SINGULARITIES IN POSITIVE CHARACTERISTIC

A. Benito, A. Bravo and O. Villamayor U.

Universidad Autónoma de Madrid

RIMS Kyoto, December 2008

Outline

1. Introduction.
   - Multiplicity and the τ-invariant
   - Local presentation
   - Strategy in characteristic zero
   - Strategy in positive characteristic
2. Rees Algebras
   - Rees algebras and main invariants
   - Resolution of Rees algebras
   - Rees algebras and integral closure
3. Elimination
   - Multiplicity of hypersurfaces
   - Elimination and differentials
   - Absolute and relative differential structure
4. The monomial case.
   - Stage B'

Outline

1. Introduction.
   - Multiplicity and the τ-invariant
   - Local presentation
   - Strategy in characteristic zero
   - Strategy in positive characteristic
2. Rees Algebras
   - Rees algebras and main invariants
   - Resolution of Rees algebras
   - Rees algebras and integral closure
3. Elimination
   - Multiplicity of hypersurfaces
   - Elimination and differentials
   - Absolute and relative differential structure
4. The monomial case.
   - Stage B'

$$k[x_1,\dots,x_n]$$

$$g(x_1,\dots,x_n)=G_b(x_1,\dots,x_n)+G_{b+1}(x_1,\dots,x_n)+\dots+G_N(x_1,\dots,x_n)$$

$G_b\neq 0 \iff \nu_x(g)=b$ at a local ring of the origin.

$\mathrm{Spec}(k[x_1,\dots,x_n])$, $X=\{g=0\}\supset F_b$ set of $b-$fold points.

$$\begin{array}{ccccc} X & & X_1 & & X_s \\ V & \longleftarrow & V_1 & \longleftarrow \dots \longleftarrow & V_s \\ F_b & & F_b^1 & & F_b^s \end{array} \qquad (1)$$

Problem: Given $F_b\subset X\subset V$ find a sequence as (1) so that $F_b^s=\emptyset$.

$$k[x_1,\dots,x_n]$$

$$g(x_1,\dots,x_n)=G_b(x_1,\dots,x_n)+G_{b+1}(x_1,\dots,x_n)+\cdots+G_N(x_1,\dots,x_n)$$

$$G_b\neq 0 \iff \nu_x(g)=b \text{ at a local ring of the origin.}$$

$\mathrm{Spec}(k[x_1,\dots,x_n])$, $X=\{g=0\}\supset F_b$ set of $b-$ fold points.

$$\begin{array}{ccccccc} X & & X_1 & & & & X_s \\ V & \longleftarrow & V_1 & \longleftarrow & \dots & \longleftarrow & V_s \\ F_b & & F_b^1 & & & & F_b^s \end{array} \qquad (1)$$

Problem: Given $F_b\subset X\subset V$ find a sequence as (1) so that $F_b^s=\emptyset$.

$$k[x_1,\dots,x_n]$$

$$g(x_1,\dots,x_n)=G_b(x_1,\dots,x_n)+G_{b+1}(x_1,\dots,x_n)+\cdots+G_N(x_1,\dots,x_n)$$

$$G_b\neq 0 \qquad \Longleftrightarrow \qquad \nu_x(g)=b \text{ at a local ring of the origin.}$$

$$\text{Spec}(k[x_1,\dots,x_n]),\ X=\{g=0\}\supset F_b \text{ set of } b-\text{fold points.}$$

$$\begin{array}{ccccc} X & & X_1 & & X_s \\ V & \longleftarrow & V_1 & \longleftarrow \dots \longleftarrow & V_s \\ F_b & & F_b^1 & & F_b^s \end{array} \qquad (1)$$

Problem: Given $F_b\subset X\subset V$ find a sequence as (1) so that $F_b^s=\emptyset$.

$$k[x_1,\dots,x_n]$$

$$g(x_1,\dots,x_n) = G_b(x_1,\dots,x_n)+G_{b+1}(x_1,\dots,x_n)+\cdots+G_N(x_1,\dots,x_n)$$

$G_b \neq 0 \iff \nu_x(g) = b$ at a local ring of the origin.

$\mathrm{Spec}(k[x_1,\dots,x_n])$, $X = \{g=0\} \supset F_b$ set of $b-$ fold points.

$$\begin{array}{ccccccc} X & & X_1 & & & & X_s \\ V & \longleftarrow & V_1 & \longleftarrow & \dots & \longleftarrow & V_s \\ F_b & & F_b^1 & & & & F_b^s \end{array} \qquad (1)$$

Problem: Given $F_b \subset X \subset V$ find a sequence as (1) so that $F_b^s = \emptyset$.

$V^{(d)}$ smooth scheme of dimension d over k

$X \subset V^{(d)}$ hypersurface, b the highest multiplicity of X

$$F_b := \{x \in X \mid \mathrm{mult}_x(X) = b\}$$

$x \in F_b$ if and only $\nu_x(I(X)) = b$

Notation: ν_x denotes the order in the local regular ring $\mathcal{O}_{V^{(d)},x}$

Fix X, $I(X) \subset \mathcal{O}_{V^{(d)},x}$

$$\{x_1, \dots, x_d\} \text{ r.s.p.}$$

$$gr_{\mathfrak{M}_x}(\mathcal{O}_{V^{(d)}}) \cong k'[X_1, \dots, X_d]$$

$$\mathbb{I}_{X,x} \text{ initial ideal of } I(X) \text{ in } gr_{\mathfrak{M}_x}(\mathcal{O}_{V^{(d)}})$$

τ_X : least number of variables needed to express a generator of $\mathbb{I}_{X,x}$.

Notation: $\tau_X = \tau$

Outline

1. Introduction.
   - Multiplicity and the τ-invariant
   - Local presentation
   - Strategy in characteristic zero
   - Strategy in positive characteristic
2. Rees Algebras
   - Rees algebras and main invariants
   - Resolution of Rees algebras
   - Rees algebras and integral closure
3. Elimination
   - Multiplicity of hypersurfaces
   - Elimination and differentials
   - Absolute and relative differential structure
4. The monomial case.
   - Stage B'

LOCAL PRESENTATION

$(V^{(d)}, x) \xrightarrow{\beta} (V^{(d-\tau)}, x_\tau)$ a composition of smooth morphisms

$$(V^{(d)}, x) \longrightarrow (V^{(d-1)}, x_1) \longrightarrow \dots \longrightarrow (V^{(d-\tau)}, x_\tau)$$

A local presentation of X (at x) is defined by

(1) Positive integers $0 \leq e_1 \leq e_2 \leq \cdots \leq e_\tau$.

(2) Monic polynomials,

$$f_1^{(p^{e_1})}(z_1) = z_1^{p^{e_1}} + a_1^{(1)} z_1^{p^{e_1}-1} + \cdots + a_{p^{e_1}}^{(1)} \in \mathcal{O}_{V^{(d-1)}}[z_1]$$

$$\vdots$$

$$f_\tau^{(p^{e_\tau})}(z_\tau) = z_\tau^{p^{e_\tau}} + a_1^{(\tau)} z_\tau^{p^{e_\tau}-1} + \cdots + a_{p^{e_\tau}}^{(\tau)} \in \mathcal{O}_{V^{(d-\tau)}}[z_\tau].$$

(3) $I^{(s)}$: an ideal in $\mathcal{O}_{V^{(d-\tau)}}$ and a positive integer s.

$$F_b = \bigcap_{i=1}^{\tau} \{x \in V^{(d)} \mid \nu_x(f_i^{(p^{e_i})}) \geq p^{e_i}\} \cap \{x \in V^{(d)} \mid \nu_x(\beta^*(I^{(s)})) \geq s\}$$

Local presentation

$(V^{(d)}, x) \xrightarrow{\beta} (V^{(d-\tau)}, x_\tau)$ a composition of smooth morphisms

$$(V^{(d)}, x) \longrightarrow (V^{(d-1)}, x_1) \longrightarrow \dots \longrightarrow (V^{(d-\tau)}, x_\tau)$$

A local presentation of X (at x) is defined by

(1) Positive integers $0 \leq e_1 \leq e_2 \leq \dots \leq e_\tau$.

(2) Monic polynomials,

$$f_1^{(p^{e_1})}(z_1) = z_1^{p^{e_1}} + a_1^{(1)} z_1^{p^{e_1}-1} + \dots + a_{p^{e_1}}^{(1)} \in \mathcal{O}_{V^{(d-1)}}[z_1]$$

$$\vdots$$

$$f_\tau^{(p^{e_\tau})}(z_\tau) = z_\tau^{p^{e_\tau}} + a_1^{(\tau)} z_\tau^{p^{e_\tau}-1} + \dots + a_{p^{e_\tau}}^{(\tau)} \in \mathcal{O}_{V^{(d-\tau)}}[z_\tau].$$

(3) $I^{(s)}$: an ideal in $\mathcal{O}_{V^{(d-\tau)}}$ and a positive integer s.

$$F_b = \bigcap_{i=1}^{\tau} \{x \in V^{(d)} \mid \nu_x(f_i^{(p^{e_i})}) \geq p^{e_i}\} \cap \{x \in V^{(d)} \mid \nu_x(\beta^*(I^{(s)})) \geq s\}$$

LOCAL PRESENTATION

$(V^{(d)},x) \xrightarrow{\beta} (V^{(d-\tau)},x_\tau)$ a composition of smooth morphisms

$$(V^{(d)},x) \longrightarrow (V^{(d-1)},x_1) \longrightarrow \dots \longrightarrow (V^{(d-\tau)},x_\tau)$$

A local presentation of X (at x) is defined by

(1) Positive integers $0 \leq e_1 \leq e_2 \leq \cdots \leq e_\tau$.

