K)$ to the category $\mathrm{Mod}(\mathbb{C})$ of $\mathbb{C}$-vector spaces. Let

[5] We may write here $\mathrm{Hom}^{\mathrm{top}}_{U(\mathfrak{g})}(M, \mathrm{C}^\infty(G_\mathbb{R}))$, but we use this notation in order to emphasize that it is calculated not on $\mathrm{Mod}(U(\mathfrak{g}))$ but on $\mathrm{Mod}_f(\mathfrak{g},K)$.

$$\Gamma_c(G_{\mathbb{R}};\mathscr{D}ist_{G_{\mathbb{R}}})\overset{\mathbf{L}}{\otimes}_{(\mathfrak{g},K_{\mathbb{R}})}\bullet : \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}_f(\mathfrak{g},K)) \to \mathrm{D}^{\mathrm{b}}(\mathbb{C}) \tag{9.5.2}$$

be its left derived functor.

For any $M \in \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}_f(\mathfrak{g},K))$, we can take a quasi-isomorphism $P^\bullet \to M$ such that each P^n has a form $U(\mathfrak{g}) \otimes_{\mathfrak{k}} V^n$ for a finite-dimensional K-module V^n. Then, we have

$$\begin{aligned}\Gamma_c(G_{\mathbb{R}};\mathscr{D}ist_{G_{\mathbb{R}}}) \otimes_{(\mathfrak{g},K_{\mathbb{R}})} P^n &\simeq \Gamma_c(G_{\mathbb{R}};\mathscr{D}ist_{G_{\mathbb{R}}}) \otimes_{U(\mathfrak{k})} V^n \\ &\simeq (\Gamma_c(G_{\mathbb{R}};\mathscr{D}ist_{G_{\mathbb{R}}}) \otimes_{\mathbb{C}} V^n)^{K_{\mathbb{R}}},\end{aligned}$$

where the superscript $K_{\mathbb{R}}$ means the set of $K_{\mathbb{R}}$-invariant vectors. The $K_{\mathbb{R}}$-module structure on $\Gamma_c(G_{\mathbb{R}};\mathscr{D}ist_{G_{\mathbb{R}}})$ is by the right action of $K_{\mathbb{R}}$ on $G_{\mathbb{R}}$. By the left action of $G_{\mathbb{R}}$ on $G_{\mathbb{R}}$, $G_{\mathbb{R}}$ acts on $\Gamma_c(G_{\mathbb{R}};\mathscr{D}ist_{G_{\mathbb{R}}}) \otimes_{(\mathfrak{g},K_{\mathbb{R}})} P^n$. Hence it belongs to $\mathbf{DFN}_{G_{\mathbb{R}}}$. The object $\Gamma_c(G_{\mathbb{R}};\mathscr{D}ist_{G_{\mathbb{R}}}) \otimes_{(\mathfrak{g},K_{\mathbb{R}})} P^\bullet \in \mathrm{D}^{\mathrm{b}}(\mathbf{DFN}_{G_{\mathbb{R}}})$ does not depend on the choice of a quasi-isomorphism $P^\bullet \to M$, and we denote it by $\Gamma_c(G_{\mathbb{R}};\mathscr{D}ist_{G_{\mathbb{R}}})\overset{\mathbf{L}}{\otimes}_{(\mathfrak{g},K_{\mathbb{R}})}M$. Thus we have constructed a functor:

$$\Gamma_c(G_{\mathbb{R}};\mathscr{D}ist_{G_{\mathbb{R}}})\overset{\mathbf{L}}{\otimes}_{(\mathfrak{g},K_{\mathbb{R}})}\bullet : \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}_f(\mathfrak{g},K)) \to \mathrm{D}^{\mathrm{b}}(\mathbf{DFN}_{G_{\mathbb{R}}}). \tag{9.5.3}$$

If we forget the topology and the equivariance, (9.5.3) reduces to (9.5.2).

We have

$$\Gamma_c(G_{\mathbb{R}};\mathscr{D}ist_{G_{\mathbb{R}}})\overset{\mathbf{L}}{\otimes}_{(\mathfrak{g},K_{\mathbb{R}})}M \simeq \Big(\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{(\mathfrak{g},K_{\mathbb{R}})}(M, \mathrm{C}^\infty(G_{\mathbb{R}}))\Big)^* \tag{9.5.4}$$

in $\mathrm{D}^{\mathrm{b}}(\mathbf{DFN}_{G_{\mathbb{R}}})$. (Here, we fix an invariant measure on $G_{\mathbb{R}}$.)

In general, $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{(\mathfrak{g},K_{\mathbb{R}})}(M, \mathrm{C}^\infty(G_{\mathbb{R}}))$ and $\Gamma_c(G_{\mathbb{R}};\mathscr{D}ist_{G_{\mathbb{R}}})\overset{\mathbf{L}}{\otimes}_{(\mathfrak{g},K_{\mathbb{R}})}M$ are not strict (see Theorems 10.4.1 and 10.4.2).

9.6 Integral transformation formula

Since X has finitely many K-orbits, the Riemann-Hilbert correspondence (Theorem 7.14.1) implies the following theorem.

Theorem 9.6.1. *The de Rham functor gives an equivalence of categories:*

$$\mathrm{DR}_X\colon \mathrm{D}^{\mathrm{b}}_{K,\,\mathrm{coh}}(\mathscr{D}_{X,\lambda}) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{K^{\mathrm{an}},\tau_{-\lambda},\mathbb{C}\text{-}c}(\mathbb{C}_{X^{\mathrm{an}}}). \tag{9.6.1}$$

Recall that the de Rham functor is defined by

$$\mathrm{DR}_X\colon \mathscr{M} \mapsto \mathbf{R}\mathscr{H}om_{\mathscr{D}_{X^{\mathrm{an}},\lambda}}(\mathscr{O}_{X^{\mathrm{an}}}(\lambda), \mathscr{M}^{\mathrm{an}}),$$

where $\mathscr{M}^{\mathrm{an}} = \mathscr{D}_{X^{\mathrm{an}},\lambda} \otimes_{\mathscr{D}_{X,\lambda}} \mathscr{M}$. Similarly to (9.4.1), we have the equivalence of categories:

$$\mathrm{Ind}_K^G\colon \mathrm{D}^{\mathrm{b}}_{K,\mathrm{coh}}(\mathscr{D}_{X,\lambda}) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{G,\mathrm{coh}}(\mathscr{D}_{X\times S,\lambda}) \tag{9.6.2}$$

and a quasi-commutative diagram

$$\begin{array}{ccc} \mathrm{D}^{\mathrm{b}}_{K,\mathrm{coh}}(\mathscr{D}_{X,\lambda}) & \xrightarrow{\mathrm{Ind}_K^G} & \mathrm{D}^{\mathrm{b}}_{G,\mathrm{coh}}(\mathscr{D}_{X\times S,\lambda}) \\ {\scriptstyle \mathrm{DR}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{DR}_{X\times S}} \\ \mathrm{D}^{\mathrm{b}}_{K^{\mathrm{an}},\tau_{-\lambda},\mathbb{C}\text{-c}}(\mathbb{C}_{X^{\mathrm{an}}}) & \xrightarrow{\mathrm{Ind}_{K^{\mathrm{an}}}^{G^{\mathrm{an}}}} & \mathrm{D}^{\mathrm{b}}_{G^{\mathrm{an}},\tau_{-\lambda},\mathbb{C}\text{-c}}(\mathbb{C}_{(X\times S)^{\mathrm{an}}}). \end{array}$$

