VANISHING CYCLE SHEAVES AND HOLONOMIC SYSTEMS OF DIFFERENTIAL EQUATIONS

By

M. Kashiwara
Research Institute for Mathematical Sciences
Kyoto University

1. Let X be a complex manifold and f a holomorphic function on X. Then, for a complex of sheaves $\underline{F}^\cdot$ on X, we can define a "vanishing cycle sheaf" $\mathbb{R}\Psi\underline{F}^\cdot$ (in Deligne's notation) on $f^{-1}(0)$ (See [3], [1]). The purpose of this paper is to give a corresponding holonomic system when $\underline{F}^\cdot$ is given as a de Rham complex of a regular holonomic system.

2. Let X be a smooth complex manifold and Y a smooth submanifold of X. We denote by $\mathcal{O}_X$ and $\underline{I}_Y$ the sheaf of holomorphic functions on X and the defining Ideal of Y. We denote by $\mathcal{A}$ the graded $\mathcal{O}_X$-Algebra $\bigoplus_{k\in\mathbb{Z}} \underline{I}_Y{}^k t^{-k} \subset \mathcal{O}_X[t,t^{-1}]$. Here, $\underline{I}_Y{}^k$ stands for $\mathcal{O}_X$ if $k \leq 0$. We denote by $\pi:\tilde{X} \to X$ the space Specan $\mathcal{A}$ over X. Then $\tilde{X}$ is smooth and t defines a hypersurface of $\tilde{X}$ isomorphic to the normal bundle T_YX of Y.

Let $\tilde{\mathbb{C}}$ be the real manifold $(\mathbb{C}-\{0\}) \sqcup S^1$ with the boundary $S^1 = \mathbb{C}^\times/\mathbb{R}^+$, with the obvious projection $\tilde{\mathbb{C}} \to \mathbb{C}$. For a complex of sheaves $\underline{F}^\cdot$, we define

(2.1) $$\nu_Y(\underline{F}^\cdot) = i^{-1}\mathbb{R}j_*p^{-1}\underline{F}^\cdot.$$

Here p is the projection $\tilde{X} - T_Y X = (\mathbb{C}-\{0\}) \underset{\mathbb{C}}{\times} \tilde{X} \to X$ and j: $\tilde{X}-T_Y X \hookrightarrow \tilde{\mathbb{C}} \underset{\mathbb{C}}{\times} \tilde{X}$, which are given by $t: \tilde{X} \to \mathbb{C}$. The map i is the inclusion $T_Y X \hookrightarrow S^1 \times T_Y X \hookrightarrow \tilde{\mathbb{C}} \underset{\mathbb{C}}{\times} \tilde{X}$ given by $(1 \bmod \mathbb{R}^+) \in S^1$.

By using a local coordinate system $(x_1,\ldots,x_\ell,\ldots x_n)$ of X such that Y is $x' = (x_1,\ldots,x_\ell) = 0$, the stalks of $\nu_Y(\underline{F}^{\cdot})$ are described as follows. For $(x_0,v) \in T_Y X$ $(x_0 \in \mathbb{C}^{n-\ell}, v \in \mathbb{C}^\ell)$, we have

$$(2.2) \qquad H^j(\nu_Y(\underline{F}^{\cdot}))_{(x_0,v)} = \varinjlim_{U} H^j(U;\underline{F}^{\cdot}).$$

Here, U runs over the set of open subsets of X which contain $\{x=(x',x'')\in\mathbb{C}^\ell\times\mathbb{C}^{n-\ell}; |x'|<\varepsilon, |x''-x_0|<\varepsilon, x' \in \Gamma\}$ for some $\varepsilon > 0$ and an open cone $\Gamma \ni v$ of $\mathbb{C}^\ell$.

3. Let $\mathcal{D}_X$ be the sheaf of differential operators on X and $\mathfrak{M}$ a regular holonomic $\mathcal{D}_X$-Module. We shall then construct a regular holonomic $\mathcal{D}_{T_Y X}$-Module $\mathfrak{M}'$ such that

$$\nu_Y(\mathbb{R}\,\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{O}_X)) = \mathbb{R}\,\underline{\mathrm{Hom}}_{\mathcal{D}_{T_Y X}}(\mathfrak{M}', \mathcal{O}_{T_Y X}).$$

If such an $\mathfrak{M}'$ exists, it is unique up to an isomorphism. We shall denote it by $\nu_Y(\mathfrak{M})$.

