On the number of periodic orbits in Hamiltonian dynamics

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Notations

Let (M, ω) be a (closed) symplectic manifold and $H : M \to \mathbb{R}$ be a Hamiltonian function.

Hamiltonian vector field

The Hamiltonian vector field X_H is defined by

$$\omega(X_H, \cdot) = -dH.$$

In the standard Eucledean space $(\mathbb{R}^{2n}, \omega_0)$, X_H is written as follows.

$$X_H(x_1, \cdots, x_n, y_1, \cdots, y_n) = \sum_{i=1}^{n} \left\{ -\frac{\partial H}{\partial y_i} \frac{\partial}{\partial x_i} + \frac{\partial H}{\partial x_i} \frac{\partial}{\partial y_i} \right\}$$

This is a local model of Hamiltonian vector fields (Darboux theorem).

We use S^1-dependent (1-periodic) Hamiltonian functions and Hamiltonian vector fields. Let $\{\phi_H^t\}$ be a flow of a periodic Hamiltonian vector field.

$$H : S^1 \times M \to \mathbb{R}$$

$$\phi_H^0(x) = x$$

$$\frac{d}{dt}\phi_H^t(x) = X_{H_t}(\phi_H^t(x))$$

The Hamiltonian diffeomorphism group is the set of time 1 flow of such vector fields.

Hamiltonian diffeomorphim group

$$\mathrm{Ham}(M,\omega) = \left\{ \phi_H^1 \in \mathrm{Symp}_0(M,\omega) \;\middle|\; H : S^1 \times M \to \mathbb{R} \right\}$$

$Symp_0(M,\omega)$ is the identity component of the symplectic diffeomorphism group $Symp(M,\omega) = \{\phi \in \mathrm{Diff}(M) \mid \phi^*\omega = \omega\}$.

Periodic orbits

A l-periodic orbit ($l \in \mathbb{N}$) of a Hamiltonian flow $\{\phi_H^t\}$ is a loop which satisfies the following conditions.

$$x : \mathbb{R}/l \cdot \mathbb{Z} \longrightarrow M$$
$$\frac{d}{dt}x(t) = X_H(x(t)).$$

Note that any iteration of a periodic orbit is a periodic orbit.

$$y : \mathbb{R}/kl \cdot \mathbb{Z} \longrightarrow M$$
$$y(t) = x(t)$$

is a kl-periodic orbit. We study the number of periodic orbits, so we have to exclude iterated periodic orbits of lower periods.

(simple) periodic orbits

A l-periodic orbit $x(t)$ is called "not simple" if there is another l'-periodic orbit $y(t)$ such that l' is a divisor of l and

$$x(t) = y(t) \quad (\forall t \in \mathbb{R})$$

holds. In other words, $x(t)$ is a iteration of $y(t)$. Non-simple periodic orbits are not essential.

homotopy class of periodic "points"

Let H and G be periodic Hamiltonian functions so that their time one flows coincide ($\phi_H^1 = \phi_G^1$). There is one to one correspondence between periodic orbits of $\{\phi_H^t\}$ and $\{\phi_G^t\}$. Assume that $x(t)$ and $y(t)$ be l-periodic orbits of their flows such that $x(0) = y(0)$ holds. Then a loop

$$\gamma(t) = \begin{cases} x(t) & 0 \leq t \leq l \\ y(l-t) & l \leq t \leq 2l \end{cases}$$

is contractible. So $x(t)$ and $y(t)$ are homotopic.This is a consequence of the Arnold conjecture. If $\{\psi^t\}_{0 \leq t \leq 1}$ is a loop of Hamiltonian diffeomorphisms, then every loop $\{\psi^t(x)\}_{0 \leq t \leq 1}$ $(x \in M)$ is contactible.

Remark

A homotopy class of periodic points of a Hamiltonian diffeomorphim $\phi \in \mathrm{Ham}(M, \omega)$ is well-defined. It does not depend on the choice of a Hamiltonian function H and its flow such that $\phi = \phi_H^1$ holds.