Consider the diagram:

$$\begin{array}{ccccc} & & X\times S & & \\ & {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} & \\ X & & & & S \xleftarrow[i]{} S_{\mathbb{R}}. \end{array}$$

Let us take $\mathscr{L}\in \mathrm{D}^{\mathrm{b}}_{K,\mathrm{coh}}(\mathscr{D}_{X,\lambda})$ and set $\mathscr{L}_0=\mathrm{Ind}_K^G\mathscr{L}\in\mathrm{D}^{\mathrm{b}}_G(\mathscr{D}_{X\times S,\lambda})$. Set $L=\mathrm{DR}_X(\mathscr{L}):=\mathbf{R}\mathscr{H}om_{\mathscr{D}_{X^{\mathrm{an}},\lambda}}(\mathscr{O}_{X^{\mathrm{an}}}(\lambda),\mathscr{L}_0^{\mathrm{an}})\in\mathrm{D}^{\mathrm{b}}_{K^{\mathrm{an}},\tau_{-\lambda},\mathbb{C}\text{-c}}(\mathbb{C}_{X^{\mathrm{an}}})$ and $L_0=\mathrm{Ind}_{K^{\mathrm{an}}}^{G^{\mathrm{an}}}L=\mathrm{DR}_{X\times S}(\mathscr{L}_0)\in\mathrm{D}^{\mathrm{b}}_{G^{\mathrm{an}},\tau_{-\lambda},\mathbb{C}\text{-c}}(\mathbb{C}_{(X\times S)^{\mathrm{an}}})$. Let $\mathscr{M}\in\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_{X,-\lambda},G)$.

Then Theorem 8.2.1 (see Remark 8.2.2) immediately implies the following result.

Proposition 9.6.2. *For $\mathscr{M}\in\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_{X,-\lambda},G)$ and $\mathscr{L}\in\mathrm{D}^{\mathrm{b}}_{K,\mathrm{coh}}(\mathscr{D}_{X,\lambda})$ and $L=\mathrm{DR}_X(\mathscr{L})\in\mathrm{D}^{\mathrm{b}}_{K^{\mathrm{an}},\tau_{-\lambda},\mathbb{C}\text{-}c}(\mathbb{C}_{X^{\mathrm{an}}})$, we have*

$$\begin{aligned}(9.6.3)\qquad &\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_S}\big((\mathscr{M}\overset{\mathbf{D}}{\circ}\mathrm{Ind}_K^G(\mathscr{L}))\otimes i_*i^!\mathbb{C}_{S^{\mathrm{an}}},\mathscr{O}_{S^{\mathrm{an}}}\big)\\ &\quad\simeq\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_{X,-\lambda}}(\mathscr{M}\otimes\Phi(L),\mathscr{O}_{X^{\mathrm{an}}}(-\lambda))[d_X]\end{aligned}$$

in $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$.

Let us recall the equivalence $\mathbf{L}j^*\colon\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_S,G)\xrightarrow{\sim}\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}_f(\mathfrak{g},K))$ in (9.3.1). Since $\mathbf{L}j^*(\mathscr{M}\overset{\mathbf{D}}{\circ}\mathrm{Ind}_K^G(\mathscr{L}))$ has Harish-Chandra modules as cohomologies by Proposition 9.3.2, the isomorphism (9.5.1) reads as

$$\begin{aligned}(9.6.4)\qquad &\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_S}(\mathscr{M}\overset{\mathbf{D}}{\circ}\mathrm{Ind}_K^G(\mathscr{L})\otimes i_*i^!\mathbb{C}_{S^{\mathrm{an}}},\mathscr{O}_{S^{\mathrm{an}}})\\ &\quad\simeq\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{(\mathfrak{g},K_{\mathbb{R}})}(\mathbf{L}j^*(\mathscr{M}\overset{\mathbf{D}}{\circ}\mathrm{Ind}_K^G(\mathscr{L})),\mathrm{C}^\infty(G_{\mathbb{R}}))\end{aligned}$$

in $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$. Thus we obtain the following proposition.

Proposition 9.6.3. *For $\mathscr{M}\in\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_{X,-\lambda},G)$ and $\mathscr{L}\in\mathrm{D}^{\mathrm{b}}_{K,\,\mathrm{coh}}(\mathscr{D}_{X,\lambda})$, we have*

$$\begin{aligned}(9.6.5)\qquad &\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{(\mathfrak{g},K_{\mathbb{R}})}\big(\mathbf{L}j^*(\mathscr{M}\overset{\mathbf{D}}{\circ}\mathrm{Ind}_K^G\mathscr{L}),\mathrm{C}^\infty(G_{\mathbb{R}})\big)\\ &\quad\simeq\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_{X,-\lambda}}(\mathscr{M}\otimes\Phi(\mathrm{DR}_X(\mathscr{L})),\mathscr{O}_{X^{\mathrm{an}}}(-\lambda))[d_X].\end{aligned}$$

Now let us take as $\mathscr{M}$ the quasi-G-equivariant $\mathscr{D}_{X,-\lambda}$-module

$$\mathscr{M}_0 := \mathscr{D}_{X,-\lambda} \otimes \Omega_X^{\otimes -1}.$$

Then we have by Proposition 9.3.1

$$\mathbf{L}j^*(\mathscr{M}_0 \overset{\mathbf{D}}{\circ} \operatorname{Ind}_K^G(\mathscr{L})) \simeq \mathbf{R}\Gamma(X;\mathscr{L}).$$

On the other hand, we have

$$\begin{aligned} &\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_{X,-\lambda}}(\mathscr{M}_0 \otimes \Phi(L), \mathscr{O}_{X^{\mathrm{an}}}(-\lambda)) \\ &\qquad \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathbb{C}}(\Phi(L), \Omega_{X^{\mathrm{an}}} \otimes_{\mathscr{O}_{X^{\mathrm{an}}}} \mathscr{O}_{X^{\mathrm{an}}}(-\lambda)) \\ &\qquad \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathbb{C}}(\Phi(L), \mathscr{O}_{X^{\mathrm{an}}}(-\lambda + 2\rho)). \end{aligned}$$

Here the last isomorphism follows from $\Omega_X \simeq \mathscr{O}_X(2\rho)$.

Thus we obtain the following theorem.