4. Keeping X and Y as in the preceding section, we shall define the filtration $F^{\cdot} = F^{\cdot}(\mathcal{D}_X)$ of $\mathcal{D}_X$ by

$$(4.1) \qquad F^k(\mathcal{D}_X) = \{P \in \mathcal{D}_X;\ P(\underline{I}_Y^j) \subset \underline{I}_Y^{j+k} \text{ for any } j\}.$$

Then, one can show easily the following

Proposition 1. (1) $F^k(\mathcal{D}_X)/F^{k+1}(\mathcal{D}_X)$ *is isomorphic to the sheaf of* differential operators on $T_Y X$ homogeneous of degree k. Hence its graduation $gr_{F^{\cdot}}(\mathcal{D}_X)$ is a subring of $\mathcal{D}_{T_Y X}$.

(2) There exists (locally) a vector field θ tangent to Y acting on I_Y/I_Y^2 as the identity.

5. Now, let $\mathfrak{M}$ be a coherent $\mathcal{D}_X$-Module. A filtration $F_I^{\cdot}$ of $\mathfrak{M}$ is called a good filtration of $\mathfrak{M}$ with respect to $F^{\cdot}(\mathcal{D}_X)$ if it satisfies

(5.1) $$F^k(\mathcal{D}_X)F_I^j \subset F_I^{k+j} \quad \text{for any } k \text{ and } j$$

(5.2) $$F^k(\mathcal{D}_X)F_I^j = F_I^{k+j} \quad \text{if } j \gg 0 \text{ and } k \geqq 0 \text{ or if } j \ll 0 \text{ and } k \leqq 0.$$

(5.3) F_I^j is a coherent $F^0(\mathcal{D}_X)$-Module.

(5.4) $$\mathfrak{M} = \cup F_I^j.$$

The following proposition is proved in [2].

Proposition 2. Let $\mathfrak{M}$ be a regular holonomic system. Then there exist locally a coherent $\mathcal{O}_X$ sub-Module $\mathcal{F}$ of $\mathfrak{M}$ and a non-zero polynomial $b(\theta)$ such that

(5.5) $$b(\theta)\mathcal{F} \subset (\mathcal{D}_X(\deg b) \cap F^1(\mathcal{D}_X))\mathcal{F}$$

(5.6) $$\mathfrak{M} = \mathcal{D}_X \mathcal{F} .$$

Here $\mathcal{D}_X(m)$ denotes the sheaf of differential operators of order $\leq m$, and θ is the one given in Proposition 1.

6. Let $\mathfrak{R}$ be the abelian category of coherent $\mathcal{D}_X$-Modules satisfying the conclusion in Proposition 2. Let G be a subset of $\mathbb{C}$ satisfying the following condition:

(6.1) For any $a \in \mathbb{C}$, $G \cap (a+\mathbb{Z})$ consists of a single point.

Then we have the following

Theorem 1. (1) *For any $\mathfrak{M} \in \mathfrak{R}$, there exists a good filtration $F_G^{\cdot}(\mathfrak{M})$ of $\mathfrak{M}$ satisfying the following condition: there exists a polynomial $b(\theta)$ such that $b^{-1}(0) \subset G$ and $b(\theta-k)F_G^k(\mathfrak{M}) \subset F_G^{k+1}(\mathfrak{M})$ for any k.*

Moreover such a filtration is unique.

(2) *For $\mathfrak{M} \in \mathfrak{R}$, $gr_{F_G^{\cdot}}(\mathfrak{M})$ does not depend on the choice of G as a* (not graded) $gr_{F^{\cdot}}(\mathcal{D})$-*Module. We shall denote it by $gr\,\mathfrak{M}$.*

(3) *$\mathfrak{M} \mapsto gr\,\mathfrak{M}$ is an exact functor from $\mathfrak{R}$ into the* category *of coherent $gr_{F^{\cdot}}(\mathcal{D})$-Modules.*