Conley conjecture

Hamiltonian diffeomorphisms tend to have infinitely many simple contractible periodic orbits.

Conley conjecture (Conley 1984)

Any Hamiltonian diffeomorphism on the standard torus $(\mathbb{T}^{2n}, \omega_0)$ has infinitely many simple contractible periodic orbits.

This original conjecture was already proved and it is known that this is also true for weakly exact symplectic manifolds ($\omega|_{\pi_2(M)} = 0$), symplectic manifolds with vanishing c_1 ($c_1|_{\pi_2(M)} = 0$) and negatively monotone symplectic manifolds. Today, it is believed that this theorem holds for "almost all" (not "all") symplectic manifolds. This is today's Conley conjecture.

Counterexample

An irrational rotation of the 2-sphere ϕ_θ only has 2 simple periodic orbits (the North pole and the South pole). So Conley conjecture is false on S^2.

For $(x, y, z) \in S^2$ and $\theta \in \mathbb{R} \backslash \mathbb{Q}$, the irrational rotation is defined as follows.

$$\phi^t_\theta(x, y, z) = (\cos \frac{\theta t}{2\pi} x + \sin \frac{\theta t}{2\pi} y, -\sin \frac{\theta t}{2\pi} x + \cos \frac{\theta t}{2\pi} y, z)$$

The periodic points of ϕ_θ are $N = (0, 0, 1)$ and $S = (0, 0, -1)$.

Remark

In dimension two, the Conley conjecture holds for $\Sigma_{g \geq 1}$.

Generic Conley conjecture

We also have another variant of the Conley conjecture. That is the following generic Conley conjecture.

C^∞-generic Conley conjecture

For any closed symplectic manifold (M, ω), C^∞-generic Hamiltonian diffeomorphisms have infinitely many simple contractible periodic orbits.

In dimension 2, the following strong results is known.

C^∞-generic density theorem (Asaoka-Irie)

Periodic points are dense C^∞-generically in dimension two.

Main theorem 1

We proved C^∞-generic Conley conjecture for "almost all" symplectic manifolds.

Main theorem 1

Let (M^{2n}, ω) be a closed symplectic manifold which satisfies at least one of the following three conditions.

1. n is odd
2. $H_{odd}(M : \mathbb{R}) \neq 0$
3. $N \neq 1$ (N: minimum Chern number)

Then, there exists a C^∞-generic subset $\mathcal{U} \subset \mathrm{Ham}(M, \omega)$ such that any element of $\mathcal{U}$ has infinitely many simple contractible periodic orbits.

The tangent bundle $\pi : TM \longrightarrow M$ has (comtatible) complex structures.
Let $c_1(TM) \in H^2(M : \mathbb{Z})$ be the uniquely determined first Chern class.
The minimum Chern number $N \in \mathbb{N} \cup \{+\infty\}$ is the positive generator of the image of the following map.

$$\begin{aligned} c_1 : \pi_2(M) &\longrightarrow \mathbb{Z} \\ [u] &\mapsto \langle c_1(TM), [u] \rangle \end{aligned}$$

If this map is trivial, N is $+\infty$.

Sketch of the proof

We apply the following local theorem.

Theorem (Birkhoff, Moser)

Let ϕ be a symplectic map defined in an open neighborhood of the origin in $(\mathbb{R}^{2n}, \omega_0)$ and the origin is a fixed point. Let $\{\lambda_1, \cdots, \lambda_m, \lambda_1^{-1}, \cdots, \lambda_m^{-1}\}$ be the all engenvalues of the differential map

$$d\phi_{(0,\cdots,0)} : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$$

on the unit circle in $\mathbb{C}$ ($|\lambda_i| = 1$). Assume that ϕ satisfies the following conditions.