Theorem 9.6.4. *For $\mathscr{L} \in \mathrm{D}^{\mathrm{b}}_K(\mathscr{D}_{X,\lambda})$, we have an isomorphism*

$$\begin{aligned} &\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{(\mathfrak{g},K_{\mathbb{R}})}(\mathbf{R}\Gamma(X;\mathscr{L}), \mathrm{C}^\infty(G_{\mathbb{R}})) \\ &\qquad \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathbb{C}}(\Phi(\mathrm{DR}_X(\mathscr{L})), \mathscr{O}_{X^{\mathrm{an}}}(-\lambda + 2\rho))[\mathrm{d}_X] \end{aligned} \tag{9.6.6}$$

in $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$.

Taking their dual, we obtain the following theorem.

Theorem 9.6.5. *For $\mathscr{L} \in \mathrm{D}^{\mathrm{b}}_K(\mathscr{D}_{X,\lambda})$, we have an isomorphism*

$$\begin{aligned} &\Gamma_c(G_{\mathbb{R}}; \mathscr{D}ist_{G_{\mathbb{R}}}) \overset{\mathbf{L}}{\otimes}_{(\mathfrak{g},K_{\mathbb{R}})} \mathbf{R}\Gamma(X;\mathscr{L}) \\ &\qquad \simeq \mathbf{R}\Gamma^{\mathrm{top}}_{\mathrm{c}}(X^{\mathrm{an}}; \Phi(\mathrm{DR}_X(\mathscr{L})) \otimes \mathscr{O}_{X^{\mathrm{an}}}(\lambda)) \end{aligned} \tag{9.6.7}$$

in $\mathrm{D}^{\mathrm{b}}(\mathbf{DFN}_{G_{\mathbb{R}}})$.

These results are conjectured in [14, Conjecture 3].

10 Vanishing Theorems

10.1 Preliminary

In this section, let us show that, for any Harish-Chandra module M, the object $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{(\mathfrak{g},K_{\mathbb{R}})}(M, \mathrm{C}^\infty(G_{\mathbb{R}}))$ of $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$ is strict and

$$\mathrm{Ext}^n_{(\mathfrak{g},K_{\mathbb{R}})}(M, \mathrm{C}^\infty(G_{\mathbb{R}})) := H^n\bigl(\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{(\mathfrak{g},K_{\mathbb{R}})}(M, \mathrm{C}^\infty(G_{\mathbb{R}}))\bigr) = 0 \quad \text{for } n \neq 0.$$

In order to prove this, we start by the calculation of the both sides of (9.6.6) for a K-equivariant holonomic $\mathscr{D}_{X,\lambda}$-module $\mathscr{L}$ such that

$$\mathscr{L} \xrightarrow{\sim} j_* j^{-1} \mathscr{L}, \tag{10.1.1}$$

where $j\colon Kx_0 \hookrightarrow X$ is the open embedding of the open K-orbit Kx_0 into X. There exists a cartesian product

$$\begin{array}{ccc} Kx_0 & \xhookrightarrow{\ j\ } & X \\ \downarrow & \square & \downarrow{\scriptstyle \pi} \\ Kx_0^{\min} & \hookrightarrow & X_{\min}. \end{array}$$

Since $Kx_0^{\min} \cong K/M$ is an affine variety, $Kx_0^{\min} \to X_{\min}$ is an affine morphism, and hence $j\colon Kx_0 \hookrightarrow X$ is an affine morphism. Therefore

$$\mathbf{D}^n j_* j^{-1} \mathscr{M} = 0 \quad \text{for } n \neq 0 \text{ and an arbitrary } \mathscr{M} \in \mathrm{Mod}(\mathscr{D}_X). \tag{10.1.2}$$

Hence by the hypothesis (10.1.1), we have

$$\mathscr{L} \xrightarrow{\sim} \mathbf{D} j_* j^{-1} \mathscr{L}. \tag{10.1.3}$$

Let V be the stalk $\mathscr{L}(x_0)$ regarded as a $(K \cap B)$-module. Then its infinitesimal action coincides with $\mathfrak{k} \cap \mathfrak{b} \to \mathfrak{b} \xrightarrow{\lambda} \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(V)$ by Lemma 9.1.1.

Hence, if $\mathscr{L} \neq 0$, then we have

$$\lambda|_{\mathfrak{t}\cap\mathfrak{k}} \text{ is integral, in particular } \langle \alpha^\vee, \lambda \rangle \in \mathbb{Z} \text{ for any } \alpha \in \Delta_k^+. \tag{10.1.4}$$

Recall that we say $\lambda|_{\mathfrak{t}\cap\mathfrak{k}}$ is integral if $\lambda|_{\mathfrak{t}\cap\mathfrak{k}}$ is the differential of a character of $K \cap T = M \cap T$.

Conversely, for a $(K \cap B)$-module V whose infinitesimal action coincides with $\mathfrak{k} \cap \mathfrak{b} \to \mathfrak{b} \xrightarrow{\lambda} \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(V)$, there exists a K-equivariant $\mathscr{D}_{X,\lambda}$-module $\mathscr{L}$ such that it satisfies (10.1.1) and $\mathscr{L}(x_0) \simeq V$ (see by Lemma 9.1.1).

10.2 Calculation (I)

Let $\mathscr{L}$ be a K-equivariant coherent $\mathscr{D}_{X,\lambda}$-module satisfying (10.1.1).

Recall that $\pi\colon X \simeq G/B \to X_{\min} = G/P$ is a canonical morphism. Let $s\colon X_0 := \pi^{-1}(x_0^{\min}) \to X$ be the embedding. Then $X_0 \simeq P/B \simeq M/(M \cap B)$ is the flag manifold of M. Note that $\mathscr{L}|_{Kx_0}$ is a locally free $\mathscr{O}_{Kx_0}$-module ($Kx_0 = \pi^{-1}(Kx_0^{\min})$ is an open subset of X). Hence we have $\mathbf{D}s^*\mathscr{L} \simeq s^*\mathscr{L}$. Since X_0 is the flag manifold of M and $s^*\mathscr{L}$ is a $\mathscr{D}_{X_0,\lambda}$-module, we have by Theorem 9.2.2

$$H^n(X_0; s^*\mathscr{L}) = 0 \quad \text{for } n \neq 0 \tag{10.2.1}$$

under the condition:

$$\lambda|_{\mathfrak{t}\cap\mathfrak{k}} \text{ is integral and } \langle\alpha^\vee,\lambda\rangle \in \mathbb{Z}_{\le 0} \text{ for } \alpha\in\Delta_k^+, \tag{10.2.2}$$

Hence $R^n\pi_*\mathscr{L}\big|_{Kx_0^{\min}} = 0$ for $n \neq 0$, and we have

$$H^n(X;\mathscr{L}) = H^n(Kx_0;\mathscr{L}) = H^n(Kx_0^{\min};\pi_*(\mathscr{L})). \tag{10.2.3}$$