(4) $\nu_Y(\mathbb{R}\,\underline{Hom}_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{O}_X)) = \mathbb{R}\,\underline{Hom}_{\mathcal{D}_{T_Y X}}(\mathcal{D}_{T_Y X} \underset{gr\mathcal{D}_X}{\otimes} gr\,\mathfrak{M}, \mathcal{O}_{T_Y X})$

$\nu_Y(\mathbb{R}\,\underline{Hom}_{\mathcal{D}_X}(\mathcal{O}_X, \mathfrak{M})) = \mathbb{R}\,\underline{Hom}_{\mathcal{D}_{T_Y X}}(\mathcal{O}_{T_Y X}, \mathcal{D}_{T_Y X} \underset{gr\mathcal{D}_X}{\otimes} gr\,\mathfrak{M})$.

(5) *If $\mathfrak{M}$ is regular holonomic, so is* $\mathcal{D}_{T_Y X} \underset{gr\mathcal{D}}{\otimes} gr\,\mathfrak{M}$.

We shall indicate the proof of the theorem.

Proof of (1). By using Proposition 2, there exists a good filtration $F_I^{\cdot}$ of $\mathfrak{M}$ and a non-zero polynomial b such that

(6.2) $$b(\theta-k)F_I^k \subset F_I^{k+1} \quad \text{for any } k.$$

In fact, setting $F_I^k = F^k(\mathcal{D})\mathcal{F}$ we apply the following lemma.

Lemma 1. For any $f(\theta)\in\mathbb{C}[\theta]$ and $P\in F^k(\mathcal{D})$, $f(\theta)P - Pf(\theta+k)\in F^{k+1}(\mathcal{D})$.

Now, assume that $b(\theta)$ in (6.2) is a product of two polynomials $b_1(\theta)$ and $b_2(\theta)$ and we set $F^k_{II} = F^{k+1}_I + b_1(\theta-k)F^k_I$. Then F_{II} is a good filtration satisfying $b_1(\theta-k-1)b_2(\theta-k)F^k_{II}\subset F^{k+1}_{II}$. Repeating this procedure, we can show the existence of $F^{\cdot}_G$.

The uniqueness of $F^{\cdot}_G$ is proved as follows.

Let F_I and F_{II} be two good filtrations and $b_I(\theta)$ and $b_{II}(\theta)$ two polynomials satisfying $b_J(\theta-k)F^k_J\subset F^{k+1}_J$ and $b_J^{-1}(0)\subset G$ for $J = I, II$. There exists $N\geq 1$ such that $F^k_I\subset F^{k-N}_{II}$ for any k. Then $b_I(\theta-k)F^k_I\subset F^{k+1}_I\subset F^{k-N+1}_{II}$ and $b_{II}(\theta-k+N)F^k_I\subset b_{II}(\theta-k+N)F^{k-N}_{II}\subset F^{k-N+1}_{II}$. Since $b_I(s-k)$ and $b_{II}(s-k+N)$ have no common root, $F^k_I\subset F^{k-N+1}_{II}$. Repeating this, we finally obtain $F^k_I\subset F^k_{II}$. (3) is proved by a similar discussion.

Proof of (2). Let G and G' be two subsets of $\mathbb{C}$ satisfying (6.1). We shall show $\mathrm{gr}\,F_G\simeq \mathrm{gr}\,F_{G'}$. We may assume $G\ni\lambda$ and $G' = (G-\{\lambda\})\cup\{\lambda+1\}$. We write $\mathrm{gr}\,F_G$ for $\mathrm{gr}_{F_G}\mathfrak{M}$.

Let $b(\theta)$ be a polynomial such that $b^{-1}(0)\subset G$ and $b(\theta-k)F^k_G\subset F^{k+1}_G$. Set $b(\theta) = (\theta-\lambda)^m a(\theta)$ with $a(\lambda)\neq 0$. Then $F^k_{G'} = (\theta-\lambda-k)^m F^k_G + F^{k+1}_G$. Let us take $\varphi,\psi\in\mathbb{C}[\theta]$ satisfying

$$
\begin{aligned}
(6.3)\qquad \varphi &\equiv 0 \mod (\theta-\lambda)^m(\theta-\lambda-1)^m,\\
\varphi &\equiv 1 \mod a(\theta),\\
\psi &\equiv 0 \mod a(\theta)a(\theta-1),\\
\psi &\equiv 1 \mod (\theta-\lambda)^m.
\end{aligned}
$$

We shall define $f\colon \mathrm{gr}\,F_G\to \mathrm{gr}\,F_{G'}$ and $g\colon \mathrm{gr}\,F_{G'}\to\mathrm{gr}\,F_G$ as follows.

