Program on resolution of singularities in characteristic p

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Canonical Resolution of Singularities in char.0
(Hironaka), (simplifications by Bierstone-Milman, Villamayor and others)

Thm Let X be an algebraic variety (char 0).
There exists $f : \widetilde{X} \to X$ such that
(1) $\widetilde{X}$ is smooth,
(2) f is projective
(3) f isomorphism over X_{ns},
(4) $f^{-1}(V(\text{Sing}X))$ is normal crossings,
(5) functorial on smooth morphisms
(6) functorial on field extensions.

Embedded Desingularization(Hironaka)
Let X be a subvariety on smooth M (char. 0). There exists **embed. desingular.** of $X \subset M$: seq. of blow-ups σ_i with smooth centers C_{i-1}

$$M = M_0 \xleftarrow{\sigma_1} M_1 \longleftarrow \dots M_i \longleftarrow \dots M_r = \widetilde{M}$$

(i) The exc. divisors E_i of has only SNC and C_i has SNC with E_i.
(ii) The strict transform $\widetilde{X} := X_r$ of X is smooth and have SNC with the except. div.
(iii) $(M, X) \leftarrow (\widetilde{M}, \widetilde{X})$ is a functor:
-commutes with smooth morphisms
-commutes with embeddings of ambient varieties
-commutes with field extensions

Canonical Principalization(Hironaka)

Let $\mathcal{I}$ sheaf of ideals on smooth M (char. 0).
There exists a **principalization** of $\mathcal{I}$:
seq. of blow-ups σ_i with smooth centers C_{i-1}

$$M = M_0 \xleftarrow{\sigma_1} M_1 \longleftarrow \ldots M_i \longleftarrow \ldots M_r = \widetilde{M}$$

(i) The exc. divisors E_i of has only SNC and C_i has SNC with E_i.

(ii) The total transform $\widetilde{\mathcal{I}} := (\sigma_r^* \ldots \sigma_1^*)(\mathcal{I})$ is the ideal of SNC divisor with comp. in E_r

(iii) $(M, \mathcal{I}) \leftarrow (\widetilde{M}, \widetilde{\mathcal{I}})$ is a functor:

-commutes with smooth morphisms

-commutes with embeddings of ambient varieties

-commutes with field extensions

Hironaka resolution principle
(used by Villamayor in his proof)

(1) (Canonical) Principalization of the sheaves $\mathcal{I}$ on M

$$\Downarrow$$

(2) (Canonical) Embedded Desingularization of subvarieties $X \subset M$

$$\Downarrow$$

(3) (Canonical) Desingularization.

Let $\mathcal{I}$ be a sheaf of ideals on smooth X.

Definition: order of $\mathcal{I}$ at $x \in M$. :

$$\text{ord}_x(\mathcal{I}) = max\{k : m_x^k \supset \mathcal{I}\}$$

Definition: Let $\sigma : M' \to M$ be the blow-up at smooth center. The weak transform of $\mathcal{I}$ is

$$\mathcal{I}(D)^{-\mu}\sigma^*(\mathcal{I})$$

where $\mathcal{I}(D)^{\mu}$ is a maximal power of the exc. divisor which divides $\sigma^*(\mathcal{I})$

The main strategy of principalization :

Reduce the maximal order of the weak transform $\mathcal{I}'$ of $\mathcal{I}$

$\text{maxord}(\mathcal{I}) := \max\{\text{ord}_x(\mathcal{I}) : x \in M\}$

The main tool **marked ideal** (idealisistic exponent, basic object):

$$(\mathcal{I}, \mu)$$

"Part of $\mathcal{I}$ where the order of $\mathcal{I}$ is $\geq \mu$"

The **support** of marked ideal

$$\mathrm{supp}(\mathcal{I}, \mu) = \{x \in M \mid \mathrm{ord}_x(\mathcal{I}) \geq \mu\}$$

Definition: Let $\sigma : M' \to M$ be the blow-up at smooth center $C \subset \mathrm{supp}(\mathcal{I}, \mu)$. The <u>controlled transform</u> of $(\mathcal{I}, \mu)$ is

$$\sigma^{\mathrm{c}}(\mathcal{I}, \mu) := \mathcal{I}(D)^{-\mu}\sigma^*(\mathcal{I})$$

Rephrasing the main strategy:
Resolve marked ideal:

$$\mathrm{supp}(\mathcal{I}, \mu) = \emptyset.$$

Key observation

If $C : \overline{x} = 0 \subset \mathrm{supp}(\mathcal{I}, \mu)$,

then at any point $p \in C$ the functions $f \in \mathcal{I}$ can be written in the form

$$f = \sum_{|\alpha| \geq \mu} c_\alpha(\overline{y})\overline{x}^\alpha$$

After the blow- up

$$\overline{x} = (x_1, \ldots x_k) = (z = x'_1, zx'_2, \ldots zx'_k),$$

where z is the exceptional divisor.

$$f = \sum_{|\alpha| \geq \mu} c_\alpha(\overline{y})\overline{x}^\alpha$$

$$\downarrow$$

$$\sigma^*(f) = \sum_{|\alpha| \geq \mu} c_\alpha(\overline{y}) z^{|\alpha|}\overline{x'}^\alpha$$

Hironaka resolution principle

(0) (Canonical) Resolution of marked ideals $(\mathcal{I}, \mu)$

$$\Downarrow$$

(1) (Canonical) Principalization of the sheaves $\mathcal{I}$ on M

$$\Downarrow$$

(2) (Canonical) Embedded Desingularization of subvarieties $X \subset M$

$$\Downarrow$$

(3) (Canonical) Desingularization.

p-order ord^p in characteristic p.

Let $a = a_0 + a_1 p + \ldots + a_k p^k$ be the (reverse p-adic expansion of $a \in \mathbf{N}$.)
Set $[a] := (a_0, a_1, \ldots, a_k, 0, \ldots) \in \mathbf{N}^{finite}$

1. Put $\text{ord}^p(x^a) = [a]$

Example $char.K = 2$
$\text{ord}^p(x^2) = (0, 1, 0, \ldots)$ $\quad (2 = 0 \cdot 1 + 1 \cdot 2)$

$\text{ord}^p(y^3) = (1, 1, 0, 0, \ldots),$ $\quad (3 = 1 \cdot 1 + 1 \cdot 2)$

$\text{ord}^p(z^4) = (0, 0, 1, 0, \ldots)$ $\quad (4 = 0 \cdot 1 + 0 \cdot 2 + 1 \cdot 2^2)$

2. Put $\text{ord}^p(x_1^{b_1} \cdot \ldots \cdot x_n^{b_n}) = \text{ord}^p(x_1^{b_1}) + \ldots + \text{ord}^p(x_n^{b_n}) = [b_1] + \ldots + [b_n] \in \mathbf{N}^{finite}$

Example
$\text{ord}^p(x^2 y^3 z^4) = (0, 1, 0, \ldots) + (1, 1, 0, \ldots) + (0, 0, 1, 0, \ldots)$
$(1, 2, 1, 0, \ldots)$