(2) Monic polynomials,

$$f_1^{(p^{e_1})}(z_1) = z_1^{p^{e_1}} + a_1^{(1)} z_1^{p^{e_1}-1} + \cdots + a_{p^{e_1}}^{(1)} \in \mathcal{O}_{V^{(d-1)}}[z_1]$$

$$\vdots$$

$$f_\tau^{(p^{e_\tau})}(z_\tau) = z_\tau^{p^{e_\tau}} + a_1^{(\tau)} z_\tau^{p^{e_\tau}-1} + \cdots + a_{p^{e_\tau}}^{(\tau)} \in \mathcal{O}_{V^{(d-\tau)}}[z_\tau].$$

(3) $I^{(s)}$: an ideal in $\mathcal{O}_{V^{(d-\tau)}}$ and a positive integer s.

$$F_b = \bigcap_{i=1}^{\tau} \{x \in V^{(d)} \mid \nu_x(f_i^{(p^{e_i})}) \geq p^{e_i}\} \cap \{x \in V^{(d)} \mid \nu_x(\beta^*(I^{(s)})) \geq s\}$$

LOCAL PRESENTATION

$(V^{(d)}, x) \xrightarrow{\beta} (V^{(d-\tau)}, x_\tau)$ a composition of smooth morphisms

$$(V^{(d)}, x) \longrightarrow (V^{(d-1)}, x_1) \longrightarrow \dots \longrightarrow (V^{(d-\tau)}, x_\tau)$$

A local presentation of X (at x) is defined by

(1) Positive integers $0 \leq e_1 \leq e_2 \leq \cdots \leq e_\tau$.

(2) Monic polynomials,

$$f_1^{(p^{e_1})}(z_1) = z_1^{p^{e_1}} + a_1^{(1)} z_1^{p^{e_1}-1} + \cdots + a_{p^{e_1}}^{(1)} \in \mathcal{O}_{V^{(d-1)}}[z_1]$$

$$\vdots$$

$$f_\tau^{(p^{e_\tau})}(z_\tau) = z_\tau^{p^{e_\tau}} + a_1^{(\tau)} z_\tau^{p^{e_\tau}-1} + \cdots + a_{p^{e_\tau}}^{(\tau)} \in \mathcal{O}_{V^{(d-\tau)}}[z_\tau].$$

(3) $I^{(s)}$: an ideal in $\mathcal{O}_{V^{(d-\tau)}}$ and a positive integer s.

$$F_b = \bigcap_{i=1}^{\tau} \{x \in V^{(d)} \mid \nu_x(f_i^{(p^{e_i})}) \geq p^{e_i}\} \cap \{x \in V^{(d)} \mid \nu_x(\beta^*(I^{(s)})) \geq s\}$$

Local presentation

$(V^{(d)}, x) \xrightarrow{\beta} (V^{(d-\tau)}, x_\tau)$ a composition of smooth morphisms

$$(V^{(d)}, x) \longrightarrow (V^{(d-1)}, x_1) \longrightarrow \dots \longrightarrow (V^{(d-\tau)}, x_\tau)$$

A local presentation of X (at x) is defined by

(1) Positive integers $0 \leq e_1 \leq e_2 \leq \cdots \leq e_\tau$.

(2) Monic polynomials,

$$f_1^{(p^{e_1})}(z_1) = z_1^{p^{e_1}} + a_1^{(1)} z_1^{p^{e_1}-1} + \cdots + a_{p^{e_1}}^{(1)} \in \mathcal{O}_{V^{(d-1)}}[z_1]$$

$$\vdots$$

$$f_\tau^{(p^{e_\tau})}(z_\tau) = z_\tau^{p^{e_\tau}} + a_1^{(\tau)} z_\tau^{p^{e_\tau}-1} + \cdots + a_{p^{e_\tau}}^{(\tau)} \in \mathcal{O}_{V^{(d-\tau)}}[z_\tau].$$

(3) $I^{(s)}$: an ideal in $\mathcal{O}_{V^{(d-\tau)}}$ and a positive integer s.

$$F_b = \bigcap_{i=1}^{\tau} \{x \in V^{(d)} \mid \nu_x(f_i^{(p^{e_i})}) \geq p^{e_i}\} \cap \{x \in V^{(d)} \mid \nu_x(\beta^*(I^{(s)})) \geq s\}$$

LOCAL PRESENTATION

$(V^{(d)}, x) \xrightarrow{\beta} (V^{(d-\tau)}, x_\tau)$ a composition of smooth morphisms

$$(V^{(d)}, x) \longrightarrow (V^{(d-1)}, x_1) \longrightarrow \dots \longrightarrow (V^{(d-\tau)}, x_\tau)$$

A local presentation of X (at x) is defined by

(1) Positive integers $0 \leq e_1 \leq e_2 \leq \cdots \leq e_\tau$.

(2) Monic polynomials,

$$f_1^{(p^{e_1})}(z_1) = z_1^{p^{e_1}} + a_1^{(1)} z_1^{p^{e_1}-1} + \cdots + a_{p^{e_1}}^{(1)} \in \mathcal{O}_{V^{(d-1)}}[z_1]$$

$$\vdots$$

$$f_\tau^{(p^{e_\tau})}(z_\tau) = z_\tau^{p^{e_\tau}} + a_1^{(\tau)} z_\tau^{p^{e_\tau}-1} + \cdots + a_{p^{e_\tau}}^{(\tau)} \in \mathcal{O}_{V^{(d-\tau)}}[z_\tau].$$

(3) $I^{(s)}$: an ideal in $\mathcal{O}_{V^{(d-\tau)}}$ and a positive integer s.

$$F_b = \bigcap_{i=1}^{\tau} \{x \in V^{(d)} \mid \nu_x(f_i^{(p^{e_i})}) \geq p^{e_i}\} \cap \{x \in V^{(d)} \mid \nu_x(\beta^*(I^{(s)})) \geq s\}$$

Local presentation and permissible transformations

$Y \subset F_b$ permissible center, $Y \cong \beta(Y)$

$$\begin{array}{rccc}
 & X & & X_1 \\
Y \subset & (V^{(d)}, x) & \xleftarrow{\ \pi\ } & (V_1^{(d)}, x') \\
 & \Big\downarrow \beta & \circlearrowleft & \Big\downarrow \beta_1 \\
\beta(Y) \subset & (V^{(d-\tau)}, x_\tau) & \xleftarrow{\ \widetilde{\pi}\ } & (V_1^{(d-\tau)}, x'_\tau)
\end{array}$$

$$I^{(s)}\mathcal{O}_{V_1^{(d-\tau)}} = I(H_1^{(d-\tau)})^s \cdot I_1^{(s)},$$

where $H_1^{(d-\tau)}$ is the exceptional locus of $\widetilde{\pi}$

Since

$$\beta : (V^{(d)}, x) \longrightarrow (V^{(d-1)}, x_1) \longrightarrow \dots \longrightarrow (V^{(d-\tau)}, x_\tau)$$

the blow-up induces

$$\beta_1 : (V_1^{(d)}, x') \longrightarrow (V_1^{(d-1)}, x'_1) \longrightarrow \dots \longrightarrow (V_1^{(d-\tau)}, x'_\tau)$$

(1)' Positive integers $0 \leq e_1 \leq e_2 \leq \dots \leq e_\tau$ as before.

(2)' Monic polynomials, $g_i^{(p^{e_i})}$ for $i = 1, \dots, \tau$ (the strict transforms of the monic polynomials $f_i^{(p^{e_i})}$).

(3)' The ideal $I_1^{(s)}$ in $\mathcal{O}_{V_1^{(d-\tau)}}$ with the same positive integer s.

(1)'+(2)'+(3)' define a local presentation.

Since

$$\beta : (V^{(d)}, x) \longrightarrow (V^{(d-1)}, x_1) \longrightarrow \dots \longrightarrow (V^{(d-\tau)}, x_\tau)$$

the blow-up induces

$$\beta_1 : (V_1^{(d)}, x') \longrightarrow (V_1^{(d-1)}, x_1') \longrightarrow \dots \longrightarrow (V_1^{(d-\tau)}, x_\tau')$$

(1)' Positive integers $0 \leq e_1 \leq e_2 \leq \cdots \leq e_\tau$ as before.

(2)' Monic polynomials, $g_i^{(p^{e_i})}$ for $i = 1, \dots, \tau$ (the strict transforms of the monic polynomials $f_i^{(p^{e_i})}$).

(3)' The ideal $I_1^{(s)}$ in $\mathcal{O}_{V_1^{(d-\tau)}}$ with the same positive integer s.

(1)'+(2)'+(3)' define a local presentation.

A sequence of permissible transformations

$$
\begin{array}{ccccc}
X & & X_1 & & & X_r \\
V^{(d)} & \xleftarrow{\pi_1} & V_1^{(d)} & \longleftarrow \cdots & \xleftarrow{\pi_\tau} & V_r^{(d)}
\end{array}
$$

induces

$$
\begin{array}{ccccc}
V^{(d)} & \xleftarrow{\pi_1} & V_1^{(d)} & \longleftarrow \cdots & \xleftarrow{\pi_\tau} & V_r^{(d)} \\
\downarrow \beta & & \downarrow \beta_1 & & & \downarrow \beta_\tau \\
V^{(d-\tau)} & \xleftarrow{\widetilde{\pi}_1} & V_1^{(d-\tau)} & \longleftarrow \cdots & \xleftarrow{\widetilde{\pi}_\tau} & V_r^{(d-\tau)}
\end{array}
$$

and

$$
\begin{array}{ccccc}
V^{(d-\tau)} & \longleftarrow & V_1^{(d-\tau)} & \longleftarrow \cdots & \longleftarrow & V_r^{(d-\tau)} \\
I^{(s)} & & I_1^{(s)} & & & I_r^{(s)}
\end{array}
$$

A sequence of permissible transformations

$$\begin{array}{ccccc} X & & X_1 & & X_r \\ V^{(d)} & \xleftarrow{\pi_1} & V_1^{(d)} & \longleftarrow \cdots \xleftarrow{\pi_\tau} & V_r^{(d)} \end{array}$$

induces

$$\begin{array}{ccccc} V^{(d)} & \xleftarrow{\pi_1} & V_1^{(d)} & \longleftarrow \cdots \xleftarrow{\pi_\tau} & V_r^{(d)} \\ \downarrow \beta & & \downarrow \beta_1 & & \downarrow \beta_\tau \\ V^{(d-\tau)} & \xleftarrow{\widetilde{\pi}_1} & V_1^{(d-\tau)} & \longleftarrow \cdots \xleftarrow{\widetilde{\pi}_\tau} & V_r^{(d-\tau)} \end{array}$$

and

$$\begin{array}{ccccc} V^{(d-\tau)} & \longleftarrow & V_1^{(d-\tau)} & \longleftarrow \cdots \longleftarrow & V_r^{(d-\tau)} \\ I^{(s)} & & I_1^{(s)} & & I_r^{(s)} \end{array}$$