(6.4) $$f: \mathrm{gr}\, F_G = \oplus F_G^k / F_G^{k+1} \ni \sum u_k \mapsto \sum v_k \in \mathrm{gr}\, F_{G'} = \oplus F_{G'}^k / F_{G'}^{k+1}$$

$$v_k = \varphi(\theta - k) u_k + \psi(\theta - k - 1) u_{k+1}$$

(6.5) $$g: \mathrm{gr}\, F_{G'} \ni \sum v_k \mapsto \sum u_k \in \mathrm{gr}\, F_G$$

by $u_k = v_k + \psi(\theta - k) v_{k-1}$.

Then one can easily show that f and g are inverses to each other.

(4) is shown by reducing the problem to the following special case, which is easy to prove.

Proposition 2. *Let* $b(\theta)$ *be a non-zero polynomial of degree* m *with* $b^{-1}(0) \subset G$ *and* P *an* $N \times N$ *matrix of differential operators in* $F^1(\mathcal{D}) \cap \mathcal{D}(m)$.

Set $\mathfrak{M} = \mathcal{D}^N / \mathcal{D}^N(b(\theta) - P)$. *Then* (4) *in Theorem* 1 *is true for* $\mathfrak{M}$.

(5) is proved in [2].

7. Suppose that Y is a smooth hypersurface of X given by $f = 0$. Then, for a complex of sheaves $F^\bullet$ whose cohomology groups are constructible, one can define $\mathbb{R}\psi$ and $\mathbb{R}\varphi$ and can: $\mathbb{R}\Phi \to \mathbb{R}\Psi$ and Var: $\mathbb{R}\Psi \to \mathbb{R}\Phi$ (See [3]). If we take a vector field ∂ such that $\partial f \equiv 1 \mod \underline{I}_Y$, then $\theta = f\partial$ and $\mathrm{gr}^0_{F^\bullet}(\mathcal{D})$ is isomorphic to $\mathcal{D}_Y$. Suppose $\underline{F}^\bullet = \mathbb{R}\,\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{O}_X)$ for a regular holonomic $\mathcal{D}_X$-Module $\mathfrak{M}$. Then we have the following

Theorem 2. *Assume* $G \subset \mathbb{C}$ *satisfies* (6.1) *and contains* 0.

(0) $gr_G^k \mathfrak{M}$ <u>is a regular holonomic</u> $\mathcal{D}_Y$-<u>Module</u>.

(1) $\mathbb{R}\Psi = \mathbb{R}\,\underline{\mathrm{Hom}}_{\mathcal{D}_Y}(gr_G^0 \mathfrak{M}, \mathcal{O}_Y)$ <u>and</u> $\mathbb{R}\Phi = \mathbb{R}\,\underline{\mathrm{Hom}}_{\mathcal{D}_Y}(gr_G^{-1} \mathfrak{M}, \mathcal{O}_Y)$.

(2) can <u>is given by</u> $f: gr_G^{-1}\mathfrak{M} \longmapsto gr_G^0 \mathfrak{M}$ <u>and</u> Var <u>is given by</u>

$\partial \frac{e^{2\pi i\theta}-1}{\theta} : gr_G^0 \mathfrak{M} \to gr_G^{-1} \mathfrak{M}$.

<u>Remark 1</u>. We can replace in (2), f and $\partial \frac{e^{2\pi i\theta}-1}{\theta}$ with $\frac{e^{2\pi i\theta}-1}{\theta} f$ and ∂.

<u>Remark 2</u>. If we replace $\mathbb{R}\,\underline{\mathrm{Hom}}_{\mathcal{D}_X}(*, \mathcal{O}_X)$ with $\mathbb{R}\,\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{O}_X, *)$, then (1) holds by replacing $\mathbb{R}\,\underline{\mathrm{Hom}}_{\mathcal{D}_X}(*, \mathcal{O}_Y)$ with $\mathbb{R}\,\underline{\mathrm{Hom}}_{\mathcal{D}_Y}(\mathcal{O}_Y, *)$. Accordingly, (2) holds by exchanging Var and can.