1. $m \geq 1$
2. $\Pi_{k=1}^{m} \lambda_k^{j_k} \neq 1$ for $1 \leq \sum_{k=1}^{m} |j_k| \leq 4$
3. The Taylor coefficient of ϕ up to order three satisfies a "non-degenerate" condition.

Then ϕ possesses infinitely many periodic orbits in any neighborhood of 0.

1. Let $\mathcal{U} \subset \mathbb{R}^2$ and $\phi(x, y) = (u, v)$ be as follows.

$$\begin{cases} u = x \cos \Phi - y \cos \Phi + f_1 \\ v = x \sin \Phi + y \cos \Phi + f_2 \\ \Phi = \alpha + \beta(x^2 + y^2) \end{cases}$$

The orders of the error terms f_i are ≥ 4. "non-degenerate" means $\beta \neq 0$.

2. Let $\mathcal{U} \subset \mathbb{R}^{2n}$ and $\phi(x_1, , x_n, y_1, \cdots, y_n) = (u_1, \cdots, u_n, v_1 \cdots, v_n)$ be as follows.

$$\begin{cases} u_k = x_k \cos \Phi_k - y_k \sin \Phi_k + f_k \\ v_k = x_k \sin \Phi_k - y_k \cos \Phi_k + f_{n+k} \\ \Phi_k = \alpha_k + \sum_{l=1}^{n} \beta_{kl}(x_l^2 + y_l^2) \end{cases}$$

"Non-degenerate" means the matrix $\{\beta_{kl}\}$ is non-singular.

Our proof is based on Birkhoff-Moser theorem and Hamiltonian Floer homology. We divide $\mathrm{Ham}(M,\omega)$ into three subsets.

$$\mathcal{H}^{(1)} = \left\{ \phi \in \mathrm{Ham}(M,\omega) \;\middle|\; \begin{array}{c} \text{contractible periodic orbits are finite} \\ \text{all periodic orbits are hyperbolic} \end{array} \right\}$$

$$\mathcal{H}^{(2)} = \left\{ \phi \in \mathrm{Ham}(M,\omega) \;\middle|\; \begin{array}{c} \text{contractible periodic orbits are finite} \\ \text{there is non-hyperbolic orbit} \end{array} \right\}$$

$$\mathcal{H}^{(3)} = \left\{ \phi \in \mathrm{Ham}(M,\omega) \;\middle|\; \text{contractible periodic orbits are infinite} \right\}$$

We first use Floer theory to prove that $\mathcal{H}^{(1)}$ is empty under our assumptions. Next we apply Birkhoff-Moser theorem to construct a generic subset $\mathcal{U} \subset Ham(M,\omega)$.

Further directions

Question 1

Let ϕ be a Hamiltonian diffeomorphism with finite contractible periodic orbits. Is there at least non-hyperbolic orbit? Are all periodic orbits elliptic?

If this is correct, $\mathcal{H}^{(1)}$is empty and generic Conley conjectre holds for all symplectic manifolds.

Question 2 (generic density theorem?)

Is there C^∞-generic subset $\mathcal{U} \subset \mathrm{Ham}(M, \omega)$ such that periodic points of $\forall \phi \in \mathcal{U}$ are dense in M?

This is a generalization of Asaoka-Irie's theorem for higher dimensions.

Non-contractible periodic orbits

There is a big difference between contractible periodic orbits and non-contractible periodic orbits.

Remark

The set of contractible periodic orbits of a Hamiltonian diffeomorphism is never empty. This is a consequence of the Arnold conjecture. However, the set of non-contractible periodic orbits may be empty.

For example, there is no non-contractible periodic orbits if M is simply connected. If the Hamiltonian function H is C^2-small, every one peroidic orbits of ϕ_H^1 are contractible. However, it seems that there are infinitely many simple non-contractible periodic orbits if there is at least one non-contractible periodic orbits.

Hofer-Zehnder conjecture

Theorem (Frank)

Any area preserving homeomorphism on the 2-sphere S^2 with more than two fixed points has infinitely many periodic points.

