Positive mass theorem with low-regularity Riemannian metrics

Weimin Sheng
with W. Jiang and H. Zhang

Zhejiang University

Pacific Rim Complex & Symplectic Geometry Conference
Aug. 1-5, 2022

Positive Mass Theorem

One of the most famous results on manifolds with nonnegative scalar curvature is the positive mass theorem proved by Scheon and Yau: 1979, 1981 (for $n < 8$), 2017.

- The ADM mass of each end of an n-dimensional asymptotically flat manifold with nonnegative scalar curvature is nonnegative.
- If the ADM mass of an end is zero, then the manifold is isometric to the Euclidean space.
- If the manifold is spin, Witten in 1981 proved the positive mass theorem by a different method.

All the results are assumed to be smooth.

Problem-PMT with singularity

It is natural to ask:

If the manifold admits singularity in a subset Σ, what is the conditions of Σ such that the positive mass theorem still holds?

It is necessary to assume that the metric is continuous:

In 2018, Shi and Tam has constructed an asymptotically flat metric with a cone singularity and nonnegative scalar curvature, but with negative ADM mass.

PMT with singularity

There are many results

- Miao (2002) and Shi-Tam (2002) proved a PMT with Lipschitz metric and Σ is a hypersurface satisfying certain conditions on the mean curvatures of Σ. Shi-Tam used this result to prove the positivity of the Brown-York quasilocal mass.
- McFeron and Székelyhidi (2012) used Ricci flow giving a new proof of positivity of Miao's result and proved the rigidity when the ADM mass is zero.
- Dan Lee (2013) considered a PMT for (M^n, g) with bounded $W^{1,p}_{loc}$-metric for $n < p \leq \infty$ and are smooth away from a singular set Σ with $\frac{n}{2}(1 - n/p)$-dimensional Minkowski content vanishing. This result was improved by Shi-Tam 2018 where they showed that the singular set Σ only requires $(n-2)$-dimensional Minkowski content vanishing.

PMT with singularity

- If (M^n, g) is spin, Lee and LeFloch (2015) were able to prove a PMT for $C^0 \cap W^{1,n}_{loc}$ metric, where the metric could be singular. Their theorem can be applied to all previous results for non-smooth metrics under the additional assumption that the manifold is spin.
- Li-Mantoulidis 2019 considered bounded metric with skeleton singularities along a codimensional 2 submanifolds. Without further derivative assumptions on metric, they proved a PMT in dimensional 3 with isolated singularity.

Our theorem

We improve and recover some results of Shi-Tam 2018, Lee 2013 and Lee-LeFloch 2015.

Theorem

Let M^n $(n \geq 3)$ be a smooth manifold. Let $g \in C^0 \cap W^{1,p}_{loc}(M)$ $(n \leq p \leq \infty)$ be a complete asymptotically flat metric on M. Assume g is smooth away from a bounded closed subset Σ with $\mathcal{H}^{n-\frac{p}{p-1}}(\Sigma) < \infty$ if $n \leq p < \infty$ or $\mathcal{H}^{n-1}(\Sigma) = 0$ if $p = \infty$, assume that $R_g \geq 0$ on $M \setminus \Sigma$. Then the ADM mass of g of each end is nonnegative. Moreover, the ADM mass of one end is zero if and only if when (M, g) is isometric to Euclidean space.

Remarks

Remark

For the rigidity part, we will show such space has nonnegative Ricci curvature in RCD sense provided that the mass is zero. The rigidity would follow by the volume rigidity of nonnegative Ricci curvature.

Remark

For the case $p = \infty$, the condition $\mathcal{H}^{n-1}(\Sigma) = 0$ is optimal, since one can construct counterexample if Σ is a hypersurface. In particular, this confirms a conjecture of Lee (2013).

