SINGULARITIES IN POSITIVE CHARACTERISTIC III

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Outline

1. Transversality and elimination algebra.
   - Multiplicity and transversality
   - Elimination algebra
2. Overview

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Multiplicity

A smooth k-algebra.

$$f(Z) = Z^n + a_1 Z^{n-1} + \cdots + a_n \in A[Z],$$

$F_n = \{x \in \mathrm{Spec}(A[Z]) \mid \nu_x(f) \geq n\}$. Set $B = A[Z]/f(Z)$.

$$\begin{array}{ccc} F_n \subset & \mathrm{Spec}(B) \\ \| & \downarrow \beta \\ \beta(F_n) \subset & \mathrm{Spec}(A) \end{array}$$

Zariski's multiplicity formula.

$$[B : A] e(\mathfrak{q}) = \sum_{i \geq 1} [k(P_i) : k(\mathfrak{q})] e_{B_{P_i}}(\mathfrak{q} B_{P_i}).$$

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Transversality

$$V^{(d)} = \mathrm{Spec}(A[Z]) \text{ and } V^{(d-1)} = \mathrm{Spec}(A)$$
$$V^{(d)} \xrightarrow{\beta} V^{(d-1)}$$
$$X = \mathrm{Spec}(A[Z]/\langle Z^n + a_1 Z^{n-1} + \cdots + a_n \rangle)$$

$$\begin{array}{ccc} x \in F_n \subset X & \hookrightarrow & V^{(d)} \\ \downarrow & \swarrow & \\ V^{(d-1)} & & \end{array}$$

$$\mathcal{L}_{X,x} \subset \mathcal{C}_{X,x} \subset \mathbb{T}_{V^{(d)},x}$$

Definition

β is *transversal* to X in x if the tangent line ℓ to $\beta^{-1}(\beta(x))$ is not included in $\mathcal{L}_{X,x}$.

If C is smooth and $C \subset F_n \subset X$, then $\mathcal{L}_{C,x} \subset \mathcal{L}_{X,x}$ and

$$C \cong \beta(C)$$

$$\begin{array}{ccc} X & & X_1 \\ V^{(d)} & \xleftarrow{\pi_C} & V_1^{(d)} \\ \downarrow \beta & & \downarrow \beta_1 \\ V^{(d-1)} & \xleftarrow{\pi_{\beta(C)}} & V_1^{(d-1)} \end{array}$$

Multiplicity and transversality

$V^{(d)} \xrightarrow{\beta} V^{(d-1)}$

$\mathcal{G} \subset \mathcal{O}_{V^{(d)}}[W]$, $x \in \mathrm{Sing}(\mathcal{G}) \subset V^{(d)}$, $\tau_{\mathcal{G},x} \geq 1$.

$$\mathcal{L}_{\mathcal{G},x} \subset \mathcal{C}_{\mathcal{G},x} \subset \mathbb{T}_{V^{(d)},x}$$

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Definition

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$$\begin{array}{ccccccc}
 & & & \mathcal{G} & & \mathcal{G}_1 \\
C \subset & \mathrm{Sing}(\mathcal{G}) \subset & & V^{(d)} & \xleftarrow{\pi_C} & V_1^{(d)} \\
\| & & & \downarrow \beta & & \downarrow \beta_1 \\
\beta(C) \subset & & & V^{(d-1)} & \xleftarrow{\pi_{\beta(C)}} & V_1^{(d-1)}
\end{array}$$

Outline

$$V^{(d)} \xrightarrow{\beta} V^{(d-1)}$$

i) $\mathcal{G} \subset \mathcal{O}_{V^{(d)}}[W]$ so that $\tau_{\mathcal{G}} \geq 1$
ii) $\mathcal{G}$ is a β-relative differential algebra.
iii) β transversal at x.
We now define a Rees algebra, say $\mathcal{R}_{\mathcal{G},\beta} \subset \mathcal{O}_{V^{(d-1)}}[W]$, such that

$$\begin{array}{rcl} & & \mathcal{G} \\ \mathrm{Sing}(\mathcal{G}) & \subset & V^{(d)} \\ & & \downarrow \beta \\ \beta(\mathrm{Sing}(\mathcal{G})) \subset \mathrm{Sing}(\mathcal{R}_{\mathcal{G},\beta}) & \subset & V^{(d-1)} \\ & & \mathcal{R}_{\mathcal{G},\beta} \end{array}$$

ELIMINATION AND PERMISSIBLE TRANSFORMATIONS

$C \subset \mathrm{Sing}(\mathcal{G})$

$$\begin{array}{ccccc} & \mathcal{G} & & \mathcal{G}_1 \\ C \subset \mathrm{Sing}(\mathcal{G}) \subset & V^{(d)} & \xleftarrow{\pi_C} & V_1^{(d)} \\ & \downarrow \beta & & \downarrow \beta_1 \\ \beta(C) \subset \mathrm{Sing}(\mathcal{R}_{\mathcal{G},\beta}) \subset & V^{(d-1)} & \xleftarrow{\pi_{\beta(C)}} & V_1^{(d-1)} \\ & \mathcal{R}_{\mathcal{G},\beta} & & (\mathcal{R}_{\mathcal{G},\beta})_1 \end{array}$$

