Mirror symmetry of Fano manifolds via toric degenerations

Fumihiko Sanda

Gakushuin University

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Laurent mirror symmetry : Symplectic geometry

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Definition (Regularized quantum period)

The regularized quantum period $\widehat{G}_X(t)$ is defined as follows :

$$\widehat{G}_X(t) := 1 + \sum_{2\leq k} \sum_{\substack{d\in H_2(X;\mathbb{Z}) \\ c_1(d)=k}} k!\langle \psi^{k-2}[\mathrm{pt}]\rangle_{0,1,d} t^k, \text{where}$$

$$\langle \psi^{k-2}[\mathrm{pt}]\rangle_{0,1,d} = \int_{[X_{0,1,d}]^{\mathrm{vir}}} \psi_1^{k-2} \cup \mathrm{ev}_1^*[\mathrm{pt}]$$

is the gravitational Gromov-Witten invariant which is defined by counting holoomorphic spheres with homology class d.

Remark

Let $J_X(z)$ be the J function of X which is a “solution” of the quantum D-module of X. Then, by setting $t = 1/z$, we have

$$G_X(t) := \langle J_X(z), [\mathrm{pt}] \rangle = 1 + \sum_{2 \leq k} \sum_{\substack{d \in H_2(X;\mathbb{Z}) \\ c_1(d) = k}} \langle \psi^{k-2}[\mathrm{pt}] \rangle_{0,1,d} t^k,$$

which is called a quantum period.

J functions are calculated for many examples (e.g., complete intersections in toric Fano manifolds, Grassmannians).

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Laurent mirror symmetry : Complex/Singularity theory

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Definition (period)

The preiod $\pi_f(t)$ of f is defined as follows:

$$\pi_f(t) := \frac{1}{(2\pi\sqrt{-1})^n} \int_{|z_1|=\cdots|z_n|=\epsilon} \frac{1}{1-tf}\Omega$$
$$= \sum_{0 \le k} \mathrm{Const}(f^k) t^k,$$

where ϵ is a enough small positive real number and $\mathrm{Const}(f^k)$ is the constant term of f^k

Laurent mirror symmetry : Definition

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Definition (Laurent mirror)

Let X ba a n-dim Fano manifold.
A normal Laurent polynomial f of n variables is called a Laurent mirror of X if they satisfy

$$\widehat{G}_X(t) = \pi_f(t)$$

Remark

- If f is a Laurent mirror of X, then constant term of f is 0.
- A Laurent mirror of X is not unique.

An Example of Laurent mirror

- Let $X := \mathbb{C}P^2$.
  Then, by Givental's mirror theorem, we have

$$\widehat{G}_X(t) = \sum_{d=0}^{\infty} \frac{(3d)!}{(d!)^3} t^{3d}$$

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Set $f = x + y + 1/xy$, then we easily see that

$$\mathrm{Const}(f^k) = \begin{cases} \frac{(3d)!}{(d!)^3} & (k = 3d) \\ 0 & (k \neq 3d). \end{cases}$$

Thus f is a Laurent mirror of X.

Monotone Lagrangian submanifolds

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Let L be a Lagrangian submanifold of X and $\mu_L \in H^2(X, L; \mathbb{Z})$ be the Maslov class of L (defined later).

Remark

The natural morphism $H^2(X, L; \mathbb{Z}) \to H^2(X; \mathbb{Z})$ takes the Maslov class μ_L to $2c_1$. Hence μ_L is a "relative version" of the first Chern class.

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Since $\omega|_L = 0$, ω gives a cohomology class $[\omega] \in H^2(X, L; \mathbb{R})$.

Definition

L is called monotone if $[\omega] = k\mu_L$ for some positive real number k.

