Singular Demailly–Skoda-type theorem

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Setting

X : complex manifold of $\dim n$
$E \to X$: holomorphic vector bundle of $\operatorname{rank} r$
h : (possibly singular) Hermitian metric on E
Θ_h : Chern curvature of (E, h)

$(z_1, \ldots, z_n)$: holomorphic local coordinate of X
$(e_1, \ldots, e_r)$: orthonormal frame of E at some fixed point $x_0 \in X$

$$\sqrt{-1}\Theta_h = \sum_{\substack{1 \leq j,k \leq n \\ 1 \leq \lambda,\mu \leq r}} c_{j\bar{k}\lambda\mu} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k \otimes e_\lambda^\star \otimes e_\mu.$$

Definition

$(E,h) \to X$: smooth Hermitian vetor bundle

$$\sqrt{-1}\Theta_h = \sum_{\substack{1\leq j,k\leq n\\ 1\leq\lambda,\mu\leq r}} c_{j\bar{k}\lambda\mu} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k \otimes e_\lambda^\star \otimes e_\mu.$$

We naturally identify the curvature tensor with a Hermitian form $\widetilde{\Theta}_h$ on $T_X \otimes E$ defined by

$$\widetilde{\Theta}_h(\tau,\tau) = \sum_{\substack{1\leq j,k\leq n\\ 1\leq\lambda,\mu\leq r}} c_{j\bar{k}\lambda\mu} \tau_{j\lambda} \bar{\tau}_{k\mu}$$

for $\tau = \sum_{j,\lambda} \tau_{j\lambda} \frac{\partial}{\partial z_j} \otimes e_\lambda \in T_X \otimes E$ at $x_0 \in X$, where $(e_1, \ldots, e_r)$ is orthonormal.

Definition (Griffiths and Nakano positivity)

Keep the notation.
We say that:

1. h is *Nakano positive* if

$$\widetilde{\Theta}_h(\tau, \tau) = \sum c_{j\bar{k}\lambda\mu}\tau_{j\lambda}\bar{\tau}_{k\mu} > 0$$

   for all non-zero elements $\tau \in T_X \otimes E$, where $\tau = \sum_{j,\lambda} \tau_{j\lambda} \frac{\partial}{\partial z_j} \otimes e_\lambda$.

2. h is *Griffiths positive* if

$$\widetilde{\Theta}_h(v \otimes s, v \otimes s) = \sum c_{j\bar{k}\lambda\mu} v_j \bar{v}_k s_\lambda \bar{s}_\mu > 0$$

   for all non-zero decomposable elements $v \otimes s \in T_X \otimes E$, where $v = \sum_j v_j \frac{\partial}{\partial z_j} \in T_X$ and $s = \sum_\lambda s_\lambda e_\lambda \in E$.

Note that Nakano positivity is stronger than Griffiths positivity.

For example, the following vanishing theorem holds:
Let X be a projective manifold and $(E, h) \to X$ be a smooth Nakano positive vector bundle. Then the following qth cohomology group vanishes

$$H^q(X, K_X \otimes E) = 0$$

for $q > 0$, which is known as the *Nakano vanishing theorem*.

However, for $n > 1$

$$H^{n-1}(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes T_{\mathbb{P}^n}) \neq 0.$$

Here $T_{\mathbb{P}^n}$ admits a Griffiths positive smooth Hermitian metric.

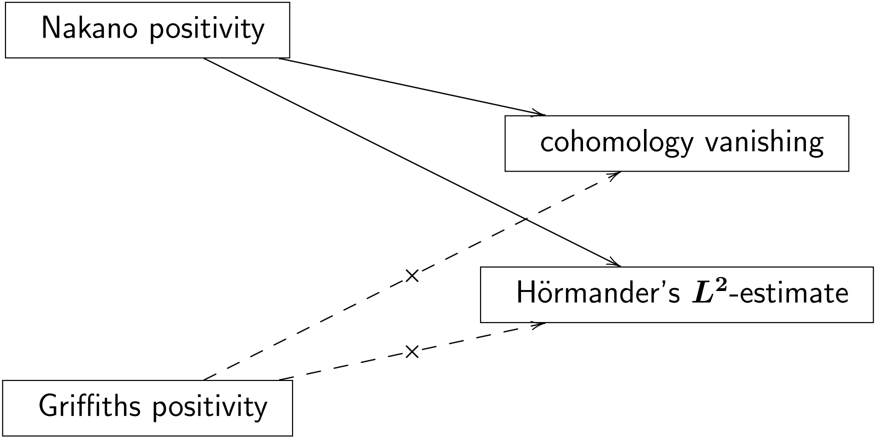
We can also obtain the following L^2-estimate for Nakano positive vector bundles.