Hofer-Zehnder conjecture is a generalization of Frank's theorem.

Hofer-Zehnder conjecture

Every Hamiltonian map on a compact symplectic manifold (M, ω) possessing more fixed points than necessarily required by the V. Arnold conjecture possesses always infinitely many periodic orbits.

This conjecture was stated in their famous book "Symplectic invariants and Hamiltonian dynamics".

Hofer-Zehnder conjecture implies that a Hamiltonian diffeomorphism possesses infinitely many contractible periodic orbits if it possesses more than the sum of the Betti numbers. Non-contractible version of Hofer-Zehnder conjecture can be written as follows.

Conjecture

Any Hamiltonian diffeomorphism possesses infinitely many non-contractible periodic orbits if it possesses at least one non-contractible periodic orbit.

Note that the required number of non-contractible periodic orbits is zero because the Floer homology of non-contractible orbits is trivial. So the sum of the Betti numbers is replaced by zero.

Previous research

1. A symplectic manifold (M, ω) is called symplectically atoroidal if
$$\int_{\mathbb{T}^2} u^*\omega = 0 \quad (\ \forall u : \mathbb{T}^2 \to M \ )$$
holds. Gurel proved the conjecture for atoroidal symplectic manifolds.
2. Orita proved the conjecture for the standard torus $(\mathbb{T}^{2n}, \omega_0)$. Note that $(\mathbb{T}^{2n}, \omega_0)$ is not symplectically atoroidal. $(\mathbb{T}^{2n}, \omega_0)$ is the simplest example of non-atoroidal and non simply connected symplectic manifold.

Weakly monotone symplectic manifold

A symplectic manifold is called weakly monotone if it satisfies one of the following conditions.

1. (M, ω) is a monotone symplectic manifold. There is a constant $\lambda \geq 0$ such that
$$\int_{S^2} u^*\omega = \lambda \int_{S^2} u^* c_1 \quad (\, \forall u : S^2 \to M \,)$$
holds.
2. $c_1(A) = 0$ holds for $\forall A \in \pi_2(M)$
3. $N \geq n - 2$

Every symplectic manifold (M^{2n}, ω) $(2n \leq 6)$ is a weakly monotone symplectic manifold.

Remark

If (M, ω) is weakly monotone, We can define the Floer homology without using so-called virtual technique. We can avoid "bubbling phenomena". In particular, Floer homology can be defined over $\mathbb{Z}$-coefficient. The virtual technique works over $\mathbb{Q}$-coefficient.

Main theorem 2

Main theorem 2

Let (M, ω) be a closed weakly monotone symplectic manifold. Let $\phi \in \mathrm{Ham}(M, \omega)$ be a Hamiltonian diffeomorphism such that 1-periodic orbits in the class $\gamma \neq 0 \in H_1(M : \mathbb{Z})/\mathrm{Tor}$ is finite, not empty and the local Floer homology $HF^{loc}(\phi, x)$ of at least one of them is not zero. Then, for sufficiently large prime p, ϕ possesses p-periodic or p'-periodic orbit in the class $p \cdot \gamma$. Here, p' is the first prime number greater than p. In particular, there are infinitely many simple non-contractible periodic orbits.

The assumption "weakly monotone" is a purely technical assumption. If we can define Floer theory over $\mathbb{Z}$-coefficient, we can remove this assumption.

We use the local Floer homology of periodic orbits to define "reasonable homological count of periodic orbits". If a periodic orbit x is "non-degenerate" (generic case),

$$\dim HF^{loc}(\phi, x) = 1$$

This suggests that the reasonable homological count of periodic orbit x is $\dim HF^{loc}(\phi, x)$ (not 1 in general). So the condition "the local Floer homology $HF^{loc}(\phi, x)$ of at least one of them is not zero" means that "there is at least one non-contractible periodic orbit". If the local Floer homology of all periodic orbits are trivial, they should be regarded as empty.