Potential functions for Lagrangian submanifolds 1

- X : a compact monotone symplectic manifold with a symplectic form ω
- L : a compact monotone oriented spin Lagrangian torus in X.
- $N := H_1(L; \mathbb{Z}) \cong \mathbb{Z}^n$

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Let $\beta \in H_2(X, L; \mathbb{Z})$ with $\mu_L(\beta) = 2$. Choose a point $p \in L$.
We define $n_\beta \in \mathbb{Z}$ by counting holomorphic disks $f : (D, \partial D) \to (X, L)$ with homology class β which through the point p

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Since $N \cong \mathbb{Z}^n$, the group ring $\mathbb{C}[N]$ is a ring of Laurent polynomials.
The monomial corresponding to $v \in N$ is denoted by z^v.

Definition (potential function of L)

We define a potential function $W_L \in \mathbb{C}[N]$ as follows:

$$W_L = \sum_{\mu_L(\beta)=2} n_\beta z^{\partial\beta}$$

Potential functions for Lagrangian submanifolds 2

Theorem (Tonkonog)

Let X be a monotone symplectic manifold and L be a monotone oriented spin Lagrangian torus. Then they satisfy

$$\widehat{G}_X(t) = \pi_{W_L}(t).$$

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Some previous results on computation of W_L

- Toric Fano manifolds (Cho-Oh, Fukaya-Oh-Ohta-Ono)
- Flag varieties (Nisninou-Nohara-Ueda)
- Del Pezzo surfaces (Vianna, Tonkonog-Pascareff)
- Quadrics (Kim)

Notes on singularity of toric variety

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Note that

- X_Σ is Fano $\Leftrightarrow$ all primitive ray generators are vertices of Δ_Σ.
- X_Σ has at worst canonical singularities $\Leftrightarrow \mathrm{int}(\Delta_\Sigma) \cap N = \{0\}$.
- X_Σ has at worst terminal singularities $\Leftrightarrow \Delta_\Sigma \cap N = \{0\} \cup \{\text{vertices of } \Delta_\Sigma\}$.
- X_Σ is Gorenstein $\Leftrightarrow \Delta_\Sigma$ is reflexive $\Rightarrow$ canonical

Main result 1

- $\mathcal{X}$: an $n+1$-dim reduced irreducible Cohen-Macaulay $\mathbb{Q}$-Gorenstein complex analytic space
- $\pi : \mathcal{X} \to D^2_\epsilon = \{z \in \mathbb{C} \mid |z| < \epsilon\}$: a proper flat morphism.
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Suppose that

- X_t is smooth for $t \neq 0$
- X_0 is isomorphic to a normal toric Fano variety X_Σ.
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Take a enough small t. Then we have the following:

Main result 2

Theorem

- There exists a symplectic form ω_t on X_t with $[\omega_t] = c_1$
- There exists a oriented spin monotone Lagrangian torus $L_t \subset X_t$

such that

- $N_{W_L} = \Delta_\Sigma$ where N_{W_L} is the Newton polytope and Δ_Σ is the fan polytope
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Remark

- Recently Galkin-Mikhalkin showed a similar statement for "$\mathbb{Q}\Gamma$-deformations".(Arxiv:2203.10043).
- If X_Σ has at worst terminal singularity, then coefficients of non-constant term are 1.
- Nishinou-Nohara-Ueda showed the theorem under the condition that X_Σ has a small toric resolution.

Examples of computation : Quadrics

- Let Q^n be an n-dim quadric $(2 \leq n)$. Then Q^n admits a toric degeneration. Using the theorem, a potential function is given by

$$W_L = \frac{1}{z_1} + \frac{z_1}{z_2} + \cdots + \frac{z_{n-1}}{z_n} + az_{n-1} + z_{n-1}z_n,$$

  where $a \in \mathbb{Q}$ is some constant.
- We know that critical value of W_L is an eigenvalue of the quantum multiplication by $c_1 = nH$.
- By using Givental's mirror theorem, quantum cohomology ring of Q^n is $\mathbb{C}[H]/\langle H(H^n - 4)\rangle$ and the eigenvalues are

$$\{n\eta \mid \eta^n = 4\} \cup \{0\}.$$

- By direct calcuration, critical values of W_L are

$$\{n\eta \mid \eta^n = a \pm 2,\ \eta \neq 0\}.$$

- Hence we have $a = 2$.