Some fundament definitions

Definition (asymptotically flat)

Let M be a smooth n-manifold, and g be a C^0 metric on M. We say that (M, g) is asymptotically flat if $\exists K \subset M$ a compact subset, s. t. g is C^2 on $M \backslash K$, $M \backslash K$ has finite many components, says Σ_l, $l = 1, \dots, p$, and for each component Σ_l, $\exists$ smooth diffeomorphism $\Phi_l : \Sigma_l \to \mathbb{R}^n \setminus B$, B a ball, s. t. if we see Φ_l as a coordinate system on Σ_l, then

$$g_{ij} - \delta_{ij} = O(|x|^{-\delta})$$

$$g_{ij,k} = O(|x|^{-\delta-1})$$

$$g_{ij,kl} = O(|x|^{-\delta-2}),$$

where $\delta > (n-2)/2$ some constant, and the commas denote partial derivatives. We also call each components Σ_l as an end of M.

Definition (ADM mass)

Given an asymptotically flat manifold (M, g), we define the ADM mass of each end Σ_l as the limit

$$\lim_{r\to\infty} \frac{1}{2(n-1)}\omega_{n-1} \int_{S_r} \sum_{i,j=1}^{n} (g_{ij,i} - g_{ii,j})\nu^j d\mu,$$

where S_r: the coordinate sphere in (Σ_l, Φ_l) of radius r, ν: the unit outward normal vector of S_r, ω_{n-1}: the volume of the unit $(n-1)$-dim sphere, $d\mu$: the volume form of S_r in Euclidean metric. In a given end Σ_l, we will denote $r = \Phi_l^*\left(\sqrt{\Sigma_{i=1}^n (x^i)^2}\right)$.

Bartnik: this limit exists provided that the scalar curvature of g is integrable, and it is a geometric invariant.

We only check one end of M, other ends holds in the same way

We choose an arbitrary end, and denote the mass by $m(g)$.

We fix a smooth background metric h on M, $C^{-1}h \le g \le Ch$ for $C > 1$, and $h = g$ outside some compact subset of M. All the convergences are taken w.r.t. h. $\tilde{\nabla}$: the covariant derivative taken w. r. t. h.

Definition

Let M^n smooth, g: $C^0 \cap W^{1,p}_{\mathrm{loc}}$ metric on M, h: a fixed smooth metric as above. For a family of smooth function $\{f_\delta\}$, we say that f_δ converge to a function f locally in $W^{1,p}$-norm, if $\forall \epsilon > 0$, $\forall 0 < r < 1$, $\exists \delta_0 > 0$, s. t. $\forall \delta \in (0, \delta_0), x \in M$, we have

$$\int_{B_r(x)} |f_\delta - f|^p d\mu_h < \epsilon, \int_{B_r(x)} |\tilde{\nabla} f_\delta - \tilde{\nabla} f|^p d\mu_h < \epsilon,$$

the norm are taken with respect to h. f_δ converge to a function f locally in C^0-norm, if $\forall \epsilon > 0$, $\forall r > 0$, $\exists \delta_0 > 0$, s. t. $\forall \delta \in (0, \delta_0), x \in M$, we have $\sup_{B_r(x)} |f_\delta - f| d\mu_h < \epsilon$.

Definition

g be $C^0 \cap W^{1,n}_{loc}$ metric on M, and h is a fixed smooth metric as above. We can define the scalar curvature distribution as Lee and LeFloch 2015,

$$\langle R_g, \varphi \rangle := \int_M \left(-V \cdot \tilde{\nabla} \left(\varphi \frac{d\mu_g}{d\mu_h} \right) + F \varphi \frac{d\mu_g}{d\mu_h} \right) d\mu_h \tag{1}$$

for any compactly supported $\varphi \in W^{1,n/(n-1)}$, where "$\cdot$" w.r.t. h, V: a vector field, F: a scalar field,

$$\Gamma^k_{ij} := \frac{1}{2} g^{kl} \left(\tilde{\nabla}_i g_{jl} + \tilde{\nabla}_j g_{il} - \tilde{\nabla}_l g_{ij} \right), \tag{2}$$

$$V^k := g^{ij} \Gamma^k_{ij} - g^{ik} \Gamma^j_{ji} = g^{ij} g^{k\ell} (\tilde{\nabla}_j g_{i\ell} - \tilde{\nabla}_\ell g_{ij}), \tag{3}$$

$$F := \mathrm{tr}_g \widetilde{Ric} - \tilde{\nabla}_k g^{ij} \Gamma^k_{ij} + \tilde{\nabla}_k g^{ik} \Gamma^i_{ji} + g^{ij} \left(\Gamma^k_{k\ell} \Gamma^\ell_{ij} - \Gamma^k_{j\ell} \Gamma^\ell_{ik} \right), \tag{4}$$

and μ_h: the Lebesgue measure w.r.t. h.