Sketch of the proof

We fix $\phi \in \mathrm{Ham}(M, \omega)$ and $\gamma \neq 0 \in H_1(M : \mathbb{Z})/\mathrm{Tor}$. Let $P(\phi, \gamma)$ be the set of 1-periodic orbits in γ.

$$P(\phi, \gamma) = \left\{ x \in P(\phi) \;\middle|\; [x] = \gamma \right\}$$

We define the Floer homology for such periodic orbits. We use so-called Novikov ring Λ and its subring Λ_0 over the ground field $\mathbb{Z}_p$.

$$\Lambda = \left\{ \sum_{i=1}^{l} a_i T^{\lambda_i} \;\middle|\; a_i \in \mathbb{Z}_p, \lambda_i \in \mathbb{R}, \lambda_i \nearrow \infty \ (\text{if } l = \infty) \right\}$$

$$\Lambda_0 = \left\{ \sum_{i=1}^{l} a_i T^{\lambda_i} \in \Lambda \;\middle|\; \lambda_i \geq 0 \right\}$$

The Floer chain complex is a module generated by periodic orbits $P(\phi, \gamma)$.

$$CF(\phi, \gamma : \Lambda_0) = \bigoplus_{x \in P(\phi,\gamma)} \Lambda_0 \cdot x$$

There is a boundary operator d_{Floer} and the Floer homology is the homology of this chain complex.

$$HF(\phi, \gamma : \Lambda_0) = H(CF(\phi, \gamma : \Lambda_0), d_{Floer})$$

Remark

We use the Floer homology over Λ_0 (not over Λ). $HF(\phi, \gamma : \Lambda)$ is always trivial and not interesting. However, $HF(\phi, \gamma : \Lambda_0)$ is not necessarily trivial. In general, $HF(\phi, \gamma : \Lambda_0)$ has a torsion.

There is a sequence $0 < \beta_1 \leq \cdots \leq \beta_m$ so that

$$HF(\phi, \gamma : \Lambda_0) \cong \bigoplus_{i=1}^{m} \Lambda_0 / T^{\beta_i} \Lambda_0$$

holds. The behavior of β_1 under iterations is important for our purpose. β_1 stands for the "minimum energy" of the Floer boundary operator d_{Floer}. We also fix a prime number p and a sequence $0 < \delta_1 \leq \cdots \leq \delta_{m'}$ so that

$$HF(\phi^p, p\gamma : \Lambda_0) \cong \bigoplus_{i=1}^{m'} \Lambda_0 / T^{\delta_i} \Lambda_0$$

holds. Our next purpose is to compare $p \cdot \beta_1$ and δ_1.

Assume that there is no simple p-periodic orbits in $p\gamma$. This means every element of $P(\phi^p, p\gamma)$ is an iteration of an element of $P(\phi, \gamma)$. Then we can prove the following proposition.

Proposition

Under this assumption, $p \cdot \beta_1 \leq \delta_1$ holds.

Combining this proposition with "filtered" Floer homology, we can prove that $P(\phi^{p'}, p\gamma)$ is not empty. Note that any element of $P(\phi^{p'}, p\gamma)$ is simple.

Remark

In the proof of this proposition, we used the $\mathbb{Z}_p$-equivariant Floer homology. Here we used the Floer theory over $\mathbb{Z}_p$-coefficient.

Furhter directions

We assumed that (M, ω) is a weakly monotone symplectic manifold. This was a purely technical assumption. In general, Floer homology is defined over $\mathbb{Q}$. If we can define the Floer thoery over $\mathbb{Z}$ for every closed symplectic manifold, we can remove this assumption.

Question

Can we construct $\mathbb{Z}$ or $\mathbb{Z}_p$-coefficient Floer homology and $\mathbb{Z}_p$-equivariant Floer homology for every closed symplectic manifold?

Constructions of such theories could be very important for other problems (e.g. Hofer-Zehnder conjecture).