A sequence of permissible transformations

$$\begin{array}{ccccc} X & X_1 & & & X_r \\ V^{(d)} \xleftarrow{\pi_1} & V_1^{(d)} & \longleftarrow \cdots \xleftarrow{\pi_\tau} & & V_r^{(d)} \end{array}$$

induces

$$\begin{array}{ccccc} V^{(d)} & \xleftarrow{\pi_1} & V_1^{(d)} & \longleftarrow \cdots \xleftarrow{\pi_\tau} & V_r^{(d)} \\ \downarrow \beta & & \downarrow \beta_1 & & \downarrow \beta_\tau \\ V^{(d-\tau)} & \xleftarrow{\widetilde{\pi}_1} & V_1^{(d-\tau)} & \longleftarrow \cdots \xleftarrow{\widetilde{\pi}_\tau} & V_r^{(d-\tau)} \end{array}$$

and

$$\begin{array}{ccccc} V^{(d-\tau)} & \longleftarrow & V_1^{(d-\tau)} & \longleftarrow \cdots \longleftarrow & V_r^{(d-\tau)} \\ I^{(s)} & & I_1^{(s)} & & I_r^{(s)} \end{array}$$

Outline

1. Introduction.
   - Multiplicity and the τ-invariant
   - Local presentation
   - Strategy in characteristic zero
   - Strategy in positive characteristic
2. Rees Algebras
   - Rees algebras and main invariants
   - Resolution of Rees algebras
   - Rees algebras and integral closure
3. Elimination
   - Multiplicity of hypersurfaces
   - Elimination and differentials
   - Absolute and relative differential structure
4. The monomial case.
   - Stage B'

Strategy in characteristic zero

A local presentation of X (at x) when $char(k) = 0$ is

(1) Positive integers $0 = e_1 = e_2 = \cdots = e_\tau$.

(2) Regular parameters

$$f_1^{(1)}(z_1) = z_1 \in \mathcal{O}_{V^{(d-1)}}[z_1]$$
$$\vdots$$
$$f_\tau^{(1)}(z_\tau) = z_\tau \in \mathcal{O}_{V^{(d-\tau)}}[z_\tau].$$

(3) $I^{(s)}$: an ideal in $\mathcal{O}_{V^{(d-\tau)}}$ and a positive integer s.

Stage A): Reduction to the monomial case.

$$\begin{array}{ccccccc} X & & X_1 & & & & X_r \\ V^{(d)} & \longleftarrow & V_1^{(d)} & \longleftarrow & \cdots & \longleftarrow & V_r^{(d)} \end{array}$$

is defined so that, setting

$$V^{(d-\tau)} \longleftarrow V_1^{(d-\tau)} \longleftarrow \ldots \longleftarrow V_r^{(d-\tau)}$$

and $I^{(s)}$ as above, then

$$I_r^{(s)} = I(H_1^{(d-\tau)})^{\alpha_1} \cdot I(H_2^{(d-\tau)})^{\alpha_2} \ldots I(H_r^{(d-\tau)})^{\alpha_r} \subset \mathcal{O}_{V_r^{(d-\tau)}}.$$

Stage B): Resolution of the monomial case.

$$\begin{array}{ccccc} X_r & X_{r+1} & & & X_N \\ V_r^{(d)} \longleftarrow & V_{r+1}^{(d)} & \longleftarrow & \cdots \longleftarrow & V_N^{(d)} \end{array}$$

is defined so that setting

$$\begin{array}{ccccc} V_r^{(d-\tau)} \longleftarrow & V_{r+1}^{(d-\tau)} & \longleftarrow & \cdots \longleftarrow & V_N^{(d-\tau)} \\ I_r^{(s)} & I_{r+1}^{(s)} & & & I_N^{(s)} \end{array}$$

and $I_i^{(s)} \subset \mathcal{O}_{V_i^{(d-\tau)}}$ as before, then

$$\{x \in V_N^{(d-\tau)} \mid \nu_x(I_N^{(s)}) \geq s\} = \emptyset \text{ (easy)}.$$

Key Point

The local presentation in char zero will allow us to lift sequence

$$\begin{array}{ccccc} V_r^{(d-\tau)} & \longleftarrow V_{r+1}^{(d-\tau)} & \longleftarrow \cdots \longleftarrow & V_N^{(d-\tau)} \\ I_r^{(s)} & I_{r+1}^{(s)} & & I_N^{(s)} \end{array}$$

to a resolution

$$\begin{array}{ccccc} X_r & X_{r+1} & & X_N \\ V_r^{(d)} & \longleftarrow V_{r+1}^{(d)} & \longleftarrow \cdots \longleftarrow & V_N^{(d)} \end{array}$$

Since $f_i = x_i$ and

$$\{x \in V^{(d)} \mid \nu_x(x_i) \geq 1\} = \{x_1 = 0, \ldots, x_\tau = 0\}$$

is naturally identified with $V^{(d-\tau)}$.

Outline

1 Introduction.
- Multiplicity and the τ-invariant
- Local presentation
- Strategy in characteristic zero
- Strategy in positive characteristic

2 Rees Algebras
- Rees algebras and main invariants
- Resolution of Rees algebras
- Rees algebras and integral closure

3 Elimination
- Multiplicity of hypersurfaces
- Elimination and differentials
- Absolute and relative differential structure

4 The monomial case.
- Stage B'

Strategy in positive characteristic

$$\beta : V^{(d)} \longrightarrow V^{(d-1)} \longrightarrow \dots \longrightarrow V^{(d-\tau)}$$

A local presentation of X (at x) is defined by

(1) Positive integers $0 \leq e_1 \leq e_2 \leq \cdots \leq e_\tau$.

(2) Monic polynomials,

$$f_1^{(p^{e_1})}(z_1) = z_1^{p^{e_1}} + a_1^{(1)} z_1^{p^{e_1}-1} + \cdots + a_{p^{e_1}}^{(1)} \in \mathcal{O}_{V^{(d-1)}}[z_1]$$

$$\vdots$$

$$f_\tau^{(p^{e_\tau})}(z_\tau) = z_\tau^{p^{e_\tau}} + a_1^{(\tau)} z_\tau^{p^{e_\tau}-1} + \cdots + a_{p^{e_\tau}}^{(\tau)} \in \mathcal{O}_{V^{(d-\tau)}}[z_\tau].$$

(3) $I^{(s)}$: an ideal in $\mathcal{O}_{V^{(d-\tau)}}$ and a positive integer s.

Theorem (Stage A): Reduction to the monomial case.

$$\begin{array}{ccccccc} X & & X_1 & & & & X_r \\ V^{(d)} & \longleftarrow & V_1^{(d)} & \longleftarrow & \dots & \longleftarrow & V_r^{(d)} \end{array}$$

is defined so that, setting

$$V^{(d-\tau)} \longleftarrow V_1^{(d-\tau)} \longleftarrow \dots \longleftarrow V_r^{(d-\tau)}$$

and $I^{(s)}$ as above, then

$$I_r^{(s)} = I(H_1^{(d-\tau)})^{\alpha_1} \cdot I(H_2^{(d-\tau)})^{\alpha_2} \dots I(H_r^{(d-\tau)})^{\alpha_r} \subset \mathcal{O}_{V_r^{(d-\tau)}}.$$

THEOREM (STAGE B'): THE MONOMIAL CASE.

Given $I_r^{(s)} = I(H_1^{(d-\tau)})^{\alpha_1} \cdot I(H_2^{(d-\tau)})^{\alpha_2} \dots I(H_r^{(d-\tau)})^{\alpha_r} \subset \mathcal{O}_{V_r^{(d-\tau)}}$, there exists (a τ-monomial):

$$I_r'^{(s)} = I(H_1^{(d-\tau)})^{h_1} \cdot I(H_2^{(d-\tau)})^{h_2} \dots I(H_r^{(d-\tau)})^{h_r}$$

with exponents $0 \leq h_i \leq \alpha_i$, so that a combinatorial resolution:

$$\begin{array}{ccccccc} V_p^{(d-\tau)} & \longleftarrow & V_{p+1}^{(d-\tau)} & \longleftarrow & \cdots & \longleftarrow & V_N^{(d-\tau)} \\ I_r'^{(s)} & & I_{r+1}'^{(s)} & & & & I_N'^{(s)} \end{array}$$

can be lifted to a permissible sequence

$$\begin{array}{ccccccc} X & & X_1 & & & & X_N \\ V^{(d)} & \longleftarrow & V_1^{(d)} & \longleftarrow & \cdots & \longleftarrow & V_N^{(d)}. \end{array}$$

Theorem (Stage B'): The monomial case.