Since $Kx_0^{\min} \cong K/M$ is an affine variety, we obtain

$$\begin{aligned} &H^n(X;\mathscr{L}) = 0 \quad \text{for } n\neq 0\\ &\qquad\qquad \text{under the conditions (10.1.1) and (10.2.2).}\end{aligned} \tag{10.2.4}$$

Now let us calculate $\Gamma(Kx_0;\mathscr{L})$. The sheaf $\mathscr{L}$ is a K-equivariant vector bundle on Kx_0. We have $Kx_0 = K/(K\cap B)$. Hence $\mathscr{L}$ is determined by the isotropy representation of $K\cap B$ on the stalk $V := \mathscr{L}(x_0)$ of $\mathscr{L}$ at x_0. We have as a K-module

$$\Gamma(X;\mathscr{L}) = \Gamma(Kx_0;\mathscr{L}) \cong (\mathscr{O}_K(K)\otimes V)^{K\cap B}. \tag{10.2.5}$$

Here the action of $K\cap B$ on $\mathscr{O}_K(K)\otimes V$ is the diagonal action where the action on $\mathscr{O}_K(K)$ is through the right multiplication of $K\cap B$ on K. The superscript $K\cap B$ means the space of $(K\cap B)$-invariant vectors. The K-module structure on $(\mathscr{O}_K(K)\otimes V)^{K\cap B}$ is through the left K-action on K.

Thus we obtain the following proposition.

Proposition 10.2.1. *Assume that λ satisfies* (10.2.2) *and a K-equivariant holonomic $\mathscr{D}_{X,\lambda}$-module $\mathscr{L}$ satisfies* (10.1.1)*, and set $V = \mathscr{L}(x_0)$. Then we have*

$$H^n(X;\mathscr{L}) \cong \begin{cases} (\mathscr{O}_K(K)\otimes V)^{K\cap B} & \text{for } n=0,\\ 0 & \text{for } n\neq 0\end{cases} \tag{10.2.6}$$

as a K-module.

For a $(\mathfrak{g},K)$-module M, we shall calculate $\mathrm{Hom}_{(\mathfrak{g},K)}(M,\Gamma(X;\mathscr{L}))$. We have the isomorphism $\mathrm{Hom}_K(M,\Gamma(X;\mathscr{L})) \xrightarrow{\sim} \mathrm{Hom}_{K\cap B}(M,V)$ by the evaluation map $\psi\colon \Gamma(X;\mathscr{L}) \to \mathscr{L}(x_0) = V$. Since $\mathscr{L}$ is a $\mathscr{D}_{X,\lambda}$-module, we have

$$\psi(At) = \langle\lambda, A\rangle\psi(t) \quad \text{for any } A\in\mathfrak{b} \text{ and } t\in\Gamma(X;\mathscr{L}). \tag{10.2.7}$$

Indeed, $L_X(A) - \langle\lambda,A\rangle \in \mathfrak{m}_{x_0}\mathscr{D}_{X,\lambda}$ for any $A\in\mathfrak{b}$, where $\mathfrak{m}_{x_0}$ is the maximal ideal of $(\mathscr{O}_X)_{x_0}$.

Lemma 10.2.2. *For any $(\mathfrak{g},K)$-module M, and $\mathscr{L}\in\mathrm{Mod}_K(\mathscr{D}_{X,\lambda})$ satisfying* (10.1.1)*, we have*

$$\begin{aligned}&\mathrm{Hom}_{(\mathfrak{g},K)}(M,\Gamma(X;\mathscr{L}))\\ &\quad\cong \big\{f\in\mathrm{Hom}_{K\cap B}(M,\mathscr{L}(x_0))\,;\\ &\qquad\qquad f(As) = \langle\lambda,A\rangle f(s) \text{ for any } A\in\mathfrak{b} \text{ and } s\in M\big\}.\end{aligned} \tag{10.2.8}$$

Proof. Set $V = \mathscr{L}(x_0)$.

For $h \in \mathrm{Hom}_{(\mathfrak{g},K)}(M, \Gamma(X;\mathscr{L}))$, let $f \in \mathrm{Hom}_{K\cap B}(M, V)$ be the element $\psi\circ h$. Since h is $\mathfrak{g}$-linear, (10.2.7) implies that f satisfies the condition: $f(As) = \psi(h(As)) = \psi(Ah(s)) = \langle\lambda, A\rangle\psi(h(s)) = \langle\lambda, A\rangle f(s)$ for any $A \in \mathfrak{b}$ and $s \in M$.

Conversely, for $f \in \mathrm{Hom}_{K\cap B}(M, V)$ such that $f(As) = \langle\lambda, A\rangle f(s)$ for $A \in \mathfrak{b}$ and $s \in M$, let $h \in \mathrm{Hom}_K(M, \Gamma(X;\mathscr{L}))$ be the corresponding element: $\psi(h(s)) = f(s)$.

Then, we obtain

$$h(As) = Ah(s) \quad \text{at } x = x_0 \text{ for any } A \in \mathfrak{g}. \tag{10.2.9}$$

Indeed, we have $\mathfrak{g} = \mathfrak{k} + \mathfrak{b}$. The equation (10.2.9) holds for $A \in \mathfrak{k}$ by the K-equivariance of h, and also for $A \in \mathfrak{b}$ because

$$f(As) = \langle\lambda, A\rangle f(s) = \langle\lambda, A\rangle\psi(h(s)) = \psi(Ah(s)).$$

Since h is K-equivariant, $h(As) = Ah(s)$ holds at any point of Kx_0. Therefore we have $h(As) = Ah(s)$. Q.E.D.

10.3 Calculation (II)

Let $\mathscr{L} \in \mathrm{Mod}_{K,\mathrm{coh}}(\mathscr{D}_{X,\lambda})$, and set $L = \mathrm{DR}_X(\mathscr{L}) \in \mathrm{D}^{\mathrm{b}}_{K^{\mathrm{an}},\tau_{-\lambda}}(\mathbb{C}_{X^{\mathrm{an}}})$. Now, we shall calculate $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathbb{C}}(\Phi(L), \mathscr{O}_{X^{\mathrm{an}}}(-\lambda+2\rho))[\mathrm{d}_X]$, the right-hand side of (9.6.6), under the conditions (10.1.1) and (10.2.2). We do it forgetting the topology and the equivariance.

By the assumption (10.2.2), we can decompose $\lambda = \lambda_1 + \lambda_0$ where λ_1 is integral and $\lambda_0|_{\mathfrak{k}\cap\mathfrak{t}} = 0$. Then λ_0 may be regarded as a P-invariant map $\mathrm{Lie}(P) = \mathfrak{m}\oplus\mathfrak{a}\oplus\mathfrak{n} \to \mathbb{C}$. Hence, we can consider the twisting data $\tau_{\lambda_0, X^{\mathrm{an}}_{\mathrm{min}}}$ on $X^{\mathrm{an}}_{\mathrm{min}}$. Then, the twisting data τ_{λ_0} on X^{an} is isomorphic to $\pi^*\tau_{\lambda_0, X^{\mathrm{an}}_{\mathrm{min}}}$. Since the twisting data τ_{λ_1} is trivial, we have $\tau_\lambda \cong \pi^*\tau_{\lambda_0, X^{\mathrm{an}}_{\mathrm{min}}}$.