Theorem

Let (X, ω) be a complete Kähler manifold and $(E, h) \to X$ be a Nakano positive vector bundle such that $\widetilde{\Theta}_{E,h}(\tau, \tau) \geq \delta |\tau|^2_{\omega,h}$ for $\delta > 0$ and $\tau \in T_X \otimes E$. Then for any $\overline{\partial}$-closed E-valued (n, q)-form u with finite L^2-norm, there exists an E-valued $(n, q-1)$-form α satisfying $\overline{\partial}\alpha = u$ and

$$\int_X |\alpha|^2_{\omega,h} \mathrm{d}V_\omega \leq \frac{1}{\delta q} \int_X |u|^2_{\omega,h} \mathrm{d}V_\omega.$$

This type of result is often called Hörmander's or a Hörmander-type L^2-estimate due to the original work of Hörmander.


Nakano positivity
cohomology vanishing
Hörmander's L^2-estimate
Griffiths positivity

Theorem (Demailly–Skoda'79)

If (E, h) is a Griffiths (semi-)positive smooth Hermitian vector bundle, $(E \otimes \det E, h \otimes \det h)$ is Nakano (semi-)positive.

Thanks to the generalizations by Berndtsson'09 and Liu–Sun–Yang'13, it is also known that if (E, h) is Griffiths (semi-)positive, $(S^m E \otimes \det E, S^m h \otimes \det h)$ is Nakano (semi-)positive for all $m \in \mathbb{N}$, where $S^m E$ is the mth symmetric power of E.

In this talk, we only focus on the case that $m = 1$ for simplicity.

The Demailly–Skoda theorem has many important applications.

Theorem

Let X be a projective manifold and $(E, h) \to X$ be a Griffiths positive vector bundle. Then the qth cohomology group vanishes

$$H^q(X, K_X \otimes E \otimes \det E) = 0$$

for $q > 0$.

This theorem is the so-called Griffiths vanishing theorem. By using the Demailly–Skoda theorem and the Nakano vanishing theorem, we can easily see that this result holds since $(E \otimes \det E, h \otimes \det h)$ is Nakano positive.

We would like to consider the generalization of the Demailly–Skoda theorem for singular Hermitian vector bundles.

In the next section, I will explain a notion of singular Hermitian metrics.

Definition

Let $E \to X$ be a holomorphic vector bundle over a complex manifold X. We say that h is a *singular Hermitian metric* on E if:

1. h is a **measurable map** from X to the space of non-negative Hermitian forms on the fibers, that is, $|u|_h$ is a measurable function whenever $U \subset X$ is open and $u \in H^0(U, E)$.
2. h is finite and positive definite almost everywhere.

Roughly speaking, a singular Hermitian metric is just a measurable metric. When a metric has either positive or negative curvature, it has stronger regularity properties than being measurable.

- It is known that the Chern curvature current Θ_h cannot be defined with measure coefficients in general (Raufi'15).
- For this reason, we cannot generally define the positivity or negativity of a singular Hermitian metric on a vector bundle by using the Chern curvature current.
- We need to define positivity and negativity notions **without** using the Chern curvature current.

There is a definition of Griffiths positivity and negativity for singular Hermitian metrics in the following form.

Definition

Let h be a singular Hermitian metric on E. We say that:

1. h is *Griffiths semi-negative* if $|u|_h^2$ (or $\log |u|_h^2$) is psh, where $\forall u \in H^0(U, E)$ and $U \subset X$ is an open subeset.
2. h is *Griffiths semi-positive* if the dual metric $h^\star$ on $E^\star$ is Griffiths semi-negative.

I show some example.

There is also an approach to Nakano positivity for singular Hermitian metrics.

Based on the work of Hosono–I.('21) and Deng–Ning–Wang–Zhou('22), we can define Nakano positivity for singular Hermitian metrics.