Given $I_r^{(s)} = I(H_1^{(d-\tau)})^{\alpha_1} \cdot I(H_2^{(d-\tau)})^{\alpha_2} \dots I(H_r^{(d-\tau)})^{\alpha_r} \subset \mathcal{O}_{V_r^{(d-\tau)}}$, there exists (a τ-monomial):

$$I_r'^{(s)} = I(H_1^{(d-\tau)})^{h_1} \cdot I(H_2^{(d-\tau)})^{h_2} \dots I(H_r^{(d-\tau)})^{h_r}$$

with exponents $0 \leq h_i \leq \alpha_i$, so that a combinatorial resolution:

$$\begin{array}{ccccccc} V_p^{(d-\tau)} & \longleftarrow & V_{p+1}^{(d-\tau)} & \longleftarrow \cdots \longleftarrow & V_N^{(d-\tau)} \\ I_r'^{(s)} & & I_{r+1}'^{(s)} & & I_N'^{(s)} \end{array}$$

can be lifted to a permissible sequence

$$\begin{array}{ccccccc} X & & X_1 & & X_N \\ V^{(d)} & \longleftarrow & V_1^{(d)} & \longleftarrow \cdots \longleftarrow & V_N^{(d)}. \end{array}$$

Outline

1. Introduction.
   - Multiplicity and the τ-invariant
   - Local presentation
   - Strategy in characteristic zero
   - Strategy in positive characteristic
2. Rees Algebras
   - Rees algebras and main invariants
   - Resolution of Rees algebras
   - Rees algebras and integral closure
3. Elimination
   - Multiplicity of hypersurfaces
   - Elimination and differentials
   - Absolute and relative differential structure
4. The monomial case.
   - Stage B'

$$B \text{ smooth over } k; \quad V = Spec(B)$$

$$\mathcal{G} = B[f_1 W^{n_1}, f_2 W^{n_2}, \dots, f_s W^{n_s}] = \oplus_{n \geq 0} I_n W^n \subset B[W]; \quad I_0 = \mathcal{O}_V$$

$$\boxed{\mathrm{Sing}(\mathcal{G}) = \{x \in V \mid \nu_x(I_n) \geq n\}}$$

Definition (Hironaka)

$$ord : \mathrm{Sing}(\mathcal{G}) \to \mathbb{Q}$$

$$ord(x) = min\left\{\frac{\nu_x(I_k)}{k}\right\}$$

Remark: $ord(x) = min\left\{\frac{\nu_x(f_i)}{n_i}, i = 1, \dots, s\right\}$

$$B \text{ smooth over } k; \quad V = Spec(B)$$

$$\mathcal{G} = B[f_1 W^{n_1}, f_2 W^{n_2}, \dots, f_s W^{n_s}] = \oplus_{n \geq 0} I_n W^n \subset B[W]; \quad I_0 = \mathcal{O}_V$$

$$\boxed{\text{Sing}(\mathcal{G}) = \{x \in V \mid \nu_x(I_n) \geq n\}}$$

Definition (Hironaka)

$$ord : \text{Sing}(\mathcal{G}) \to \mathbb{Q}$$

$$ord(x) = min\left\{\frac{\nu_x(I_k)}{k}\right\}$$

Remark: $ord(x) = min\left\{\frac{\nu_x(f_i)}{n_i}, i = 1, \dots, s\right\}$

$$B \text{ smooth over } k; \quad V = Spec(B)$$

$$\mathcal{G} = B[f_1 W^{n_1}, f_2 W^{n_2}, \ldots, f_s W^{n_s}] = \oplus_{n \geq 0} I_n W^n \subset B[W]; \quad I_0 = \mathcal{O}_V$$

$$\boxed{\text{Sing}(\mathcal{G}) = \{x \in V \mid \nu_x(I_n) \geq n\}}$$

Definition (Hironaka)

$$ord : \text{Sing}(\mathcal{G}) \to \mathbb{Q}$$

$$ord(x) = min\left\{ \frac{\nu_x(I_k)}{k} \right\}$$

Remark: $ord(x) = min\left\{ \frac{\nu_x(f_i)}{n_i}, i = 1, \ldots, s \right\}$

Outline

1 Introduction.
- Multiplicity and the τ-invariant
- Local presentation
- Strategy in characteristic zero
- Strategy in positive characteristic

2 Rees Algebras
- Rees algebras and main invariants
- Resolution of Rees algebras
- Rees algebras and integral closure

3 Elimination
- Multiplicity of hypersurfaces
- Elimination and differentials
- Absolute and relative differential structure

4 The monomial case.
- Stage B'

Smooth center $Y \subset \mathrm{Sing}(\mathcal{G}) = \{x \in V \mid \nu_x(I_n) \geq n\}$

$$V \xleftarrow{\pi} V_1 \qquad H = \pi^{-1}(Y)$$

$$I_n \mathcal{O}_{V_1^{(d)}} = I(H)^n I'_n$$

Definition (Transformation)

$\mathcal{G}_1 = \oplus_{n \geq 0} I'_n W^n$ is the transform of $\mathcal{G} = \oplus_{n \geq 0} I_n W^n$

$$(V, \mathcal{G}) \longleftarrow (V_1, \mathcal{G}_1)$$
$$Y \subset \mathrm{Sing}(\mathcal{G})$$

Smooth center $Y \subset \mathrm{Sing}(\mathcal{G}) = \{x \in V \mid \nu_x(I_n) \geq n\}$

$$V \xleftarrow{\pi} V_1 \qquad H = \pi^{-1}(Y)$$

$$I_n \mathcal{O}_{V_1^{(d)}} = I(H)^n I'_n$$

Definition (Transformation)

$\mathcal{G}_1 = \oplus_{n \geq 0} I'_n W^n$ is the transform of $\mathcal{G} = \oplus_{n \geq 0} I_n W^n$

$$(V, \mathcal{G}) \longleftarrow (V_1, \mathcal{G}_1)$$
$$Y \subset \mathrm{Sing}(\mathcal{G})$$

Resolution

$$V \text{ smooth } /k, \quad \mathcal{G} = \oplus I_k W^k \subset \mathcal{O}_{V^{(d)}}$$

Find a sequence of permissible transformations:

$$(V, \mathcal{G}) \longleftarrow (V_1, \mathcal{G}_1) \longleftarrow \dots \longleftarrow (V_n, \mathcal{G}_n)$$

with

$$\boxed{\text{Sing}(\mathcal{G}_n) = \emptyset}$$

OUTLINE

1 INTRODUCTION.
- Multiplicity and the τ-invariant
- Local presentation
- Strategy in characteristic zero
- Strategy in positive characteristic

2 REES ALGEBRAS
- Rees algebras and main invariants
- Resolution of Rees algebras
- Rees algebras and integral closure

3 ELIMINATION
- Multiplicity of hypersurfaces
- Elimination and differentials
- Absolute and relative differential structure

4 THE MONOMIAL CASE.
- Stage B'

Rees Algebras and integral closure

$$\mathcal{G}_1 = \oplus_{n\geq 0} I_n W^n, \qquad \mathcal{G}_2 = \oplus_{n\geq 0} J_n W^n, \subset \mathcal{O}_{V^{(d)}}[W]$$

$$\mathcal{G}_1 \sim \mathcal{G}_2 \text{ if same integral closure in } \mathcal{O}_{V^{(d)}}[Z]$$

Then:

- $\mathcal{G}_1 \sim \mathcal{O}_{V^{(d)}}[I_s W^s]$ for some s.
- $\mathrm{Sing}(\mathcal{G}_1) = \mathrm{Sing}(\mathcal{G}_2)$.
- Hironaka's functions coincide.
- Equivalence is stable by transformations.

Rees Algebras and integral closure

$$\mathcal{G}_1 = \oplus_{n\geq 0} I_n W^n, \qquad \mathcal{G}_2 = \oplus_{n\geq 0} J_n W^n, \subset \mathcal{O}_{V^{(d)}}[W]$$

$$\mathcal{G}_1 \sim \mathcal{G}_2 \text{ if same integral closure in } \mathcal{O}_{V^{(d)}}[Z]$$

Then:

- $\mathcal{G}_1 \sim \mathcal{O}_{V^{(d)}}[I_s W^s]$ for some s.
- $\mathrm{Sing}(\mathcal{G}_1) = \mathrm{Sing}(\mathcal{G}_2)$.
- Hironaka's functions coincide.
- Equivalence is stable by transformations.

Rees Algebras and integral closure

$$\mathcal{G}_1 = \oplus_{n\geq 0} I_n W^n, \qquad \mathcal{G}_2 = \oplus_{n\geq 0} J_n W^n, \subset \mathcal{O}_{V^{(d)}}[W]$$

$\mathcal{G}_1 \sim \mathcal{G}_2$ if same integral closure in $\mathcal{O}_{V^{(d)}}[Z]$

Then:

- $\mathcal{G}_1 \sim \mathcal{O}_{V^{(d)}}[I_s W^s]$ for some s.
- $\text{Sing}(\mathcal{G}_1) = \text{Sing}(\mathcal{G}_2)$.
- Hironaka's functions coincide.
- Equivalence is stable by transformations.

Rees Algebras and integral closure

$$\mathcal{G}_1 = \oplus_{n\geq 0} I_n W^n, \qquad \mathcal{G}_2 = \oplus_{n\geq 0} J_n W^n, \subset \mathcal{O}_{V^{(d)}}[W]$$

$\mathcal{G}_1 \sim \mathcal{G}_2$ if same integral closure in $\mathcal{O}_{V^{(d)}}[Z]$

Then:

- $\mathcal{G}_1 \sim \mathcal{O}_{V^{(d)}}[I_s W^s]$ for some s.
- $\text{Sing}(\mathcal{G}_1) = \text{Sing}(\mathcal{G}_2)$.
- Hironaka's functions coincide.
- Equivalence is stable by transformations.

Rees Algebras and integral closure

$$\mathcal{G}_1 = \oplus_{n\geq 0} I_n W^n, \qquad \mathcal{G}_2 = \oplus_{n\geq 0} J_n W^n, \subset \mathcal{O}_{V^{(d)}}[W]$$

$$\mathcal{G}_1 \sim \mathcal{G}_2 \text{ if same integral closure in } \mathcal{O}_{V^{(d)}}[Z]$$

Then:

- $\mathcal{G}_1 \sim \mathcal{O}_{V^{(d)}}[I_s W^s]$ for some s.
- $\mathrm{Sing}(\mathcal{G}_1) = \mathrm{Sing}(\mathcal{G}_2)$.
- Hironaka's functions coincide.
- Equivalence is stable by transformations.

REES ALGEBRAS AND INTEGRAL CLOSURE

$$\mathcal{G}_1 = \oplus_{n\geq 0} I_n W^n, \qquad \mathcal{G}_2 = \oplus_{n\geq 0} J_n W^n, \subset \mathcal{O}_{V^{(d)}}[W]$$

$\mathcal{G}_1 \sim \mathcal{G}_2$ if same integral closure in $\mathcal{O}_{V^{(d)}}[Z]$

Then:

- $\mathcal{G}_1 \sim \mathcal{O}_{V^{(d)}}[I_s W^s]$ for some s.
- $\mathrm{Sing}(\mathcal{G}_1) = \mathrm{Sing}(\mathcal{G}_2)$.
- Hironaka's functions coincide.
- Equivalence is stable by transformations.