Since $\mathscr{L} \simeq \mathbf{D}j_*j^{-1}\mathscr{L}$, we have $L \simeq \mathbf{R}j_*j^{-1}L$. Hence, (9.4.4) implies that

$$\Phi(L) = j^a_*(L|_{G_{\mathbb{R}}x_0}). \tag{10.3.1}$$

Here, $j^a\colon G_{\mathbb{R}}x_0 \hookrightarrow X$ is the closed embedding. We can regard $L|_{G_{\mathbb{R}}x_0}$ as a $G_{\mathbb{R}}$-equivariant $(\pi^*\tau_{-\lambda_0, X^{\mathrm{an}}_{\mathrm{min}}})$-twisted local system

Then there exists a $G_{\mathbb{R}}$-equivariant $(\tau_{-\lambda_0, X^{\mathrm{an}}_{\mathrm{min}}})$-twisted local system $\widetilde{L}$ on $X^{\mathbb{R}}_{\mathrm{min}}$ such that $L|_{G_{\mathbb{R}}x_0} \simeq (\pi^{\mathrm{an}})^{-1}\widetilde{L}$, because the fiber of π^{an} is simply connected.

Hence, we have

$$\begin{aligned}
&\mathbf{R}\mathrm{Hom}_{\mathbb{C}}\left(\Phi(L), \mathscr{O}_{X^{\mathrm{an}}}(-\lambda+2\rho)\right)[\mathrm{d}_X] \\
&\simeq \mathbf{R}\mathrm{Hom}_{\mathbb{C}}\left((\pi^{\mathrm{an}})^{-1}\widetilde{L}, \mathscr{O}_{X^{\mathrm{an}}}(-\lambda+2\rho)\right)[\mathrm{d}_X] \\
&\simeq \mathbf{R}\mathrm{Hom}_{\mathbb{C}}\left(\widetilde{L}, \mathbf{R}(\pi^{\mathrm{an}})_*\mathscr{O}_{X^{\mathrm{an}}}(-\lambda+2\rho)\right)[\mathrm{d}_X].
\end{aligned}$$

On the other hand, we have

$$\mathbf{R}(\pi^{\mathrm{an}})_*\mathscr{O}_{X^{\mathrm{an}}}(-\lambda+2\rho)\simeq \mathscr{O}_{X^{\mathrm{an}}_{\mathrm{min}}}(-\lambda_0)\otimes \mathbf{R}(\pi^{\mathrm{an}})_*\mathscr{O}_{X^{\mathrm{an}}}(-\lambda_1+2\rho),$$

and we have, by the Serre-Grothendieck duality,

$$\begin{aligned}\mathbf{R}\pi_*\mathscr{O}_X(-\lambda_1+2\rho)[\mathrm{d}_X] &\simeq \mathbf{R}\pi_*\,\mathbf{R}\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X(\lambda_1),\Omega_X)[\mathrm{d}_X]\\ &\simeq \mathbf{R}\mathscr{H}om_{\mathscr{O}_{X_{\mathrm{min}}}}(\mathbf{R}\pi_*\mathscr{O}_X(\lambda_1),\Omega_{X_{\mathrm{min}}})[\mathrm{d}_{X_{\mathrm{min}}}].\end{aligned}$$

Since $\lambda_1|_{\mathfrak{k}\cap\mathfrak{t}}=\lambda|_{\mathfrak{k}\cap\mathfrak{t}}$ is anti-dominant, $\mathbf{R}\pi_*\mathscr{O}_X(\lambda_1)$ is concentrated at degree 0 by Theorem 9.2.2 (ii), and $\mathcal{V}=\pi_*\mathscr{O}_X(\lambda_1)$ is the G-equivariant locally free $\mathscr{O}_{X_{\mathrm{min}}}$-module associated with the representation

$$P\to MA\to \mathrm{Aut}(V_{\lambda_1}),$$

where V_{λ_1} is the irreducible (MA)-module with lowest weight λ_1 (see (9.2.3)).

Thus we obtain

$$\begin{aligned}&\mathbf{R}\mathrm{Hom}_{\mathbb{C}}(\Phi(L),\mathscr{O}_{X^{\mathrm{an}}}(-\lambda+2\rho))[\mathrm{d}_X]\\ &\quad\simeq \mathbf{R}\mathrm{Hom}_{\mathscr{O}_{X_{\mathrm{min}}}}(\mathbf{R}\pi_*\mathscr{O}_X(\lambda_1)\otimes\mathscr{O}_{X^{\mathrm{an}}_{\mathrm{min}}}(\lambda_0)\otimes\widetilde{L},\Omega_{X^{\mathrm{an}}_{\mathrm{min}}})[\mathrm{d}_{X_{\mathrm{min}}}]\\ &\quad\simeq \mathbf{R}\mathrm{Hom}_{\mathscr{O}_{X^{\mathrm{an}}_{\mathrm{min}}}}(\mathcal{V}^{\mathrm{an}}\otimes\mathscr{O}_{X^{\mathrm{an}}_{\mathrm{min}}}(\lambda_0)\otimes\widetilde{L},\Omega_{X^{\mathrm{an}}_{\mathrm{min}}})[\mathrm{d}_{X_{\mathrm{min}}}].\end{aligned}$$

On the other hand, since $\widetilde{L}$ is supported on $X^{\mathbb{R}}_{\mathrm{min}}$,

$$\begin{aligned}\mathbf{R}\mathscr{H}om_{\mathbb{C}}(\widetilde{L},\mathscr{O}_{X^{\mathrm{an}}_{\mathrm{min}}})\;[\mathrm{d}_{X_{\mathrm{min}}}] &\simeq \mathbf{R}\mathscr{H}om_{\mathbb{C}}(\widetilde{L},\mathbf{R}\,\Gamma_{X^{\mathbb{R}}_{\mathrm{min}}}(\mathscr{O}_{X^{\mathrm{an}}_{\mathrm{min}}}))[\mathrm{d}_{X_{\mathrm{min}}}]\\ &\simeq \mathbf{R}\mathscr{H}om_{\mathbb{C}}(\widetilde{L},\mathscr{B}_{X^{\mathbb{R}}_{\mathrm{min}}}\otimes or_{X^{\mathbb{R}}_{\mathrm{min}}}).\end{aligned}$$