Examples of comuputation : cubic surface

- Let $Q = \mathbb{C}P^3 \cap (3)$ be a cubic surface. Then Q admits a toric degeneration. Using the theorem and $\mathbb{Z}/3\mathbb{Z}$-symmetry, a potential function is given by
$$\frac{z_1^2}{z_2} + \frac{z_2^2}{z_1} + \frac{1}{z_1 z_2} + a(z_2 + \frac{1}{z_1} + \frac{z_1}{z_2}) + b(z_1 + \frac{1}{z_2} + \frac{z_2}{z_1}),$$
where a, b are some rational numbers.
- The period of W_L is
$$1 + (6ab)t^2 + (6 + 18ab + 6a^3 + 6b^3)t^3 + \cdots$$
- By Givental's mirror theorem, quantum period is given by
$$G_Q(t) = e^{-6t} \sum_{d=0}^{\infty} \frac{(3d)!}{(d!)^3} t^d$$
and $\widehat{G}_Q(t) = 1 + 54t^2 + 492t^3 + \cdots$
- By Tonkonog's theorem, we have $ab = 9$, $a^3 + b^3 = 54$, which implies $a = b = 3$ (Since a, b are rational).

Strategy of proof

1. Generalize Maslov class to (singular) $\mathbb{Q}$-Gorenstein complex spaces
2. Counting holomorphic disks with maslov number two in singular toric Fano varieties.
3. Show "adjunction formula" for Maslov class (Maslov class is a "relative anticanonical class").
4. Show "Gromov compactnes" for singular spaces by using embeded resolution of singularities.
5. Construct a monotone Lagrangian torus in general fibers of a toric degeneration by using the gradient Hamiltonian flow.
6. Compare a general fiber and a central fiber of the toric degeneration using adjunction formula and Gromov compactness (Similar to Nishinou-Nohara-Ueda).

Maslov index

- X: a reduced normal $\mathbb{Q}$-Gorenstein complex space such that K_X^{-2r} is Cartier.
- L : a totally real submanifold in the smooth part X^{sm}.
  $\Rightarrow (\det_{\mathbb{R}} TL)^{\otimes 2r} \otimes \mathbb{C} \cong K_X^{-2r}|_L$.
- $s_L := \mathrm{vol}_L^{\otimes 2r}$: a section of $(\det_{\mathbb{R}} TL)^{\otimes 2r}$

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Let $\beta \in \pi_2(X, L)$ and take a continuous map $f : (D, \partial D) \to (X, L)$ which represents β. Then the pull back $f^* K_X^{-2r}$ is a trivial complex line bundle and fix a trivialization $f^* K_X^{-2r} \cong D \times \mathbb{C}$.
The section s_L gives a continuous map $\tilde{s}_L : \partial D \cong S^1 \to \mathbb{C}$.

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Definition (Maslov class)

The Maslov class $\mu_L : \pi_2(X, L) \to \mathbb{Q}$ is defined by
$\mu_L(\beta) = \frac{1}{r}(\text{the rotation nunber of } \tilde{s}_L) \in \mathbb{Q}$.

If X is a closed analytic subspace of a complex manifold, then $\mu_L \in H^2(X, L; \mathbb{Q})$.

Adjunction formula

Moreover we assume that X is Cohen-Macaulay.
Let $\pi : X \to D^2_\epsilon$ be a flat holomorphic map. Set $Y := \pi^{-1}(t)$ and $L_t := Y \cap L$.