<u>Sketch of proof</u>. The theorem is essentially equivalent to the following one-dimensional case. Let $X = \mathbb{C}$ and $Y = \{0\}$. Let V_0 and V_{-1} be two vector spaces and let $A: V_0 \to V_{-1}$ and $B: V_{-1} \to V_0$ be two homomorphisms. Let $\mathfrak{M}$ be a $\mathcal{D}_X$-Module generated by $V_0 \oplus V_{-1}$ with the fundamental relation:

$$xu = Bu \quad \text{for} \quad u \in V_{-1}$$

$$\partial v = Av \quad \text{for} \quad v \in V_0.$$

If we assume the eigen-values of AB are contained in G, then $gr_G^k \mathfrak{M} = V_k$ for $k = 0, -1$.

Let U be a non-empty convex cone in $\mathbb{C}$ such that $U \not\ni 0$. Then we have

$$\mathbb{R}\Psi = \mathrm{Hom}_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{O}_X(U)) \qquad \text{and}$$

$$\mathbb{R}\Phi = \mathrm{Hom}_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{O}_X(U)/\mathcal{O}_X(\mathbb{C})).$$

The homomorphism can is given by $\mathcal{O}_X(U) \to \mathcal{O}_X(U)/\mathcal{O}_X(\mathbb{C})$. The homomorphism Var is given as follows: for $\varphi \in \mathrm{Hom}_{\mathcal{D}_X}(\mathfrak{m}, \mathcal{O}(U)/\mathcal{O}(X))$ and $s \in \mathfrak{m}$, let us choose a representative $u \in \mathcal{O}(U)$ of $\varphi(s)$. Then u can be continued to a multi-valued holomorphic function on $\mathbb{C}-\{0\}$, so that we can obtain the holomorphic function Tu defined on U by the analytic continuation of u along a path around the origin. Then $Tu-u$ does not depend on the choice of a representative u and $s \mapsto Tu-u$ gives a homomorphism from $\mathfrak{m}$ to $\mathcal{O}_X(U)$. This is the homomorphism Var.

Now, $\mathbb{R}\Psi$ and $\mathbb{R}\Phi$ are isomorphic to V_0^* and V_{-1}^* as follows:

$$V_0^* \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(\mathfrak{m}, \mathcal{O}_X(U)), \quad V_1^* \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(\mathfrak{m}, \mathcal{O}_X(U)/\mathcal{O}_X(\mathbb{C}))$$

by $V_0^* \ni \alpha \mapsto \varphi$ and $V_{-1}^* \ni \beta \mapsto \psi$, where $\varphi(u) = \langle\alpha, x^{BA-1}Bu\rangle$, $\varphi(v) = \langle\alpha, x^{BA}v\rangle$ and $\psi(u) = \langle\beta, x^{AB-1}\Gamma(1-AB)u\rangle$, $\psi(v) = -\langle\beta, Ax^{BA}\Gamma(-BA)v\rangle$ for $u \in V_{-1}$ and $v \in V_0$.

Remark that $x^\lambda\Gamma(\lambda)$ defines well an element of $\mathcal{O}(U)/\mathcal{O}(\mathbb{C})$ by the analytic continuation on λ (e.g. $x^\lambda\Gamma(\lambda) = \log x$ at $\lambda = 0$ and $x^\lambda\Gamma(\lambda) = \log x + ((\log x)^2/2 - \gamma\log x)N + ((\log x)^3/6 - \gamma(\log x)^2/2 + (\pi^2/3 + \gamma^2/2)\log x)N^2$ at $\lambda = N$ with $N^3 = 0$; γ is the Euler constant).

Thus with these identification, can is given by $\alpha \mapsto \alpha B(\Gamma(1-AB))^{-1}$ and Var is given by $\beta \mapsto \beta(2\pi i A e^{\pi i BA}/\Gamma(1+BA))$. Finally it is enough to note that x, ∂, θ correspond to B, A and BA (or $AB-1$) and $(\Gamma(1-AB))^{-1}$ is invertible under the condition on the eigenvalues of AB.

References

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