Outline

1. Introduction.
   - Multiplicity and the τ-invariant
   - Local presentation
   - Strategy in characteristic zero
   - Strategy in positive characteristic
2. Rees Algebras
   - Rees algebras and main invariants
   - Resolution of Rees algebras
   - Rees algebras and integral closure
3. Elimination
   - Multiplicity of hypersurfaces
   - Elimination and differentials
   - Absolute and relative differential structure
4. The monomial case.
   - Stage B'

$$A \text{ smooth}/k, \quad V = Spec(A[Z])$$

$$f = Z^n + a_1 Z^{n-1} + \cdots + a_n \in A[Z]$$

$$\mathcal{G} = \mathcal{O}_V[fW^n], \ \ \text{Sing}(\mathcal{G}) = F_n = \text{n-fold points of } f.$$

$\beta(F_n)$?

$$A \text{ smooth}/k, \quad V = Spec(A[Z])$$

$$f = Z^n + a_1 Z^{n-1} + \cdots + a_n \in A[Z]$$

$$\mathcal{G} = \mathcal{O}_V[fW^n], \ \ \text{Sing}(\mathcal{G}) = F_n = \text{n-fold points of } f.$$

$$\begin{array}{ccc} \text{Spec}(A[Z]) & \xrightarrow{\beta} & \text{Spec}(A) \\ \uparrow & \nearrow & \\ \text{Spec}(A[Z]/\langle f \rangle) & & \end{array}$$

$$\beta(F_n)?$$

$$A \text{ smooth}/k, \quad V = Spec(A[Z])$$

$$f = Z^n + a_1 Z^{n-1} + \cdots + a_n \in A[Z]$$

$$\mathcal{G} = \mathcal{O}_V[fW^n], \ \text{Sing}(\mathcal{G}) = F_n = \text{n-fold points of } f.$$

$$\begin{array}{ccc} \text{Spec}(A[Z]) & \xrightarrow{\beta} & \text{Spec}(A) \\ \uparrow & \nearrow & \\ \text{Spec}(A[Z]/\langle f \rangle) & & \end{array}$$

$$\beta(F_n)?$$

UNIVERSAL SETTING

$$(Z - Y_1)(Z - Y_2)\cdots(Z - Y_n) \in k[Y_1, \dots, Y_n][Z]$$

$$F_n(Z) = Z^n - s_1 Z^{n-1} + \cdots + (-1)^n s_n \in k[s_1, \dots, s_n][Z]$$

$$\begin{array}{ccc} k[s_1, \dots, s_n][Z]/\langle F_n(Z)\rangle & \longrightarrow & A[Z]/\langle Z^n + a_1 Z^{n-1} + \cdots + a_n\rangle \\ \uparrow & & \uparrow \\ k[s_1, \dots, s_n] & \longrightarrow & A \\ s_i & \longmapsto & (-1)^i a_i. \end{array}$$

Universal setting

$$(Z - Y_1)(Z - Y_2)\cdots(Z - Y_n) \in k[Y_1, \dots, Y_n][Z]$$
$$F_n(Z) = Z^n - s_1 Z^{n-1} + \cdots + (-1)^n s_n \in k[s_1, \dots, s_n][Z]$$

$$\begin{array}{ccc}
k[s_1, \dots, s_n][Z]/\langle F_n(Z)\rangle & \longrightarrow & A[Z]/\langle Z^n + a_1 Z^{n-1} + \cdots + a_n\rangle \\
\uparrow & & \uparrow \\
k[s_1, \dots, s_n] & \longrightarrow & A \\
s_i & \longmapsto & (-1)^i a_i.
\end{array}$$

Invariants under change of variables

$$F_n(Z) = (Z - Y_1)(Z - Y_2) \cdots (Z - Y_n) \in k[Y_1, \dots, Y_n][Z]$$
$$L = V(Y_i - Y_j, 1 \leq i, j, \leq n)$$

$$\boxed{k[Y_1, \dots, Y_n]^L = k[Y_i - Y_j ; 1 \leq i, j, \leq n]}$$

S_n acts linearly on $k[Y_1, \dots, Y_n]^L \subset k[Y_1, \dots, Y_n]$

$$(k[Y_1, \dots, Y_n]^L)^{S_n} \subset (k[Y_1, \dots, Y_n])^{S_n}$$

$$\boxed{(k[Y_1, \dots, Y_n]^L)^{S_n} = k[H_1, \dots, H_r]}$$

$H_j = H_j(Y_1, \dots, Y_n)$ homog of degree d_j
$H_j = H_j(s_1, \dots, s_n)$ w. homog. of degree d_j

INVARIANTS UNDER CHANGE OF VARIABLES

$$F_n(Z) = (Z - Y_1)(Z - Y_2)\cdots(Z - Y_n) \in k[Y_1, \dots, Y_n][Z]$$
$$L = V(Y_i - Y_j, 1 \leq i, j, \leq n)$$

$$\boxed{k[Y_1, \dots, Y_n]^L = k[Y_i - Y_j; 1 \leq i, j, \leq n]}$$

S_n acts linearly on $k[Y_1, \dots, Y_n]^L \subset k[Y_1, \dots, Y_n]$

$$(k[Y_1, \dots, Y_n]^L)^{S_n} \subset (k[Y_1, \dots, Y_n])^{S_n}$$

$$\boxed{(k[Y_1, \dots, Y_n]^L)^{S_n} = k[H_1, \dots, H_r]}$$

$H_j = H_j(Y_1, \dots, Y_n)$ homog of degree d_j
$H_j = H_j(s_1, \dots, s_n)$ w. homog. of degree d_j

Invariants under change of variables

$$F_n(Z) = (Z - Y_1)(Z - Y_2)\cdots(Z - Y_n) \in k[Y_1, \dots, Y_n][Z]$$
$$L = V(Y_i - Y_j, 1 \leq i, j, \leq n)$$

$$\boxed{k[Y_1, \dots, Y_n]^L = k[Y_i - Y_j; 1 \leq i, j, \leq n]}$$

S_n acts linearly on $k[Y_1, \dots, Y_n]^L \subset k[Y_1, \dots, Y_n]$

$$(k[Y_1, \dots, Y_n]^L)^{S_n} \subset (k[Y_1, \dots, Y_n])^{S_n}$$

$$\boxed{(k[Y_1, \dots, Y_n]^L)^{S_n} = k[H_1, \dots, H_r]}$$

$H_j = H_j(Y_1, \dots, Y_n)$ homog of degree d_j
$H_j = H_j(s_1, \dots, s_n)$ w. homog. of degree d_j

$$\begin{array}{ccc} k[s_1,\dots,s_n][Z]/\langle F_n(Z)\rangle & & A[Z]/\langle Z^n + a_1 Z^{n-1} + \dots + a_n\rangle \\ \uparrow & & \uparrow \\ k[s_1,\dots,s_n] & \xrightarrow{\phi} & (A,M)(loc., reg.) \\ s_i & \longmapsto & (-1)^i a_i \in M_i. \end{array}$$

$$\boxed{\phi(H_j) = H_j(a_1,\dots,a_n) = h_j \in M^{d_i}}$$

- $\langle h_j\rangle$ invariant by $Z \to uZ + a \qquad u \in U(A); a \in A$.
- $p \in V(\langle h_1,\dots,h_{n_i}\rangle) \subset Spec(A)$ iff $B \otimes \overline{k(p)}$ is local.
- $\cap\{P \in Spec(A) : \nu_P(h_i) \geq d_i\} = \beta(F_n)$ if $n \neq 0$ in k.

$$\begin{array}{ccc} k[s_1,\dots,s_n][Z]/\langle F_n(Z)\rangle & & A[Z]/\langle Z^n+a_1Z^{n-1}+\cdots+a_n\rangle \\ \uparrow & & \uparrow \\ k[s_1,\dots,s_n] & \xrightarrow{\phi} & (A,M)(loc.,reg.) \\ s_i & \longmapsto & (-1)^i a_i \in M_i. \end{array}$$

$$\boxed{\phi(H_j)=H_j(a_1,\dots,a_n)=h_j\in M^{d_i}}$$

- $\langle h_j\rangle$ invariant by $Z\to uZ+a$ $\quad u\in U(A); a\in A$.
- $p\in V(\langle h_1,\dots,h_{n_i}\rangle)\subset Spec(A)$ iff $B\otimes\overline{k(p)}$ is local.
- $\cap\{P\in Spec(A): \nu_P(h_i)\geq d_i\}=\beta(F_n)$ if $n\neq 0$ in k.

$$
\begin{array}{ccc}
k[s_1,\dots,s_n][Z]/\langle F_n(Z)\rangle & & A[Z]/\langle Z^n + a_1 Z^{n-1} + \cdots + a_n\rangle \\
\uparrow & & \uparrow \\
k[s_1,\dots,s_n] & \xrightarrow{\ \phi\ } & (A,M)(loc., reg.) \\
s_i & \longmapsto & (-1)^i a_i \in M_i.
\end{array}
$$

$$\boxed{\phi(H_j) = H_j(a_1,\dots,a_n) = h_j \in M^{d_i}}$$

- $\langle h_j\rangle$ invariant by $Z \to uZ + a \qquad u \in U(A); a \in A.$
- $p \in V(\langle h_1,\dots,h_{n_i}\rangle) \subset Spec(A)$ iff $B \otimes \overline{k(p)}$ is local.
- $\cap\{P \in Spec(A) : \nu_P(h_i) \geq d_i\} = \beta(F_n)$ if $n \neq 0$ in k.

$$
\begin{array}{ccc}
k[s_1,\ldots,s_n][Z]/\langle F_n(Z)\rangle & & A[Z]/\langle Z^n + a_1 Z^{n-1} + \cdots + a_n\rangle \\
\uparrow & & \uparrow \\
k[s_1,\ldots,s_n] & \xrightarrow{\phi} & (A,M)(loc., reg.) \\
s_i & \longmapsto & (-1)^i a_i \in M_i.
\end{array}
$$

$$
\boxed{\phi(H_j) = H_j(a_1,\ldots,a_n) = h_j \in M^{d_i}}
$$

- $\langle h_j\rangle$ invariant by $Z \to uZ + a$ $\quad u \in U(A); a \in A$.
- $p \in V(\langle h_1,\ldots,h_{n_i}\rangle) \subset Spec(A)$ iff $B \otimes \overline{k(p)}$ is local.
- $\cap\{P \in Spec(A) : \nu_P(h_i) \geq d_i\} = \beta(F_n)$ if $n \neq 0$ in k.