For any $\alpha := (a_0, a_1, \ldots, a_k, 0, \ldots) \in \mathbf{N}^{finite}$ set

$$|\alpha| := a_0 + a_1 p + \ldots + a_k p^k$$

Then $\text{ord}(x^\beta) = |\,\text{ord}^p(x^\beta)|$

The order on $\mathbf{N}^{finite}$: $\alpha < \alpha'$ if
$|\alpha| < |\alpha'|$ or $|\alpha| = |\alpha'|$ but $\alpha <_{lexicograph.} \alpha'$

Example $\text{ord}^p(xy) = (2, 0, \ldots) < \text{ord}^p(x^2) = (0, 1, 0 \ldots) < \text{ord}^p(xyz) = (3, 0, 0, 0, \ldots)$

3. For $f = \sum_\alpha c_\alpha x_\alpha$, put

$$\text{ord}^p(f) = \min\{\text{ord}^p(x^\alpha)) \mid \alpha \neq 0\}$$

Example: char K=2

$\text{ord}^p(x^2 - y^2) = (0, 1, 0, \ldots) > \text{ord}^p(xy) = (2, 0, \ldots)$

Properties of ord^p

A. $\mathrm{ord}^p(f_1 + f_2) \geq min\{\mathrm{ord}^p(f_1), \mathrm{ord}^p(f_2)\}$

B. $\mathrm{ord}^p(f_1 \cdot f_2) \geq \mathrm{ord}^p(f_1) + \mathrm{ord}^p(f_2)$

C. $\mathrm{ord}^p(f) \geq (\mathrm{ord}(f), 0, \ldots, 0,)$

D. $\mathrm{ord}^p((f^{p^k}) \geq (0, \ldots, 0, \mathrm{ord}(f)_k, \ldots)$

Independence of coordinates. Let ϕ be an automorphism defining the coordinate change

$$\phi^* : x_1, \ldots, x_k \mapsto x'_1, \ldots, x'_k$$

(1) Let $a \in \mathbf{N}$ write $a = a_0 + a_1 p + \ldots + a_k p^k$

$\mathrm{ord}^p(\phi^*(x^a)) = \mathrm{ord}^p(\phi^*(x^{a_0} \cdot (x^{a_1})^p \ldots \cdot (x^{a_k})^{p^k}) \geq$

$\mathrm{ord}^p(\phi^*(x^{a_0})) + \mathrm{ord}^p(\phi^*(x^{a_1})^p) \ldots + \mathrm{ord}^p(\phi^*(x^{a_k})^{p^k}) \geq$

$(a_0, 0, \ldots) + (0, a_1, 0) + ... = (a_0, \ldots, a_k) = \mathrm{ord}^p(x^a)$.

(2) Now

$$\mathrm{ord}^p(\phi^*(x_1^{b_1}\cdot\ldots\cdot x_k^{b_k})) = \mathrm{ord}^p(\phi^*(x_1^{b_1})\cdot\ldots\cdot\phi^*(x_k^{b_k})) \geq$$

$$\mathrm{ord}^p(\phi^*(x_1^{b_1}))+\mathrm{ord}^p(\phi^*(x_2^{b_2})+\ldots+\mathrm{ord}^p(\phi^*(x_k^{b_k})) \geq$$

$$\mathrm{ord}^p((x_1^{b_1}))+\ldots+\mathrm{ord}^p((x_k^{b_k})) = \mathrm{ord}^p(x_1^{b_1}\cdot\ldots\cdot x_k^{b_k})$$

(3) In general for $f = \sum_\alpha c_\alpha x_\alpha$,

$$\mathrm{ord}^p(\phi^*(f)) \geq \min\{\mathrm{ord}^p(\phi^*(x^\alpha)) \mid c_\alpha \neq 0)\} \geq$$

$$\min\{\mathrm{ord}^p(x^\alpha)) \mid c_\alpha \neq 0)\} = \mathrm{ord}^p(f)$$

Derivations and support (in char. 0)
(Giraud, Villamayor)

Let $\mathcal{I}$ be a coh. sheaf of ideals on sm. var. M.
$\mathcal{D}(\mathcal{I})$ locally gener. by $f \in I, \partial f/\partial x$.

$$\mathcal{D}(\mathcal{I},\mu) := (\mathcal{D}(\mathcal{I}),\mu-1)$$

(i) If $\mathrm{ord}_x(\mathcal{I}) = \mu$ and $i \leq \mu - 1$ then

$$\mathrm{ord}_x(\mathcal{D}^i(\mathcal{I})) = \mu - i.$$

In particular $\mathrm{ord}_x(\mathcal{D}^{\mu-1}(\mathcal{I})) = 1$

(ii) $\mathrm{supp}(\mathcal{I},\mu) = \mathrm{supp}(\mathcal{D}^i(\mathcal{I}),\mu-i))$ ($i \leq \mu-1$,)

In particular $\mathrm{supp}(\mathcal{I},\mu) = \mathrm{supp}(\mathcal{D}^{\mu-1}(\mathcal{I}),1))$

Hasse-Dieudonne derivations in positive characteristic

Let $x_1, \ldots, x_n$ local system of coordinates. Let $\alpha = (a_1, \ldots, a_n)$. Then

$$D_x^\alpha := 1/\alpha! \frac{\partial}{\partial x_1^{a_1}} \ldots \frac{\partial}{\partial x_n^{a_n}}$$

is defined in characteristic p. $D_x^\alpha(x^\beta) = \binom{\beta}{\alpha} x^{\beta-\alpha}$.

In general

$$\mathrm{ord}_x(\mathcal{D}^i(\mathcal{I})) \neq \mathrm{ord}_x(\mathcal{I}) - i.$$

The order of derivation D_x^α is equal to

$$\mathrm{ord}(x^\alpha) = |\alpha| = a_1 + \ldots + a_n$$

and its p-order is equal to

$$\mathrm{ord}^p(\alpha) := \mathrm{ord}^p(x^\alpha) = \mu(x_1^{a_1} \cdot \ldots \cdot x_n^{a_n})$$

Derivations and support in positive characteristic

Let $\mathcal{I}$ be a coh. sheaf of ideals. $\alpha \in \mathbf{N}^{finite}$ $\mathcal{D}^{\alpha}(\mathcal{I})$ locally gener. by $D^{\alpha'}(f)$, where $\alpha' \leq \alpha$.

Lemma 1 If $\mathrm{ord}_x^p(\mathcal{I}) = \mu^p$ and $\mu^p - \alpha \in \mathbf{N}^{finite}$ then

$$\mathrm{ord}_x^p(\mathcal{D}^{\alpha}(\mathcal{I})) = \mu^p - \alpha$$

$$\mathrm{ord}_x(\mathcal{D}^{|\alpha|}(\mathcal{I})) = |\mu^p| - |\alpha|$$

In particular let $\mu^p = (0, \ldots, 0, a_k, a_k + 1, \ldots,)$. Set $1_k := (0, \ldots, 0, 1_k, 0, \ldots,)$

$$\mathrm{ord}_x^p(\mathcal{D}^{\mu - 1_k})(\mathcal{I}) = 1_k$$

Thus

$$\mathrm{ord}_x(\mathcal{D}^{|\mu^p| - p^k})(\mathcal{I}) = p^k$$

Note that

Lemma 2

(ii) supp$(\mathcal{I}, a) \subseteq$ supp$(\mathcal{D}^b(\mathcal{I}), a-b))$ ($b < a$,)

In particular

(ii) supp$(\mathcal{I}, a) \subseteq$ supp$(\mathcal{D}^{a-p^k}(\mathcal{I}), p^k))$

Hironaka's hypersurfaces of maximal contact in char. 0

-(i) contain support of marked ideal
-(ii) property (i) is persistent with resp. to any mult. test blow-up

Idea of hypersurface of maximal contact: Reduce the problem of resolution to lower dimensions.