- By Lee-Lefloch, $\langle R_g, \varphi \rangle$ is independent of h,
- $\langle R_g, \varphi \rangle = \int_M R_g \varphi d\mu_h$ when g is C^2 and R_g is defined in classical sense.
- We say that g has weakly nonnegative scalar curvature if for any nonnegative test function φ, $\langle R_g, \varphi \rangle \geq 0$.

Smoothing the metric

The following mollification lemma could be found in Grant-Tassotti 2014 Lemma 4.1(for $W^{2,\frac{n}{2}}$ version), our version could be proved in the same manner.

Lemma (2.1)

Let M^n smooth manifold, $g \in C^0 \cap W^{1,n}_{\mathrm{loc}}$ on M, then $\exists$ smooth metrics $\{g_\delta\}$, $\delta > 0$, s. t. g_δ converge to g locally both in C^0-norm and in $W^{1,n}$ norm. Moreover, if g is smooth away from a compact subset, then we can take g_δ such that $g_\delta = g$ on $M \setminus K$ some compact set K independent of δ.

Remark

Since we take g_δ such that $g_\delta = g$ on $M \setminus K$ some compact set K independent of δ, by the definition of ADM mass, $m(g_\delta) = m(g)$.

From this mollification, the scalar curvature distribution has an approximation.

Lemma (2.2)

Let M^n be smooth, $g \in C^0 \cap W^{1,n}_{\mathrm{loc}}$ on M. Suppose that the L^2 Sobolev constant of (M, g) has an upper bound C_s. Let $\{g_\delta\}$ the mollification of g. Suppose $g_\delta = g$ on $M \setminus K$ some compact set K. Then

$$|\langle R_{g_\delta}, u^2\rangle - \langle R_g, u^2\rangle| \leq \Psi(\delta) \int_M |\nabla u|^2 d\mu_g, \forall u \in C_0^\infty(M),$$

where $\lim_{\delta \to 0} \Psi(\delta) = 0$. Here $\Psi(\delta)$ depends only on the Sobolev constant C_s and the $W^{1,n}$-norm of $|g - g_\delta|$.

The L^2 Sobolev constant condition means: $\forall u \in C_0^\infty(M)$, $\left(\int_M u^{\frac{2n}{n-2}} d\mu_g\right)^{\frac{n-2}{n}} \leq C_s \int_M |\nabla u|^2 d\mu_g$. This always holds for asymptotically flat manifolds.

Remark

In the same condition of Lemma 2.2, we can calculate in the same manner and get that

$$|\langle R_{g_\delta}, u\rangle - \langle R_g, u\rangle| \leq \Psi(\delta)\left(\int_M |\nabla u|^{\frac{n}{n-1}} d\mu_{g_\delta}\right)^{\frac{n-1}{n}}, \forall u \in C_0^\infty(M),$$

where $\Psi(\delta)$ is independent of u and $\Psi(\delta) \to 0$ as $\delta \to 0$.

Weak nonnegative scalar curvature

Under the same assumption about Σ as in Theorem, we can check that R_g is weakly nonnegative.

Lemma (2.3)

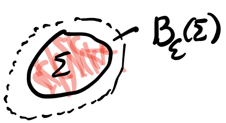
Let M^n be a smooth manifold with $g \in C^0 \cap W^{1,p}_{loc}(M)$ with $n \le p \le \infty$. Assume g is smooth away from a closed subset Σ with $\mathcal{H}^{n-\frac{p}{p-1}}(\Sigma) < \infty$ if $n \le p < \infty$ or $\mathcal{H}^{n-1}(\Sigma) = 0$ if $p = \infty$, and assume $R_g \ge 0$ on $M \setminus \Sigma$, then $\langle R_g, u\rangle \ge 0$ for any nonnegative, compactly supported $u \in W^{1,p/(p-1)}$.