Outline

1. Introduction.
   - Multiplicity and the τ-invariant
   - Local presentation
   - Strategy in characteristic zero
   - Strategy in positive characteristic
2. Rees Algebras
   - Rees algebras and main invariants
   - Resolution of Rees algebras
   - Rees algebras and integral closure
3. Elimination
   - Multiplicity of hypersurfaces
   - Elimination and differentials
   - Absolute and relative differential structure
4. The monomial case.
   - Stage B'

TAYLOR MORPHISM

$$\begin{array}{rccc} Tay: & A[Z] & \to & A[Z,T] \\ & Z & \to & Z+T \end{array}$$

$Tay(f(Z)) = \sum_{r\in\mathbb{N}} b_r(X)T^r;$ $\boxed{\Delta^r(f(X)) = b_r(X)}$

$p \in Spec(S[Z]),$ if $\nu_p(f(Z)) \geq n$, then $\nu_p(\Delta^r(f(Z))) \geq n-r$

$$A[Z][fW^n] \subset A[Z][\Delta^{n-1}(f)W^1, \ldots, \Delta^1(f)W^{n-1}, fW^n]$$

$$\boxed{\mathrm{Sing}(A[Z][fW^n]) = \mathrm{Sing}(A[Z][\Delta^k(f)W^{n-k}, fW^n])}$$

TAYLOR MORPHISM

$$\begin{array}{rccc} Tay: & A[Z] & \to & A[Z,T] \\ & Z & \to & Z+T \end{array}$$

$Tay(f(Z)) = \sum_{r\in\mathbb{N}} b_r(X)T^r$; $\boxed{\Delta^r(f(X)) = b_r(X)}$

$p \in Spec(S[Z])$, if $\nu_p(f(Z)) \geq n$, then $\nu_p(\Delta^r(f(Z))) \geq n-r$

$$A[Z][fW^n] \subset A[Z][\Delta^{n-1}(f)W^1, \dots, \Delta^1(f)W^{n-1}, fW^n]$$

$$\boxed{\mathrm{Sing}(A[Z][fW^n]) = \mathrm{Sing}(A[Z][\Delta^k(f)W^{n-k}, fW^n])}$$

INVARIANTS AND DIFFERENTIAL OPERATORS

$$F_n(Z) = \prod(Z - Y_i) = Z^n - s_1 Z^{n-1} + \dots s_n \in k[s_1, \dots, s_n]$$

$$\begin{array}{ccc} k[Y_1, \dots, Y_n] & & k[s_1, \dots, s_n] \\ \cup & \xrightarrow{S_n} & \cup \\ k[Y_i - Y_j] & & k[H_1, \dots, H_{n_i}] \end{array}$$

$$\begin{array}{ccc} k[Z - Y_1, \dots, Z - Y_n] & & k[\Delta^r F_n, F_n] \\ \cup & \xrightarrow{S_n} & \cup \\ k[Y_i - Y_j] & & k[H_1, \dots, H_{n_i}] \end{array}$$

Each H_i is weighted hom. on the $\Delta^r F_n$ (weight $n - r$).

INVARIANTS AND DIFFERENTIAL OPERATORS

$$F_n(Z) = \prod(Z - Y_i) = Z^n - s_1 Z^{n-1} + \dots s_n \in k[s_1, \dots, s_n]$$

$$\begin{array}{ccc} k[Y_1, \dots, Y_n] & & k[s_1, \dots, s_n] \\ \cup & \xrightarrow{S_n} & \cup \\ k[Y_i - Y_j] & & k[H_1, \dots, H_{n_i}] \end{array}$$

$$\begin{array}{ccc} k[Z - Y_1, \dots, Z - Y_n] & & k[\Delta^r F_n, F_n] \\ \cup & \xrightarrow{S_n} & \cup \\ k[Y_i - Y_j] & & k[H_1, \dots, H_{n_i}] \end{array}$$

Each H_i is weighted hom. on the $\Delta^r F_n$ (weight $n - r$).

Weighted Pull-backs of algebras

- Set $k[Z-Y_1,\dots,Z-Y_n]^{S_n}=k[\Delta^r F_n, F_n]$. The weighted pull-back $k[\Delta^r F_n W^{n-r}, F_n W^n]$ is

$$A[Z][f_n W^n, \Delta^r f_n W^{n-r}].$$

- The weighted-Pull-Back of $k[H_1 W^{d_1},\dots,H_s W^{d_s}]$ is

$$A[h_1 W^{d_1},\dots,h_s W^{d_s}]\subset A[W].$$

$$\boxed{A[h_1 W^{d_1},\dots,h_s W^{d_s}]\ \subset A[Z][f_n W^n, \Delta^r f_n W^{n-r}]}$$

Definition

$A[h_1 W^{d_1},\dots,h_s W^{d_s}]$ elimination alg. of $A[Z][f_n W^n, \Delta^r f_n W^{n-r}]$.

Weighted Pull-backs of algebras

- Set $k[Z-Y_1,\dots,Z-Y_n]^{S_n}=k[\Delta^r F_n,F_n]$. The weighted pull-back $k[\Delta^r F_n W^{n-r},F_n W^n]$ is
$$A[Z][f_n W^n,\Delta^r f_n W^{n-r}].$$

- The weighted-Pull-Back of $k[H_1 W^{d_1},\dots,H_s W^{d_s}]$ is
$$A[h_1 W^{d_1},\dots,h_s W^{d_s}]\subset A[W].$$

$$\boxed{A[h_1 W^{d_1},\dots,h_s W^{d_s}]\ \subset A[Z][f_n W^n,\Delta^r f_n W^{n-r}]}$$

Definition

$A[h_1 W^{d_1},\dots,h_s W^{d_s}]$ elimination alg. of $A[Z][f_n W^n,\Delta^r f_n W^{n-r}]$.

Weighted Pull-backs of algebras

- Set $k[Z-Y_1,\dots,Z-Y_n]^{S_n}=k[\Delta^r F_n, F_n]$. The weighted pull-back $k[\Delta^r F_n W^{n-r}, F_n W^n]$ is
$$A[Z][f_n W^n, \Delta^r f_n W^{n-r}].$$
- The weighted-Pull-Back of $k[H_1 W^{d_1},\dots,H_s W^{d_s}]$ is
$$A[h_1 W^{d_1},\dots,h_s W^{d_s}]\subset A[W].$$

$$\boxed{A[h_1 W^{d_1},\dots,h_s W^{d_s}]\ \subset A[Z][f_n W^n, \Delta^r f_n W^{n-r}]}$$

Definition

$A[h_1 W^{d_1},\dots,h_s W^{d_s}]$ elimination alg. of $A[Z][f_n W^n, \Delta^r f_n W^{n-r}]$.

OUTLINE

1. INTRODUCTION.
   - Multiplicity and the τ-invariant
   - Local presentation
   - Strategy in characteristic zero
   - Strategy in positive characteristic
2. REES ALGEBRAS
   - Rees algebras and main invariants
   - Resolution of Rees algebras
   - Rees algebras and integral closure
3. ELIMINATION
   - Multiplicity of hypersurfaces
   - Elimination and differentials
   - Absolute and relative differential structure
4. THE MONOMIAL CASE.
   - Stage B'

$$\beta_{d,d-\tau} : V^{(d)} \longrightarrow V^{(d-\tau)} \text{ smooth } \tau \leq d$$

$Diff^{r}_{\beta_{d,d-\tau}}$ sheaf of relative differential operators.

Definition

$\mathcal{G} = \oplus_{n\geq 0} I_n W^n \subset \mathcal{O}_{V^{(d)}}[W]$ is a relative Diff-algebra if

$$Diff^{r}_{\beta_{d,d-\tau}}(I_n) \subset I_{n-r}.$$

Properties:

- There is a smallest extension $\mathcal{G} \subset G_{\beta_{d,d-\tau}}(\mathcal{G})$, where $G_{\beta_{d,d-\tau}}(\mathcal{G})$ is a relative (or absolute) Diff-algebra.
- $\text{Sing}(\mathcal{G}) = \text{Sing}(G_{\beta_{d,d-\tau}}(\mathcal{G}))$.
- $G_{\beta_{d,d-\tau}}$ compatible with integral closure.

$$\beta_{d,d-\tau} : V^{(d)} \longrightarrow V^{(d-\tau)} \text{ smooth } \tau \leq d$$

$Diff^r_{\beta_{d,d-\tau}}$ sheaf of relative differential operators.

DEFINITION

$\mathcal{G} = \oplus_{n\geq 0} I_n W^n \subset \mathcal{O}_{V^{(d)}}[W]$ is a relative Diff-algebra if

$$Diff^r_{\beta_{d,d-\tau}}(I_n) \subset I_{n-r}.$$

Properties:

- There is a smallest extension $\mathcal{G} \subset G_{\beta_{d,d-\tau}}(\mathcal{G})$, where $G_{\beta_{d,d-\tau}}(\mathcal{G})$ is a relative (or absolute) Diff-algebra.
- $\text{Sing}(\mathcal{G}) = \text{Sing}(G_{\beta_{d,d-\tau}}(\mathcal{G}))$.
- $G_{\beta_{d,d-\tau}}$ compatible with integral closure.

$$\beta_{d,d-\tau} : V^{(d)} \longrightarrow V^{(d-\tau)} \text{ smooth } \tau \leq d$$

$Diff^r_{\beta_{d,d-\tau}}$ sheaf of relative differential operators.