Definition(Villamayor) We say that a marked ideal is of **maximal order** if

$$\max\{\operatorname{ord}_x(\mathcal{I}) \mid x \in M\} = \mu$$

Existence of hypersurface of maximal contact in char. 0

A marked ideal of maximal order $(M, \mathcal{I}, \emptyset, \mu)$ admits locally hypersurfaces of maximal contact.

Lemma (Giraud) Let $(\mathcal{I}, \mu)$ max. order. Let $u \in \mathcal{D}^{\mu-1}(\mathcal{I})$ $\quad \mathrm{ord}_x(u) = 1$. Then

$V(u) = \mathrm{supp}(u, 1) \supseteq \mathrm{supp}(\mathcal{D}^{\mu-1}(\mathcal{I}), 1) = \mathrm{supp}(\mathcal{I}, \mu)$

(ii) Let $\sigma : M \leftarrow M'$ be a blow-up at C
Set $u' := \sigma^c(u) = u \circ \sigma / y$. (y-exceptional div).
Then $u' \in \sigma^c(\mathcal{D}^{\mu-1}(\mathcal{I})) \subset \mathcal{D}^{\mu-1}(\sigma^c(\mathcal{I}, \mu))$ and $\mathrm{ord}_{x'}(u') = 1$. Hence

$$V(u') \supseteq \mathrm{supp}(\sigma^c(\mathcal{I}, \mu))$$

Existence of hypersurface of maximal contact in char. p

Lemma Let $(\mathcal{I}, \mu)$ max. order. Then

$$\text{ord}_x^p = \mu^p = (0, \ldots, 0, a_k, a_k + 1, \ldots,)$$

Let $\mathcal{U} \in \mathcal{D}^{\mu - p^k}(\mathcal{I})$ $\quad$ $\text{ord}^p{}_x(\mathcal{U}) = 1_k$. Then

$$\mathcal{U} = u^{p^k} + f,$$

where $ord_x(f) > p^k$

$$\text{supp}(\mathcal{U}, p^k) \supseteq \text{supp}(\mathcal{D}^{\mu - p^k}(\mathcal{I}), p^k) \supseteq \text{supp}(\mathcal{I}, \mu)$$

(ii) Let $\sigma : M \leftarrow M'$ be a blow-up at C
Set $\mathcal{U}' := \sigma^c(\mathcal{U}) \in \sigma^c(\mathcal{D}^{\mu - p^k}(\mathcal{I})) \subset \mathcal{D}^{\mu - p^k}(\sigma^c(\mathcal{I}, \mu))$ and $\mu_{x'}(\mathcal{U}') = 1$. Hence

$$\text{supp}(\mathcal{U}', p^k) \supseteq \text{supp}(\sigma^c(\mathcal{I}, \mu))$$

Remark. The scheme $\text{supp}(\mathcal{U}, p^k)$ is a "hypersurface of maximal contact" for $(\mathcal{I}, \mu)$ in positive characteristic.

Restriction to the hypersurface of maximal contact

1. $\mathbf{char}(K) = 0$. Let $V(u)$ be a hyp. of max. contact. Let $u, x_1, \ldots, x_k$ be a local system of coord. at p. Write a function g as a formal power series at p.

$$g = c_0 + c_1 u + \ldots + c_i u^i + \ldots,$$

where $c_i = c_i(x_1, \ldots, x_n)$ are formal power series in $x_1, \ldots, x_k$. Then

$$g_{|V(u)} = (c_0)_{|V(u)} \leftrightarrow c_0$$

is the restriction of g to $V(u)$.

2. $\mathbf{char}(K) = p$. Let $\mathcal{U} = u^{p^k} + f$. Write

$$g = c_0 + c_1 \mathcal{U} + \ldots + c_i \mathcal{U}^i + \ldots,$$

where $\mathrm{deg}_u(c_i) < p^k$. Then

$$g_{|\mathrm{supp}(\mathcal{U}, p^k)} := c_0$$

is the **restriction of** g to $\mathrm{supp}(\mathcal{U}, p^k)$.

Partial restriction in positive characteristic

There are some minor disadvantages of the restriction.
In view of Example **A2** we will be using more natural **partial restriction**. Write g as a finite sum

$$g = c_0 + c_1\mathcal{U} + \ldots + c_i\mathcal{U}^i + \ldots + c_{p^k-1}\mathcal{U}^{p-1}$$

where c_i contains u^l, where

$$l = j + ap^{k+1},$$

$j < p^k$, and $a \in \mathbf{N} \cup \{0\}$. Then

$$g_{||\mathrm{supp}(\mathcal{U},p^k)} := c_0$$

is the **partial restriction of** g to supp$(\mathcal{U}, p^k)$.

A. Constraints for maximal contacts

A1 Consider hypersurface of maximal contact in A^{p+2}

$$\mathcal{U} = (1+X)u^p + Xv_1 \cdot \ldots \cdot v_p,$$

where X and $X' = 1+X$ are both equations of exceptional divisors (passing through different points)

In the neighorhood U of $u = v_1 = \ldots = v_p = X = 0$ it has a form

$$\mathcal{U} = u^p + Xv_1 \cdot \ldots \cdot v_p.$$

In the neighorhood U' of $u = v_1 = \ldots = v_p = X' = 0$ it has a form

$$X'u^p + v_1 \cdot \ldots \cdot v_p.$$

Possible centers for U and V:

$u = v_1 = \ldots = v_p = 0$ for U and V.
$u = v_1 = \ldots = v_p = X = 0$ for U,
$(u = v_1 = \ldots = v_p = X' = 0$ for $V)$

The second blow-up does not change singularity.

To resolve the singularity one needs to perform the blow-up at the center

$$u = v_1 = \ldots = v_p = 0.$$

The generic points of the first center has singularity

$$\mathcal{U} = u^p + v_1 \cdot \ldots \cdot v_p$$

The singularities along the center are the same from the point of view of the algorithm. That is the algorithm considers the following singularities to be the same.

$$\mathcal{U} = u^p + Xv_1 \cdot \ldots \cdot v_p.$$

$$\mathcal{U}' = v_1 \cdot \ldots \cdot v_p + X'u^p$$

$$\mathcal{U} = u^p + v_1 \cdot \ldots \cdot v_p$$

Condition 1- Coherency of the algorithm. The maximal contact $\mathcal{U} = u^p + Xv_1 \cdot \ldots \cdot v_p$ at the point 0 still remains a maximal contact in the neighborhood of 0. That is

$$\mathcal{U} = u^p + v_1 \cdot \ldots \cdot v_p$$

is a maximal contact in the neihgborhood. To fullfill the condition for the maximal contact

$$\mathcal{U} = u^{p^k} + X^a f$$

we allow a weaker condititon $\mathrm{ord}_x(X^a f) \geq p^k$. The stronger condition $\mathrm{ord}_x(X^a f) > p^k$ is valid only at certain points and is not valid along the centers of blow-ups.