Construct a cut-off function $\eta_\epsilon \ge 0$ is key (Cheeger 2003) but standard, based on the condition $\mathcal{H}^{n-\frac{p}{p-1}}(\Sigma) < \infty$ if $n \le p < \infty$ or $\mathcal{H}^{n-1}(\Sigma) = 0$ if $p = \infty$.

$\eta_\varepsilon \equiv 1$ in $B_{\varepsilon/2}(\Sigma) \supset \Sigma$

$\operatorname{supp} \eta_\varepsilon \subset B_\varepsilon(\Sigma)$, $0 \le \eta_\varepsilon \le 1$

$\lim_{\varepsilon \to 0} \int_M |\nabla \eta_\varepsilon|^{p/(p-1)} d\mu_h = 0$



$$\langle R_g, u\rangle = \langle R_g, \eta_\epsilon u\rangle + \langle R_g, (1-\eta_\epsilon)u\rangle.$$

$$\langle R_g, (1-\eta_\epsilon)u\rangle = \int_{M\setminus\Sigma} R_g(1-\eta_\epsilon)u\,d\mu_g \ge 0, \lim_{\epsilon \Rightarrow 0} |\langle R_g, \eta_\epsilon u\rangle| = 0.$$

Schoen-Yau's Lemma

Let us recall a lemma essentially proved in Schoen-Yau 1979.

Lemma (3.1)

Let $(M, \tilde{g})$ be a complete smooth asymptotically flat manifold, f, h be smooth functions with compact support on M, then there exists an $\epsilon > 0$, such that if f satisfies

In S-Y, $f = f_+ - f_-$ $\left(\int_M |f_-|^{\frac{n}{2}}\right)^{\frac{2}{n}} < \varepsilon$

$$\int_M f\xi^2 d\mu_{\tilde{g}} \geq -\epsilon \int_M |\nabla \xi|^2 d\mu_{\tilde{g}}, \forall \xi \in C_0^\infty(M),$$

then the equation

$$\Delta_{\tilde{g}} v - fv = h \tag{5}$$

has a solution v satisfying $v = O(r^{2-n})$ as $r \to \infty$. Moreover,

$$v = A/r^{n-2} + \omega,$$

where A is a constant, $\omega = O(r^{1-n})$ and $|\partial\omega| = O(r^{-n})$.

The proof is almost the same as Schoen-Yau 1979.
Now under assumption of Theorem ($g \in C^0 \cap W^{1,p}_{loc}(M)$, $n \leq p \leq \infty$, $\cdots$), we consider the equation (see also Li-Mantoulidis 2019)

$$\begin{cases} \Delta_{g_\delta} u_\delta - c_n \varphi^2 R_{g_\delta} u_\delta = 0 \quad on \quad M \\ \lim_{r\to\infty} u_\delta = 1 \end{cases}, \tag{6}$$

where $\varphi : M \to [0,1]$ be a smooth cut-off function such that $\varphi = 1$ on K, $\varphi = 0$ outside some neighborhood of K, a compact set $K \supset \Sigma$, $c_n = \frac{n-2}{4(n-1)}$.

Corollary (3.2)

There exists $\delta_0 > 0$ such that the equation (6) has a positive solution for all $\delta \in (0, \delta_0)$. Moreover, we have

$u_\delta - 1 = v_\delta$

$$u_\delta = 1 + \frac{A_\delta}{r^{n-2}} + \omega,$$

where A_δ is a constant, $\omega = O(r^{1-n})$ and $|\partial\omega| = O(r^{-n})$.