Here, $or_{X^{\mathbb{R}}_{\mathrm{min}}}$ is the orientation sheaf of $X^{\mathbb{R}}_{\mathrm{min}}$, and $\mathscr{B}_{X^{\mathbb{R}}_{\mathrm{min}}}=or_{X^{\mathbb{R}}_{\mathrm{min}}}\otimes \mathbf{R}\mathscr{H}om_{\mathbb{C}}(\mathbb{C}_{X^{\mathbb{R}}_{\mathrm{min}}},\mathscr{O}_{X^{\mathrm{an}}_{\mathrm{min}}})[\mathrm{d}_{X_{\mathrm{min}}}]$ is the sheaf of hyperfunctions. Thus we obtain

$$\begin{aligned}&\mathbf{R}\mathrm{Hom}_{\mathbb{C}}\big(\Phi(L),\mathscr{O}_{X^{\mathrm{an}}}(-\lambda+2\rho)\big)[\mathrm{d}_X]\\ &\simeq \mathbf{R}\mathrm{Hom}_{\mathscr{O}_{X_{\mathrm{min}}}}(\mathcal{V}\otimes\Omega_{X_{\mathrm{min}}}^{\otimes-1}\otimes\mathscr{O}_{X^{\mathrm{an}}_{\mathrm{min}}}(\lambda_0)\otimes\widetilde{L}\otimes or_{X^{\mathbb{R}}_{\mathrm{min}}},\mathscr{B}_{X^{\mathbb{R}}_{\mathrm{min}}}).\end{aligned}$$

Note that $\mathscr{O}_{X^{\mathrm{an}}_{\mathrm{min}}}(\lambda_0)$ is a $\tau_{\lambda_0,X^{\mathrm{an}}_{\mathrm{min}}}$-twisted sheaf and $\widetilde{L}$ is a $\tau_{-\lambda_0,X^{\mathrm{an}}_{\mathrm{min}}}$-twisted sheaf. Hence $\mathscr{O}_{X^{\mathrm{an}}_{\mathrm{min}}}(\lambda_0)\otimes\widetilde{L}$ is a (non-twisted) locally free $\mathscr{O}_{X^{\mathrm{an}}_{\mathrm{min}}}|_{X^{\mathbb{R}}_{\mathrm{min}}}$-module. Hence, so is $\mathcal{V}^{\mathrm{an}}\otimes\Omega_{X_{\mathrm{min}}}^{\otimes-1}\otimes\mathscr{O}_{X^{\mathrm{an}}_{\mathrm{min}}}(\lambda_0)\otimes\widetilde{L}\otimes or_{X^{\mathbb{R}}_{\mathrm{min}}}$. Since $\mathscr{B}_{X^{\mathbb{R}}_{\mathrm{min}}}$ is a flabby sheaf, we have

$$H^n\big(\mathbf{R}\mathrm{Hom}_{\mathscr{O}_{X^{\mathrm{an}}_{\mathrm{min}}}}(\mathcal{V}^{\mathrm{an}}\otimes\Omega_{X_{\mathrm{min}}}^{\otimes-1}\otimes\mathscr{O}_{X^{\mathrm{an}}_{\mathrm{min}}}(\lambda_0)\otimes\widetilde{L}\otimes or_{X^{\mathbb{R}}_{\mathrm{min}}},\mathscr{B}_{X^{\mathbb{R}}_{\mathrm{min}}})\big)=0\text{ for }n\neq 0.$$

Hence, we obtain

$$H^n\big(\mathbf{R}\mathrm{Hom}_{\mathbb{C}}(\Phi(L),\mathscr{O}_{X^{\mathrm{an}}}(-\lambda+2\rho)[\mathrm{d}_X])\big)=0\text{ for }n\neq 0.$$

Proposition 10.3.1. *Assume that* $\lambda \in \mathfrak{t}^*$ *satisfies* (10.2.2)*, and let* $\mathscr{L}$ *be a* K*-equivariant* $\mathscr{D}_{X,\lambda}$*-module satisfying* (10.1.1)*. Then we have*

(i) $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{(\mathfrak{g},K_{\mathbb{R}})}(\Gamma(X;\mathscr{L}),\mathrm{C}^\infty(G_{\mathbb{R}})) \in \mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$ *is strict, and*
(ii) $H^n\big(\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{(\mathfrak{g},K_{\mathbb{R}})}(\Gamma(X;\mathscr{L}),\mathrm{C}^\infty(G_{\mathbb{R}}))\big) = 0$ *for* $n \neq 0$*.*

Proof. Set $M = \Gamma(X;\mathscr{L})$. By (9.6.6), we have

$$\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{(\mathfrak{g},K_{\mathbb{R}})}(M,\mathrm{C}^\infty(G_{\mathbb{R}})) \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathbb{C}}\big(\Phi(L),\mathscr{O}_{X^{\mathrm{an}}}(-\lambda+2\rho)[\mathrm{d}_X]\big).$$

Hence, forgetting the topology and the equivariance, the cohomology groups of $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{(\mathfrak{g},K_{\mathbb{R}})}(M,\mathrm{C}^\infty(G_{\mathbb{R}}))$ are concentrated at degree 0. On the other hand, $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{(\mathfrak{g},K_{\mathbb{R}})}(M,\mathrm{C}^\infty(G_{\mathbb{R}}))$ is represented by a complex in $\mathbf{FN}_{G_{\mathbb{R}}}$ whose negative components vanish. Hence it is a strict complex. Q.E.D.

10.4 Vanishing theorem

By using the result of the preceding paragraph, we shall prove the following statement.

Theorem 10.4.1. *Let* N *be a Harish-Chandra module. Then we have*

(i) $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{(\mathfrak{g},K_{\mathbb{R}})}(N,\mathrm{C}^\infty(G_{\mathbb{R}})) \in \mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_{\mathbb{R}}})$ *is strict,*
(ii) $H^n(\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{(\mathfrak{g},K_{\mathbb{R}})}(N,\mathrm{C}^\infty(G_{\mathbb{R}}))) = 0$ *for* $n \neq 0$*.*

Proof. Since $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{(\mathfrak{g},K_{\mathbb{R}})}(N,\mathrm{C}^\infty(G_{\mathbb{R}}))$ is represented by a complex in $\mathbf{FN}_{G_{\mathbb{R}}}$ whose negative components vanish, it is enough to show that, forgetting topology,

(10.4.1) $$\mathrm{Ext}^n_{(\mathfrak{g},K_{\mathbb{R}})}(N,\mathrm{C}^\infty(G_{\mathbb{R}})) = 0 \quad \text{for } n \neq 0.$$

We shall prove this by the descending induction on n. If $n \gg 0$, this is obvious because the global dimension of $\mathrm{Mod}(\mathfrak{g},K)$ is finite.

We may assume that N is simple without the loss of generality.