Condition 2- Commutativity of maximal contacts

The algorithm (and the invariant) does not distinguish between the maximal contact $\mathcal{U}$, in U, where

$$\mathcal{U} = u^p + Xv_1 \cdot \ldots \cdot v_p.$$

and the maximal contacts $(v_i, 1)$ in U',

$$\mathcal{U}' = v_1 \cdot \ldots \cdot v_p + X'u^p.$$

In particular the order of maximal contact **is not a part of the invariant which is constant along the center.**

The finer invariant controlling the algorithm may lead to the infinite loop in the algorithm.

Condition 3 -Restriction vs partial restriction to maximal contact.

A2 Consider hypersurface of maximal contact in A^{p+1}

$$\mathcal{U} = (1+X)u^p + Xv_1 \cdot \ldots \cdot v_{p-1}(u+v_p^2)$$

where X and $X' = 1+X$ are both equations of exceptional divisors passing through different points.

It has two different forms along the center

$$\mathcal{U} = u^p + Xv_1 \cdot \ldots \cdot v_{p-1}(u+v_p^2).$$

$$\mathcal{U}' = v_1 \cdot \ldots \cdot v_{p-1}(u+v_p^2) + X'u^p$$

It "follows" from commutativity of max. cont.

$$\mathcal{U}, v_1, \ldots, v_{p-1}, u+v_p^2$$

that partial restrtriction shall be used. We do not "kill" $\mathcal{U}$ by partial restricting to $u+v_p^2$.

Normal form of the maximal contact

It follows from the previous **Example** that Giraud form

$$\mathcal{U} = u^{p^k} + f, \quad \text{ord}_x(f) > p^k$$

is not preserved along the center.

Instead we introduce:
Definition. The maximal contact $\mathcal{U} = u^{p^k} + f$ in the **normal form** if $\text{ord}_x(f) \geq p^k$, and u is transversal to exceptional divisors. Moreover the following conditions are satisfied:

-If $\mathcal{U} = u^{p^k} + f$ and $\text{ord}_x(f) > p^k$ then $\mathcal{U}$ is in the normal form.

-If D is the exceptional divisor then $\mathcal{U} = u^{p^k} + f$ is in normal form if $\mathcal{U}_{|\mathbb{D}}$ is in the normal form

-If $\mathcal{U} = u^{p^k} + f$, where $\text{ord}_x(f) = p^k$ and $\mathcal{V} \in \mathcal{T}(\mathcal{U})$ is maximal contact of $\mathcal{U}$ through x then $\mathcal{U}$ is in the normal form if $\mathcal{U}_{||\mathcal{V}}$ is in the normal form.

Standard approach to the resolution of $(\mathcal{U}, p^k)$

Write,

$$\mathcal{U} = u^{p^k} + X^a f$$

where a and $ord(f)$ are maximal possible.

Natural invariant - classic approach

$$\mu_{2,p}(\mathcal{U}) = \mathrm{ord}_p(f).$$

Gives a good control on the singularity.
Not well controlled under the blow-ups-
requires further modifications. There are major problems to overcome.

B . Frobenius phenomenon.- Ambiguity of order and jumping phenomen

B0. $\mathcal{U} = u^p + v^{kp}w$, where $k \geq 1$

Automorphism acting on $(\mathcal{U}, p)$
$u \to u + tv^k$
$w \to w - t^p, \quad v \to v$

The only possible center of blow-up: $u = v = 0$ (by canonicity)

What is the μ_2- order of $\mathcal{U}$?

B1. $\mathcal{U} = u^p + v^{2p}w + t_1^{1000p+1}t_2$.

We blow-up $u = v = t_1 = t_2 = 0$. After the blow-up ($X = t_2$) it becomes

$$\sigma^*\mathcal{U} = X^p u^p + X^{2p}v^{2p}w + X^{1000p+2}t_1^{1000p+1}.$$

$$\mu_2(\mathcal{U}) = 2p\,?$$

B2. $\mathcal{U} = u^p + v^{2p}w + w^{1000p+1}t_2$.

We blow-up $u = v = w = t_2 = 0$. After the blow-up ($X = t_2$) it becomes

$\sigma^*\mathcal{U} = X^p u^p + X^{2p+1} v^p w + X^{1000p+2} w^{1000p+1}$.

$$\mu_2(\mathcal{U}) = 2p + 1\,?$$

B3. $\mathcal{U} = u^p + v_1^{2p} w_1 + v_2^{2p} w_2 + HOT$?

B4. Increase of μ_2

$\mathcal{U} = u^{p^2} + v^{p^2}(w_1 + w_2^p) + w_1^{p^3+1} + w_3^{p^3+1}$

$\mu_2 = p^2 + 1$.

Possible blow-up $u = v = w_1 = w_3 = 0$. (support of $\mathcal{U}$) . After blow-up

$\mathcal{U} = u^{p^2} + v^{p^2}(Zw_1 + w_2^p) + Z^{p^3+1-p^2} w_1^{p^3+1} + Z^{p^3+1-p^2}$

$\mu_2 = p^2 + 2$ increases after the blow-up.

C. Moh- Seidenberg phenomenon

$\mathcal{U} = u^3 + X^2Y^3(Y^2 + X^3)$

After blow-up at the $u = X = Y = 0$

$\mathcal{U} = u^3 + X^4Y^2(Y^2 + X)$

$\mathcal{U} = u^3 + X^4Y^4(X + Y)$ $\mu = 1$ (μ drops by 1)

$\mathcal{U} = u^3 + Z^6(-1 + v^2 - v^3 + v^5)$

$u \mapsto u + Z^3v^2$

$\mathcal{U} = u^3 + Z^6(v^2 - v^3 + v^5)$ (μ increases by 1)

Virtual marked ideals

Introducing new invariant leads to the new objects- **virtual marked ideals** considered from the perspective of this invariant. For a maximal contact $\mathcal{U} = u^{p^k} + X^a f$ we introduce

$$[\mathcal{U}]_{p^k} = [X^a f]_{p^k} := \mathcal{U} + \mathcal{O}^{p^k}$$

to be the class of the element

$$\mathcal{U} + \mathcal{O}^{p^k}$$

in the quotient $\mathcal{O}^{p^k}$- module $\mathcal{O}/\mathcal{O}^{p^k}$.

Virtual marked ideals have form $([g]_{p^k}, \mu)$, where μ is "virtual marking". They are controlled by "the virtual order -vord".

The virtual objects behave nicely with respect to the logarithmic derivations XD_X, D_u. In particular

$$D^{p^l}[X^a f]_{p^k} = D^{p^l}(X^a f), \text{for, } l < k$$

$$D^{p^l}[X^a f]_{p^k} = [D^{p^l}(X^a f)]_{p^k}, \text{for, } l \geq k$$

In view of **Examples B, C** the definition of order of the virtual element [U] requires important adjustments.

Let $\mathcal{U} = u^{p^k} + X^a f$.