Definition (I.'22)

Let h be a Griffiths semi-positive singular Hermitian metric on a vector bundle $E \to X$ over a complex manifold X. We say that h is *Nakano semi-positive* if $\forall(\Omega, \iota)$ Stein coordinate such that $E|_{\iota(\Omega)}$ is trivial, $\forall\omega_\Omega$ Kähler form, $\forall\psi \in C^\infty(\Omega) \cap \mathbf{SPSH}(\Omega)$ and $\forall\overline{\partial}$-closed $u \in L^2_{(n,q)}(\Omega, \iota^\star E; \omega, (\iota^\star h)e^{-\psi})$, there exists $\alpha \in L^2_{(n,q-1)}(\Omega, \iota^\star E; \omega_\Omega, (\iota^\star h)e^{-\psi})$ satisfying $\overline{\partial}\alpha = u$ and

$$\int_\Omega |u|^2_{\omega_\Omega, \iota^\star h} e^{-\psi} dV_{\omega_\Omega} \leq \int_\Omega \langle [\sqrt{-1}\partial\overline{\partial}\psi \otimes \mathrm{Id}_E, \Lambda_{\omega_\Omega}]^{-1} f, f \rangle_{\omega_\Omega, \iota^\star h} e^{-\psi} dV_{\omega_\Omega}.$$

- $\iota : \Omega \to X$ is a holomorphic injective map.
- $[\cdot, \cdot]$ is the graded Lie bracket.
- Λ_{ω_Ω} is the adjoint operator of $L_{\omega_\Omega} = \omega_\Omega \wedge \cdot$.

Apart from the detailed definitions, we have shown the following theorem.

Theorem (I.'22)

Let h be a Griffiths semi-positive singular Hermitian metric on E. Then $h \otimes \det h$ is Nakano semi-positive in the sense of singular Hermitian metrics on $E \otimes \det E$.

This is a singular version of the Demailly–Skoda theorem. For example, we can prove the following Hörmander-type L^2-estimate.

Theorem (I.'20)

Let (X,ω) be a Stein or projective manifold and $E \to X$ be a holomorphic vector bundle. We also let h be a strictly Griffiths δ_ω-positive singular Hermitian metric on E for $\delta > 0$. Then for any $\overline{\partial}$-closed $E \otimes \det E$-valued (n,q)-form u with finite L^2-norm, there is an $E \otimes \det E$-valued $(n,q-1)$-form α such that $\overline{\partial}\alpha = u$ and

$$\int_X |\alpha|^2_{\omega,h\otimes\det h} dV_\omega \le \frac{1}{\delta q r}\int_X |u|^2_{\omega,h\otimes\det h} dV_\omega.$$

We say that h is *strictly Griffiths δ_ω-positive* for $\delta > 0$ if for any open set U in X and for any Kähler potential φ of ω on U, that is, $\sqrt{-1}\partial\overline{\partial}\varphi = \omega$ on U, $he^{\delta\varphi}$ is Griffiths semi-positive. This is a singular version of $\widetilde{\Theta}_{E,h}(v\otimes s, v\otimes s) \ge \delta |v|^2_\omega |s|^2_h$ for $v \in T_X$ and $s \in E$ ($\sqrt{-1}\Theta_{he^{\delta\varphi}} = \sqrt{-1}\Theta_h - \delta\sqrt{-1}\partial\overline{\partial}\varphi \otimes \mathrm{Id}_E$).

Before discussing the applications of the main theorems, we introduce a notion of a higher rank analogue of a multiplier ideal sheaf, which is a subsheaf of the locally free sheaf $\mathcal{O}_X(E)$ associated with E.

Definition

Let (E, h) be a singular Hermitian vector bundle over a complex manifold X. Then we define the *higher rank analogue of the multiplier ideal sheaf* $\mathcal{E}(h)$ by

$$\mathcal{E}(h)_x = \{s \in \mathcal{O}_X(E) \mid |s|_h^2 \text{ is locally integrable around } x \in X\}.$$

The multiplier ideal sheaf was originally introduced by Nadel in the case that E is a line bundle.

Definition (Nadel'90)

Let $(L, h = e^{-\varphi})$ be a singular Hermitian line bundle over a complex manifold X. Then we define the *multiplier ideal sheaf* $\mathcal{I}(h)$ by

$$\mathcal{I}(h)_x = \{f_x \in \mathcal{O}_{X,x} \mid |f|_h^2 = |f|^2 e^{-\varphi} \text{ is } L^1_{\mathrm{loc}} \text{ around } x\}.$$

As a corollary of Hörmander's L^2-estimate for $(E \otimes \det E, h \otimes \det h)$, we obtain the following vanishing theorem.