Now, we define the conformal metrics

$$\tilde{g}_\delta = u_\delta^{\frac{4}{n-2}} g_\delta$$

Then the standard conformal transformation formula shows

$$\begin{aligned}\tilde{R}_{\tilde{g}_\delta} &= -c_n^{-1} u_\delta^{-\frac{n+2}{n-2}} (\Delta_{g_\delta} u_\delta - c_n R_{g_\delta} u_\delta) \\ &= u_\delta^{1-\frac{n+2}{n-2}} (R_{g_\delta} - \varphi^2 R_{g_\delta}) \\ &\geq 0,\end{aligned}$$

since $\varphi = 1$ on K and $R_{g_\delta} = R(g) \geq 0$ on $M \backslash K$.

Lemma (3.4)

The lower limit of mass of g_δ is no less than the lower limit of mass of $\tilde{g}_\delta$.

By the definition of mass, we can calculate straightforwardly and get the following equation (see P. Miao 2002),

$$m(\tilde{g}_\delta) = m(g_\delta) + (n-1)A_\delta \tag{7}$$

We can show

$$A_\delta = \frac{1}{(2-n)\omega_n} \int_M \left(|\nabla_{g_\delta} u_\delta|^2 + c_n \varphi^2 R_{g_\delta} u_\delta^2\right) d\mu_{g_\delta}$$

and

$$\overline{\lim_{\delta \to 0^+}} A_\delta \le 0.$$

Thus

$$\underline{\lim_{\delta \to 0^+}} m(g_\delta) \ge \underline{\lim_{\delta \to 0^+}} m(\tilde{g}_\delta).$$

Proof of the inequality part of Theorem

Proof of the inequality part of Theorem.

Since $g_\delta = g$ on $M \backslash K$, we have

$$m(g_\delta) = m(g)$$

Since $\tilde{g}_\delta$ has nonnegative scalar curvature, by the classical positive mass theorem, we have

$$m(\tilde{g}_\delta) \geq 0$$

Thus by Lemma 3.4, we have

$$m(g) \geq 0$$

□

Outline of the proof of rigidity part

Let us outline the idea of the proof.

- First, we follow the idea of Shi-Tam 2018 to show that the manifold is Ricci flat away from Σ.
- Then we show that the manifold has nonnegative Ricci curvature in RCD sense.
- Noting that the manifold is asymptotically flat, by volume convergence and volume comparison, we get the rigidity result.

We first prove that $m(g) = 0$ implies Ricci curvature vanishing away from the singular set Σ.

Lemma (4.1)

Assume as Theorem. If $m(g) = 0$, then $\mathrm{Ric}_g \equiv 0$ on $M \backslash \Sigma$.

The idea of the argument comes from Shi-Tam 2018. We can argue it by a contradiction through a local perturbation.

RCD space

We show that the singular space has nonnegative Ricci curvature in RCD sense providing the manifold is Ricci flat away from the singular set.

Theorem (4.2)

Let (M^n, g) $(n \geq 3)$ be a smooth manifold with $g \in C^0 \cap W^{1,p}_{loc}(M)$ with $n \leq p \leq \infty$. Assume g is smooth and Ricci flat away from a closed subset Σ with $\mathcal{H}^{n-\frac{p}{p-1}}(\Sigma) < \infty$ when $n \leq p < \infty$ and $\mathcal{H}^{n-1}(\Sigma) = 0$ when $p = \infty$, and assume g is asymptotically flat. Then (M^n, g) as a metric measure space with Lebesgue measure has nonnegative Ricci curvature in the sense of RCD.

Definition (4.3 RCD Ricci lower bound)

Let K be some real constant. For a Riemannian manifold (M^n, g) with $g \in C^0(M)$, we say that it is a RCD(K, n) space, or say it has Ricci curvature not less than K in the sense of RCD, if

(1) it is infinitesimally Hilbertian,

(2) for some $C > 0$, and some point $p \in M$, it holds $\mu_g(B_r(x)) \leq e^{Cr^2}$, for any $r > 0$, where μ_g is Lebesgue measure taken w. r. t. g,

(3) for any $f \in W^{1,2}(M)$ satisfying $|\nabla f| \in L^\infty(M)$, it admits a Lipschitz representative $\tilde{f}$ with $\mathrm{Lip}(\tilde{f}) \leq \|\nabla f\|_{L^\infty(M)}$,