1. **"Pure case"** $f = f_0 + HOT$, where f_0 is the initial form of f and $f_0 \notin (X_1, \ldots, X_l)$.

1.1 **"Pure irregular case"**

$$\overline{\text{ord}}_x([\mathcal{U}]_{p^k}) = \overline{\text{ord}}_x([X^a f]) := lp^k + \delta_r,$$

where $\delta_r = 0, p^r$

if - $\text{vord}_x(f_0) = lp^k$ and $f_0 \in \mathcal{O}^{p^k}$, and $X^a \in \mathcal{O}^{p^r}$

or

- $\text{vord}_x(f_0) = lp^k$ and f_0 contains $(v^b)^{p^k} u^{p^r}$, where v, u are local coordinates.

1.2. **"Pure regular case"** .

$$\text{vord}_x([\mathcal{U}]) = \overline{\text{vord}}_x([X^a f]) := \text{ord}_x f_0 = \text{ord}_x f$$

What is a virtual order "vord" ?

Conceptual definition: Two possible definitions of order for an ideal $\mathcal{I}$: **Standard order**

$$\text{ord}_x(\mathcal{I}) = max\{k : m_x^k \supset \mathcal{I}\}$$

Effective order along center.

Let $\sigma : M' \to M$ be the blow-up at smooth center C and exceptional divisor. Then $\text{vord}_{x,C,}\mathcal{I}$ is the maximal power of the exc. divisor which divides $\sigma^*(\mathcal{I})$

$$\mathcal{I}(D)^\mu \supset \sigma^*(\mathcal{I})$$

Definition. $\text{vord}_x([\mathcal{U}])$ is a function defined on the permissible centers of blow-ups.

What is a monomial mod $\mathcal{O}^{p^k}$?

According to our definition of virtual order

$$\mathrm{vord}[X^a]_{p^k} = \delta_r,$$

where $X^a \in \mathcal{O}^{p^r} \setminus \mathcal{O}^{p^{r+1}}$. The monomial $[X^a]_{p^k}$ behave as if its relative order is 0. However -passing to a neighborhood or blow-up may easily transform to

$$[X^{(bp^k)} u^{p^r}]_{p^k}$$

and then to

$$[X^{(bp^k)} Z^{p^r} u^{p^r}]_{p^k}$$

and more generally to $[X^{ap^r} u^{p^r}]_{p^k}$.

Conceptual definition of monomial

Monomial $[X^a]_{p^k}$ is a coherent object stable under permissible blow-ups.

In particular

$[X^a]_{p^k}$, $[X^{(bp^k)}u^{p^r}]_{p^k}$ $[X^{(bp^k)}Z^{p^r}u^{p^r}]_{p^k}$ $[X^{ap^r}u^{p^r}]_{p^k}$.

are different incarnations of monomial.

Instability of monomial forms .

Key Observation: The monomial form $[X^a]_{p^k}$ is unstable.

To assure stability of the

$$([x^{(bp^k)}u^{p^r}]_{p^k}, \delta_r)$$

we introduce the condition of SNC of the monomial with the center. We say that

$$([x^{(bp^k)}u^{p^r}]_{p^k}, \delta_r)$$

has SNC with the center of blow-up C if either

-$[u^{p^r}]_{p^k} \in \mathcal{I}_C^{p^r} + \mathcal{O}^{p^k}$. We write $\delta_r(C) = p^r$
or
$[u^{p^r}]_{p^k}$ is transversal to C i.e. for every $x \in C$,

$$\mathrm{vord}_x([u^{p^r}]_{p^k}{}_{|C}) = \delta_r.$$

We write $\delta_r(C) = 0$.

If

$$[x^{(bp^k)}u^{p^r}]_{p^k}$$

does not have SNC with C then after a single blow-up we may loose a monomial form

$$[x^{(bp^k)}u^{p^r}]_{p^k}$$

Example. Suppose u has a form $u = u' + v^2$, where $u' \in \mathcal{I}_C$ and v transversal to C. Then

$$[x^{(bp^k)}(u' + v^2)^{p^r}]_{p^k}$$

transforms to

$$[x^{(bp^k)}(Zu' + v^2)^{p^r}]_{p^k}$$

Since $[x^a]_{p^k}$ $[x^{(bp^k)}(u)^{p^r}]_{p^k}$ and $[x^{(ap^r)}(u)^{p^r}]_{p^k}$ are not distinguished the notion of the SNC can be considered with respect to all three forms. It can be generalized to a set of monomials.

Permissibilty of centers.

In the resolution procedure we shall use the following objects:

-marked ideals $(\mathcal{I}_i, \mu)$
-virtual marked ideals $([f_j]_{p^k}, \mu)$
-virtual monomial marked ideals $([x_l^\alpha]_{p^k}, \delta_*)$

The center C will be called **permissible** if

1. $C \subset \bigcap$ supp$(\mathcal{I}_i, \mu) \cap$
   $\bigcap$ supp$([f_j]_{p^k}, \mu) \cap \quad \bigcap$ supp$([x_l^\alpha]_{p^k}, \delta_*)$

2. C has SNC with $([x_l^\alpha]_{p^k}, \delta_*)$

3. C has SNC with the exceptional divisors E.

In the virtual monomial $([x_l^\alpha]_{p^k}, \delta_r)$ the function δ_r is defined on the set of permissible centers of blow-ups, and has values $0, p^r$.

Homogenized derivations Let $(\mathcal{I},\mu)$ be of maximal order, such that

$$\mathrm{ord}_{\log,x}(\mathcal{I}) = \mathrm{ord}_x(\mathcal{I}) \quad \text{for } x \in \mathrm{supp}(\mathcal{I},\mu).$$

There exists

$$(\mathcal{T}(\mathcal{I}) = \mathcal{D}^{\mu-p^r}(\mathcal{I}), p^r),$$

where $p^r|\mu$ is of maximal order. By the **homogenizing derivations** we mean

$$\mathcal{H}_{\mathcal{T}} := \mathcal{T}(\mathcal{I})\mathcal{D}^{p^r} \cap \mathcal{D}^{p^r}_{\log}$$

Note that

$$\mathrm{ord}_x(f) \leq \mathrm{ord}_x(H(f)),$$

where $H \in \mathcal{H}_{\mathcal{T}}$.

Homogenized ideals and tangent directions

By the **homogenized ideal** we mean the ideal $(\mathcal{H}_{\mathcal{T}}(\mathcal{I}),\mu)$ generated by functions $f\in\mathcal{I}$ and all its homogenized derivatives.

Then $(\mathcal{H}_{\mathcal{T}}(\mathcal{I}),\mu)$ is equivalent to $(\mathcal{I},\mu)$. (Have the same supports and the same resolutions.)

Moreover for any two maximal contacts $\mathcal{U}_1,\mathcal{U}_2\in\mathcal{T}(\mathcal{I}),\mu)$, the restrictions $(\mathcal{H}_{\mathcal{T}}(\mathcal{I}_{||\mathcal{U}_i}),\mu)$ are equivalent. In fact we have

Lemma

$$\mathcal{H}_{\mathcal{T}}(\mathcal{H}_{\mathcal{T}}(\mathcal{I})_{||\mathcal{U}_1}),\mu)=\mathcal{H}_{\mathcal{T}}(\mathcal{H}_{\mathcal{T}}(\mathcal{I})_{||\mathcal{U}_2}),\mu)$$

If

Coefficient ideals in positive characteristic
Let $(\mathcal{I}, \mu)$ be an ideal of maximal order. Let

$$\mathcal{U} = u^{p^k} + f$$

be a maximal contact of $(\mathcal{I}, \mu)$, where $p^k | \mu$. Denote $r := \min(p, \mu/p^k)$.