By [2, 5], $N/\tilde{\mathfrak{n}}N \neq 0$, where $\tilde{\mathfrak{n}} = [\mathfrak{b},\mathfrak{b}]$ is the nilpotent radical of $\mathfrak{b}$. Since the center $\mathfrak{z}(\mathfrak{g})$ acts by scaler on N, $N/\tilde{\mathfrak{n}}N$ is $U(\mathfrak{t})$-finite. Hence there exists a surjective $(\mathfrak{t},T\cap K)$-linear homomorphism $N/\tilde{\mathfrak{n}}N \twoheadrightarrow V$ for some one-dimensional $(\mathfrak{t},T\cap K)$-module V. Let $\lambda \in \mathfrak{t}^*$ be the character of V. Since $S/(\mathfrak{k}\cap\tilde{\mathfrak{n}})S \to V$ is a surjective homomorphism for some irreducible M-submodule S of N, $\lambda|_{\mathfrak{k}\cap\mathfrak{t}}$ is the lowest weight of S, and hence λ satisfies (10.2.2).

Let us take a K-equivariant $(\mathscr{D}_{X,\lambda})|_{Kx_0}$- module $\mathscr{L}'$ such that $\mathscr{L}'(x_0) \cong V$ as $(B\cap K)$-modules, and set $\mathscr{L} = \mathbf{D}j_*\mathscr{L}'$.

Then by Lemma 10.2.2, $\mathrm{Hom}_{(\mathfrak{g},K)}(N,\Gamma(X;\mathscr{L}))$ contains a non-zero element. Thus we obtain an exact sequence of $(\mathfrak{g},K)$-modules

$$0 \to N \to M \to M' \to 0 \quad \text{with } M = \Gamma(X;\mathscr{L}).$$

This gives an exact sequence

$$\begin{aligned}\operatorname{Ext}^n_{(\mathfrak{g},K_\mathbb{R})}(M, \mathrm{C}^\infty(G_\mathbb{R})) \to \operatorname{Ext}^n_{(\mathfrak{g},K_\mathbb{R})}(N, \mathrm{C}^\infty(G_\mathbb{R})) \\ \to \operatorname{Ext}^{n+1}_{(\mathfrak{g},K_\mathbb{R})}(M', \mathrm{C}^\infty(G_\mathbb{R})),\end{aligned}$$

in which the first term vanishes for $n > 0$ by Proposition 10.3.1 and the last term vanishes by the induction hypothesis. Thus we obtain the desired result. Q.E.D.

By duality, we obtain the following proposition.

Theorem 10.4.2. *Let N be a Harish-Chandra module. Then we have*

(i) $\Gamma_c(G_\mathbb{R}; \mathscr{D}ist_{G_\mathbb{R}}) \overset{\mathbf{L}}{\otimes}_{(\mathfrak{g},K_\mathbb{R})} N \in \mathrm{D}^\mathrm{b}(\mathbf{DFN}_{G_\mathbb{R}})$ *is strict,*

(ii) $H^n(\Gamma_c(G_\mathbb{R}; \mathscr{D}ist_{G_\mathbb{R}}) \overset{\mathbf{L}}{\otimes}_{(\mathfrak{g},K_\mathbb{R})} N) = 0$ *for* $n \neq 0$.

Recall that the maximal globalization functor $\mathrm{MG}\colon \mathrm{HC}(\mathfrak{g}, K) \to \mathbf{FN}_{G_\mathbb{R}}$ is given by

$$\mathrm{MG}(M) = H^0\big(\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{(\mathfrak{g},K_\mathbb{R})}(M^*, \mathrm{C}^\infty(G_\mathbb{R}))\big)$$

and the minimal globalization functor $\mathrm{mg}\colon \mathrm{HC}(\mathfrak{g}, K) \to \mathbf{DFN}_{G_\mathbb{R}}$ is given by

$$\mathrm{mg}(M) = H^0\big(\Gamma_c(G_\mathbb{R}; \mathscr{D}ist_{G_\mathbb{R}}) \overset{\mathbf{L}}{\otimes}_{(\mathfrak{g},K_\mathbb{R})} M\big).$$

We denote by $\mathrm{MG}_{G_\mathbb{R}}$ (resp. $\mathrm{mg}_{G_\mathbb{R}}$) the subcategory of $\mathbf{FN}_{G_\mathbb{R}}$ (resp. $\mathbf{DFN}_{G_\mathbb{R}}$) consisting of objects isomorphic to $\mathrm{MG}(M)$ (resp. $\mathrm{mg}(M)$) for a Harish-Chandra module M (see § 1.1). Then both $\mathrm{MG}_{G_\mathbb{R}}$ and $\mathrm{mg}_{G_\mathbb{R}}$ are equivalent to the category $\mathrm{HC}(\mathfrak{g}, K)$ of Harish-Chandra modules.

The above theorem together with Theorem 10.4.1 shows the following result.

Theorem 10.4.3. (i) *The functor $M \mapsto \mathrm{MG}(M)$ (resp. $M \mapsto \mathrm{mg}(M)$) is an exact functor from the category $\mathrm{HC}(\mathfrak{g}, K)$ of Harish-Chandra modules to $\mathbf{FN}_{G_\mathbb{R}}$ (resp. $\mathbf{DFN}_{G_\mathbb{R}}$).*

(ii) *Any morphism in $\mathrm{MG}_{G_\mathbb{R}}$ or $\mathrm{mg}_{G_\mathbb{R}}$ is strict in $\mathbf{FN}_{G_\mathbb{R}}$ or $\mathbf{DFN}_{G_\mathbb{R}}$ (i.e., with a closed range).*

(iii) *Any $G_\mathbb{R}$-invariant closed subspace of E in $\mathrm{MG}_{G_\mathbb{R}}$ (resp. $\mathrm{mg}_{G_\mathbb{R}}$) belongs to $\mathrm{MG}_{G_\mathbb{R}}$ (resp. $\mathrm{mg}_{G_\mathbb{R}}$).*

(iv) *$\mathrm{MG}_{G_\mathbb{R}}$ is closed by extensions in $\mathbf{FN}_{G_\mathbb{R}}$, namely, if $0 \to E' \to E \to E'' \to 0$ is a strict exact sequence in $\mathbf{FN}_{G_\mathbb{R}}$, and E' and E'' belong to $\mathrm{MG}_{G_\mathbb{R}}$, then so does E. Similar statement holds for $\mathrm{mg}_{G_\mathbb{R}}$.*

Here the exactness in (i) means that they send the short exact sequences to strictly exact sequences.

