Representations of Lie algebras and Systems of Partial Differential Equations in the framework of Nilpotent Analysis

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In the classical theory of first order partial differential equations a fundamental rôle is played by the contact form $\omega = dz - \sum p_i dx_i$ which defines on the total space M of coordinates $(x_1, \cdots, x_n, z, p_1, \cdots, p_n)$ of independent variables, unknown function and first order derivatives a distinguished subbundle $D \subset TM$ of the tangent bundle of M of rank $2n$ by the equation $\omega = 0$. The bundle D is called a contact structure and spanned by

$$\frac{\partial}{\partial x_1} + p_1 \frac{\partial}{\partial z}, \cdots, \frac{\partial}{\partial x_n} + p_n \frac{\partial}{\partial z}, \frac{\partial}{\partial p_1}, \cdots, \frac{\partial}{\partial p_n},$$

which together with a normal vector $\frac{\partial}{\partial z}$ span the tangent space $T_a M$ at every point $a \in M$ and satisfies the same bracket relations as those of the Heisenberg Lie algebra:

$$[\frac{\partial}{\partial p_i}, \frac{\partial}{\partial x_j} + p_j \frac{\partial}{\partial z}] = \delta_{ij} \frac{\partial}{\partial z}.$$

This suggest that it is sometimes more natural to consider the tangent space at a point of the contact manifold M to be this Heisenberg Lie algebra rather than to be simply a vector space. This is one of the most typical situations that we often encounter in various geometric or analytic problems and leads to the following definition.

A filtered manifold is a differential manifold M endowed with a filtration $\{\mathfrak{f}^p\}_{p \in \mathbb{Z}}$ consisting of subbundles $\mathfrak{f}^p$ of the tangent bundle TM such that

i) $\mathfrak{f}^p \supset \mathfrak{f}^{p+1}$,

ii) $\mathfrak{f}^0 TM = 0, \quad \bigcup_{p \in \mathbb{Z}} \mathfrak{f}^p = TM$,

iii) $[\underline{\mathfrak{f}}^p, \underline{\mathfrak{f}}^q] \subset \underline{\mathfrak{f}}^{p+q} \quad$ for all $p, q \in \mathbb{Z}$,

where $\underline{\mathfrak{f}}^p$ denotes the sheaf of the germs of sections of $\mathfrak{f}^p$.

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There is associated to each point x of a filtered manifold $(M, \mathfrak{f})$ a graded object

$$gr\mathfrak{f}_x = \bigoplus_{p \in \mathbb{Z}} gr_p\mathfrak{f}_x, \quad \text{with } gr_p\mathfrak{f}_x = \mathfrak{f}_x^p/\mathfrak{f}_x^{p+1},$$

which is not only a graded vector space but also has a natural Lie bracket induced from that of vector fields and proves to be a nilpotent graded Lie algebra.

Under the slogan of "nilpotent geometry"and "nilpotent analysis" we have been studying various objects and structures on filtered manifolds by letting the tangent nilpotent Lie algebras play the usual rôle of the tangent spaces, which provide us with new perspectives and methods not only in geometry but also in the theory of differential equations. In particular, we have established a general existence theorem of local analytic solutions to a weightedly involutive analytic system of PDE's on a filtered manifold satisfying the Hörmander condition (a generalization of Cauchy -Kowalewski Theorem and Cartan Kähler Theorem) . (See T. Morimoto, Lie algebras, geometric structures and differential equations on filtered manifolds, Advanced Studies in Pure Mathematics 37, 2002, 205-252.)

In this talk I will give, in the framework of nilpotent analysis, a simple and general principle which associate to each representation of a Lie algebra a system (or a class of systems) of differential equations.

Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a transitive graded Lie algebra, that is, a Lie algebra satisfying:

i) $[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}$

ii) $\dim \mathfrak{g}_- < \infty$, where $\mathfrak{g}_- = \bigoplus_{p<0} \mathfrak{g}_p$, the negative part of $\mathfrak{g}$

iii) (Transitivity) For $i \geq 0, x_i \in \mathfrak{g}_i$, if $[x_i, \mathfrak{g}_-] = 0$, then $x_i = 0$.

Let $V = \bigoplus_{q \in \mathbb{Z}} V_q$ be a graded vector space and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ on V such that

$$\rho(\mathfrak{g}_p)V_q \subset V_{p+q}.$$

Consider the cohomology group $H(\mathfrak{g}_-, V) = \bigoplus_{p,r \in \mathbb{Z}} H_r^p(\mathfrak{g}_-, V)$ associated with the representation of $\mathfrak{g}_-$ on V, namely the cohomology group of the cochain complex:

$$\xrightarrow{\partial} \mathrm{Hom}(\wedge^{p-1}\mathfrak{g}_-, V)_r \xrightarrow{\partial} \mathrm{Hom}(\wedge^{p}\mathfrak{g}_-, V)_r \xrightarrow{\partial} \mathrm{Hom}(\wedge^{p+1}\mathfrak{g}_-, V)_r \xrightarrow{\partial}$$

where the coboundary operator ∂ is defined as usual and $\mathrm{Hom}(\wedge^p\mathfrak{g}_-, V)_r$ is the set of all homogeneous p-cochain ω of degree r, i.e., $\omega(\mathfrak{g}_{a_1} \wedge \cdots \wedge \mathfrak{g}_{a_p}) \subset V_{a_1+\cdots+a_p+r}$ for any $a_1, \cdots, a_p < 0$.

Now our assertion may be roughly stated as follows: The first cohomology group $H^1(\mathfrak{g}_-, V) = \bigoplus H_r^1(\mathfrak{g}_-, V)$ represents a system of differential equations and $V = \bigoplus V_q$ the solution space.

We shall see it is in the framework of nilpotent analysis that the principle above is properly formulated and well understood, and this principle enables one to produce plenty of examples of overdetermined systems related to various geometric structures on filtered manifolds.