Theorem (I.'22)

Let (X, ω) be a projective manifold and (E, h) be a strictly Griffiths δ_ω-positive singular Hermitian metric for $\delta > 0$. Then the qth cohomology group with coefficients in the sheaf of germs of holomorphic sections of $K_X \otimes \mathcal{E}(h \otimes \det h)$ vanishes for $q > 0$

$$H^q(X, K_X \otimes \mathcal{E}(h \otimes \det h)) = 0.$$

If h is smooth, $\mathcal{E}(h \otimes \det h) = \mathcal{O}_X(E \otimes \det E)$. Hence, we can see that this theorem is a generalization of the Griffiths vanishing theorem: $H^q(X, K_X \otimes E \otimes \det E) = 0$.

Corollary

Keep the setting. If the Lelong number $\nu(-\log\det h, x)$ of $-\log\det h$ satisfies $\nu(-\log\det h, x) < 1$ for all $x \in X$, we obtain

$$H^q(X, K_X \otimes E \otimes \det E) = 0$$

for $q > 0$.

The Lelong number is an index that reflects a certain singularity of a psh function. The definition is as follows:

$$\nu(-\log\det h, x) = \liminf_{z \to x} \frac{-\log\det h(z)}{\log|z - x|}.$$

Note that $-\log\det h$ is psh since $(\det E, \det h)$ is semi-positive.

Corollary

Keep the setting. If the Lelong number $\nu(-\log\det h, x)$ *of* $-\log\det h$ *satisfies* $\nu(-\log\det h, x) < 1$ *for all* $x \in X$*, we obtain*

$$H^q(X, K_X \otimes E \otimes \det E) = 0$$

for $q > 0$*.*

Note that if $\nu(-\log\det h, x) < 1$, thanks to the result of Skoda('72), $\mathcal{E}(h \otimes \det h) = \mathcal{O}_X(E \otimes \det E)$.

Whether the higher rank analogue of the multiplier ideal sheaf is coherent or not is an important problem. In the case that E is a line bundle, Nadel showed that a multiplier ideal sheaf associated with a semi-positive singular Hermitian metric is coherent. However, in general (that is, the case that E is a vector bundle), the coherence of it is only known in few cases.

As an application of the main theorem, we can show the following result.

Theorem (I.'22)

Let h be a Griffiths semi-positive singular Hermitian metric on E. Then we have that:

1. $\mathcal{E}(h \otimes \det h)$ *is coherent.*
2. $\mathcal{E}(h)$ *is coherent if the unbounded locus* $L(\det h)$ *of* $\det h$ *is discrete.*

(Strategy of the proof)

Nadel proved the coherence of the multiplier ideal sheaf $\mathcal{I}(h)$ based on the technique of **Hörmander's L^2-estimate** for (L, h).

Hence, by using the L^2-estimate for $(E \otimes \det E, h \otimes \det h)$, we can prove the coherence of $\mathcal{E}(h \otimes \det h)$.

Here we recall the L^2-estimate for $(E \otimes \det E, h \otimes \det h)$.

Theorem (I.'20)

Let (X, ω) be a Stein or projective manifold and $E \to X$ be a holomorphic vector bundle. We also let h be a strictly Griffiths δ_ω-positive singular Hermitian metric on E for $\delta > 0$. Then for any $\overline{\partial}$-closed $E \otimes \det E$-valued (n, q)-form u with finite L^2-norm, there is an $E \otimes \det E$-valued $(n, q-1)$-form α such that $\overline{\partial}\alpha = u$ and

$$\int_X |\alpha|^2_{\omega, h\otimes \det h} dV_\omega \leq \frac{1}{\delta q r} \int_X |u|^2_{\omega, h \otimes \det h} dV_\omega.$$

(Strategy of the proof)

Nadel proved the coherence of the multiplier ideal sheaf $\mathcal{I}(h)$ based on the technique of **Hörmander's L^2-estimate**.

Hence, by using the L^2-estimate for $(E \otimes \det E, h \otimes \det h)$, we can prove the coherence of $\mathcal{E}(h \otimes \det h)$.

The second case is technically complicated, but we still use Hörmander's L^2-estimate to prove our case.

In summary, the Demailly–Skoda theorem has been important in the smooth setting. We can see that this is also true for singular Hermitian metrics. Indeed, we can show the following results (here we let (E, h) be a Griffiths positive singular Hermitian vector bundle):

1. Hörmander-type L^2-estimate for $(E \otimes \det E, h \otimes \det h)$.
2. $H^q(X, K_X \otimes \mathcal{E}(h \otimes \det h)) = 0$.
3. the coherence of $\mathcal{E}(h \otimes \det h)$.
4. the coherence of $\mathcal{E}(h)$ when h has isolated singularities.

Thank you very much.