Proof. Let us only show the statements on the maximal globalization.
(i) follows immediately from Theorem 10.4.1.
(ii) Let M, M' be Harish-Chandra modules, and let $u\colon \mathrm{MG}(M) \to \mathrm{MG}(M')$ be a morphism in $\mathbf{FN}_{G_\mathbb{R}}$. Then

$$\psi := \mathrm{HC}(u)\colon M \simeq \mathrm{HC}(\mathrm{MG}(M)) \to \mathrm{HC}(\mathrm{MG}(M')) \simeq M'$$

is a morphism in $\mathrm{HC}(\mathfrak{g}, K)$ and $\mathrm{MG}(\psi) = u$. Let I be the image of ψ, Then $\mathrm{MG}(M) \to \mathrm{MG}(I)$ is surjective and $\mathrm{MG}(I)$ is a closed subspace of $\mathrm{MG}(M')$ by (i).
(iii) Let M be a Harish-Chandra module and E a $G_\mathbb{R}$-invariant closed subspace of $\mathrm{MG}(M)$. Then $N := \mathrm{HC}(E) \subset M$ is a Harish-Chandra module and $\mathrm{MG}(N)$ is a closed subspace of $\mathrm{MG}(M)$ by (ii), and it contains N as a dense subspace. Since E is also the closure of N, $E = \mathrm{MG}(N)$.
(iv) We have an exact sequence $0 \to \mathrm{HC}(E') \to \mathrm{HC}(E) \to \mathrm{HC}(E'') \to 0$. Since $\mathrm{HC}(E')$ and $\mathrm{HC}(E'')$ are Harish-Chandra modules, so is $\mathrm{HC}(E)$. Hence we have a commutative diagram with strictly exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & E'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{MG}(\mathrm{HC}(E')) & \longrightarrow & \mathrm{MG}(\mathrm{HC}(E)) & \longrightarrow & \mathrm{MG}(\mathrm{HC}(E'')) & \longrightarrow & 0.
\end{array}$$

Since the left and right vertical arrows are isomorphisms, the middle vertical arrow is also an isomorphism. Q.E.D.

Let us denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{MG}}(\mathbf{FN}_{G_\mathbb{R}})$ the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_\mathbb{R}})$ consisting of E such that E is strict and the cohomologies of E belong to $\mathrm{MG}_{G_\mathbb{R}}$. Similarly, we define $\mathrm{D}^{\mathrm{b}}_{\mathrm{mg}}(\mathbf{DFN}_{G_\mathbb{R}})$. Then the following result follows immediately from the preceding theorem and Lemma 2.3.1.

Corollary 10.4.4. *The category $\mathrm{D}^{\mathrm{b}}_{\mathrm{MG}}(\mathbf{FN}_{G_\mathbb{R}})$ is a triangulated full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_\mathbb{R}})$, namely, it is closed by the translation functors, and closed by distinguished triangles (if $E' \to E \to E'' \to E'[1]$ is a distinguished triangle in $\mathrm{D}^{\mathrm{b}}(\mathbf{FN}_{G_\mathbb{R}})$ and E' and E belong to $\mathrm{D}^{\mathrm{b}}_{\mathrm{MG}}(\mathbf{FN}_{G_\mathbb{R}})$, then so does E'').*

This corollary together with Theorem 10.4.1 implies the following corollary.

Corollary 10.4.5. *If $M \in \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}_f(\mathfrak{g}, K))$ has Harish-Chandra modules as cohomologies, then $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{(\mathfrak{g}, K_\mathbb{R})}(M, \mathrm{C}^\infty(G_\mathbb{R}))$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{MG}}(\mathbf{FN}_{G_\mathbb{R}})$.*

Hence we obtain the following theorem.

Theorem 10.4.6. *Let $\lambda \in \mathfrak{t}^*$, $K \in \mathrm{D}^{\mathrm{b}}_{G_\mathbb{R}, \tau_{-\lambda}, \mathbb{R}\text{-}c}(\mathbb{C}_X)$ and $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_{X,\lambda}, G)$. Then we have*

(i) $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_{X,\lambda}}(\mathscr{M} \otimes K, \mathscr{O}_{X^{\mathrm{an}}}(\lambda))$ *belongs to* $\mathrm{D}^{\mathrm{b}}_{\mathrm{MG}}(\mathbf{FN}_{G_\mathbb{R}})$.

(ii) $\mathbf{R}\Gamma_c\big(X^{\mathrm{an}}; K \otimes \Omega_{X^{\mathrm{an}}}(-\lambda) \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_{X,\lambda}} \mathscr{M}\big)$ *belongs to* $\mathrm{D}^{\mathrm{b}}_{\mathrm{mg}}(\mathbf{DFN}_{G_\mathbb{R}})$.

Proof. Since (ii) is the dual statement of (i), it is enough to prove (i). By Matsuki correspondence (Theorem 9.4.2), there exists $L \in \mathrm{D}^{\mathrm{b}}_{K^{\mathrm{an}},\tau_{-\lambda},\mathbb{C}\text{-c}}(\mathbb{C}_{X^{\mathrm{an}}})$ such that $K \simeq \Phi(L)$. By Theorem 9.6.1, there exists $\mathscr{L} \in \mathrm{D}^{\mathrm{b}}_{K,\,\mathrm{coh}}(\mathscr{D}_{X,\lambda})$ such that $\mathrm{DR}_X(\mathscr{L}) \simeq L$. Then Proposition 9.6.3 implies

$$\begin{aligned}&\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_{X,\lambda}}(\mathscr{M} \otimes K, \mathscr{O}_{X^{\mathrm{an}}}(\lambda)) \\ &\qquad\qquad \simeq \mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{(\mathfrak{g},K_{\mathbb{R}})}(\mathbf{L}j^*(\mathscr{M} \overset{\mathbf{D}}{\circ} \mathrm{Ind}^G_K \mathscr{L}), \mathrm{C}^{\infty}(S_{\mathbb{R}}))[-\mathrm{d}_X].\end{aligned}$$

Then the result follows from Corollary 10.4.5 and Proposition 9.3.2. Q.E.D.

Let us illustrate Theorem 10.4.6 in the case $\mathscr{M} = \mathscr{D}_{X,\lambda}$ and K is a twisted $G_{\mathbb{R}}$-equivariant sheaf supported on a $G_{\mathbb{R}}$-orbit Z of X^{an}.

Let us take a point $x \in Z$. Let V be a finite-dimensional $G_{\mathbb{R}} \cap B(x)$-module whose differential coincides with $\mathfrak{g}_{\mathbb{R}} \cap \mathfrak{b}(x) \to \mathfrak{b}(x) \xrightarrow{\lambda} \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(V)$.

Then the Cauchy-Riemann equations give a complex

$$\Big(\mathscr{B}(G_{\mathbb{R}}) \otimes V \otimes \overset{\bullet}{\bigwedge} \mathfrak{n}(x)\Big)^{G_{\mathbb{R}} \cap B(x)}. \tag{10.4.2}$$

Then its cohomology groups belong to $\mathrm{MG}_{G_{\mathbb{R}}}$.

Indeed, if F is the τ_λ-twisted local system on Z associated with V^* (see Lemma 9.1.1), then (10.4.2) is isomorphic to $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}_{\mathscr{D}_{X,\lambda}}(\mathscr{M} \otimes i_! F, \mathscr{O}_{X^{\mathrm{an}}}(\lambda))$ (up to a shift). Here $i\colon Z \to X^{\mathrm{an}}$ is the embedding,

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List of Notations

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