Definition

$\mathcal{G} = \oplus_{n\geq 0} I_n W^n \subset \mathcal{O}_{V^{(d)}}[W]$ is a relative Diff-algebra if

$$Diff^r_{\beta_{d,d-\tau}}(I_n) \subset I_{n-r}.$$

Properties:

- There is a smallest extension $\mathcal{G} \subset G_{\beta_{d,d-\tau}}(\mathcal{G})$, where $G_{\beta_{d,d-\tau}}(\mathcal{G})$ is a relative (or absolute) Diff-algebra.
- $\text{Sing}(\mathcal{G}) = \text{Sing}(G_{\beta_{d,d-\tau}}(\mathcal{G}))$.
- $G_{\beta_{d,d-\tau}}$ compatible with integral closure.

$$\beta_{d,d-\tau} : V^{(d)} \longrightarrow V^{(d-\tau)} \text{ smooth } \tau \leq d$$

$Diff^r_{\beta_{d,d-\tau}}$ sheaf of relative differential operators.

Definition

$\mathcal{G} = \oplus_{n\geq 0} I_n W^n \subset \mathcal{O}_{V^{(d)}}[W]$ is a relative Diff-algebra if

$$Diff^r_{\beta_{d,d-\tau}}(I_n) \subset I_{n-r}.$$

Properties:

- There is a smallest extension $\mathcal{G} \subset G_{\beta_{d,d-\tau}}(\mathcal{G})$, where $G_{\beta_{d,d-\tau}}(\mathcal{G})$ is a relative (or absolute) Diff-algebra.
- $\text{Sing}(\mathcal{G}) = \text{Sing}(G_{\beta_{d,d-\tau}}(\mathcal{G}))$.
- $G_{\beta_{d,d-\tau}}$ compatible with integral closure.

Relative differentials and Elimination

Given $\mathcal{G}$ and $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$ smooth and generic.
Assume,

$$\mathcal{G} = \mathcal{O}_{V^{(d)}}[f_n W^n, \Delta^\alpha(f_n) W^{n-\alpha}]_{1 \le \alpha \le n}$$

then $\mathcal{G}$ is a relative Diff-algebra and there is an elimination algebra,

$$\mathcal{R}_{\mathcal{G},\beta} \subset \mathcal{O}_{V^{(d-1)}}[W].$$

V smooth $|_k$ $\mathcal{G} = \oplus I_k W^k \subset \mathcal{O}_V[W]$ $ord : \mathrm{Sing}(\mathcal{G}) \to \mathbb{Q}$

Assume: $ord(x) = 1$ for any $x \in \mathrm{Sing}(\mathcal{G})$.

At each $x \in \mathrm{Sing}(\mathcal{G})$:

- $C_{\mathcal{G},x} \subset \mathbb{T}_{V,x}$
- Linear subspace $L_{\mathcal{G},x} \subset C_{\mathcal{G},x}$

Definition (Hironaka)

$\tau(x)$= codim $L_{\mathcal{G},x}$ in $\mathbb{T}_{V,x}$.

Definition

$\mathcal{G}$ is of codimensional type $\geq e$ if $\tau(x) \geq e$ for all $x \in \mathrm{Sing}(\mathcal{G})$

V smooth $|_k$ $\quad \mathcal{G} = \oplus I_k W^k \subset \mathcal{O}_V[W] \quad ord : \mathrm{Sing}(\mathcal{G}) \longrightarrow \mathbb{Q}$

Assume: $ord(x) = 1$ for any $x \in \mathrm{Sing}(\mathcal{G})$.

At each $x \in \mathrm{Sing}(\mathcal{G})$:

- $C_{\mathcal{G},x} \subset \mathbb{T}_{V,x}$
- Linear subspace $L_{\mathcal{G},x} \subset C_{\mathcal{G},x}$

Definition (Hironaka)

$\tau(x)$= codim $L_{\mathcal{G},x}$ in $\mathbb{T}_{V,x}$.

Definition

$\mathcal{G}$ is of codimensional type $\geq e$ if $\tau(x) \geq e$ for all $x \in \mathrm{Sing}(\mathcal{G})$

V smooth $|_k$ $\quad \mathcal{G} = \oplus I_k W^k \subset \mathcal{O}_V[W]$ $\quad ord : \text{Sing}(\mathcal{G}) \to \mathbb{Q}$
Assume: $ord(x) = 1$ for any $x \in \text{Sing}(\mathcal{G})$.
At each $x \in \text{Sing}(\mathcal{G})$:

- $C_{\mathcal{G},x} \subset \mathbb{T}_{V,x}$
- Linear subspace $L_{\mathcal{G},x} \subset C_{\mathcal{G},x}$

Definition (Hironaka)

$\tau(x)$= codim $L_{\mathcal{G},x}$ in $\mathbb{T}_{V,x}$.

Definition

$\mathcal{G}$ is of codimensional type $\geq e$ if $\tau(x) \geq e$ for all $x \in \text{Sing}(\mathcal{G})$

$V^{(d)}$ smooth $|_k$ $\qquad \mathcal{G} = \oplus I_k W^k \subset \mathcal{O}_{V^{(d)}}$
Assume $ord(x) = 1$, and $\mathcal{G}$ is of codimensional type $\geq \tau$.

Theorem

If $\mathcal{G}$ is rel. Diff-alg. for $\beta_{d,d-\tau} : V^{(d)} \to V^{(d-\tau)}$ generic, then:

- *(Local Presentation)*
  $\mathcal{G} \sim \mathcal{O}_{V^{(d)}}[f_{n_i} W^{n_i}, \Delta^{\alpha_i} f_{n_i} W^{n_i - |\alpha_i|}] \odot \beta^*_{d,d-e}(\mathcal{G}^{(d-\tau)})$.

Local presentation

$(V^{(d)}, x) \xrightarrow{\beta} (V^{(d-\tau)}, x_\tau)$ a composition of smooth morphisms

$$(V^{(d)}, x) \longrightarrow (V^{(d-1)}, x_1) \longrightarrow \dots \longrightarrow (V^{(d-\tau)}, x_\tau)$$

A local presentation of X (at x) is defined by

(1) Positive integers $0 \leq e_1 \leq e_2 \leq \cdots \leq e_\tau$.

(2) Monic polynomials,

$$f_1^{(p^{e_1})}(z_1) = z_1^{p^{e_1}} + a_1^{(1)} z_1^{p^{e_1}-1} + \cdots + a_{p^{e_1}}^{(1)} \in \mathcal{O}_{V^{(d-1)}}[z_1]$$

$$\vdots$$

$$f_\tau^{(p^{e_\tau})}(z_\tau) = z_\tau^{p^{e_\tau}} + a_1^{(\tau)} z_\tau^{p^{e_\tau}-1} + \cdots + a_{p^{e_\tau}}^{(\tau)} \in \mathcal{O}_{V^{(d-\tau)}}[z_\tau].$$

(3) $I^{(s)}$: an ideal in $\mathcal{O}_{V^{(d-\tau)}}$ and a positive integer s.

$$F_b = \bigcap_{i=1}^{\tau} \{x \in V^{(d)} \mid \nu_x(f_i^{(p^{e_i})}) \geq p^{e_i}\} \cap \{x \in V^{(d)} \mid \nu_x(\beta^*(I^{(s)})) \geq s\}$$

$V^{(d)}$ smooth $|_k$ $\qquad \mathcal{G} = \oplus I_k W^k \subset \mathcal{O}_{V^{(d)}}$
Assume $ord(x) = 1$, and $\mathcal{G}$ is of codimensional type $\geq e$.

Theorem

If $\mathcal{G}$ is rel. Diff-alg. for $\beta_{d,d-\tau} : V^{(d)} \to V^{(d-\tau)}$ generic, then:

- *(Local Presentation)*
  $\mathcal{G} \sim \mathcal{O}_{V^{(d)}}[f_{n_i} W^{n_i}, \Delta^{\alpha_i} f_{n_i} W^{n_i - |\alpha_i|}] \odot \beta^*_{d,d-\tau}(\mathcal{G}^{(d-\tau)})$
  $\mathcal{G}^{(d-\tau)} \subset \mathcal{O}_{V^{(d-e)}}[W]$ *(Elimination algebra).*
- $\mathrm{Sing}(\mathcal{G}) = \beta_{d,d-\tau}(\mathrm{Sing}(\mathcal{G})) \subset \mathrm{Sing}(\mathcal{G}^{(d-\tau)})$
- *The natural restriction ord* : $\mathrm{Sing}(\mathcal{G}^{(d-\tau)}) \to \mathbb{Q}$ *to* $\mathrm{Sing}(\mathcal{G})$ *is independent of* $\beta_{d,d-\tau}$.

$V^{(d)}$ smooth $|_k$ $\qquad \mathcal{G} = \oplus I_k W^k \subset \mathcal{O}_{V^{(d)}}$
Assume $ord(x) = 1$, and $\mathcal{G}$ is of codimensional type $\geq e$.

THEOREM

If $\mathcal{G}$ is rel. Diff-alg. for $\beta_{d,d-\tau} : V^{(d)} \to V^{(d-\tau)}$ generic, then:

- *(Local Presentation)*
  $\mathcal{G} \sim \mathcal{O}_{V^{(d)}}[f_{n_i} W^{n_i}, \Delta^{\alpha_i} f_{n_i} W^{n_i - |\alpha_i|}] \odot \beta^*_{d,d-\tau}(\mathcal{G}^{(d-\tau)})$
  $\mathcal{G}^{(d-\tau)} \subset \mathcal{O}_{V^{(d-e)}}[W]$ *(Elimination algebra).*
- $\mathrm{Sing}(\mathcal{G}) = \beta_{d,d-\tau}(\mathrm{Sing}(\mathcal{G})) \subset \mathrm{Sing}(\mathcal{G}^{(d-\tau)})$
- *The natural restriction $ord : \mathrm{Sing}(\mathcal{G}^{(d-\tau)}) \to \mathbb{Q}$ to $\mathrm{Sing}(\mathcal{G})$ is independent of $\beta_{d,d-\tau}$.*

Theorem

- *If $\mathcal{G}$ is absolute diff. algebra:* $\text{Sing}(\mathcal{G}) = \text{Sing}(\mathcal{G}^{(d-\tau)})$
- *(Stability) If $Y \subset \text{Sing}(\mathcal{G})$ smooth, $Y \cong \pi(Y)$ and*

$$\begin{array}{ccc} (V^{(d)}, \mathcal{G}) & \longleftarrow & (V_1^{(d)}, \mathcal{G}_1) \\ \downarrow \beta & & \downarrow \beta_1 \\ (V^{(d-\tau)}, \mathcal{G}^{(d-\tau)}) & \longleftarrow & (V_1^{(d-\tau)}, \mathcal{G}_1^{(d-\tau)}) \end{array}$$

And $\mathcal{G}_1$ is relative differential.