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Let $\pi : X \to D^2_\epsilon$ be a flat holomorphic map. Set $Y := \pi^{-1}(t)$ and $L_t := Y \cap L$.
Suppose that

- $\pi(L) \subseteq \mathbb{R}$
- $L_t \subset Y^{\mathrm{sm}}$. $\Rightarrow$ L_t is also a totally real submanifold.
- Y is also a reduced normal Cohen-Macaulay $\mathbb{Q}$-Gorenstein complex space.

Proposition (Adjunction formula)

Let $i_* : \pi_2(Y, L_t) \to \pi_2(X, L)$ be the natural map. Then

$$\mu_L \circ i_* = \mu_{L_t}.$$

Gromov compactness

Let X be a reduced complex space.

- X is called kahler if there exists an embedding to a Kahler manifold $\widetilde{X}$.
- A kahler form on X is defined by the restriction of a Kahler form on $\widetilde{X}$.

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Let X be kahler and $L \subset X^{\mathrm{sm}}$ be a Lagrangian submanifold.
By iterated blow ups of $\widetilde{X}$ along smooth centers, we obtain a resolution of singularity $\widehat{X}$ of X which is also kahler.

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Moreover we can assume $\widehat{X}$ is isomorphic to X away from a small neighborhood of X^{sing} and there is a lift $\widehat{L} \subset \widehat{X}$ of L.
Applying Gromov's compactness theorem to $(\widehat{X}, \widehat{L})$, we obtain some compactness result for (X, L).

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Remark

Since sphere bubbles may be contained in the exceptional set, a precise statement may be not obvious.

Classification of disks

Let $X \cong X_\Sigma$ be a toric Fano variety and L be the monotone torus orbit.
For $v \in N$, $\chi_v : \mathbb{C}^* \to T_N$ be the corresponding cocharacter.
Take a point $p \in L$ and define a holomorphic disk $f_{v,p} : (D, \partial D) \to (X, L)$ by $f_{v,p}(z) = \chi_v(z)p$.
Then we have the following:

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Then we have the following:

Proposition

Let $\beta \in H_2(X, L; \mathbb{Z})$ be a homology class with $\mu_L(\beta) = 2$ which is realized by a stable disk f. Then

- $\partial\beta \in \Delta_\Sigma \subset N \cong H_1(L; \mathbb{Z})$
- If $\partial\beta$ is a vertex of Δ_Σ, then the image of f is contained in X_Σ^{sm}.
- Moreover if $\partial\beta$ is a vertex v and f through p, then $f = f_{v,p}$

Construction of Lagrangian and skecth of proof

- Let $(\mathcal{X}, \pi)$ be a toric degeneration and $L_0 \subset X_0 \cong X_\Sigma$ be the monotone torus orbit. For simplicity, suppose $\mathcal{X}$ is kahler.

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- Let $(\mathcal{X}, \pi)$ be a toric degeneration and $L_0 \subset X_0 \cong X_\Sigma$ be the monotone torus orbit. For simplicity, suppose $\mathcal{X}$ is kahler.
- Let φ_t be the flow of the vector field $\frac{\mathrm{gradRe}\pi}{|\mathrm{gradRe}\pi|^2}$.
  Then $\varphi_t(X_0^{\mathrm{sm}}) \subset X_t$ and set $L_t := \varphi_t(L_0) \subset X_t, \mathcal{L} = \cup L_t \subset \mathcal{X}$.
  Moreover L_t and $\mathcal{L}$ are Lagrangian submanifolds with $\pi(\mathcal{L}) \subset \mathbb{R}$.
  $\Rightarrow$ we can apply adjunction formula.

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  Moreover L_t and $\mathcal{L}$ are Lagrangian submanifolds with $\pi(\mathcal{L}) \subset \mathbb{R}$.
  $\Rightarrow$ we can apply adjunction formula.
- Using this and a compactness theorem for $(\mathcal{X}, \mathcal{L})$, we can compare disk counting of (X_t, L_t) and (X_0, L_0).
  Hence the theorem follows from the disk counting for the toric Fano variety $X_0 \cong X_\Sigma$.