Invariants under change of variables

$$F_n(Z) = (Z - Y_1)(Z - Y_2)\cdots(Z - Y_n) \in k[Y_1, \dots, Y_n][Z]$$
$$L = V(Y_i - Y_j, 1 \leq i, j, \leq n)$$

$$\boxed{k[Y_1, \dots, Y_n]^L = k[Y_i - Y_j; 1 \leq i, j, \leq n]}$$

S_n acts linearly on $k[Y_1, \dots, Y_n]^L \subset k[Y_1, \dots, Y_n]$

$$(k[Y_1, \dots, Y_n]^L)^{S_n} \subset (k[Y_1, \dots, Y_n])^{S_n}$$

$$\boxed{(k[Y_1, \dots, Y_n]^L)^{S_n} = k[H_1, \dots, H_r]}$$

$H_j = H_j(Y_1, \dots, Y_n)$ homog of degree d_j
$H_j = H_j(s_1, \dots, s_n)$ w. homog. of degree d_j

Elimination algebra

$$k[Y_i - Y_j; 1 \leq i, j, \leq n] \subset k[Z - Y_1, \dots, Z - Y_n]$$

$$k[H_1 W^{d_1}, \dots, H_r W^{d_r}] \subset k[F_n(Z) W^n, \Delta^\alpha(F_n(Z)) W^{n-\alpha}]_{1 \leq \alpha \leq n-1}$$

$$\begin{array}{ccc} k[s_1, \dots, s_n][Z]/\langle F_n(Z) \rangle & \longrightarrow & A[Z]/\langle Z^n + a_1 Z^{n-1} + \cdots + a_n \rangle \\ \uparrow & & \uparrow \\ k[s_1, \dots, s_n] & \longrightarrow & A \\ s_i & \longmapsto & (-1)^i a_i. \end{array}$$

$$\begin{array}{ccc} \mathcal{G} = A[Z][f_n(Z) W^n, \Delta^\alpha(f_n(Z)) W^{n-\alpha}]_{1 \leq \alpha \leq n-1} & \subset & A[Z][W] \\ \cup & & \cup \\ \mathcal{R}_{\mathcal{G},\beta} & \subset & A[W] \end{array}$$

Theorem (Stage A)

If $\tau_{\mathcal{G}} \geq e$ there is a well defined sequence of permissible transformations:

$$\begin{array}{ccccc} (V^{(d)}, \mathcal{G}) & \longleftarrow & \cdots & \longleftarrow & (V_r^{(d)}, \mathcal{G}_r) \\ \downarrow \beta & & & & \downarrow \beta_r \\ (V^{(d-e)}, \mathcal{G}^{(d-e)}) & \longleftarrow & \cdots & \longleftarrow & (V_r^{(d-e)}, \mathcal{G}_r^{(d-e)}) \end{array}$$

such that $\mathrm{Sing}(\mathcal{G}_r) = \emptyset$ or $\mathcal{G}_r^{(d-e)}$ is monomial:

$$\mathcal{G}_r^{(d-e)} \sim \mathcal{O}_{V_r^{(e)}}[(I(H_1)^{\alpha_1} \cdots I(H_r)^{\alpha_r})W^s]$$

$\beta : V^{(d)} \longrightarrow V^{(d-e)}$ smooth locally at x, $\mathcal{G}$, $\tau_{\mathcal{G},x} \geq e$.
Assume β is a composition of smooth morphisms

$$V^{(d)} \longrightarrow V^{(d-1)} \longrightarrow \dots \longrightarrow V^{(d-e)}$$

LEMMA

Fix $\mathcal{G} \subset \mathcal{O}_V[W]$. If $\tau_{\mathcal{G}} \geq 1$ and codimension of Sing$(\mathcal{G})$ *is* 1 *in V, then there exists $Z(\subset V)$ smooth hypersurface so that*

$$\mathcal{G} \sim \mathcal{O}_V[I(Z)W].$$

THEOREM (STAGE A)

Assume $\tau_{\mathcal{G}} \geq 1$. There is a sequence of permissible transformations

$$\begin{array}{ccccccc}
\mathcal{G} & & \mathcal{G}_1 & & & & \mathcal{G}_r \\
V^{(d)} & \xleftarrow{\pi_{C_1}} & V_1^{(d)} & \longleftarrow & \dots & \xleftarrow{\pi_{C_r}} & V_r^{(d)} \\
\downarrow \beta & & \downarrow \beta_1 & & & & \downarrow \beta_r \\
V^{(d-1)} & \xleftarrow{\pi'_{\beta(C_1)}} & V_1^{(d-1)} & \longleftarrow & \dots & \xleftarrow{\pi'_{\beta(C_r)}} & V_r^{(d-1)} \\
\mathcal{R}_{\mathcal{G},\beta} & & (\mathcal{R}_{\mathcal{G},\beta})_1 & & & & (\mathcal{R}_{\mathcal{G},\beta})_r
\end{array}$$

$\mathrm{Sing}(\mathcal{G}_r) = \emptyset$ or $(\mathcal{R}_{\mathcal{G},\beta})_r = I(H_1)^{\alpha_1} \dots I(H_r)^{\alpha_r} W^s$

$$\begin{array}{ccccccc}
\mathcal{G}_r & & \mathcal{G}_{r+1} & & & & \mathcal{G}_R \\
V_r^{(d)} & \longleftarrow & V_{r+1}^{(d)} & \longleftarrow & \dots & \longleftarrow & V_R^{(d)}
\end{array}$$

REFERENCES

- A. Benito and O. Villamayor, ‘Monoidal transformations of singularities in positive characteristic’ http://arXiv.org/abs/0811.4148 26 November 2008.
- A. Bravo and O. Villamayor, ‘Hypersurface singularities in positive characteristic and stratification of singular locus’. http://arXiv.org/abs/0807.4308 27 July 2008.
- O. Villamayor, ‘Hypersurface singularities in positive characteristic.’ *Advances in Mathematics* 213 (2007) 687-733.
- O. Villamayor U. ‘Elimination with apllications to singularities in positive characteristic’. *Publications of RIMS, Kyoto University.* Vol 44, No. 2, 2008.