THEOREM

- *If $\mathcal{G}$ is absolute diff. algebra:* $\mathrm{Sing}(\mathcal{G}) = \mathrm{Sing}(\mathcal{G}^{(d-\tau)})$
- *(Stability) If $Y \subset \mathrm{Sing}(\mathcal{G})$ smooth, $Y \cong \pi(Y)$ and*

$$\begin{array}{ccc} (V^{(d)}, \mathcal{G}) & \longleftarrow & (V_1^{(d)}, \mathcal{G}_1) \\ \downarrow \beta & & \downarrow \beta_1 \\ (V^{(d-\tau)}, \mathcal{G}^{(d-\tau)}) & \longleftarrow & (V_1^{(d-\tau)}, \mathcal{G}_1^{(d-\tau)}) \end{array}$$

And $\mathcal{G}_1$ is relative differential.

Theorem (Stage A)

There is a well defined sequence of permissible transformations:

$$\begin{array}{ccccc} (V^{(d)}, \mathcal{G}) & \longleftarrow & \cdots & \longleftarrow & (V_r^{(d)}, \mathcal{G}_r) \\ \downarrow \beta & & & & \downarrow \beta_r \\ (V^{(d-e)}, \mathcal{G}^{(d-e)}) & \longleftarrow & \cdots & \longleftarrow & (V_r^{(d-e)}, \mathcal{G}_r^{(d-e)}) \end{array}$$

such that $\mathcal{G}_r^{(d-e)}$ is monomial:

$$\mathcal{G}_r^{(d-e)} \sim \mathcal{O}_{V_r^{(e)}}[(I(H_1)^{\alpha_1} \cdots I(H_r)^{\alpha_r})W^s]$$

Outline

1. Introduction.
   - Multiplicity and the τ-invariant
   - Local presentation
   - Strategy in characteristic zero
   - Strategy in positive characteristic
2. Rees Algebras
   - Rees algebras and main invariants
   - Resolution of Rees algebras
   - Rees algebras and integral closure
3. Elimination
   - Multiplicity of hypersurfaces
   - Elimination and differentials
   - Absolute and relative differential structure
4. The monomial case.
   - Stage B'

V smooth, $E = \{H_1, \dots, H_r\}$ smooth hypersurfaces with n.c.

- Monomial ideal supported on E:
$$\mathcal{M} = I(H_1)^{\alpha_1} \cdot I(H_2)^{\alpha_2} \cdots I(H_r)^{\alpha_r}.$$

- Monomial algebra:
Rees algebra $\mathcal{O}_V[\mathcal{M}W^s]$ for $s \in \mathbb{Z}_{>0}$.

DEFINITION

$\mathcal{G} \subset \mathcal{O}_{V^{(d)}}[W]$, $x \in \mathrm{Sing}(\mathcal{G})$, $\tau_{\mathcal{G},x} \geq 1$, $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$ generic. $\mathcal{O}_{V^{(d-1)}}[\mathcal{M}W^s]$ has monomial contact with $\mathcal{G}$ if locally there is a smooth section defined by $z \in \mathcal{O}_{V^{(d)},x}$ so that
$$\mathcal{G} \subset \langle z \rangle W \odot \mathcal{M}W^s.$$

V smooth, $E = \{H_1, \dots, H_r\}$ smooth hypersurfaces with n.c.

- Monomial ideal supported on E:
$$\mathcal{M} = I(H_1)^{\alpha_1} \cdot I(H_2)^{\alpha_2} \cdots I(H_r)^{\alpha_r}.$$
- Monomial algebra:
Rees algebra $\mathcal{O}_V[\mathcal{M}W^s]$ for $s \in \mathbb{Z}_{>0}$.

DEFINITION

$\mathcal{G} \subset \mathcal{O}_{V^{(d)}}[W]$, $x \in \mathrm{Sing}(\mathcal{G})$, $\tau_{\mathcal{G},x} \geq 1$, $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$ generic. $\mathcal{O}_{V^{(d-1)}}[\mathcal{M}W^s]$ has monomial contact with $\mathcal{G}$ if locally there is a smooth section defined by $z \in \mathcal{O}_{V^{(d)},x}$ so that
$$\mathcal{G} \subset \langle z \rangle W \odot \mathcal{M}W^s.$$

MONOMIAL CONTACT IN LOCAL COORDINATES

$$\mathcal{G} \subset \langle z \rangle W \odot \mathcal{M} W^s.$$

There is a r.s.p. $\{y_1, \ldots, y_{d-1}\}$ in $\mathcal{O}_{V^{(d-1)},\beta(x)}$, then z is such that:

(I) $\{z, y_1, \ldots, y_{d-1}\}$ is a r.s.p. at $\mathcal{O}_{V^{(d)},x}$.

(II) $\mathcal{G} \subset \langle z \rangle W \odot \langle y_1^{h_1} \ldots y_j^{h_j} \rangle W^s$ (locally at x), where $y_1^{h_1} \ldots y_j^{h_j}$ generates the monomial ideal $\mathcal{M}$ at $\mathcal{O}_{V^{(d-1)},\beta(x)}$.

Set

$$\begin{array}{ccccccc}
\mathcal{G} & & \mathcal{G}_1 & & & & \mathcal{G}_r \\
V^{(d)} & \xleftarrow{\pi_1} & V_1^{(d)} & \longleftarrow & \cdots & \xleftarrow{\pi_r} & V_r^{(d)} \\
\downarrow \beta & & \downarrow \beta_1 & & & & \downarrow \beta_r \\
V^{(d-1)} & \xleftarrow{\pi'_1} & V_1^{(d-1)} & \longleftarrow & \cdots & \xleftarrow{\pi'_r} & V_r^{(d-1)} \\
\mathcal{R}_{\mathcal{G},\beta} & & (\mathcal{R}_{\mathcal{G},\beta})_1 & & & & (\mathcal{R}_{\mathcal{G},\beta})_r
\end{array}$$

so $(\mathcal{R}_{\mathcal{G},\beta})_r = I(H_1)^{\alpha_1} \dots I(H_r)^{\alpha_r} W^s = \mathcal{N} W^s$ (monomial algebra).

THEOREM (STAGE B')

There is a naturally defined sheaf of monomial ideals $\mathcal{M}$ s.t.

(i) $\mathcal{M} = I(H_1)^{h_1} \dots I(H_r)^{h_r}$ and $0 \leq h_i \leq \alpha_i$ for $i = 1, \dots, r$.

(ii) $\mathcal{M} W^s$ has monomial contact relative to β_r with $\mathcal{G}_r$ locally at any closed point in $\mathrm{Sing}(\mathcal{G}_r)$, i.e.,

$$\mathcal{G}_r \subset \langle z \rangle W \odot \mathcal{M} W^s.$$

Set

$$\begin{array}{ccccccc}
\mathcal{G} & & \mathcal{G}_1 & & & & \mathcal{G}_r \\
V^{(d)} & \xleftarrow{\pi_1} & V_1^{(d)} & \longleftarrow & \cdots & \xleftarrow{\pi_r} & V_r^{(d)} \\
\downarrow \beta & & \downarrow \beta_1 & & & & \downarrow \beta_r \\
V^{(d-1)} & \xleftarrow{\pi'_1} & V_1^{(d-1)} & \longleftarrow & \cdots & \xleftarrow{\pi'_r} & V_r^{(d-1)} \\
\mathcal{R}_{\mathcal{G},\beta} & & (\mathcal{R}_{\mathcal{G},\beta})_1 & & & & (\mathcal{R}_{\mathcal{G},\beta})_r
\end{array}$$

so $(\mathcal{R}_{\mathcal{G},\beta})_r = I(H_1)^{\alpha_1} \dots I(H_r)^{\alpha_r} W^s = \mathcal{N} W^s$ (monomial algebra).

Theorem (Stage B')

There is a naturally defined sheaf of monomial ideals $\mathcal{M}$ s.t.

(i) $\mathcal{M} = I(H_1)^{h_1} \dots I(H_r)^{h_r}$ and $0 \leq h_i \leq \alpha_i$ for $i = 1, \dots, r$.

(ii) $\mathcal{M} W^s$ has monomial contact relative to β_r with $\mathcal{G}_r$ locally at any closed point in $\mathrm{Sing}(\mathcal{G}_r)$, i.e.,

$$\mathcal{G}_r \subset \langle z \rangle W \odot \mathcal{M} W^s.$$

$$\mathcal{G}_r \subset \langle z \rangle W \odot \mathcal{M} W^s.$$

implies that

$$\mathrm{Sing}(\langle z \rangle W \odot \mathcal{M} W^s) \subset \mathrm{Sing}(\mathcal{G}_r),$$

A resolution of $\langle z \rangle W \odot \mathcal{M} W^s$, say

$$V_r^{(d)} \longleftarrow V_{r+1}^{(d)} \longleftarrow \cdots \longleftarrow V_R^{(d)}$$

induces a sequence of transformations of $\mathcal{G}_r$, say

$$\begin{array}{ccccccc} \mathcal{G}_r & & \mathcal{G}_{r+1} & & & & \mathcal{G}_R \\ V_r^{(d)} & \longleftarrow & V_{r+1}^{(d)} & \longleftarrow & \cdots & \longleftarrow & V_R^{(d)} \end{array}$$

REFERENCES

- A. Benito and O. Villamayor, 'Monoidal transformations of singularities in positive characteristic' http://arXiv.org/abs/0811.4148 26 November 2008.
- A. Bravo and O. Villamayor, 'Hypersurface singularities in positive characteristic and stratification of singular locus'. http://arXiv.org/abs/0807.4308 27 July 2008.
- O. Villamayor, 'Hypersurface singularities in positive characteristic.' *Advances in Mathematics* 213 (2007) 687-733.
- O. Villamayor U. 'Elimination with apllications to singularities in positive characteristic'. *Publications of RIMS, Kyoto University.* Vol 44, No. 2, 2008.