Assume the coordinate u can be chosen transversal to exceptional divisors.

By the **coefficient ideal** of $(\mathcal{I}, \mu)$ at $\mathcal{U} = u^{p^k} + f$ we mean

$$\mathcal{C}(\mathcal{I}, \mu)_{||\mathcal{U}} = \sum_{i=0}^{r} ((H_{\mathcal{T}}(\mathcal{D}^{p^k}_{\log})^i H_{\mathcal{T}}(\mathcal{I}))_{||\mathcal{U}}), \mu - ip^k) +$$

$$([\mathcal{U}]_{p^k}, p^k)$$

If

$$g = c_0 + c_1\mathcal{U} + \ldots + c_i\mathcal{U}^i + \ldots + c_{p^k-1}\mathcal{U}^{p-1} \in \mathcal{I}$$

then

$$c_i = c_{i||\mathcal{U}} \in H_{\mathcal{T}}(\mathcal{D}^{p^k}_{\log})^i(\mathcal{H}_{\mathcal{T}}(\mathcal{I}))_{||\mathcal{U}}$$

Conceptual definition Coefficient ideal can be characterized by 4 conditions.

1. $\text{supp}((\mathcal{U}, p^k) + (\mathcal{C}(\mathcal{I}, \mu)_{||\mathcal{U}})) = \text{supp}(\mathcal{I}, \mu)$

2. Condition 1. persistent with respect to blow-ups $C \subset \text{supp}(\mathcal{I}, \mu)$.

3. If $\sigma^c(\mathcal{C}(\mathcal{I}, \mu)_{||\mathcal{U}})$ is monomial then $\sigma^c(\mathcal{I}, \mu)$ is easily resolvable- (reduces to $(u, 1) + (x^\alpha, \mu_\alpha)$)

4. If $\sigma^c(\mathcal{C}(\mathcal{I}, \mu)_{||\mathcal{U}})$ is 0 then $(\mathcal{I}, \mu) = u^{p^k}$)

Coefficient ideals defined for the exceptional divisors. Let $(\mathcal{I},\mu)$ be an ideal of maximal order. Let

$$X := \{X_i\}_{i\in I}$$

be a coordinates of some exceptional divisors maximal contact. Consider

$$\mathcal{T}_X = (X_i)_{i\in I}$$

$$\mathcal{H}_{\mathcal{T}} = (X_i)_{i\in I}\mathcal{D} \cap \cdot\mathcal{D}_{\mathrm{log}}$$

By the **coefficient ideal** of $(\mathcal{I},\mu)$ at $X=0$ we mean

$$\mathcal{C}(\mathcal{I},\mu)_{|X} = \sum_{i=0}^{\mu}\sum_{|\alpha|=i}(\mathcal{C}_{\alpha,X},\mu-i)$$

where

$$\mathcal{C}_{\alpha,X} = 1/X^{\alpha}(f_{|X} \mid f\in\mathcal{H}_{\mathcal{T}}(\mathcal{I}), X^a|f\}$$

Derivations compatible with centers

Let $u_1, \dots, u_n$ local coordinates at $x \in C$ such that

$$C : u_1 = \dots = u_m = 0.$$

The local coordinates at $x' \in \sigma^{-1}(x)$ are of the form

$$u'_i = \frac{u_i}{u_m} \text{ for } i < m$$

and $u'_i = u_i$ for $i \geq m$, where $u_m = u'_m = Y$ is exc. divisor. Then

$$\sigma^*(\tfrac{\partial}{\partial u_i}) = \tfrac{1}{Y}\tfrac{\partial}{\partial u'_i}, \quad 1 \leq i < m;$$

$$\sigma^*(\tfrac{\partial}{\partial u_m}) = -\tfrac{1}{Y}(u'_1\tfrac{\partial}{\partial u'_1} + \dots + u'_{m-1}\tfrac{\partial}{\partial u'_{m-1}} - Y\tfrac{\partial}{\partial Y}),$$

$$\tfrac{\partial}{\partial u'_i} = \tfrac{\partial}{\partial u_i}, \quad m < i \leq n.$$

Two kinds of compatible derivations

$\frac{\partial}{\partial u_i}$, for $\quad 1 \leq i \leq m$,

compatible with C of order -1

The derivations $\frac{\partial}{\partial u_i}$ for $\quad m+1 \leq i \leq n$.,
compatible with C of order 0
for $\quad m+1 \leq i \leq n$.

Similarly $X_i \frac{\partial}{\partial X_i}$ are **compatible with C of order 0**

"Pure regular case"

$$\mathrm{vord}[\mathcal{U}]_{p^k} = [X^a f] = [X^a(f_0 + HOT)] = \mu \in N.$$

Let $\mathrm{ord}_x^p(f_0) = \mu^p = (0, \ldots, 0, a_r, \ldots, a_m, 0, \ldots)$.
Then, since we are in regular case.

$$\mu^p < (0, \ldots, 1_r, 0, \ldots, a_p = \mu - p^r, 0, 0, \ldots)$$

$$\mu^p - 1_r < (0, \ldots, 0, a_k = \mu - p^r, 0, \ldots)$$

Then $\mathcal{D}_{\log}^{\mu^p - 1_r}(\mathcal{O})^{p^k} = 0$. Set

$$\overline{\mathcal{D}}_{\log}^{\mu - p^r} := \{D \in \mathcal{D}_{\log}^{\mu - p^r} \mid D(\mathcal{O})^{p^k} = 0\} \ni \mathcal{D}_{\log}^{\mu^p - 1_r}$$

$$\overline{\mathcal{D}}_{\log}^{\mu - p^r}([\mathcal{U}]_{p^k}, p^r) = \overline{\mathcal{D}}_{\log}^{\mu - p^r}(X^a f)$$

By the above

$$(\mathcal{T}([\mathcal{U}]_{p^k}, p^r) := (1/X^a \cdot \overline{D}_{\log}^{\mu - p^r}(X^a f), p^r)$$

is the real (i.e non-virtual) marked ideal of maximal order. There exists a maximal contact

$$\mathcal{V} \in \mathcal{T}([\mathcal{U}]_{p^k}, p^r)$$

Set

$$\mathcal{H}_{\mathcal{T}} = \mathcal{T}([\mathcal{U}]) \cdot D_{\log}^{p^r}$$

Note $\overline{\mathcal{H}}_{\mathcal{T}}(\mathcal{O}^{p^r}) = 0$ and $\overline{\mathcal{H}}_{\mathcal{T}}([\mathcal{U}]) = [\mathcal{U}] \oplus \mathcal{H}_{\mathcal{T}}(X^a f)$
Consider the coefficient ideal

$$(\mathcal{C}([\mathcal{U}])_{||\mathcal{V}}) = [\mathcal{V}] \oplus [\mathcal{U}]_{||\mathcal{V}} \oplus$$

$$1/X^a \sum_{i=0}^{r} (H_{\mathcal{T}}(\mathcal{D}_{\log}^{p^k})^i \overline{H}_{\mathcal{T}}[\mathcal{U}])_{||\mathcal{V}})$$

If $\mathcal{U} = c_0 + c_1\mathcal{V} + \ldots + c_i\mathcal{V}^i + \ldots + c_{p^k-1}\mathcal{V}^{p-1}$
then $c_i = c_{i||\mathcal{V}} \in H_{\mathcal{T}}(\mathcal{D}_{\log}^{p^k})^i(\mathcal{H}_{\mathcal{T}}(\mathcal{I}))_{||\mathcal{V}}$.