(4) $\forall f \in D(\Delta)$ with $\Delta f \in W^{1,2}(M)$, $\forall \varphi \in L^\infty(M) \cap D(\Delta)$ with $\varphi \geq 0$, $\Delta\varphi \in L^\infty(M)$, the Bochner ineq

$$\frac{1}{2}\int_M |\nabla f|^2 \Delta\varphi d\mu_g \geq \frac{1}{n}\int_M (\Delta f)^2 \varphi d\mu_g + \int_M \varphi\left(\langle \nabla f, \nabla\Delta f\rangle + K|\nabla f|^2\right) d\mu_g$$

holds.

Remark

- For a Riemannian manifold (M^n, g) with volume measure and $g \in C^0(M)$, (1) and (3) in Definition 4.3 hold automatically.
- If g is asymptotically flat, then (2) holds.
- Therefore, to prove Theorem 4.2, it only needs to check a weak Bochner inequality (see Proposition 4.4).

Proposition (4.4 Weak Bochner inequality)

Assume as Theorem 4.2. Then for any $u \in D(\Delta)$ such that $\Delta u \in W^{1,2}$, and any nonnegative bounded test function $\varphi \in D(\Delta)$ with $|\Delta\varphi| \in L^\infty$, the following Bochner inequality holds

$$\frac{1}{2}\int_M \Delta\varphi|\nabla u|^2 d\mu_g - \int_M \varphi\langle\nabla\Delta u, \nabla u\rangle d\mu_g \geq \frac{1}{n}\int_M \varphi(\Delta u)^2 d\mu_g. \quad (8)$$

Outline of the Proof of Prop 4.4

Step 1 The classical Bochner inequality holds pointwisely away from the singular set for smooth functions :

$$\frac{1}{2}\Delta|\nabla u|^2 \geq \frac{1}{n}(\Delta u)^2 + \langle \nabla \Delta u, \nabla u \rangle, \text{on} \quad M \setminus \Sigma, \forall u \in C^\infty(M).$$

To get the weak Bochner inequality for $u \in C^\infty(M)$, we may multiply the cut-off function ψ_ϵ, times a test function φ to the inequality above, and integrat it, take limit as $\epsilon \to 0^+$.

Step 2 For $u \in D(\Delta)$ with $\Delta u \in W^{1,2}$, we construct a smooth approximation of u, and take limit for the W-Bochner inequality for the smooth approximation. In order to ensure the limit hold, the smooth approx. must satisfy some good property, which does not hold for standard mollification. We use the heat kernel to construct the approximation and get the gradient estimate. This is the most difficult part.

Proof of Theorem 4.2.

Noting the Bochner inequality proved in Proposition 4.4, (M^n, g) with Lebesgue measure is an RCD space with nonnegative Ricci curvature. □

Proof of rigidity part of Theorem .

Now the rigidity part of Theorem follows immediately from Theorem 4.2 and the volume stability theorem of RCD space (Lott-Villani 2009; Sturm2006). Actually, by Lemma 4.1and Theorem 4.2, the manifold (M^n, g) has nonnegative Ricci curvature in RCD sense. (To be continued.) □

Proof of rigidity part of Theorem

Continue.

Take any point $x \in M$, noting that the manifold is asymptotically flat, we have

$$\lim_{R\to\infty} \frac{\mathrm{Vol}(B_R(x))}{\omega_n R^n} = 1, \tag{9}$$

By volume comparison of RCD space with nonnegative Ricci curvature, we have

$$\mathrm{Vol}(B_R(x)) = \omega_n R^n. \tag{10}$$

By volume rigidity Theorem by De Philippis-Gigli-Nicola 2018 , we get $B_R(x)$ is isometric to $B_R(0^n) \subset \mathbb{R}^n$. This implies M is isometric to $\mathbb{R}^n$. □

Scalar curvature lower bounds preserved by Ricci flow

Theorem (5.1)

M^n compact, $g \in W^{1,p}(M) \cap C^2(M \setminus \Sigma)$, $(n < p \leq \infty)$, $\mathcal{H}^{n-\frac{p}{p-1}}(\Sigma) < \infty$ if $n < p < \infty$ or $\mathcal{H}^{n-1}(\Sigma) = 0$ if $p = \infty$. $R_g \geq a$ on $M \setminus \Sigma$ for some constant a. Let $g(t), t \in (0, T_0)$ be the Ricci flow from initial metric g. Then $R_{g(t)} \geq a$ on M, $\forall t \in (0, T_0)$.