We can write
$[\mathcal{U}] = [c_0] + c_1\mathcal{V} + \ldots + c_i\mathcal{V}^i + \ldots + c_{p^k-1}\mathcal{V}^{p-1}$,
where

$[c_0] = c_{0||\mathcal{V}} = [\mathcal{U}_{||\mathcal{V}}]$ is the "virtual part".

$c_i = c_i\mathcal{V}^i$ are elements of "real" marked ideals.

Example Let

$$\mathcal{U} = u^p + X^a(u_1 \cdot \ldots \cdot u_r + G(u_{r+1}, \ldots, u_n),$$

where $\mathrm{ord}_x(G) > r$. Then

$(\mathcal{T}([\mathcal{U}]_p, p) := (1/X^a \cdot \overline{D}^{\mu-p^r}_{\log}(X^a f), p^r) = (u_1, \ldots, u_r)$

Thus u_1 is a maximal contact.

$$(\mathcal{C}([\mathcal{U}])_{||u_1}) = [u^p + G] \oplus \sum_{i=1}^{p} ((u_2, \ldots, u_r, HOT))^{p-i}$$

Pure irregular case

Consider the case

$$\mathrm{vord}_x([\mathcal{U}]_{p^k}) = lp^k + \delta_1$$

for simpler notations. Assume for simplicity that

$$[\mathcal{U}] = [X^a(F_1^{p^k} v_1 + \ldots F_s^{p^k} v_s + G_{p^k+1} + HOT)],$$

where F_i are forms of degree l and G_{p^k+1} is a regular form of degree $p^k + 1$. Then

$$\mathcal{D}^{lp^k}_{\log}[\mathcal{U}] = ([X^a v_1], \ldots [X^a v_s], X^a v_{s+1}, \ldots, X^a \cdot HOT)])$$

Suppose the center C has SNC with

$$[X^a v_1] + \ldots + [X^a v_s]$$

That is after possible rearrangements and modifications
$\mathrm{vord}([X^a[v_i], C) = 0$ for $i \leq s_0 \leq s$
$\mathrm{vord}([X^a[v_i], C) = 1$ for $s_0 < i \leq s$.

There are compatible derivations $\frac{\partial}{\partial v_i}$ of order 0 for $i \leq s_0$ and order -1 for $s_0 < i \leq s$.

After the blow-up at C. (Y-exc. divisor)

$$\sigma^c([\mathcal{U}]) = [Y^{lp^k} X^{a'}(F_1'^{p^k} v_1' + \dots F_{s_0}^{p^k} v_{s_0}') + Y(F_1'^{p^k} v_{s_0+1}' + \dots F_s'^{p^k}] + G'_{p^k+1} + HOT))].$$

After a sequence of blow-ups with permissible centers we will keep the following form: there exist exponents $a_0 \le a_1 \le \dots \le a_m$, and corresponding indices $s_0 \le s_1 \le s_2 \dots \le s_m \le s_{m+1}$ such that

$$\sigma^c[\mathcal{U}] = [X^{a_0}(F_1^{p^k} v_1 + \dots + F_{s_0}^{p^k} v_{s_0} + X^{a_1-a_0}(F_{s_0+1}^{p^k} v_{s_0+1} + \dots + F_{s_1}^{p^k} v_{s_1} + X^{a_2-a_1}(\dots + X^{a_m-a_{m-1}}(F_{s_{m-1}+1}^{p^k} v_{s_{m-1}+1} + F_{s_m}^{p^k} v_{s_m} + G_{p^k+1} + HOT))))]$$

$$\mathcal{D}^{lp^k}_{\log}(\sigma^c[\mathcal{U}]) = [X^{a_0} v_1], \dots, [X^{a_0} v_{s_0})], [X^{a_1} v_{s_0+1})], \dots, [X^{a_1} v_{s_1}], \dots, [X^{a_m} v_{s_{m-1}+1}], \dots [X^{a_m} v_{s_m}], X^{a_m} v_{s_m+1}, \dots, X^{a_m} v_{s_{m+1}}, X^{a_m} \cdot HOT, \dots))))],$$

where $v_{s_m+1}, \dots, v_{s_{m+1}} \in \mathcal{D}^{lp^k}(\log(G_{p^k+1} + HOT)$

The permissibilty of the center means that after possible rearrangements and modifications there exist

$$0 \leq s'_0 \leq s_0 \leq s'_1 \leq s_1 \ldots \leq s'_m \leq s_m \leq s_{m+1}$$

vord$([X_i^a[v_j], C) = 0$ for $s_{i-1} < j \leq s'_i$
vord$([X^a[v_i], C) = 1$ for $s'_i < j \leq s_i$.

Moreover we assume that along the center

$$\text{ord}(X^{a_i - a_{i-1}}) + \min_{s_i < j \leq s_{i+1}} \text{vord}([X_i^a[v_j], C) \geq$$

$$\max_{s_{i-1} < j' \leq s_i} \delta_1([X_i^a[v_{j'}], C)$$

The latter condition is to assure linear order on the set

$$\{a_0, a_1, \ldots, a_m\}$$

Roughly, along the center

$$\text{"ord"}(X^{a_i - a_{i-1}}) v_j \geq \text{"ord"}(v_{j'})$$

Tilted Derivations

Define

$$\tilde{\mathcal{D}}([\mathcal{U}]) = (\frac{1}{x^{a_i}} \cdot \frac{\partial([\mathcal{U}])}{\partial v_j} \mid 0 < i \leq m, s_{i-1} < j < s_i) +$$

$$\frac{1}{x^{a_m}}(D[U] \mid D \in \mathcal{D}_{\mathsf{log}}, D([X_i^a[v_j]) = 0)_{0<i\leq m, s_{i-1}<j<s_i}$$

Observe: there are compatible derivations $\frac{\partial}{\partial v_i}$
- of order 0 for $s_{i-1} < j \leq s_i'$ and
- order -1 for $s_i' < j \leq s_i$. After blow-up these transform

$$\sigma^c(\frac{\partial}{\partial v_i}, -\delta_1) = (1/Y^{\delta_1} \cdot \sigma^*(\frac{\partial}{\partial v_i}), -\delta_1)$$

Thus

$$\tilde{\mathcal{D}}(\sigma^c[\mathcal{U}]) = \sigma^c(\tilde{\mathcal{D}}([\mathcal{U}]))$$

Consider the maximal contact $\mathcal{U}' \in \mathcal{T}(\tilde{\mathcal{D}}([\mathcal{U}]))$.

$$\mathcal{C}([\mathcal{U}])_{||\mathcal{U}'} := [\mathcal{U}]_{||\mathcal{U}'} + \mathcal{C}(\tilde{\mathcal{D}}[\mathcal{U}])_{||\mathcal{U}'}$$

"Logarithmic order". By the logarithmic order

$$\text{ord}_{\log x(\mathcal{I})} = \mu$$

we mean is the smallest natural number μ such that $\mathcal{D}^{\mu}_{\log}(\mathcal{I})$ is monomial (generated by monomials).