Remark

- Here the Ricci flow $g(t), t \in (0, T_0)$ from C^0 initial metric g means that $g(t)$ satisfies the Ricci flow equation for $t \in (0, T_0)$ and $\lim_{t \to 0^+} d_{GH}((M, g(t)), (M, g)) = 0$
- (M. Simon 2002) For a continuous metric g, there exists a flow $\tilde{g}(t), t \in (0, T_0)$ equivalent to the Ricci flow modulo the action of diffeomorphisms and C^0 converge to g.

The relation between nonnegative and positive scalar curvature

Theorem (5.2)

M^n $(n \geq 3)$ compact, $g \in W^{1,p}(M) \cap C^2(M \setminus \Sigma)$, $(n < p \leq \infty)$, $\mathcal{H}^{n-\frac{p}{p-1}}(\Sigma) < \infty$ if $n < p < \infty$ or $\mathcal{H}^{n-1}(\Sigma) = 0$ if $p = \infty$. $R_g \geq 0$ on $M \setminus \Sigma$. Then either (M, g) is isometric to a Ricci flat manifold or M admits a metric with positive scalar curvature ($R_g > 0$).

Remark

If $p = \infty$, the condition $\mathcal{H}^{n-1}(\Sigma) = 0$ is optimal. In fact, if Σ is a hypersurface, then there exists counterexamples: we can prescribe arbitrary scalar curvature on $M \setminus \Sigma$ if M is a torus.

This result is connected with the following

Conjecture (R. Schoen)

Let g be a C^0 metric on a compact manifold M which is smooth away from a submanifold $\Sigma \subset M$ with $\mathrm{codim}(\Sigma \subset M) \geq 3$ and $R_g \geq 0$ on $M \setminus \Sigma$, then either g smoothly extends to a Ricci flat metric on M, or M admits a metric with $R_g > 0$.

Theorem (Li - Mantoulidis 2019)

Schoen's conjecture holds if $n = 3$ and Σ is a finite set.

Theorem 5.2 is a corollary of Theorem 5.1 and following

Theorem (Bourguignon, before 1975)

Let (M, g) be a compact Riemannian manifold with nonnegative scalar curvature ($R_g \geq 0$), then either g is Ricci flat, or M admits a metric with positive scalar curvature ($R_g > 0$).

Torus rigidity

One of the most well known results related to scalar curvature is:

Theorem (Torus rigidity theorem, Schoen, Yau, Gromov, Lawson)

Any metrics on torus with nonnegative scalar curvature is flat.

Remark

- In 1979, Schoen and Yau proved this theorem in dimensions 3 to 7.
- In 1980, Gromov and Lawson proved this theorem in all dimensions.
- It would be useful if we strengthen this theorem replacing the torus by the connected sum of a torus and an arbitrary compact manifold of the same dimensional. Schoen and Yau used minimal surface approach and proved it in 2017.

We have

Theorem (5.3)

$M = T^n$, $g \in W^{1,p}(M) \cap C^2(M \setminus \Sigma)$, $(n < p \leq \infty)$, $\mathcal{H}^{n-\frac{p}{p-1}}(\Sigma) < \infty$ if $n < p < \infty$ or $\mathcal{H}^{n-1}(\Sigma) = 0$ if $p = \infty$. $R_g \geq 0$ on $M \setminus \Sigma$. Then (M, g) is isometric to a flat torus.

Remark

If $p = \infty$, the condition $\mathcal{H}^{n-1}(\Sigma) = 0$ is optimal. In fact, if Σ is a hypersurface, then there exists counterexamples: we can prescribe arbitrary scalar curvature on $M \setminus \Sigma$ if M is a torus.

Thanks