Lemma If F is a form of degree μ then

$$\text{ord}_{\log x(X^a f)} = \text{ord}_{\log x(f)} \leq \mu$$

. If F is a form of degree in the "pure case" then

$$\text{ord}_{\log x(X^a f)} = \text{ord}_{\log x(f)} = \mu$$

Corollary If F is the form of degree μ, and $X^a f \in \mathcal{O}^{p^r}$ then

$$\text{ord}_{\log x([X^a f)]_{p^k}} \leq \mu + p^r$$

2. **"Logarithmic case"**

Let

$$\mathcal{U} = u^{p^k} + X^a f,$$

Write $f = f_0 + HOT$, where f_0 is the initial form of f. Then $f_0 \in (X_1, \ldots, X_l)$.

2.1 **"Logarithmic regular case".**

$$\mathrm{vord}_x([\mathcal{U}]) := \mathrm{ord}_x(f_0) \geq \mathrm{ord}_{\log x}[X^a f_0]$$

Maximal contact- "old" exceptional divisors X. Consider coefficient ideal

$$\mathcal{C}([\mathcal{U}])_{|X} := [\mathcal{U}]_{|X} \oplus 1/X^a \sum_{i=1}^{r} \sum_{|\alpha|=r} \mathcal{C}_{\alpha,X}$$

$$\mathcal{C}_{\alpha,X} = 1/X^{\alpha}(f_{|X} \mid f \in \mathcal{H}_X([\mathcal{U}]), X^a|f\}$$

After removing old divisors.

$$\mathrm{vord}_x(\mathcal{U}) = \mathrm{ord}_x(f_0) \geq \mathrm{ord}_{\log x}[X^a f_0] \geq$$

$$\mathrm{ord}_{\log x}[\sigma^c(X^a f_0)] = \mathrm{ord}_x(\sigma^c(f_0)) = \mathrm{vord}_x(\sigma^c([\mathcal{U}])$$

2.2 "Logarithmic Moh-Seidenberg (kangaroo) case".

$$\mathrm{ord}_x(f_0) < \mathrm{ord}_{\log x}[X^a f_0] = \mathrm{ord}(f_0) + p^r$$

Lemma. $\frac{\partial}{\partial {u_i}^{p^j}}[X^a f_0]_{p^k} = 0$, where $j \leq k$. Thus

$$f_0 = \sum_i c_i X^{a_i} F_i^{p^k}$$

We define

$$\mathrm{vord}_x([\mathcal{U}]) := \mathrm{ord}_{\log}(f_0)) - \delta_r,$$

where $\delta_r = p^r$ when $\mathcal{U}$ in "M-S" case , and $\delta_r = 0$ when $\mathcal{U}$ in the "pure case"

"M-S" Jumping phenomenon corresponds to passing from logarithmic case to the pure case and vice versa (in case a new initial M-S form is created). Solution is similar as for "pure irregular case"

The virtual order remains constant while the order along center varies as δ_r changes its value from 0 to p^r. "Tilted initial form" is stable and improves

Set for simplicity

$$\text{vord}_x([\mathcal{U}]) = \mu - \delta_1$$

Suppose

$$[\mathcal{U}] = [X^a(F_1 + HOT)],$$

where F_1 is in "pure" form transforms to

$$[X^a(G_1 + F_1 + HOT)],$$

where G_1 in M-S form, as in the **Lemma**.

Then

$$\mathcal{D}^{\mu-1}_{\text{log}}([\mathcal{U}]) = (g_i, f_i, h_i),$$

where f_i, g_i are functions whose initial forms are of logarithmic order 1 such that

$$g_i = X^b D_{X^b}([\mathcal{U}]) = X^a(X^b D_{X^b}(G_1{+}HOT) = X^a g_i'$$

$$f_i = D_{u^b_\alpha}([\mathcal{U}]) = X^a(D_{u^b_\alpha}(F_1 + HOT)) = X^a v_i,$$

where v_i is a parameter, $|b| = \mu - 1$

There exists compatible derivations of order 0:

$$D_i = X_i \frac{\partial}{\partial X_i},$$

where X_i are coordinates of "old divisors" ,

There exists compatible derivations of order -1 of the form :

$$\frac{\partial}{\partial v_i}$$

After the blow-up we keep the following form

$$[\mathcal{U}] = [X^{a_0}(G_1+F_1+X^{a_1-a_0}(G_2+F_2+X^{a_2-a_1}(G_3+\ldots$$

$$X^{a_m-a_{m-1}}(F_{m+1} + HOT))))]$$

"Two way traffic" - new G_i may form and "move down". Then they transform into F_i. Tilted initial form expands.

$$\mathcal{D}^{lp^k}_{\log}([\mathcal{U}]) = ((X^{a_0}g'_1, \ldots, X^{a_0}g'_{s_1}, X^{a_0}v_1, \ldots, X^{a_0}v_{t_1}$$

$$X^{a_1}g'_{t_1+1}, \ldots, X^{a_1}g'_{t_2}, X^{a_1}v_{s_1+1}, \ldots, X^{a_1}v_{s_2}$$

$$, \ldots, X^{a_m}v_{s_{m-1}+1}, \ldots, X^{a_m}v_{s_m}, X^{a_m} \cdot HOT, \ldots)$$

Define

$$\tilde{\mathcal{D}}([\mathcal{U}]) = \frac{1}{x^{a_i}}(D_j[U] \mid 0 < i \leq m, s_{i-1} < j < s_i)$$

$$(\frac{1}{x^{a_i}} \cdot \frac{\partial([\mathcal{U}])}{\partial v_j} \mid 0 < i \leq m, s_{i-1} < j < s_i) +$$

$$\frac{1}{x^{a_m}}(D[U] \mid D \in \mathcal{D}_{\log}, D(X^{a_0} g_i') = 0, D(X_i^a(v_j)) = 0)$$

Then since there are compatible derivations D_j, $\frac{\partial}{\partial v_j}$ we have

$$\tilde{\mathcal{D}}(\sigma^c[\mathcal{U}]) = \sigma^c(\tilde{\mathcal{D}}([\mathcal{U}]))$$

If the form G_i exists in the above tilted initial form the maximal contact is defined by "old divisors". We consider a coefficient ideal

$$\mathcal{C}([\mathcal{U}])_{|X} := [\mathcal{U}]_{|X} + \mathcal{C}(\tilde{\mathcal{D}}[\mathcal{U}])_{|X}$$

Otherwise consider a maximal contact $\mathcal{V} \in \mathcal{T}(\tilde{\mathcal{D}}([\mathcal{U}]))$